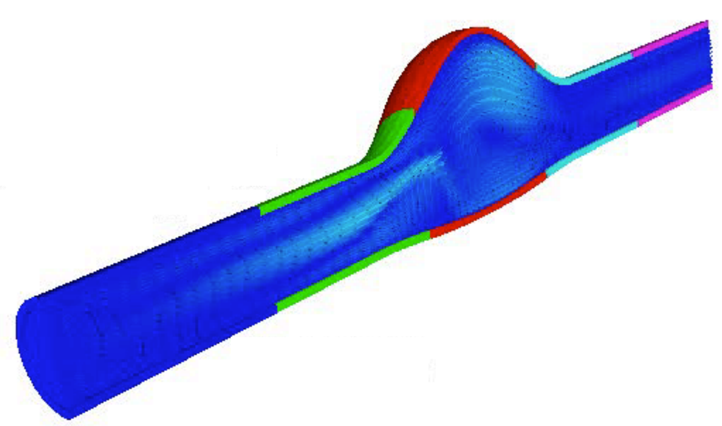
REDUCED ORDER METHODS

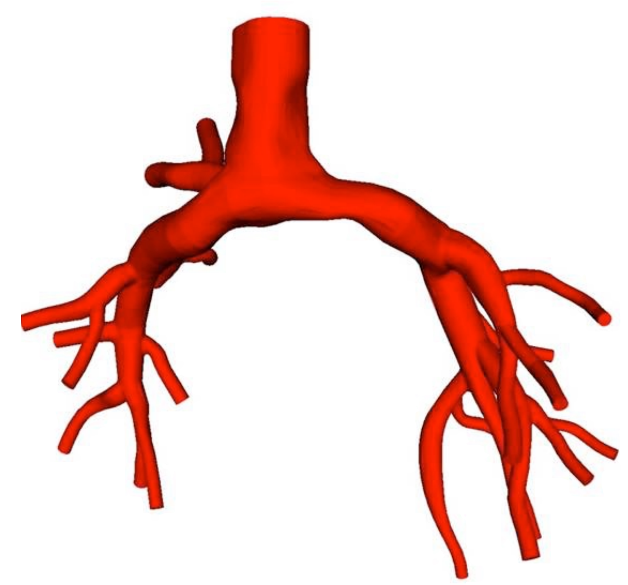
Elisa SCHENONE
Legato Team
Computational Mechanics

reo team at Inria and JLL of UPMC

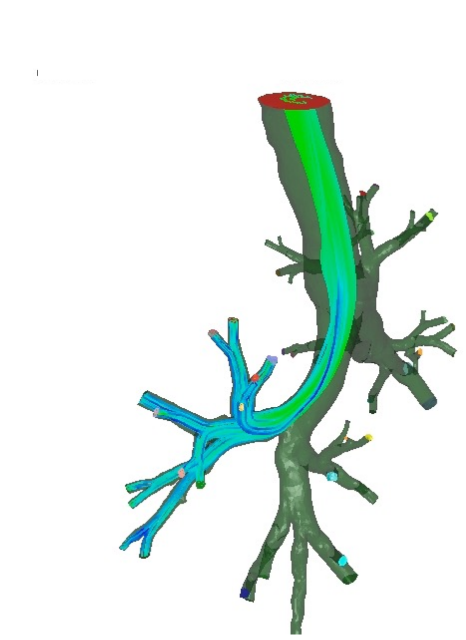
Numerical simulation of biological flows

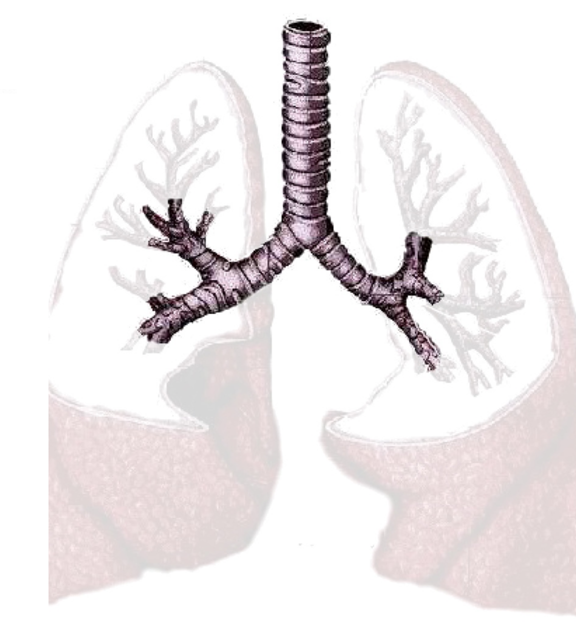
Blood flow modelling



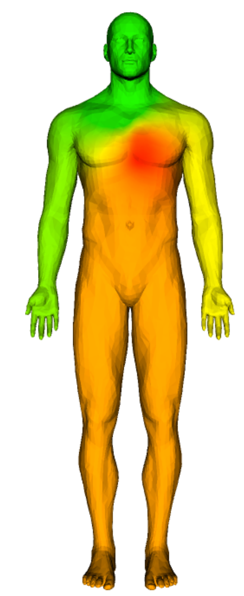


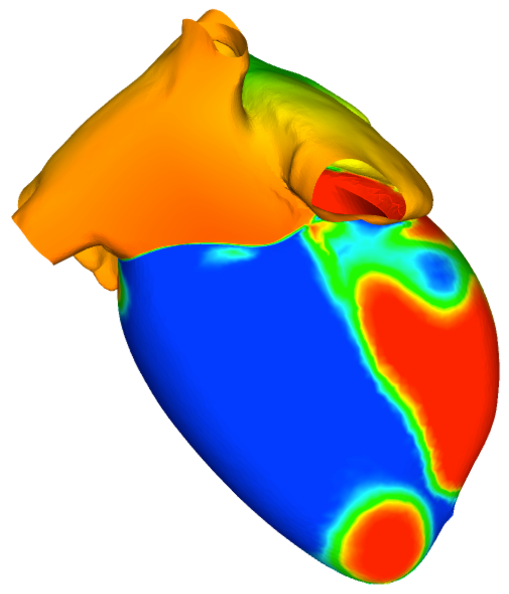
Respiration modelling

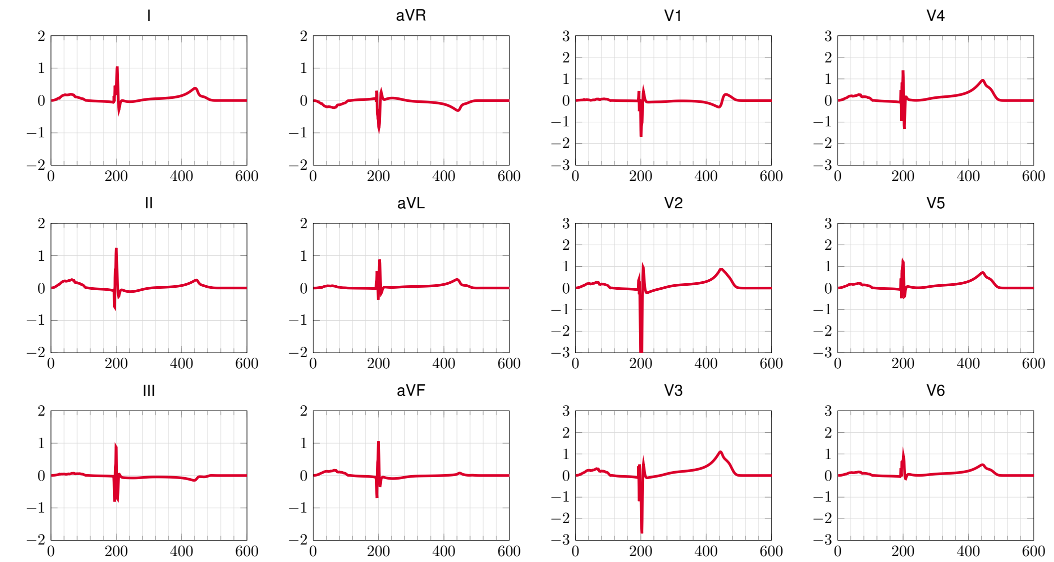




Cardiac electrophysiology







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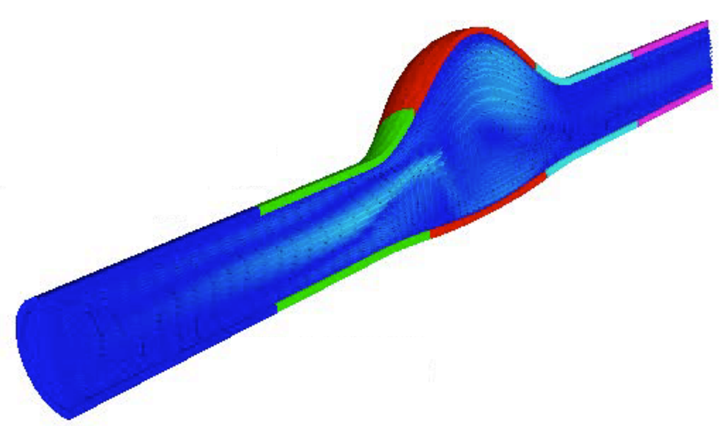

