Conservative median algebras and semilattices

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Abstract We characterize conservative median algebras and semilattices by means of forbidden substructures and by providing their representation as chains. Moreover, using a dual equivalence between median algebras and certain topological structures, we obtain descriptions of the median-preserving mappings between products of finitely many chains.

1 Introduction and preliminaries

In this paper we are interested in certain algebraic structures called median algebras. A *median algebra* is a ternary algebra $\mathbf{A} = \langle A, \mathbf{m} \rangle$ that satisfies the following equations

$$\mathbf{m}(x, x, y) = x,$$
$$\mathbf{m}(x, y, z) = \mathbf{m}(y, x, z) = \mathbf{m}(y, z, x),$$
$$\mathbf{m}(\mathbf{m}(x, y, z), t, u) = \mathbf{m}(x, \mathbf{m}(y, t, u), \mathbf{m}(z, t, u)).$$

Median algebras have been investigated by several authors (see [3,9] for early references on median algebras and see [2,10] for some surveys). For instance, it is shown in [14] that for each element a of a median algebra $\mathbf{A}$, the relation $\leq_a$ defined on A by

$$x \leq_a y \quad \Longleftrightarrow \quad \mathbf{m}(a, x, y) = x$$

is a $\wedge$-semilattice order with bottom element a. The associated operation $\wedge$ is defined by $x \wedge y = \mathbf{m}(a, x, y)$. Semilattices constructed in this way are called *median semilattices*, and can be characterized as follows.

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Theorem 1 ((3.1) in [14]) *A $\wedge$-semilattice is a median semilattice if and only if each of its principal ideal is a distributive lattice, and any three elements have a join whenever each pair of them is bounded above.*

In particular, any distributive lattice is a median semilattice. According to Theorem 1, we can define a ternary operation $\mathbf{m}_{\leq}$ called the *median operation of* $\leq$ on every median semilattice $\langle A, \leq \rangle$ by setting

$$\mathbf{m}_{\leq}(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (z \wedge y), \tag{1.1}$$

for every $x, y, z \in A$. It can be proved [1, Lemma 3 (6)] that $\mathbf{m} = \mathbf{m}_{\leq_a}$ for every median algebra $\mathbf{A} = \langle A, \mathbf{m} \rangle$, and every $a \in A$.

Here, we are particularly interested in median algebras $\mathbf{A}$ that are *conservative*, i.e., that satisfy

$$\mathbf{m}(x, y, z) \in \{x, y, z\}, \qquad x, y, z \in A. \tag{1.2}$$

Although condition (1.2) appears in §11 of [13], to the best of the authors' knowledge, the present work constitutes the first attempt of a systematic study of conservative median algebras. A median semilattice $\langle A, \leq \rangle$ whose median operation $\mathbf{m}_{\leq}$ satisfies (1.2) is called a *conservative median semilattice*. Note that a median algebra is conservative if and only if each of its subsets is a median subalgebra. Moreover, if $\mathbf{L}$ is a chain, then $\mathbf{m_L}$ satisfies (1.2); however the converse is not true. This fact was observed in §11 of [13], which presents the median operation of the four element Boolean algebra as a counter-example.

In this paper, we investigate conservative median algebras and homomorphisms between them, i.e., mappings $f \colon \mathbf{A} \to \mathbf{B}$ that are solutions of the functional equation

$$f(\mathbf{m}(x, y, z)) = \mathbf{m}(f(x), f(y), f(z)). \tag{1.3}$$

We describe such homomorphisms between conservative median algebras $\mathbf{A}$ and $\mathbf{B}$. To do so, we present a description of conservative median algebras and semilattices in terms of forbidden substructures (in complete analogy with BIRKHOFF's characterization of distributive lattices with M_5 and N_5 as forbidden substructures), and that leads to a representation of conservative median algebras (with at least five elements) as median algebras of chains. In fact, the only conservative median algebra that is not representable as a chain is the median algebra of the four element Boolean algebra.

Throughout the paper we employ the following notation. For each positive integer n, we set $[n] = \{1, \ldots, n\}$. Algebras and topological structures are denoted by bold roman capital letters $\mathbf{A}, \mathbf{B}, \mathbf{X}, \mathbf{Y}, \ldots$ and their universes by italic roman capital letters $A, B, X, Y, \ldots$ To simplify our presentation, we will keep the introduction of background to a minimum, and we will assume that the reader is familiar with the theory of lattices and ordered sets. We refer the reader to [6,8] for further background.

2 Characterizations of conservative median algebras

According to Theorem 1, a semilattice can fail to be a conservative median semilattice in tree different ways. First, it can contain a principal ideal which is not a distributive lattice, as in Fig. 1(a) that depicts the bounded lattice N_5 that is not distributive. Second, it can contain three elements b, c, d that do not have a join even though every pair of them is bounded above, such as in Fig. 1(e). Finally, it can be a median semilattice
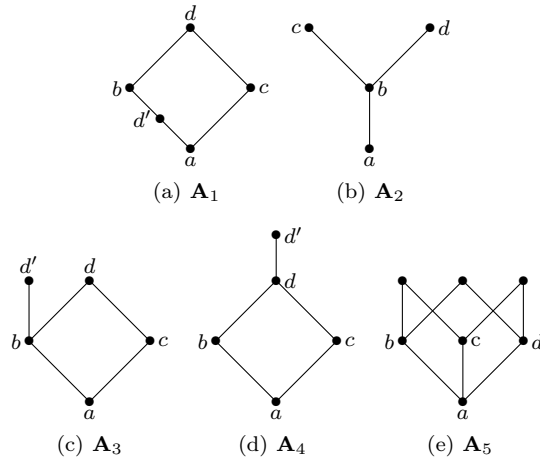
Fig. 1 Examples of ∧-semilattices that are not conservative.

that is not conservative, like $\mathbf{A}_2$ in Fig. 1(b) in which $\mathbf{m}_\leq(a, c, d) = b$, and like in Fig. 1(c)–1(d) in which the semilattices contain a copy of $\mathbf{A}_2$. Hence, we have proved the following lemma.