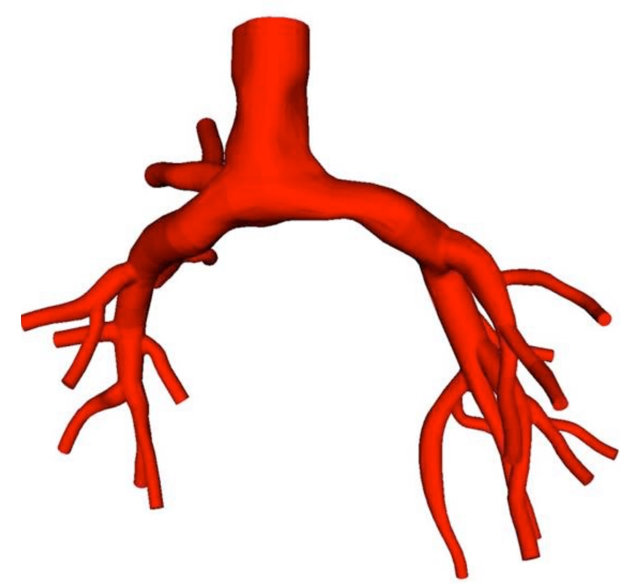

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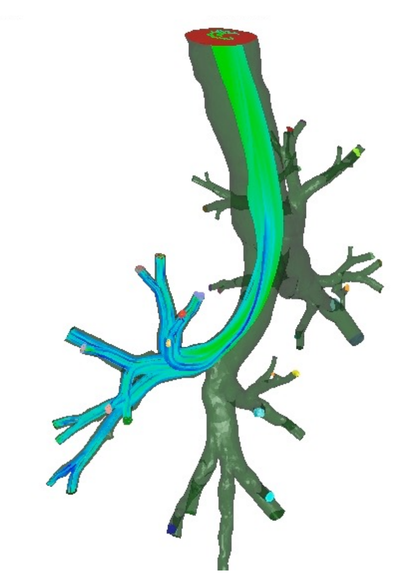
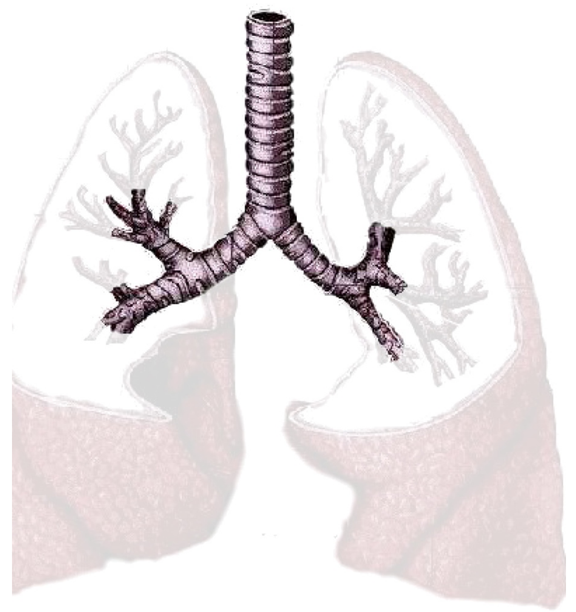
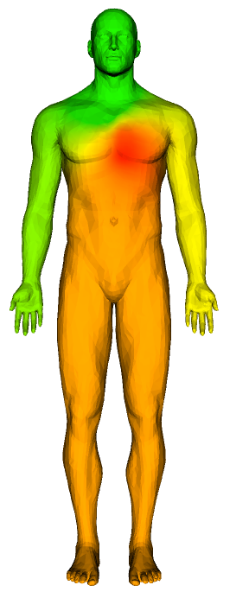
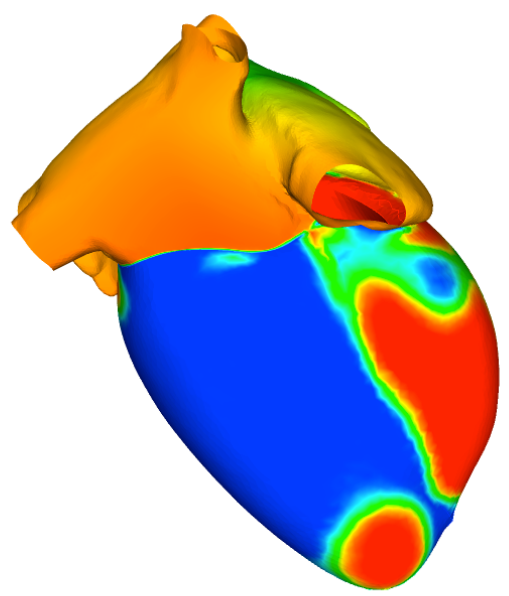
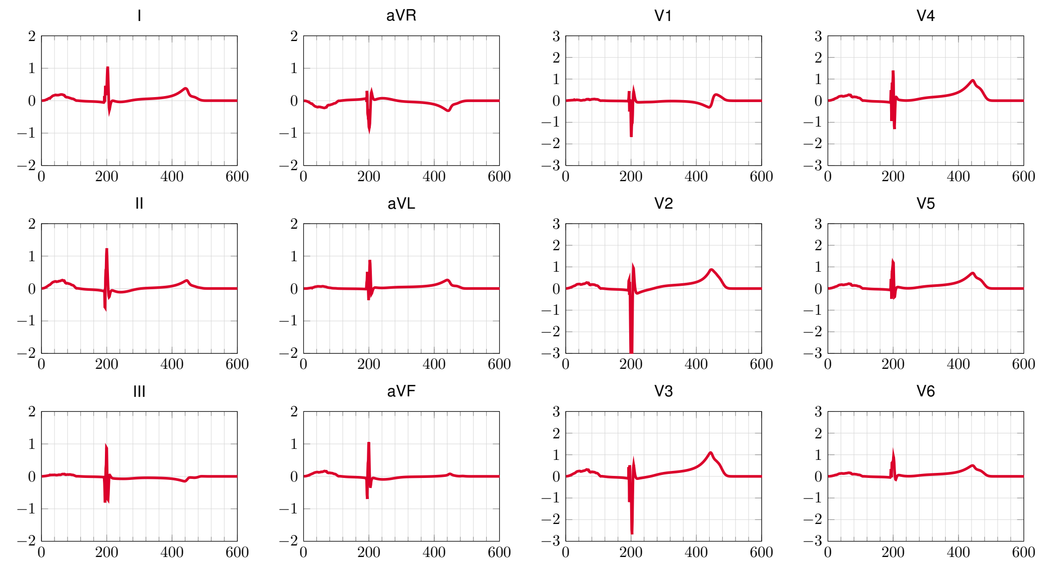




Cardiac electrophysiology







PhD project:

Inverse problems and reduced models in cardiac electrophysiology

- ✓ Supervisors:

  Muriel BOULAKIA and Jean-Frédéric GERBEAU

- ✓ Subjects:
  - ➡ Simulation of realistic Electrocardiograms
  - ➡ Parameters estimation
  - ➡ Reduced order models (RB, POD, ALP, EIM)
  - ➡ Inverse problems

Post-Doc project:

Reduced models in computational mechanics

- ✓ Supervisors:

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  - ➡ Simulation of hyperelastic materials with reduced order models

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Summary

✓ PDEs numerical approximation

➡ Finite Element Methods (overview)

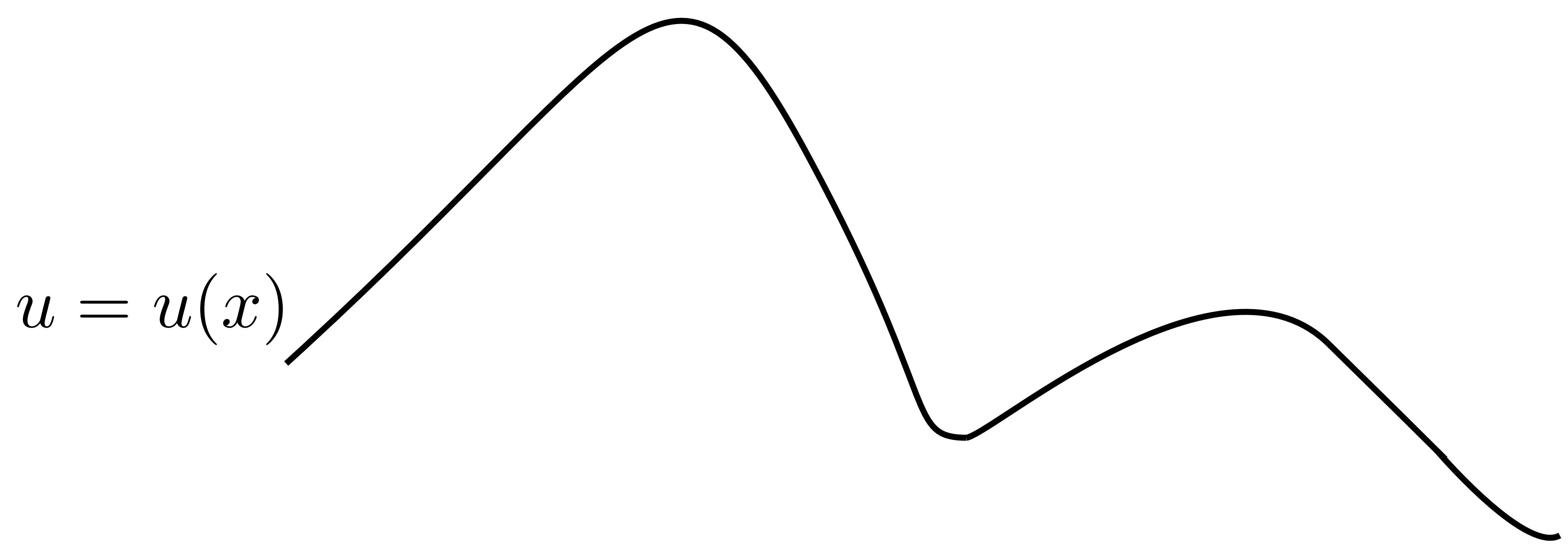
➡ Reduced Order Methods

- Reduced-Basis
- Proper Orthogonal Decomposition

✓ Numerical Results

A numerical approximation
Overview on Finite Element Method (FEM)

FEM space



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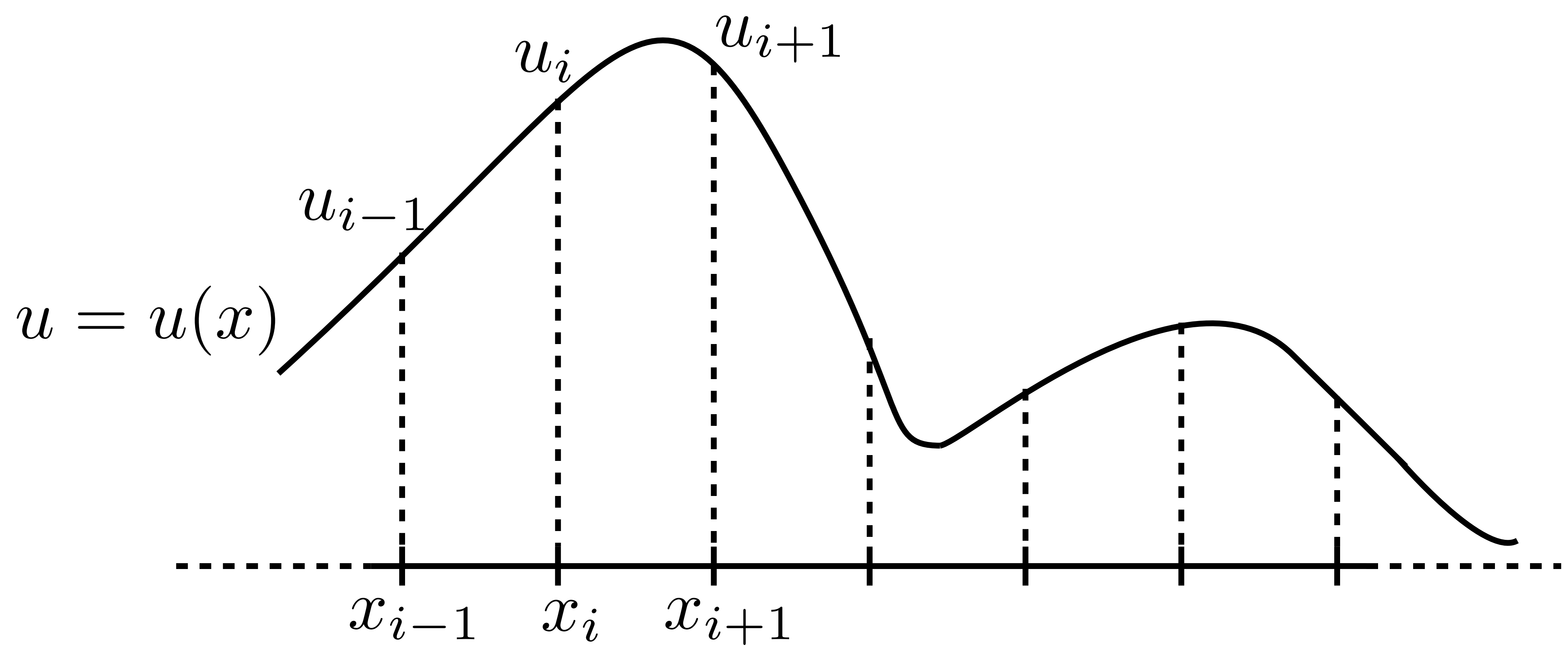

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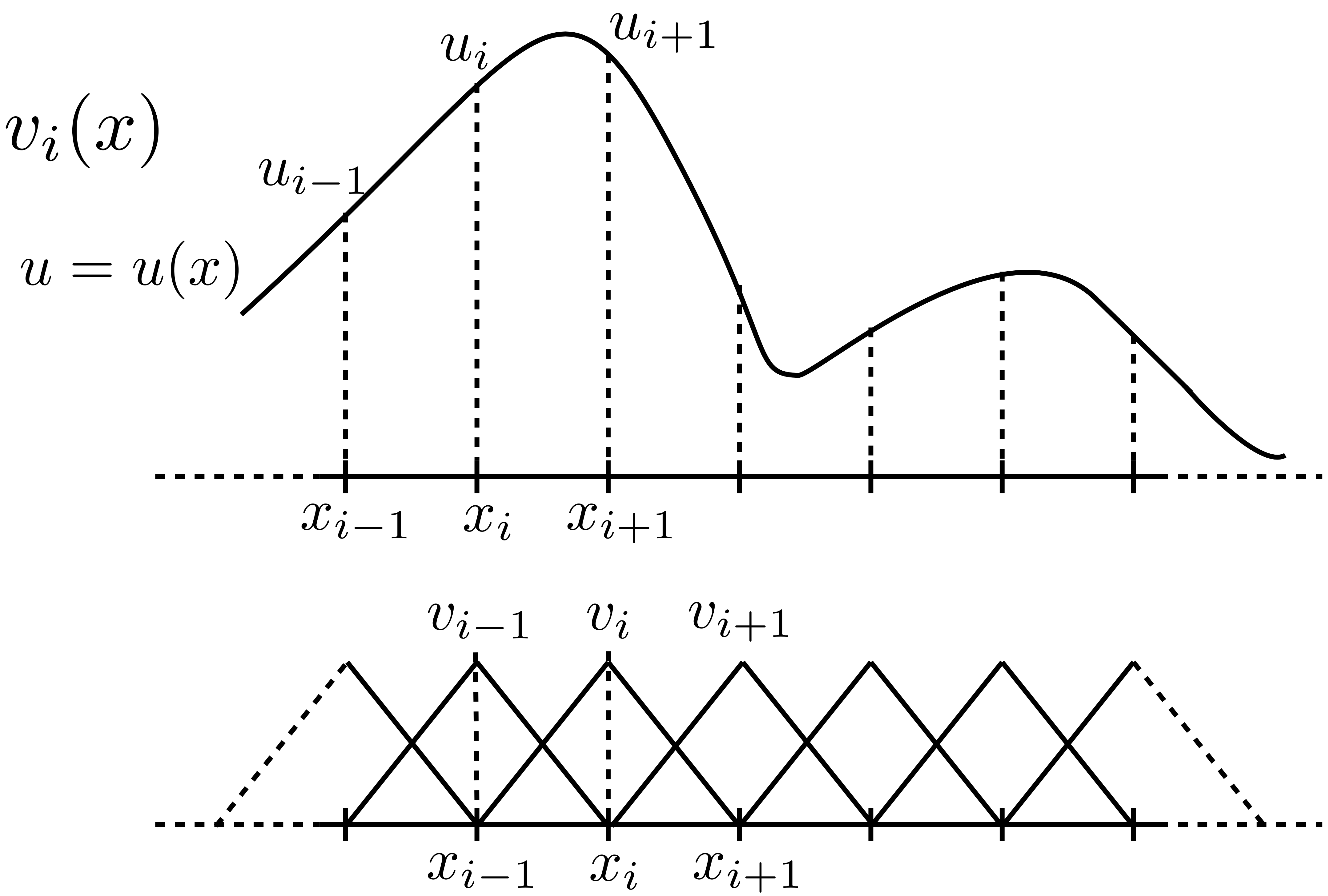
$$u(x) \simeq \sum_{i}^{\mathcal{N}} u_i v_i(x)$$



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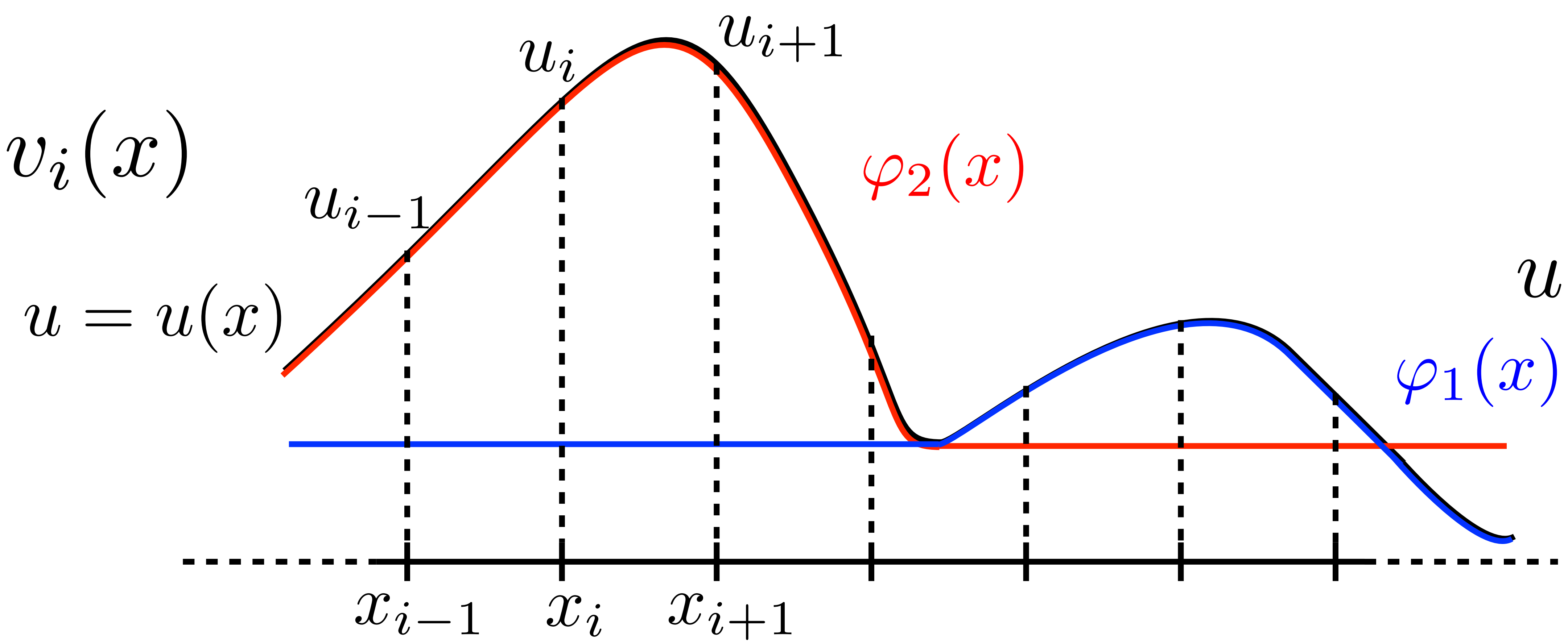


$$u(x) \simeq \tilde{u}_1 \varphi_1(x) + \tilde{u}_2 \varphi_2(x)$$

FEM versus ROM

FEM space

$$V_{\mathcal{N}} \equiv \mathrm{span}\{v_1, \dots, v_{\mathcal{N}}\} \subset V$$

$$\dim(V_{\mathcal{N}}) = \mathcal{N}$$

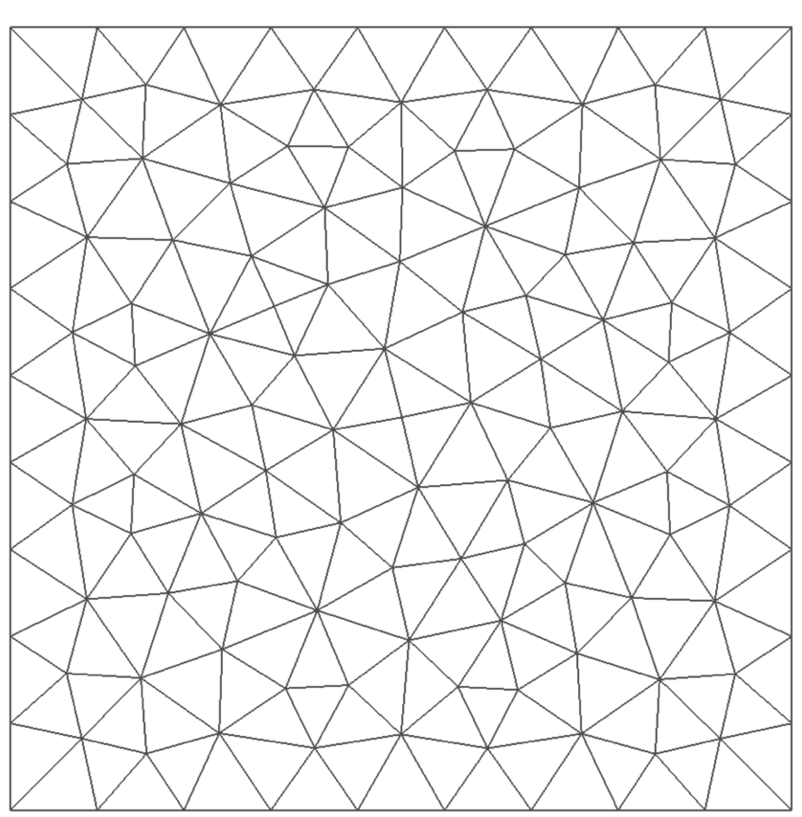
$$\mathbf{u} = \sum_i^{\mathcal{N}} u_i v_i(x), \quad u_i = \langle u, v_i \rangle$$

ROM space

$$V_N \equiv \mathrm{span}\{\varphi_1, \dots, \varphi_N\} \subset V_{\mathcal{N}}$$

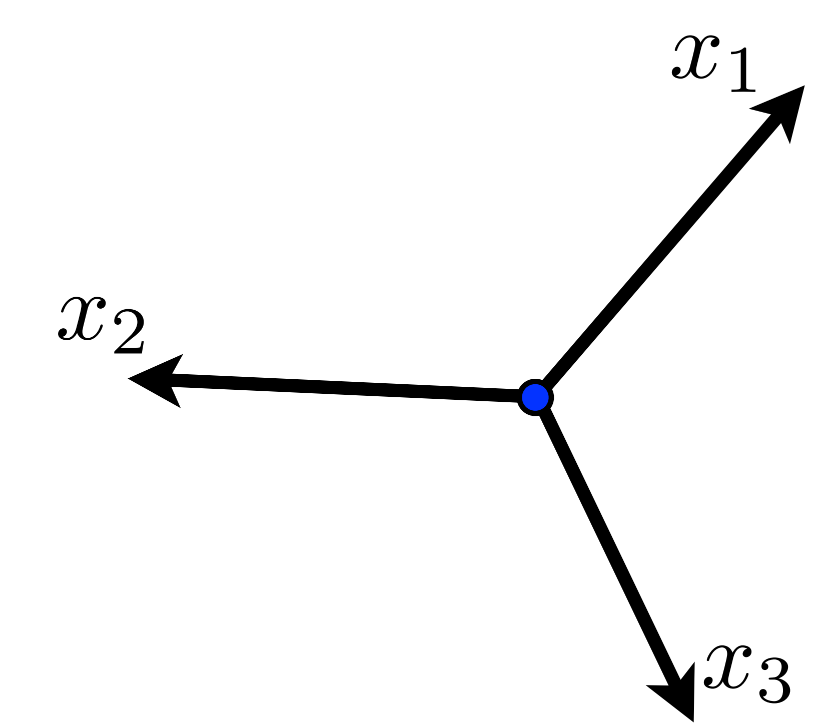
$$\dim(V_N) = N \ll \mathcal{N}$$

$$\widetilde{\mathbf{u}} = \sum_i^{N} \tilde{u}_i \varphi_i(x), \quad \tilde{u}_i = \langle u, \varphi_i \rangle$$



$$\Phi_N = [\varphi_1 \dots \varphi_N] \in \mathbb{R}^{\mathcal{N} \times N}$$

$$\widetilde{\mathbf{u}} = \Phi_N^T \mathbf{u}$$



General ROM formulation
Diffusion problem

Linear case

$$\begin{cases} -\nabla \cdot (\gamma \nabla u) = f, & \Omega \subset \mathbb{R}^d \\ \text{B.C.} & \partial\Omega \end{cases}$$

Weak formulation $\forall v \in V$

$$\int_\Omega \gamma \nabla u \nabla v + \text{B.C.} = \int_\Omega f v$$

FEM

$$K\mathbf{u} = \mathbf{f}$$

$$K \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$$
$$\mathbf{u} \in \mathbb{R}^{\mathcal{N}}$$
$$\mathbf{f} \in \mathbb{R}^{\mathcal{N}}$$

$$\Phi = [\varphi_1, \ldots, \varphi_N] \in \mathbb{R}^{\mathcal{N}\times N}$$

ROM

$$\widetilde{K}\widetilde{\mathbf{u}} = \widetilde{\mathbf{f}}$$

$$\begin{aligned} \widetilde{K} &= \Phi^T K \Phi, & \widetilde{K} &\in \mathbb{R}^{N\times N} \\ \widetilde{\mathbf{u}} &= \Phi^T \mathbf{u}, & \widetilde{\mathbf{u}} &\in \mathbb{R}^{N} \\ \widetilde{\mathbf{f}} &= \Phi^T \mathbf{f}, & \widetilde{\mathbf{f}} &\in \mathbb{R}^{N} \end{aligned}$$

Reduced-Basis Method

A "classical" approach to Reduced Order Models

- Use information given by FEM solution(s) to build a suitable basis

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1. Solve with FEM the (linear) problem(s)

$$\partial_t u = F(u, \partial_x^{(n)}, \pi), \text{ in } \Omega \times [0,T] \subset \mathbb{R}^d \times \mathbb{R}_+$$

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  2. Collect snapshots of FEM solution(s) in a matrix

$$S = (\mathbf{u}^1, \ldots, \mathbf{u}^p) \in \mathbb{R}^{\mathcal{N} \times p} \qquad \mathbf{u}^i = \begin{bmatrix} u(x_1, t_i) \\ \vdots \\ u(x_{\mathcal{N}}, t_i) \end{bmatrix} \in \mathbb{R}^{\mathcal{N}}, \ \forall i = 1, \ldots, \mathcal{N}$$