Lemma 1 *The partially ordered sets $\mathbf{A}_1, \ldots, \mathbf{A}_5$ depicted in Fig. 1 are not conservative median semilattices.*

The following theorem provides a description of conservative median algebras and semilattices in terms of forbidden substructures.

Lemma 2 *The variety of median algebras satisfies the following equations.*

$$\mathbf{m}(x, y, z) = \mathbf{m}\big(\mathbf{m}(\mathbf{m}(x, y, z), x, t), \mathbf{m}(\mathbf{m}(x, y, z), z, t), \mathbf{m}(\mathbf{m}(x, y, z), y, t)\big). \quad (2.1)$$

$$\mathbf{m}(x, y, \mathbf{m}(x, y, z)) = \mathbf{m}(x, y, z), \quad (2.2)$$

Proof Every median algebra is isomorphic to a subalgebra of a power of the median algebra $\mathbf{2} = \langle\{0, 1\}, \mathbf{m}\rangle$, where $\mathbf{m}$ is the majority ternary operation on $\{0, 1\}$ (see [2, Theorem 1.5]). Moreover, equations (2.2) and (2.1) are satisfied in $\mathbf{2}$.

Theorem 2 *For every median algebra $\mathbf{A}$, the following conditions are equivalent.*

1. *$\mathbf{A}$ is conservative.*
2. *$\mathbf{A}$ does not contain the median algebra $\mathbf{A}_2$ depicted in Fig. 1(b) as a subalgebra.*
3. *For every $a \in A$, the median semilattice $\langle A, \leq_a \rangle$ does not contain a copy of the poset depicted in Fig. 1(b).*

Proof First note that any median semilattice with at most four elements is conservative, with the exception of the poset depicted in Fig. 1(b). Hence, we assume that $|A| \geq 5$.

(2) $\iff$ (3): Follows directly from the definition of $\leq_a$.

(1) $\implies$ (3): Follows from Lemma 1.

(3) $\implies$ (1): Suppose that $\mathbf{A}$ is not conservative, that is, there are $a, b, c, d \in A$ such that $d := \mathbf{m}(a, b, c) \notin \{a, b, c\}$. Clearly, a, b and c must be pairwise distinct. By (2.2), a and b are $\leq_c$-incomparable, and $d <_c a$ and $d <_c b$. Moreover, $c <_c d$ and thus $\langle\{a, b, c, d\}, \leq_c\rangle$ is a copy of $\mathbf{A}_2$ in $\langle A, \leq_c\rangle$.

Let $\mathbf{C}_0 = \langle C_0, \leq_0, c_0 \rangle$ and $\mathbf{C}_1 = \langle C_1, \leq_1, c_1 \rangle$ be chains with bottom elements c_0 and c_1. The $\perp$-*coalesced sum* $\mathbf{C}_0 \perp \mathbf{C}_1$ of $\mathbf{C}_0$ and $\mathbf{C}_1$ is the poset obtained by amalgamating c_0 and c_1 in the disjoint union of C_0 and C_1. Formally,

$$\mathbf{C}_0 \perp \mathbf{C}_1 = \langle C_0 \sqcup C_1 \; / \equiv, \; \leq \rangle,$$

where $\sqcup$ is the disjoint union, where $\equiv$ is the equivalence generated by $\{(c_0, c_1)\}$ and where $\leq$ is defined by

$$x/\equiv \; \leq \; y/\equiv \;\; \Longleftrightarrow \;\; (x \in \{c_0, c_1\} \text{ or } x \leq_0 y \text{ or } x \leq_1 y).$$

Theorem 3 below provides descriptions of conservative median algebras and semilattices by means of representations by chains. Its proof requires the next technical result.

Lemma 3 *For every median algebra* $\mathbf{A}$ *with* $|A| \geq 5$, *the following conditions are equivalent.*

1. $\mathbf{A}$ *is conservative*
2. *There is an* $a \in A$ *and lower bounded chains* $\mathbf{C}_0$ *and* $\mathbf{C}_1$ *such that* $\langle A, \leq_a \rangle$ *is isomorphic to* $\mathbf{C}_0 \perp \mathbf{C}_1$.
3. *For every* $a \in A$, *there are lower bounded chains* $\mathbf{C}_0$ *and* $\mathbf{C}_1$ *such that* $\langle A, \leq_a \rangle$ *is isomorphic to* $\mathbf{C}_0 \perp \mathbf{C}_1$.

Proof $(1) \implies (3)$: Let $a \in A$. First, suppose that for every $b, c \in A \setminus \{a\}$ we have $\mathbf{m}(b, c, a) \neq a$. Since $\mathbf{A}$ is conservative, for every $b, c \in A$, either $b \leq_a c$ or $c \leq_a b$. Thus $\leq_a$ is a chain with bottom element a, and we can choose $\mathbf{C}_1 = \langle A, \leq_a, a \rangle$ and $\mathbf{C}_2 = \langle \{a\}, \leq_a, a \rangle$.