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3. Orthonormalization of the matrix

$$\Phi = \mathrm{orthonorm}(S)$$

- Can reproduce events described by the FEM solution(s)

RB application
Diffusion problem

1. Solve with FEM the (linear) problem(s) for different f

$$\begin{cases} -\Delta u = f, & \Omega = (0,1)^2 \\ u = 0, & \partial\Omega \end{cases}$$

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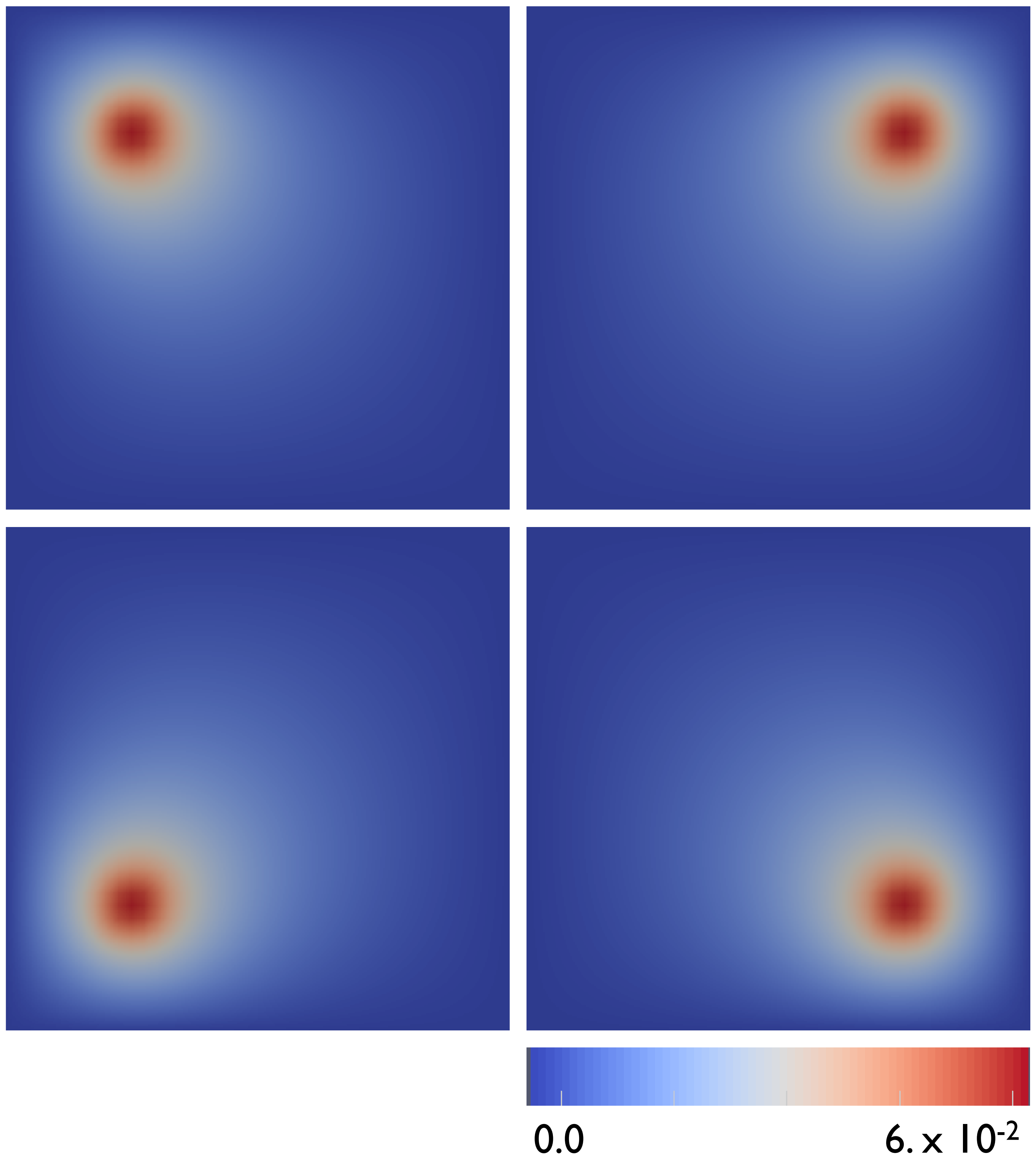
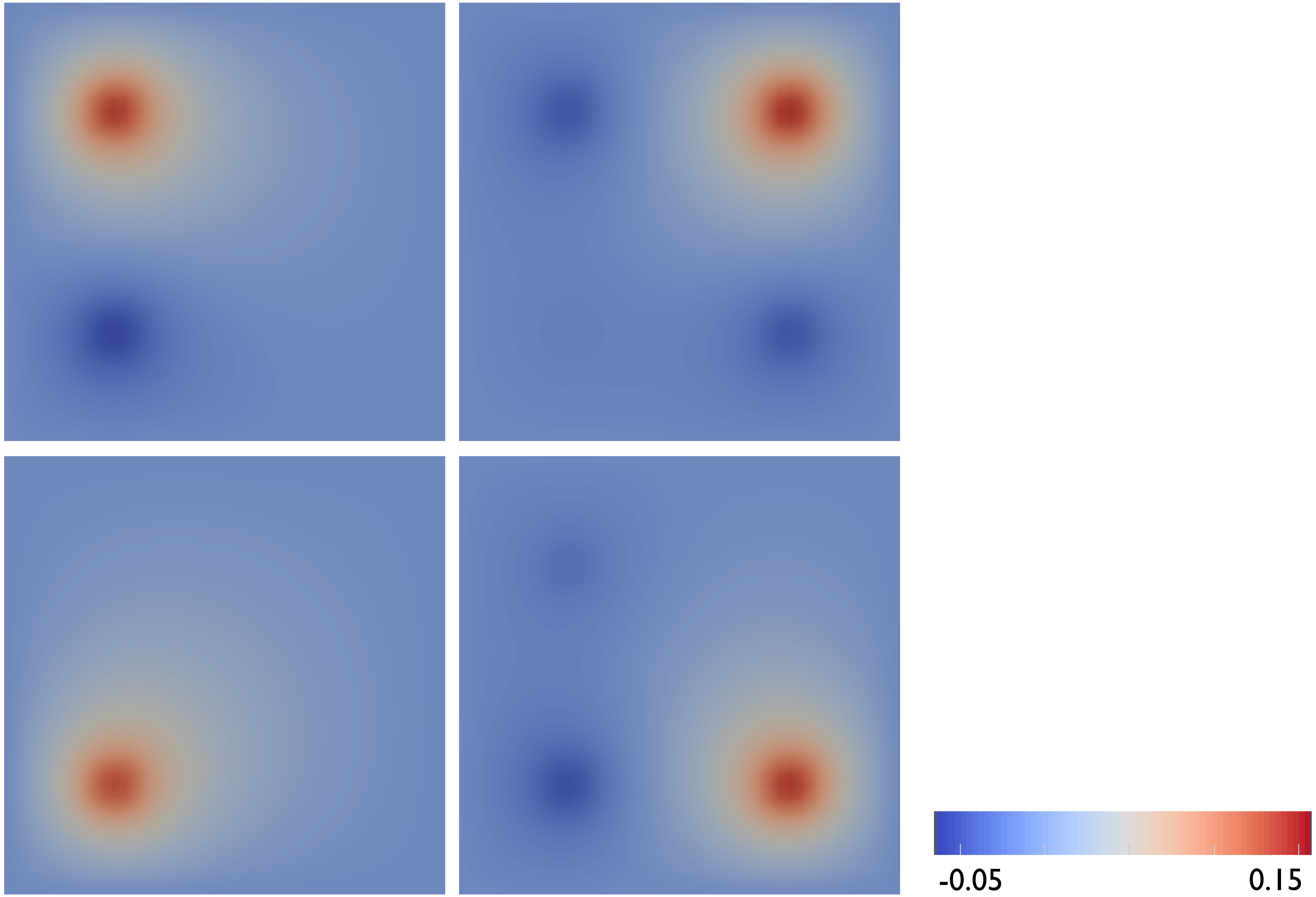
$$\Phi = [\phi_1, \ldots, \phi_4]$$



RB application
Diffusion problem

RB basis functions



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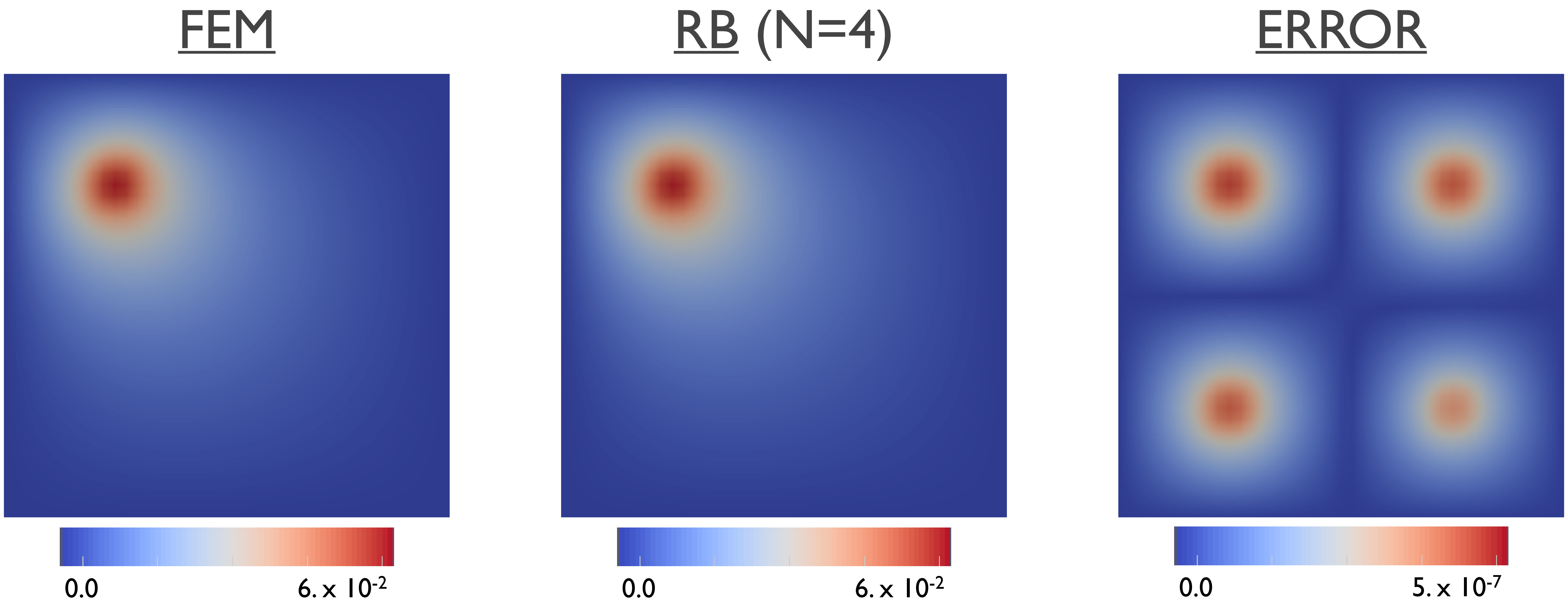