Suppose now that there are $b, c \in A \setminus \{a\}$ such that $\mathbf{m}(b, c, a) = a$, that is, $b \wedge c = a$. We show that for every $a \in A$,

$$d \neq a \implies \big(\mathbf{m}(b, d, a) \neq a \quad \text{or} \quad \mathbf{m}(c, d, a) \neq a\big). \tag{2.3}$$

For the sake of a contradiction, suppose that $\mathbf{m}(b, d, a) = a$ and $\mathbf{m}(c, d, a) = a$ for some $d \neq a$. By equation (2.1), we have

$$\mathbf{m}(b, c, d) = \mathbf{m}\big(\mathbf{m}(\mathbf{m}(b, c, d), b, a), \mathbf{m}(\mathbf{m}(b, c, d), d, a), \mathbf{m}(\mathbf{m}(b, c, d), c, a)\big). \tag{2.4}$$

Assume that $\mathbf{m}(b, c, d) = b$. Then (2.4) is equivalent to

$$b = \mathbf{m}(b, \mathbf{m}(b, d, a), \mathbf{m}(b, c, a)) = a,$$

which yields the desired contradiction. By symmetry, we derive the same contradiction in the case $\mathbf{m}(b, c, d) \in \{c, d\}$.

We now prove that for every $a \in A$,

$$d \neq a \implies \big(\mathbf{m}(b, d, a) = a \quad \text{or} \quad \mathbf{m}(c, d, a) = a\big). \tag{2.5}$$

For the sake of a contradiction, suppose that $\mathbf{m}(b, d, a) \neq a$ and $\mathbf{m}(c, d, a) \neq a$ for some $d \neq a$. Since $\mathbf{m}(b, c, a) = a$ we have that $d \notin \{b, c\}$.

If $\mathbf{m}(b, d, a) = d$ and $\mathbf{m}(c, d, a) = c$, then $c \leq_a d \leq_a b$ which contradicts $b \wedge c = a$. Similarly, if $\mathbf{m}(b, d, a) = d$ and $\mathbf{m}(c, d, a) = d$, then $d \leq_a b$ and $d \leq_a c$ which also contradicts $b \wedge c = a$. The case $\mathbf{m}(b, d, a) = b$ and $\mathbf{m}(c, d, a) = d$ leads to a similar contradiction.

Hence $\mathbf{m}(b, d, a) = b$ and $\mathbf{m}(c, d, a) = c$, and the $\leq_a$-median semilattice arising from the subalgebra $\mathbf{B} = \{a, b, c, d\}$ of $\mathbf{A}$ is the median semilattice associated with the four element Boolean algebra. Let $d' \in A \setminus \{a, b, c, d\}$. By (2.3) and symmetry we may assume that $\mathbf{m}(b, d', a) \in \{b, d'\}$. First, suppose that $\mathbf{m}(b, d', a) = d'$. Then $\langle\{a, b, c, d, d'\}, \leq_a\rangle$ is N_5 (Fig. 1(a)) which is not a median semilattice. Suppose then that $\mathbf{m}(b, d', a) = b$. In this case, the restriction of $\leq_a$ to $\{a, b, c, d, d'\}$ is depicted in Fig. 1(c) or 1(d), which contradicts Proposition 1, and the proof of (2.5) is thus complete.

Now, let $C_0 = \{d \in A \mid (b, d, a) \neq a\}$, $C_1 = \{d \in A \mid (c, d, a) \neq a\}$ and let $\mathbf{C}_0 = \langle C_0, \leq_a, a\rangle$ and $\mathbf{C}_1 = \langle C_1, \leq_a, a\rangle$. It follows from (2.3) and (2.5) that $\langle \mathbf{A}, \leq_a\rangle$ is isomorphic to $\mathbf{C}_0 \perp \mathbf{C}_1$.

(3) $\implies$ (2): Trivial.

(2) $\implies$ (1): Let $b, c, d \in \mathbf{C}_0 \perp \mathbf{C}_1$. If $b, c, d \in C_i$ for some $i \in \{0, 1\}$ then $\mathbf{m}(b, c, d) \in \{b, c, d\}$. Otherwise, if $b, c \in C_i$ and $d \notin C_i$, then $\mathbf{m}(b, c, d) \in \{b, c\}$.

Theorem 3 *Let $\mathbf{A} = \langle A, \mathbf{m}\rangle$ be a median algebra with $|A| \geq 5$. Then $\mathbf{A}$ is conservative if and only if there is a total order $\leq$ on A such that $\mathbf{m} = \mathbf{m}_{\leq}$.*

Consequently, if $\mathbf{A}$ is a conservative median algebra whose operation is not the median operation of a totally ordered set, then $\mathbf{A}$ is isomorphic to $\mathbf{2} \times \mathbf{2}$.

Proof We have already noted that if $\leq$ is a total order on A then $\langle A, \mathbf{m}_{\leq}\rangle$ is conservative. Now assume that $\mathbf{A} = \langle A, \mathbf{m}\rangle$ is a conservative median algebra with $|A| \geq 5$. Consider the universe of $\mathbf{C}_0 \perp \mathbf{C}_1$ in condition (2) of Lemma 3 endowed with $\leq$ defined by $x \leq y$ if $x \in C_1$ and $y \in C_0$ or $x, y \in C_0$ and $x \leq_0 y$ or $x, y \in C_1$ and $y \leq_1 x$. Clearly, $\leq$ is a total order and $\mathbf{m}_{\leq} = \mathbf{m}$.

For the second part of the proof, note that the only conservative median algebra with at most four elements whose median operation is not the median operation of a totally ordered set is $\mathbf{2} \times \mathbf{2}$.