NB the solution is “in” the basis

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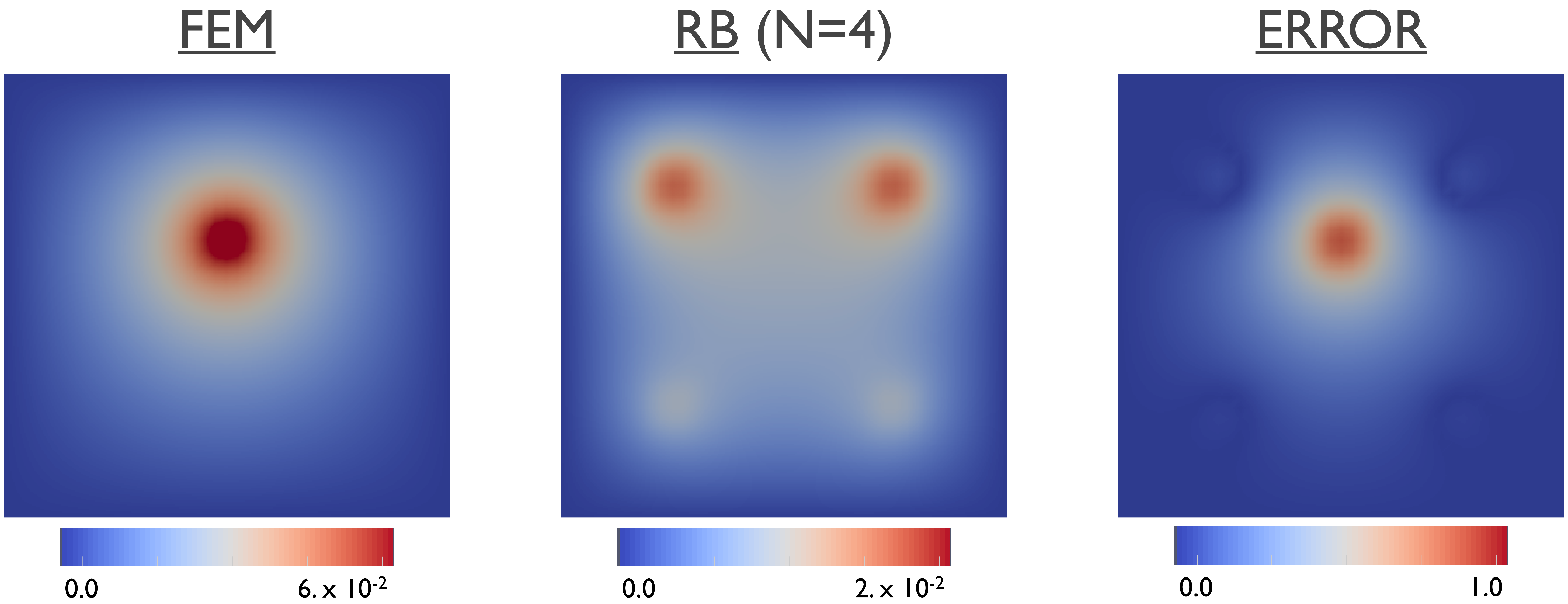

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RB application
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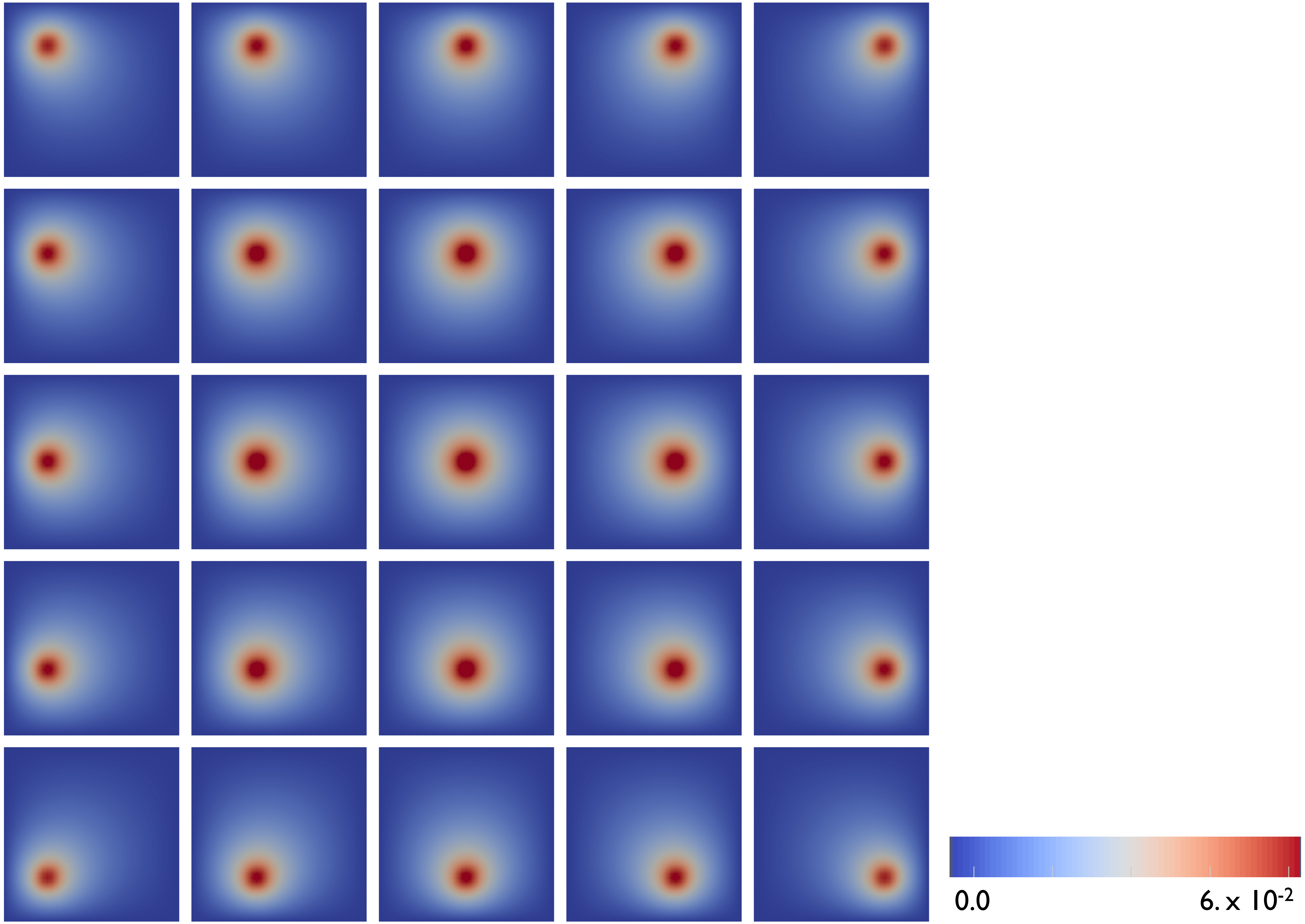
FEM snapshots
25 sources



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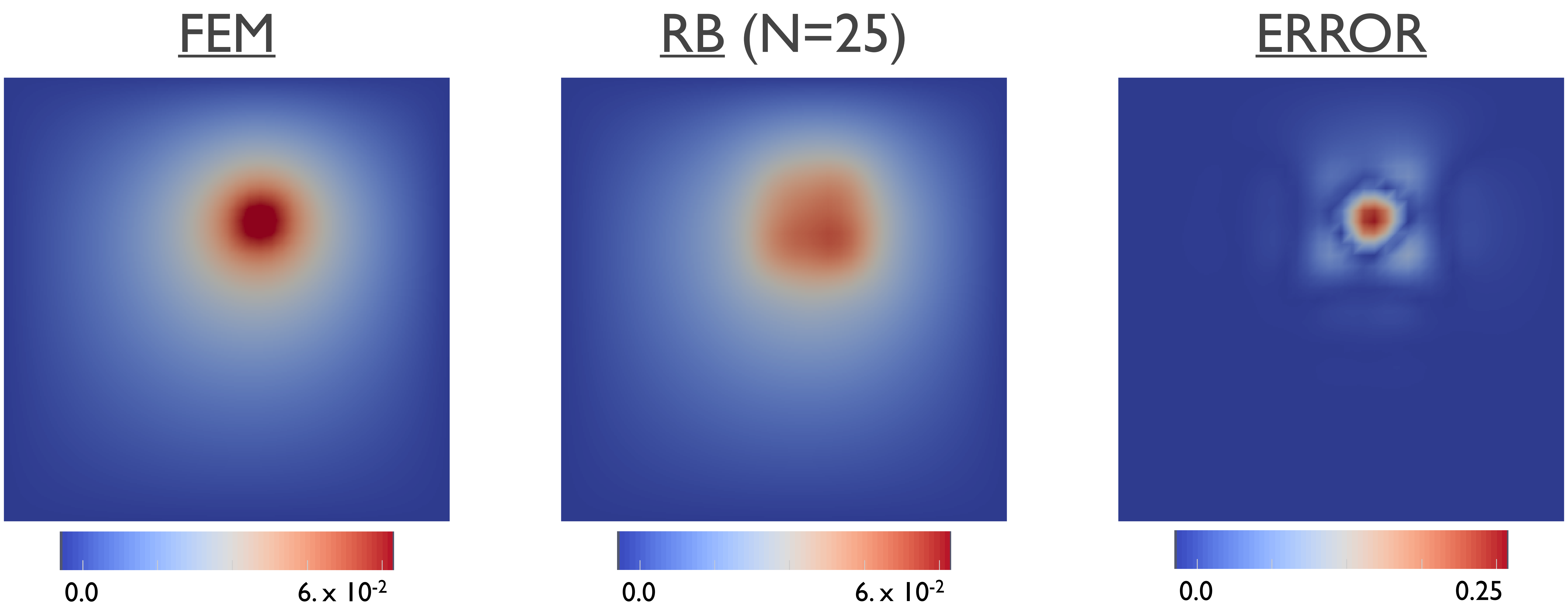

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Proper Orthogonal Decomposition (POD)

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Proper Orthogonal Decomposition (POD)

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- Use information given by FEM solution(s) to build a suitable basis
  1. Solve with FEM the problem(s)

$$\partial_t u = F(u, \partial_x^{(n)}, \pi), \text{ in } \Omega \times [0,T] \subset \mathbb{R}^d \times \mathbb{R}_+$$

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3. Compute Singular Value Decomposition (SVD) of the matrix

$$S = \Phi \, \Sigma \, \Psi^T, \quad \Sigma = \mathrm{diag}(\sigma_i)$$

4. Keep first eigenvectors as ROM basis functions

- Can reproduce events described by the FEM solution(s)

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4. Keep first eigenvectors

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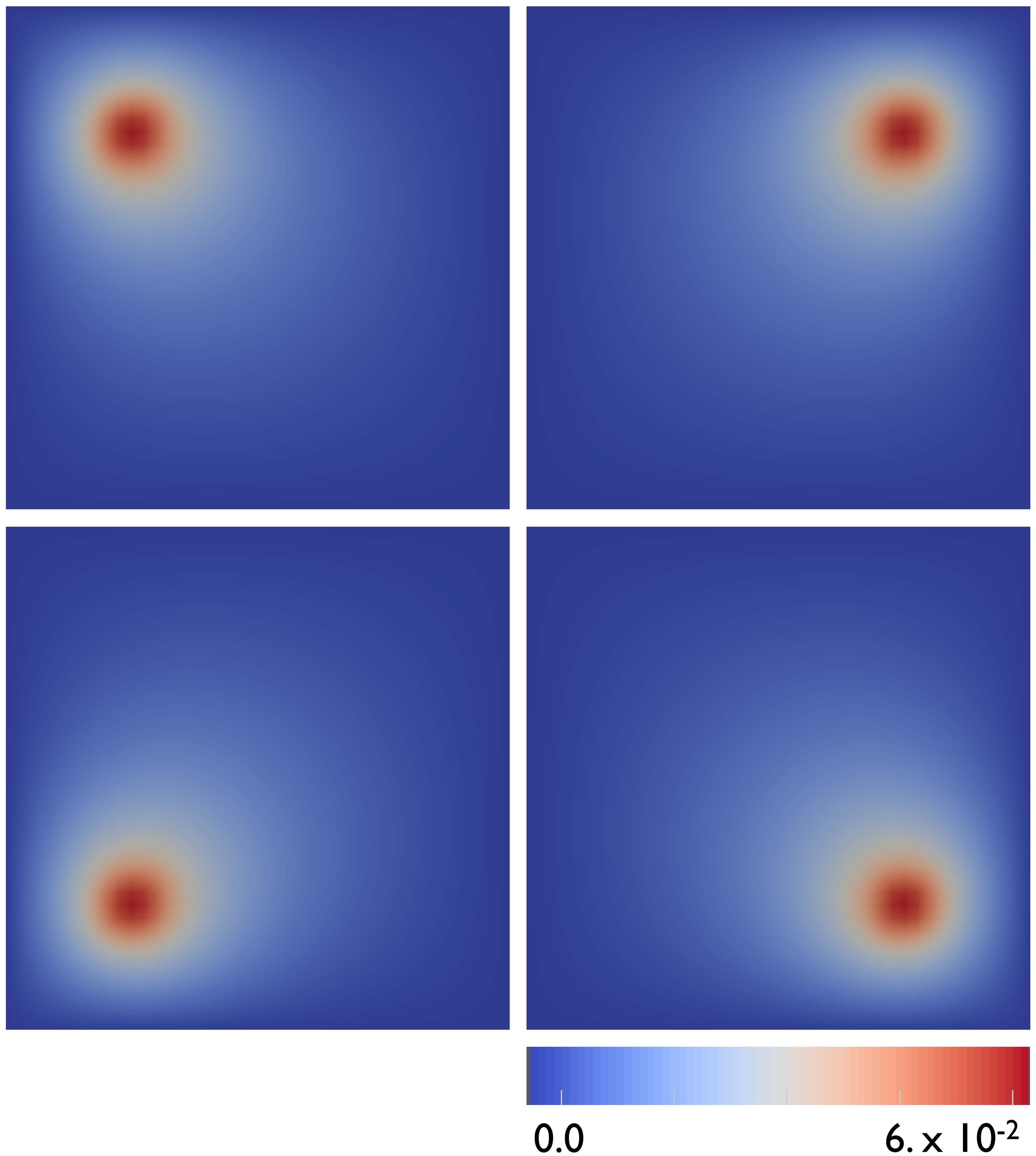
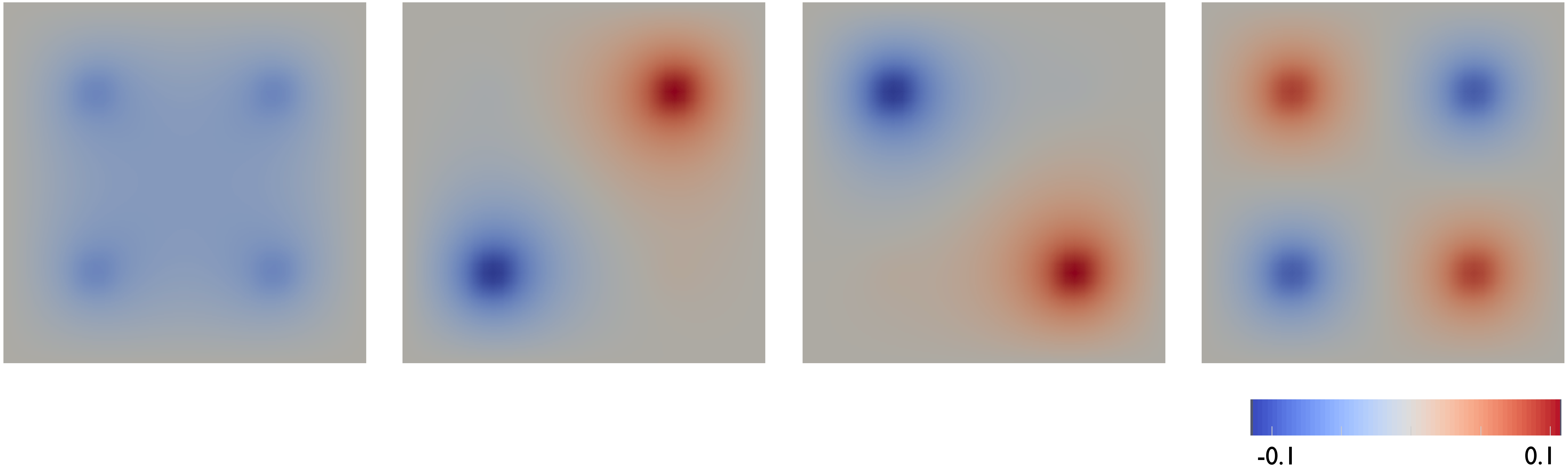


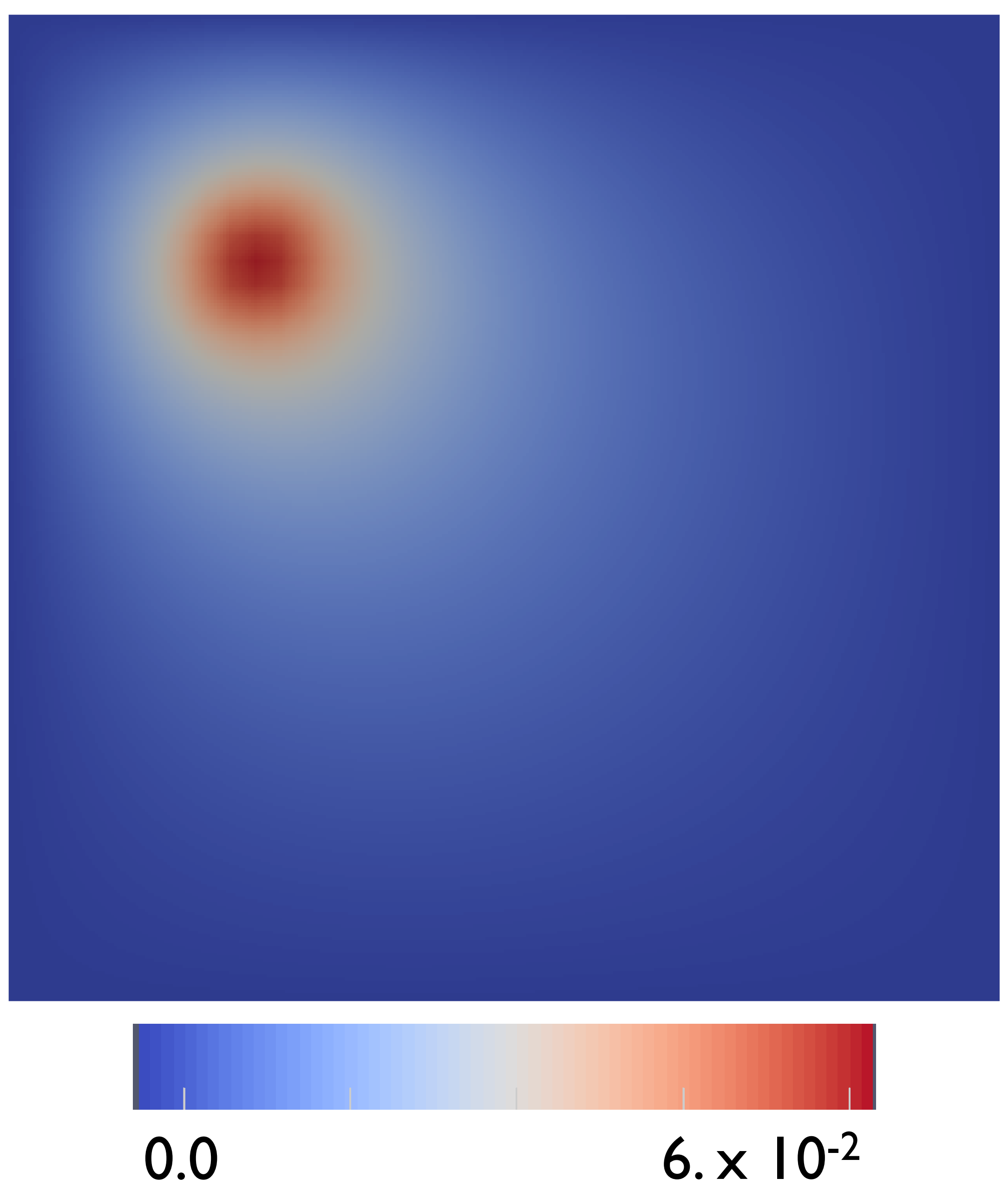
POD application
Diffusion problem

POD basis functions

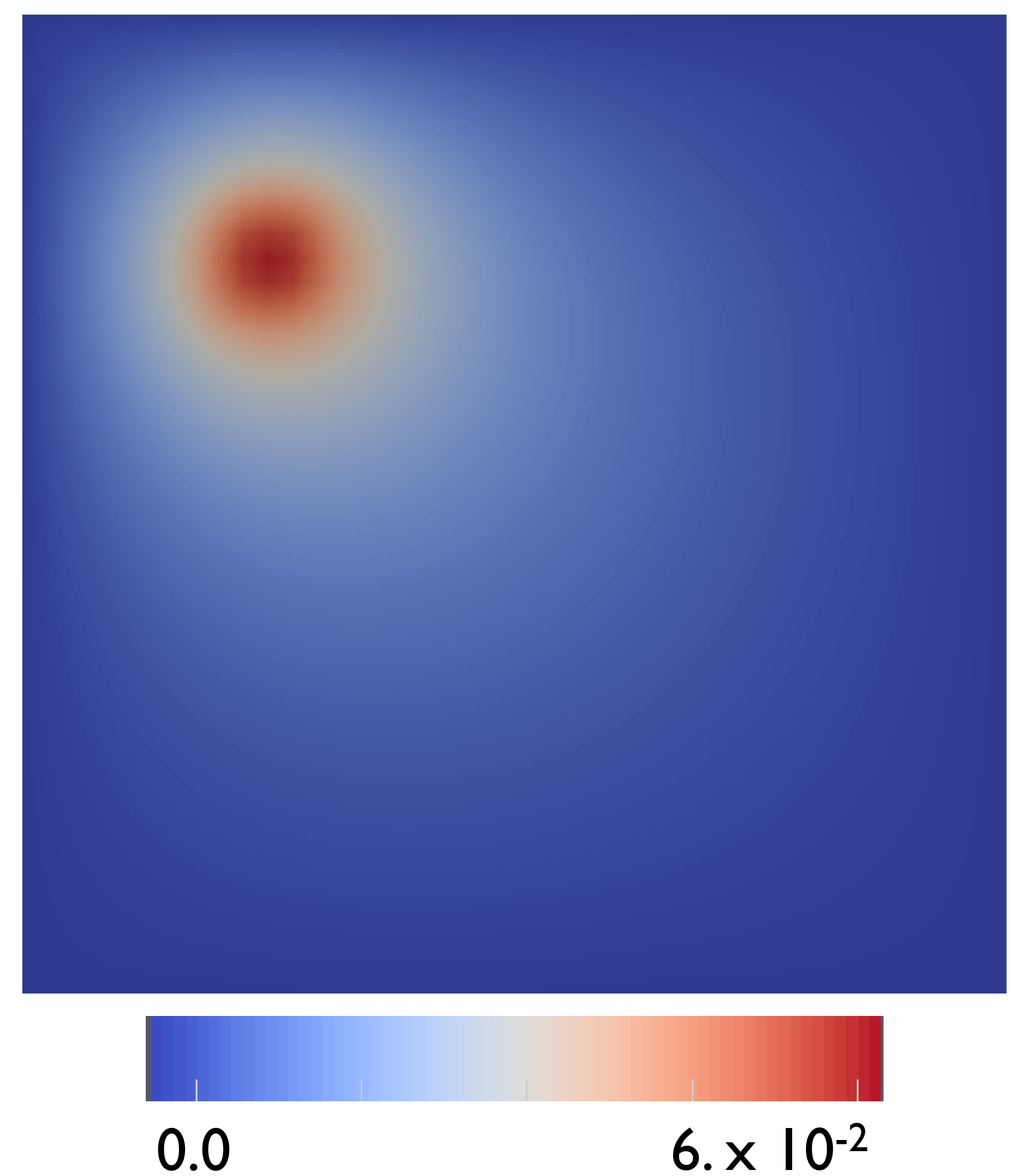


POD application
Diffusion problem

FEM



POD (N=4)