3 Duality theory toolbox

In this section, we recall a dual equivalence between the category of median algebras and a category of structured topological spaces. It was first exposed in [10] and was stated in terms of homomorphisms into the median algebra $\mathbf{2}$. It was later on recognized [15,7] as being an instance of a general scheme of dualities for finitely generated quasi-varieties of algebras known as *natural duality* [5]. This general approach, as well as its application to the variety of median algebras, is fully exposed in [5, Section 4.3].

Definition 1 ([2]) Let $\mathbf{A} = \langle A, \mathbf{m}\rangle$ be a median algebra. A subset C of A is *convex* if $\mathbf{m}(c_1, c_2, a) \in C$ whenever $c_1, c_2 \in C$ and $a \in A$. A convex subset C of A is *prime* if its complement $A \setminus C$ in A is also convex. We denote by $\mathrm{Spec}(\mathbf{A})$ the set of prime convex subsets of the median algebra $\mathbf{A}$.

Equivalently, $C \subseteq A$ is a prime convex subset if it satisfies the following condition: for every $x, y, z \in A$, the element $\mathbf{m}(x, y, z)$ belongs to C if and only if at least one of the sets $\{x, y\}, \{x, z\}, \{y, z\}$ is a subset of C.

Proposition 1 (Proposition 1.3 in [2]) *If $\mathbf{L}$ is a bounded distributive lattice, then the prime convex subsets of $\mathbf{L}$ are its prime filters and prime ideals.*

It is not difficult to check that in a median algebra $\mathbf{A}$, prime convex subsets coincide with the sets $u^{-1}(0)$ where $u \colon \mathbf{A} \to \mathbf{2}$ is a median homomorphism. It is convenient to use prime convex subsets instead of homomorphisms $u \colon \mathbf{A} \to \mathbf{2}$ in the dual equivalence we use in this paper. As noted in [10, 15], the set $\mathrm{Spec}(\mathbf{A})$ can be equipped with a topological structure that completely characterizes $\mathbf{A}$. We recall this construction in the remainder of this section. For $a \in \mathbf{A}$ we denote by r_a the set $\{I \in \mathrm{Spec}(\mathbf{A}) \mid a \notin I\}$.

Definition 2 Let $\mathbf{A}$ be a median agebra. The *dual* $\mathbf{A}^*$ *of* $\mathbf{A}$ is the topological structure $\mathbf{A}^* = \langle \mathrm{Spec}(\mathbf{A}), \subseteq, \cdot^c, \varnothing, A, \tau \rangle$ where $\cdot^c$ is the set-complement in A and τ is the topology with subbasis $\{r_a \mid a \in A\} \cup \{\mathrm{Spec}(\mathbf{A}) \setminus r_a \mid a \in A\}$.

Furthermore, for a homomorphism $f : \mathbf{A} \to \mathbf{B}$ between median algebras, let f^* the map defined on $\mathbf{B}^*$ by $f^*(I) = f^{-1}(I)$. It is not difficult to check using the definitions that f^* is valued in $\mathbf{A}^*$

Remark 1 By the *Prime Convex Theorem* [12, Theorem 13], it follows that $\{r_a \mid a \in A\} \cup \{\mathrm{Spec}(A) \setminus r_a \mid a \in A\}$ is in fact a basis.

The class of duals of median algebras can be defined as follows.

Definition 3 ([5]) A *bounded strongly complemented* PRIESTLEY *space* is a topological structure $\mathbf{X} = \langle X, \leq, \cdot^c, 0, 1, \tau \rangle$ where $\langle X, \leq, \tau \rangle$ is a PRIESTLEY space with 0 and 1 as bottom and top elements, and $\cdot^c$ is an order reversing homeomorphism that satisfies

$$x \leq x^c \implies x = 0 \quad \text{and} \quad x^{cc} = x.$$

Bounded strongly complemented PRIESTLEY spaces are called *bounded totally ordered disconnected compact spaces with an involution* in [15].

Definition 4 A *complete ideal* W of a bounded strongly complemented PRIESTLEY space $\mathbf{X}$ is a clopen downset that satisfies $x \in W$ if and only if $x^c \notin W$. With no danger of ambiguity, we also denote the set of complete ideals of $\mathbf{X}$ by $\mathrm{Spec}(\mathbf{X})$. This set is turned into the algebra $\mathbf{X}_* = \langle \mathrm{Spec}(\mathbf{X}), \mathbf{m} \rangle$ where $\mathbf{m}$ is the restriction of $\mathbf{m}_{2^X}$ to $\mathrm{Spec}(\mathbf{X})$. For a continuous structure-preserving map $\phi \colon \mathbf{X} \to \mathbf{Y}$, we define ϕ_* to be the map on $\mathbf{Y}_*$ given by $\phi_*(W) = \phi^{-1}(W)$.

The class $\mathcal{X}$ of bounded strongly complemented PRIESTLEY spaces can be thought of as a category with continuous structure-preserving maps as arrows. Likewise, the variety $\mathcal{M}$ of median algebras is thought of as a category with homomorphisms as arrows. For $\mathbf{X}, \mathbf{Y} \in \mathcal{X}$, we say that $\mathbf{Y}$ is a *substructure* of $\mathbf{X}$ if Y is a closed subset of $\langle X, \tau \rangle$ and $\mathbf{Y}$ is induced by the restriction of $\mathbf{X}$ to Y. In that case, if $\psi \colon \mathbf{Z} \to \mathbf{Y}$ is an isomorphism, we say that ψ is an embedding of $\mathbf{Z}$ into $\mathbf{X}$.