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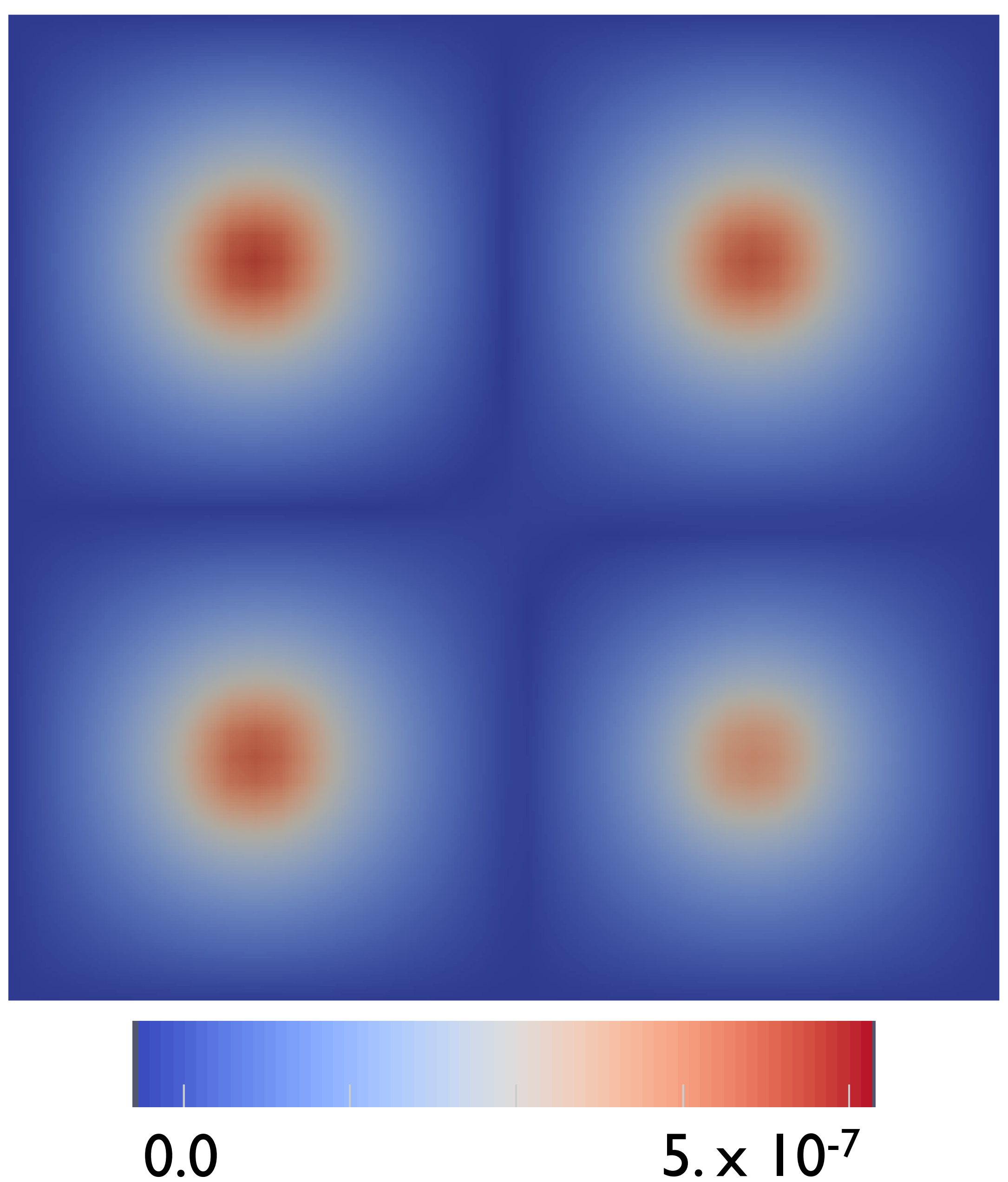
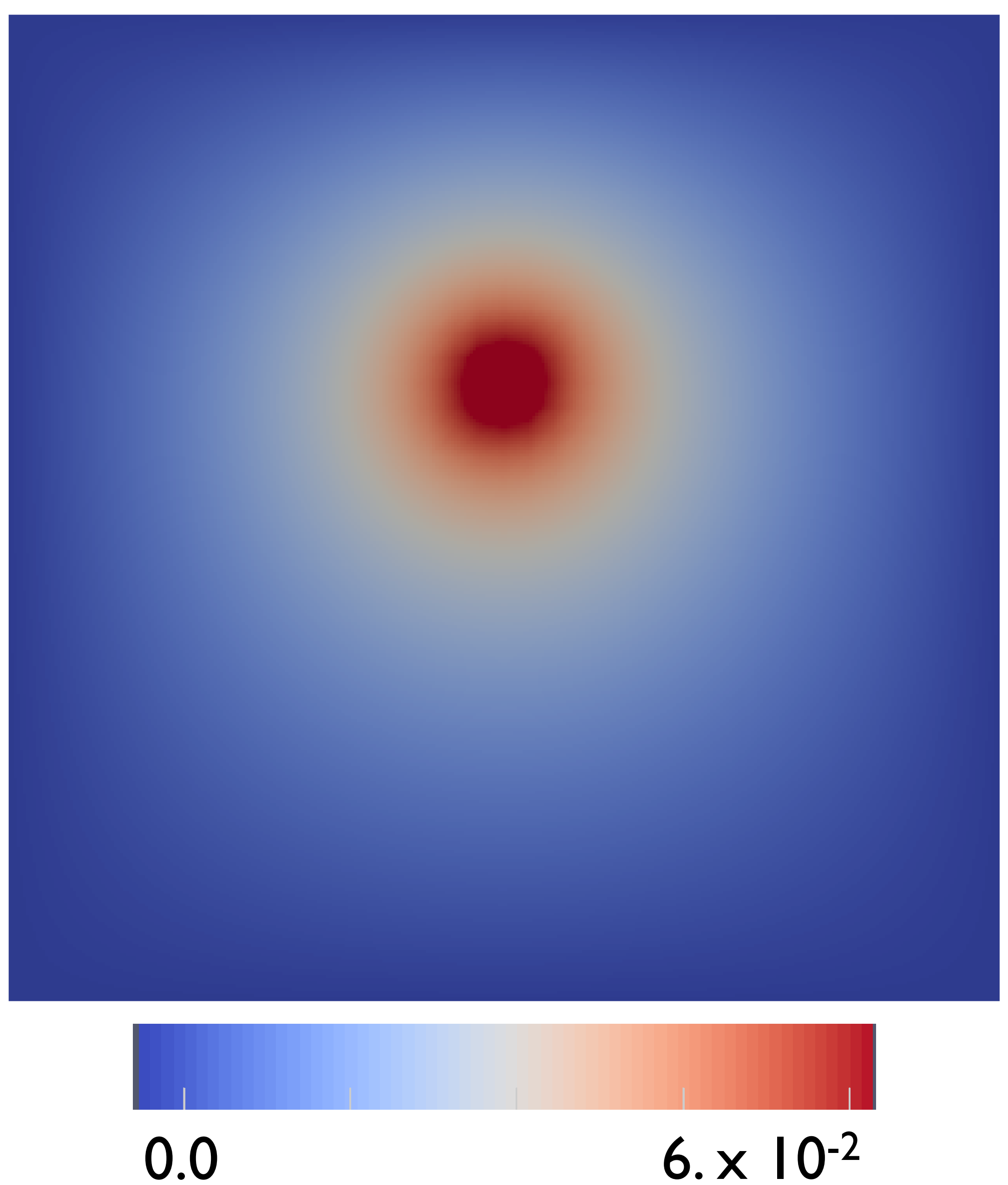


NB the solution is "in" the basis

POD application
Diffusion problem

FEM



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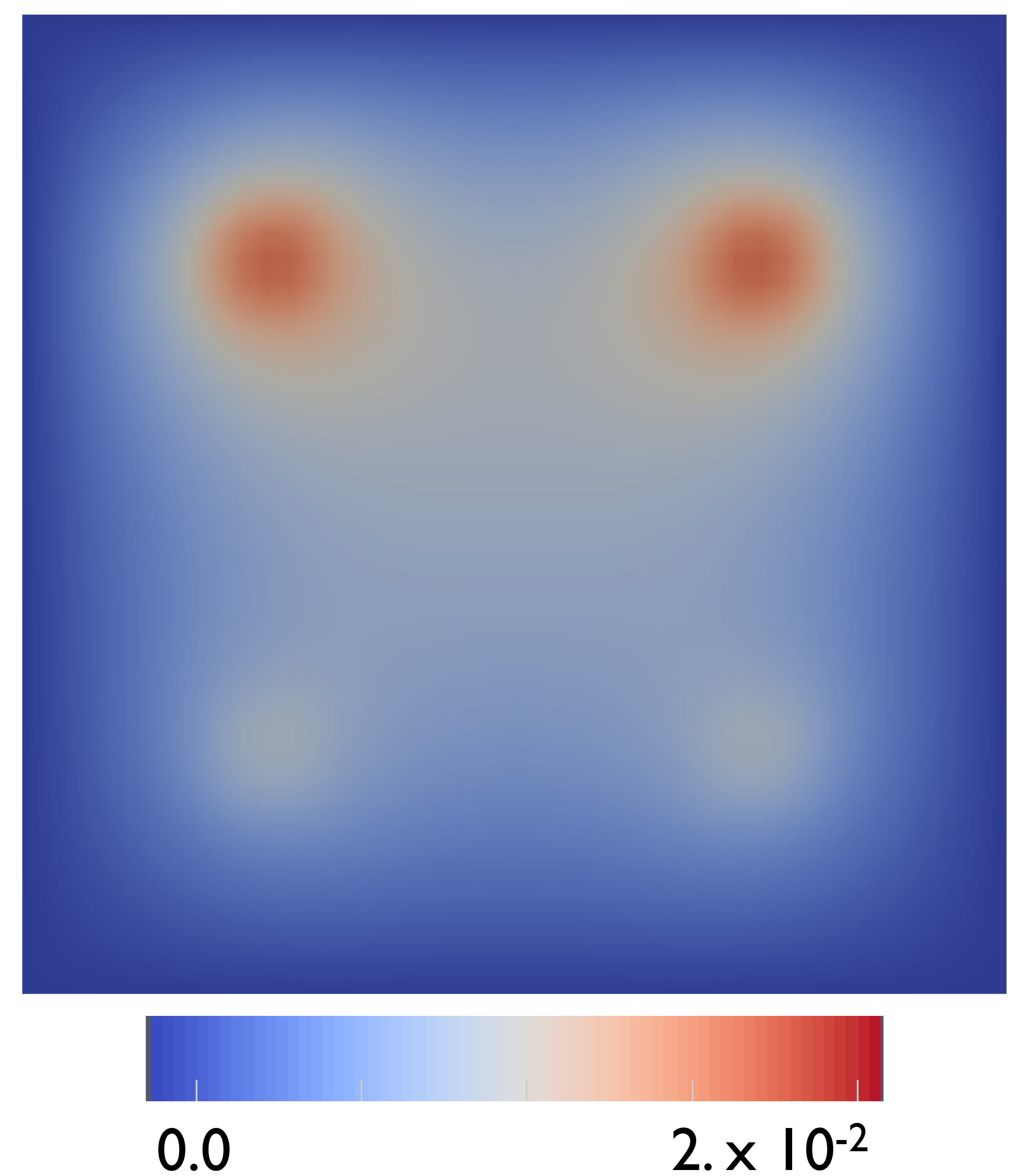

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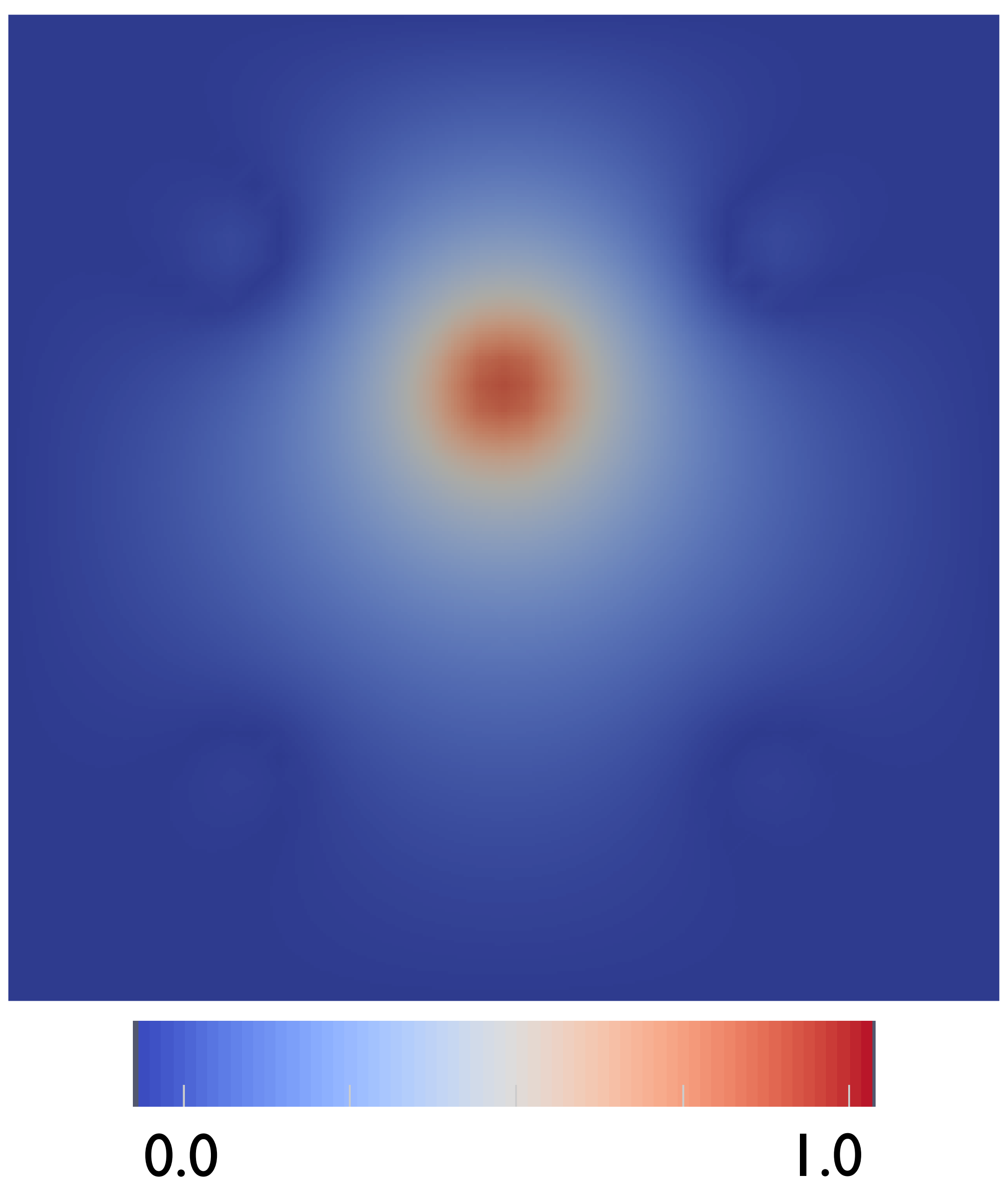