Proposition 2 ([5, 10, 15]) *The functors* $\cdot^* : \mathcal{M} \to \mathcal{X}$ *and* $\cdot_* : \mathcal{X} \to \mathcal{M}$ *define a dual equivalence between the categories* $\mathcal{M}$ *and* $\mathcal{X}$.

Remark 2 The isomorphism between $\mathbf{A}$ and $(\mathbf{A}^*)_*$ mentioned in Proposition 2 is given by $a \mapsto r_a$.

We denote by $\mathbf{X} \oplus \mathbf{Y}$ the coproduct of $\mathbf{X}, \mathbf{Y} \in \mathcal{X}$. It is not difficult to check that $\mathbf{X} \oplus \mathbf{Y}$ is realized in $\mathcal{X}$ by amalgamating 0 and 1 of $\mathbf{X}$ with 0 and 1 of $\mathbf{Y}$, respectively, in the disjoint union of $\mathbf{X}$ and $\mathbf{Y}$.

It is a general result of category theory that under a dual equivalence, products in one category correspond to coproducts in the other category (for instance, see [5, Chapter 1, Lemma 1.4]). In particular, we have

$$(\mathbf{A} \times \mathbf{B})^* \cong \mathbf{A}^* \oplus \mathbf{B}^*,$$

for every $\mathbf{A}, \mathbf{B} \in \mathcal{M}$. Moreover, we have the following useful result.

Proposition 3 *A homomorphism* $f \colon \mathbf{A} \to \mathbf{B}$ *between two median algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *is onto if and only if* $f^* \colon \mathbf{B}^* \to \mathbf{A}^*$ *is an embedding.*

Proof Stated in the language of Natural Duality, the dual equivalence of Proposition 2 is a strong duality (see [5, Chapter 4, Therorem 3.4]). Then, the proof is an application of [5, Chapter 3, Lemma 2.6].

4 Homomorphisms between conservative median algebras

We now use the duality theory apparatus recalled in Section 3 to describe median homomorphisms between (products of) conservative median algebras.

First, we characterize the duals of the conservative median algebras. Let $\mathbf{P}_0 = \langle P_0, \leq_0, 0_0, 1_0 \rangle$ and $\mathbf{P}_1 = \langle P_1, \leq_1, 0_1, 1_1 \rangle$ be two bounded posets. As in Section 2, $\mathbf{P}_0 \amalg \mathbf{P}_1$ denotes the *coalesced* sum of $\mathbf{P}_0$ and $\mathbf{P}_1$, that is, the poset obtained from the disjoint union of $\mathbf{P}_0$ and $\mathbf{P}_1$ by identifying 0_0 with 0_1, and 1_0 with 1_1. We denote by $i_{\mathbf{P}_k}$ the natural embedding $i_{\mathbf{P}_k} \colon \mathbf{P}_k \to \mathbf{P}_0 \amalg \mathbf{P}_1$ for $k \in \{0, 1\}$. To simplify notation, we often identify $\mathbf{P}_k$ with its copy $i_{\mathbf{P}_k}(\mathbf{P}_k)$ in $\mathbf{P}_0 \amalg \mathbf{P}_1$ for $k \in \{0, 1\}$.

If $\langle \mathbf{C}, \tau' \rangle$ is a bounded PRIESTLEY chain (i.e., a bounded totally ordered PRIESTLEY space, see, e.g., [6]), $\mathbf{C} \amalg \mathbf{C}^{\partial}$ can be endowed with an operation $\cdot^c$ and a topology τ, so that $\langle \mathbf{C} \amalg \mathbf{C}^{\partial}, \cdot^c, \tau \rangle$ is a bounded strongly complemented PRIESTLEY space. Indeed, it suffices to define

- τ as the final topology relative to $i_{\mathbf{C}}$ and $i_{\mathbf{C}^{\partial}}$ (*i.e.*, the finest topology that makes $i_{\mathbf{C}}$ and $i_{\mathbf{C}^{\partial}}$ continuous),
- $\cdot^c$ as the function that maps the bottom element 0 to the top element 1 and conversely, and that maps each element of $\mathbf{C} \backslash \{0, 1\}$ to its copy in $\mathbf{C}^{\partial}$ and conversely.

With no danger of ambiguity, we use $\mathbf{C} \amalg \mathbf{C}^{\partial}$ to denote $\langle \mathbf{C} \amalg \mathbf{C}^{\partial}, \cdot^c, \tau \rangle$.

For $\mathbf{X} \in \mathcal{X}$ and $Y \subseteq X$ set $Y^c = \{x^c \in X \mid x \in Y\}$. Also, for a PRIESTLEY space $\langle \mathbf{P}, \tau \rangle$, let $\mathbf{Cl}_0(\langle \mathbf{P}, \tau \rangle)$ be the set of its nonempty proper clopen downsets ordered by inclusion. Moreover, for a poset $\mathbf{P}$, let $\langle \mathbf{Up}(\mathbf{P}), \tau \rangle$ be the set of its upsets ordered by inclusion and equipped with the topology τ which has $\{\{I \in \mathbf{Up}(\mathbf{P}) \mid p \notin I\} \mid p \in P\} \cup \{\{I \in \mathbf{Up}(\mathbf{P}) \mid p \in I\} \mid p \in P\}$ as subbasis. If $\mathbf{P}$ is a chain, then $\langle \mathbf{Up}(\mathbf{P}), \tau \rangle$ is a bounded PRIESTLEY space.

Proposition 4 *Let* $\mathbf{A} = \langle A, \mathbf{m} \rangle$ *be a median algebra with* $|A| \geq 5$. *The following conditions are equivalent.*

1. $\mathbf{A}$ *is conservative.*
2. *There is a bounded* PRIESTLEY *chain* $\langle \mathbf{C}, \tau \rangle$ *such that* $\mathbf{A}^*$ *is isomorphic to* $\mathbf{C} \amalg \mathbf{C}^{\partial}$.
3. $\mathbf{A}$ *is the median algebra of the nonempty proper clopen downsets of a bounded* PRIESTLEY *chain* $\langle \mathbf{C}, \tau \rangle$.

Furthermore, if one of these conditions is satisfied and if $\mathbf{C}_0 = \langle A, \leq \rangle$ is a chain representation of $\mathbf{A}$ given by Theorem 3, then $\mathbf{A}^* \cong \mathbf{Up}(\mathbf{C}_0) \amalg \mathbf{Up}(\mathbf{C}_0)^\partial$ and $\mathbf{m}$ is the median operation of $\mathbf{Cl}_0(\mathbf{Up}(\mathbf{C}_0))$.

Proof $(1) \implies (2)$: According to Theorem 3, there is a totally ordered set $\mathbf{C}_0 = \langle A, \leq \rangle$ such that $\mathbf{A} = \langle A, \mathbf{m}_\leq \rangle$. From Proposition 1, we know that the prime convex subsets of $\langle A, \mathbf{m}_\leq \rangle$ are the prime filters and prime ideals of $\mathbf{C}_0$, that is, the upsets of $\mathbf{C}_0$ and the downsets of $\mathbf{C}_0$. Then $\mathbf{A}^*$ is isomorphic to $\mathbf{Up}(\mathbf{C}_0) \amalg \mathbf{Up}(\mathbf{C}_0)^\partial$.

$(2) \implies (3)$: The median algebra $\mathbf{A}$ is isomorphic to $(\mathbf{C} \amalg \mathbf{C}^\partial)_*$. If W is a complete ideal of $\mathbf{C} \amalg \mathbf{C}^\partial$ then $\omega_W := W \cap \mathbf{C}$ belongs to $\mathbf{Cl}_0(\langle \mathbf{C}, \tau \rangle)$. Conversely, if $\omega \in \mathbf{Cl}_0(\langle \mathbf{C}, \tau \rangle)$ then $W_\omega := \omega \cup (\mathbf{C}^\partial \setminus \omega^c)$ is a complete ideal of $\mathbf{C} \amalg \mathbf{C}^\partial$. It is not difficult to check that the maps $\omega_- : (\mathbf{C} \amalg \mathbf{C}^\partial)_* \to \mathbf{Cl}_0(\langle \mathbf{C}, \tau \rangle)$ and $W_- : \mathbf{Cl}_0(\langle \mathbf{C}, \tau \rangle) \to (\mathbf{C} \amalg \mathbf{C}^\partial)_*$ are median homomorphisms such that one is the inverse of the other. We conclude that up to isomorphism, $\mathbf{m}$ is the median operation of $\mathbf{Cl}_0(\langle \mathbf{C}, \tau \rangle)$.

$(3) \implies (1)$: Follows straightforwardly since $\mathbf{m}$ is the median operation of a chain.

The proof of the first and the second claims of the last statement are given in the proof of $(1) \implies (2)$ and $(2) \implies (3)$, respectively.

Corollary 1 *Let $\mathbf{A}$ be a median algebra. If $\mathbf{C}$ and $\mathbf{C}'$ are two chains such that $\mathbf{A} \cong \langle \mathbf{C}, \mathbf{m_C} \rangle$ and $\mathbf{A} \cong \langle \mathbf{C}', \mathbf{m_{C'}} \rangle$, then $\mathbf{C}$ is order isomorphic or dual order isomorphic to $\mathbf{C}'$.*

Given a conservative median algebra $\mathbf{A} = \langle A, \mathbf{m} \rangle$ (with $|A| \geq 5$), Theorem 3 provides with a total order $\leq_\mathbf{A}$ on A such that $\mathbf{m} = \mathbf{m}_{\leq_\mathbf{A}}$. Corollary 1 states that $\langle A, \leq_\mathbf{A} \rangle$ is unique up to isomorphisms and dual isomorphisms. We call $\leq_\mathbf{A}$ the *chain ordering* of $\mathbf{A}$ and we denote $\langle A, \leq_\mathbf{A} \rangle$ by $\mathbf{C}(\mathbf{A})$.

We use Proposition 4 to characterize median homomorphisms between conservative median algebras. Recall that a map between two posets is *monotone* if it is isotone or antitone.

Proposition 5 *Let $\mathbf{A}$ and $\mathbf{B}$ be two conservative median algebras with at least five elements. A map $f : \mathbf{A} \to \mathbf{B}$ is a median homomorphism if and only if it is monotone with respect to the chain orderings of $\mathbf{A}$ and $\mathbf{B}$.*

Proof (Necessity) We may assume that f is onto. According to Proposition 3, the map

$$f^* : \mathbf{Up}(\mathbf{C}(\mathbf{B})) \amalg \mathbf{Up}(\mathbf{C}(\mathbf{B}))^\partial \hookrightarrow \mathbf{Up}(\mathbf{C}(\mathbf{A})) \amalg \mathbf{Up}(\mathbf{C}(\mathbf{A}))^\partial$$

is a $\mathcal{X}$-embedding.

If the range of f^* is equal to $\{0, 1\}$, then $\mathbf{B}$ is the one-element median algebra and $\mathbf{C}(\mathbf{B})$ is the one-element chain, and the result follows trivially. Hence, we may assume that there is a $I \in \mathbf{Up}(\mathbf{C}(\mathbf{B}))$ such that $f^*(I) \notin \{0, 1\}$. If $f^*(I) \in \mathbf{Up}(\mathbf{C}(\mathbf{A}))$, then $f^*(\mathbf{Up}(\mathbf{C}(\mathbf{B}))) \subseteq \mathbf{Up}(\mathbf{C}(\mathbf{A}))$ since f^* is isotone. We prove that $f : \mathbf{C}(\mathbf{A}) \to \mathbf{C}(\mathbf{B})$ is isotone. Suppose that $a \leq b$ for some $a, b \in \mathbf{C}(\mathbf{A})$. Then $f^*\big([f(a))\big)$ contains b since it is an upset that contains a and $a \leq b$. It means that $f(a) \leq f(b)$, which is the desired result.

If $f^*(I) \in \mathbf{Up}(\mathbf{C}(\mathbf{A}))^\partial$, we conclude in a similar way that $f : \mathbf{C}(\mathbf{A}) \to \mathbf{C}(\mathbf{B})$ is antitone.

(Sufficiency) If $f : \mathbf{C}(\mathbf{A}) \to \mathbf{C}(\mathbf{B})$ is isotone, then it maps upsets to upsets and downsets to downsets. If it is antitone, it maps upsets to downsets and conversely. It means that f^* is valued in $\mathbf{Up}(\mathbf{C}(\mathbf{A})) \amalg \mathbf{Up}(\mathbf{C}(\mathbf{A}))^\partial$. It is then straightforwad to check that f^* is a $\mathcal{X}$-morphism.
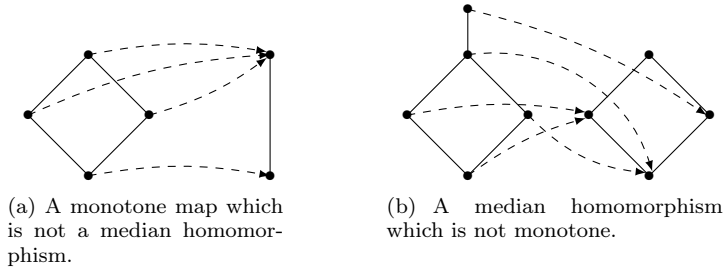
(a) A monotone map which is not a median homomorphism.

(b) A median homomorphism which is not monotone.

Fig. 2 Examples for Remark 3.

Corollary 2 *Let $\mathbf{C}$ and $\mathbf{C}'$ be two chains. A map $f : \mathbf{C} \to \mathbf{C}'$ is a median homomorphism if and only if it is monotone.*

Remark 3 Note that Corollary 2 only holds for chains. Indeed, Fig. 2(a) gives an example of a monotone map that is not a median homomorphism, and Fig. 2(b) gives an example of median homomorphism that is not monotone.

Since the class of conservative median algebras is closed under homomorphic images, we obtain the following corollary.

Corollary 3 *Let $\mathbf{A}$ and $\mathbf{B}$ be two median algebras and $f : \mathbf{A} \to \mathbf{B}$. If $\mathbf{A}$ is conservative, and if $|A|, |f(A)| \geq 5$, then f is a median homomorphism if and only if $f(\mathbf{A})$ is a conservative median subalgebra of $\mathbf{B}$ and f is monotone with respect to the chain orderings of $\mathbf{A}$ and $f(\mathbf{A})$.*

The dual equivalence between $\mathcal{M}$ and $\mathcal{X}$ turns finite products into finite coproducts. This property can be used to characterize median homomorphisms between finite products of chains. If $f_i : A_i \to A_i'$ ($i \in [n]$) is a family of maps, let $(f_1, \ldots, f_n) : A_1 \times \cdots \times A_n \to A_1' \times \cdots \times A_n'$ be defined by

$$(f_1, \ldots, f_n)(x_1, \ldots, x_n) := (f_1(x_1), \ldots, f_n(x_n)).$$

The following proposition essentially states that median homomorphisms between finite products of chains necessarily decompose componentwise.

Proposition 6 *Let $\mathbf{A} = \mathbf{C}_1 \times \cdots \times \mathbf{C}_k$ and $\mathbf{B} = \mathbf{D}_1 \times \cdots \times \mathbf{D}_n$ be two finite products of chains. Then $f : \mathbf{A} \to \mathbf{B}$ is a median homomorphism if and only if there exist $\sigma : [n] \to [k]$ and monotone maps $f_i : \mathbf{C}_{\sigma(i)} \to \mathbf{D}_i$ for $i \in [n]$ such that $f = (f_{\sigma(1)}, \ldots, f_{\sigma(n)})$.*

Proof The condition is clearly sufficient. To prove that it is necessary, let $\mathbf{A}$, $\mathbf{B}$ and f be as in the statement. The map $f^* = \mathbf{D}_1^* \oplus \cdots \oplus \mathbf{D}_n^* \to \mathbf{C}_1^* \oplus \cdots \oplus \mathbf{C}_k^*$ is an $\mathcal{X}$-morphism. Let $i \in [n]$. Since $\mathbf{D}_i^*$ is a $\mathcal{X}$-substructure of $\mathbf{B}^* \cong \mathbf{D}_1^* \oplus \cdots \oplus \mathbf{D}_n^*$, the map $f^*|_{\mathbf{D}_i^*}$ is an $\mathcal{X}$-morphism from $\mathbf{D}_i^*$ to $\mathbf{A}^* \cong \mathbf{C}_1^* \oplus \cdots \oplus \mathbf{C}_k^*$. Hence, there is a $\sigma(i) \in [k]$ such that $f^*|_{\mathbf{D}_i^*}$ is valued in $\mathbf{C}_{\sigma(i)}^*$. It follows that the diagram in Fig. 3(a) commutes, and by duality, so is the diagram in Fig. 3(b). Hence, it suffices to define f_{σ_i} as $(f^*|_{\mathbf{D}_i^*})^*$ to conclude the proof.

If $A = A_1 \times \cdots \times A_n$ and $i \in [n]$, then we denote the projection map from A onto A_i by π_i^A, or simply by π_i if there is no danger of ambiguity.

$$
\begin{array}{ccc}
\mathbf{D}_i^* & \xrightarrow{\;f^*|_{\mathbf{D}_i^*}\;} & \mathbf{C}_{\sigma(i)}^* \\
{\scriptstyle(\pi_i^B)^*}\big\uparrow & & \big\uparrow{\scriptstyle(\pi_{\sigma(i)}^A)^*} \\
\mathbf{B}^* & \xrightarrow{\;f^*\;} & \mathbf{A}^*
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbf{D}_i & \xleftarrow{\;(f^*|_{\mathbf{D}_i^*})^*\;} & \mathbf{C}_{\sigma(i)} \\
{\scriptstyle\pi_i^B}\big\uparrow & & \big\uparrow{\scriptstyle\pi_{\sigma(i)}^A} \\
\mathbf{B} & \xleftarrow{\;f\;} & \mathbf{A}
\end{array}
$$

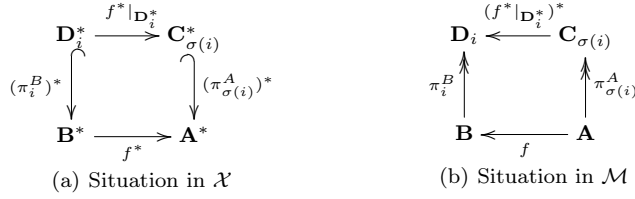(a) Situation in $\mathcal{X}$ (b) Situation in $\mathcal{M}$

Fig. 3 Proof of Proposition 6

Corollary 4 *Let $\mathbf{C}_1,\ldots,\mathbf{C}_n$ and $\mathbf{D}$ be chains. A map $f:\mathbf{C}_1\times\cdots\times\mathbf{C}_n\to\mathbf{D}$ is a median homomorphism if and only if there is a $j\in[n]$ and a monotone map $g:\mathbf{C}_j\to\mathbf{D}$ such that $f=g\circ\pi_j$.*

In the particular case of Boolean algebras, Proposition 6 can be restated as in the following corollary.

Corollary 5 *Assume that $f:\mathbf{2}^n\to\mathbf{2}^m$ is a map between two finite Boolean algebras. The map f is a median homomorphism if and only if there are $\sigma:[m]\to([n]\cup\{\bot\})$ and $\varepsilon:[m]\to\{\mathrm{id},\neg\}$ such that*

$$f:(x_1,\ldots,x_n)\mapsto(\varepsilon_1 x_{\sigma_1},\ldots,\varepsilon_m x_{\sigma_m}),$$

where $x_\bot$ is defined as the constant map 0.

Corollary 6 *1. The Boolean functions on $\mathbf{2}^n$ that are median homomorphisms are exactly the constant functions, the projection maps $\pi:\mathbf{2}^n\to\mathbf{2}$ and the negations of the projection maps.*
2. A map $f:\mathbf{2}^n\to\mathbf{2}^n$ is a median isomorphism if and only if there is a permutation σ of $[n]$ and an element ε of $\{\mathrm{id},\neg\}^n$ such that $f(x_1,\ldots,x_n)=(\varepsilon_1 x_{\sigma(1)},\ldots,\varepsilon_n x_{\sigma(n)})$ for any $(x_1,\ldots,x_n)$ in $\mathbf{2}^n$.

Remark 4 As kindly noticed by the reviewer, Corollaries 5 and 6 follow from properties of congruence distributive varieties generated by a finite simple algebra. For instance, it can be shown that if $\mathbf{A}$ is a finite simple algebra that generates a congruence distributive variety and if $f:\mathbf{A}^n\to\mathbf{A}^n$ is an isomorphism, then there exist a permutation σ of $[n]$ and automorphisms $\varepsilon_1,\ldots,\varepsilon_n$ of $\mathbf{A}$ such that $f(x_1,\ldots,x_n)=(\varepsilon_1 x_{\sigma(1)},\ldots,\varepsilon_n x_{\sigma(n)})$ for every $(x_1,\ldots,x_n)\in\mathbf{A}^n$. Since the variety of median algebras has a near-unanimity term, it is congruence distributive (see [11, Theorem 2]) and hence Corollary 6.2 can be obtained from the latter result.

5 Concluding remarks and further research directions

In this paper we have described conservative median algebras and semilattices with at least five elements in terms of forbidden configurations and have given a representation by chains. We have also characterized median homomorphisms between finite products of these algebras, showing that they are essentially determined componentwise. The next step in this line of research is to extend our results to larger classes of median algebras and their ordered counterparts. The topological duality for the variety of median algebras recalled in this paper may again turn out to be a valuable tool.

Another research direction would be to turn the representation theorem stated in Proposition 4 into a dual equivalence, and to use this equivalence to describe existentially and algebraically closed elements in the category of conservative median algebras by following the ideas developed in [5, Chapter 5].

Acknowledgment

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