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POD application
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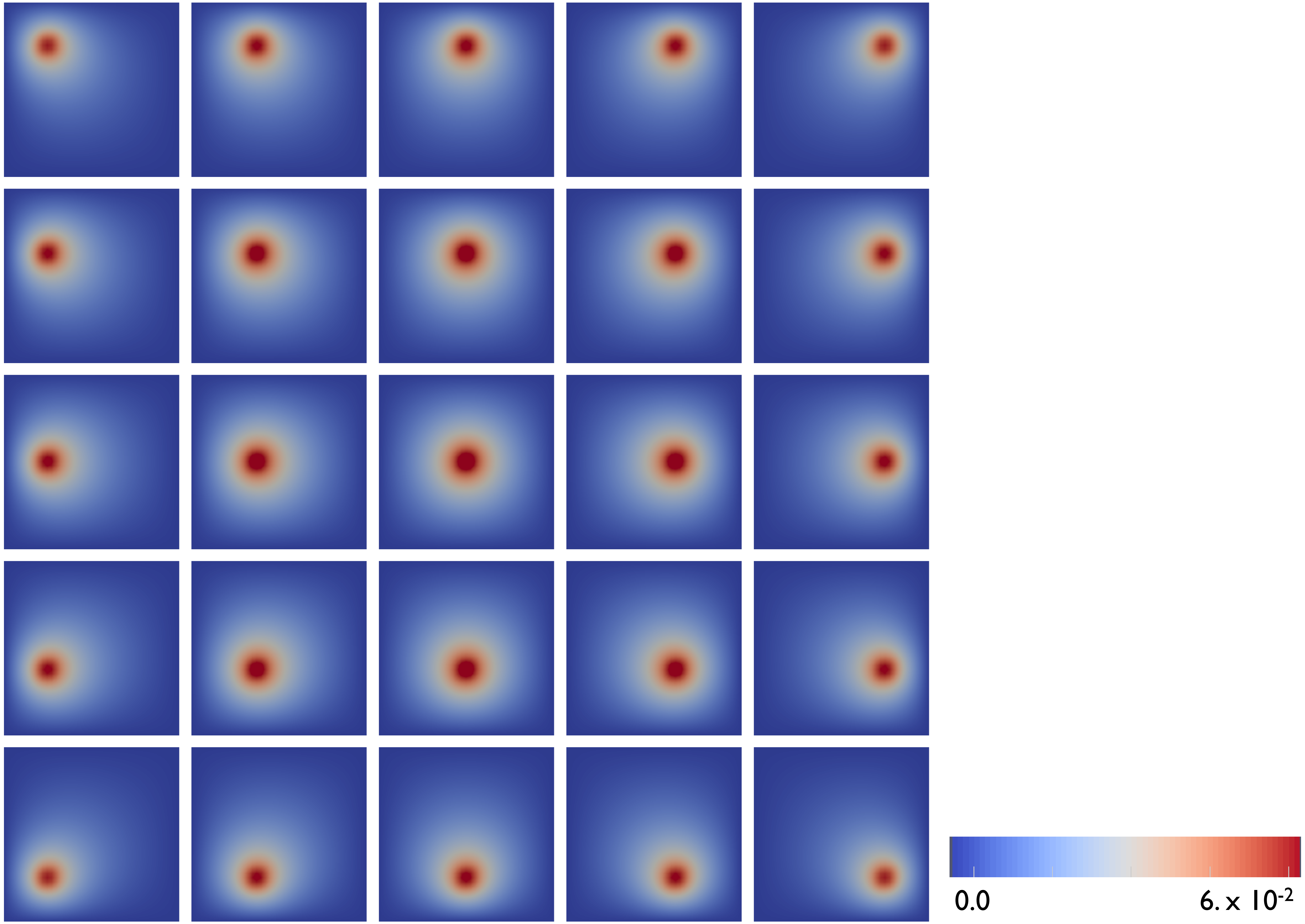
FEM snapshots
25 sources



POD application
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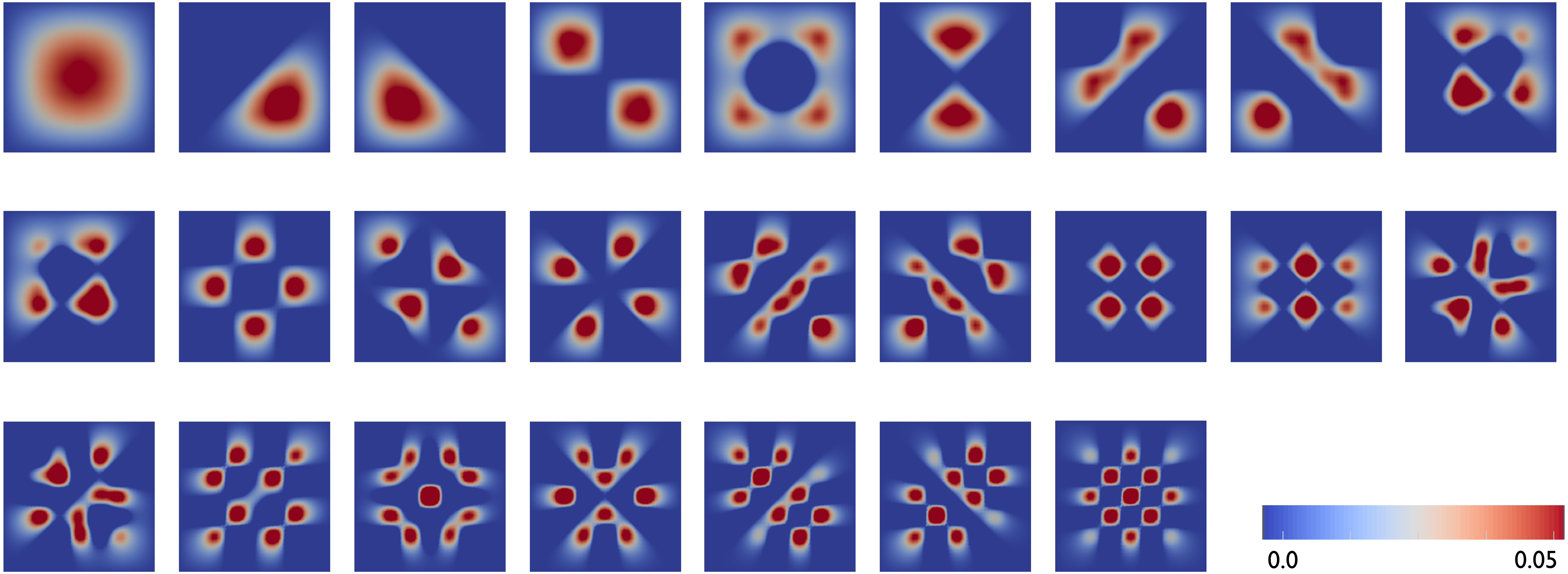
POD basis functions



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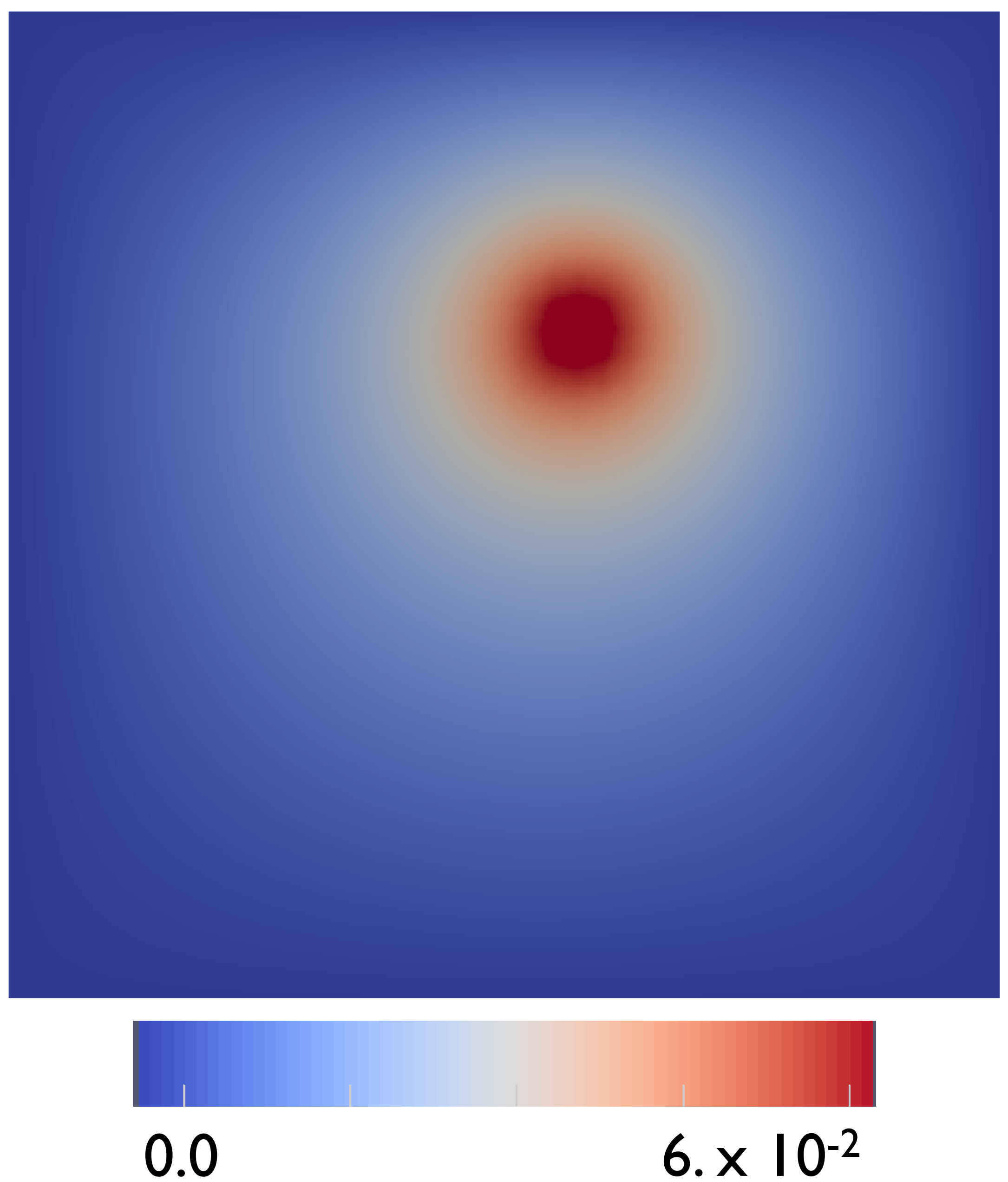
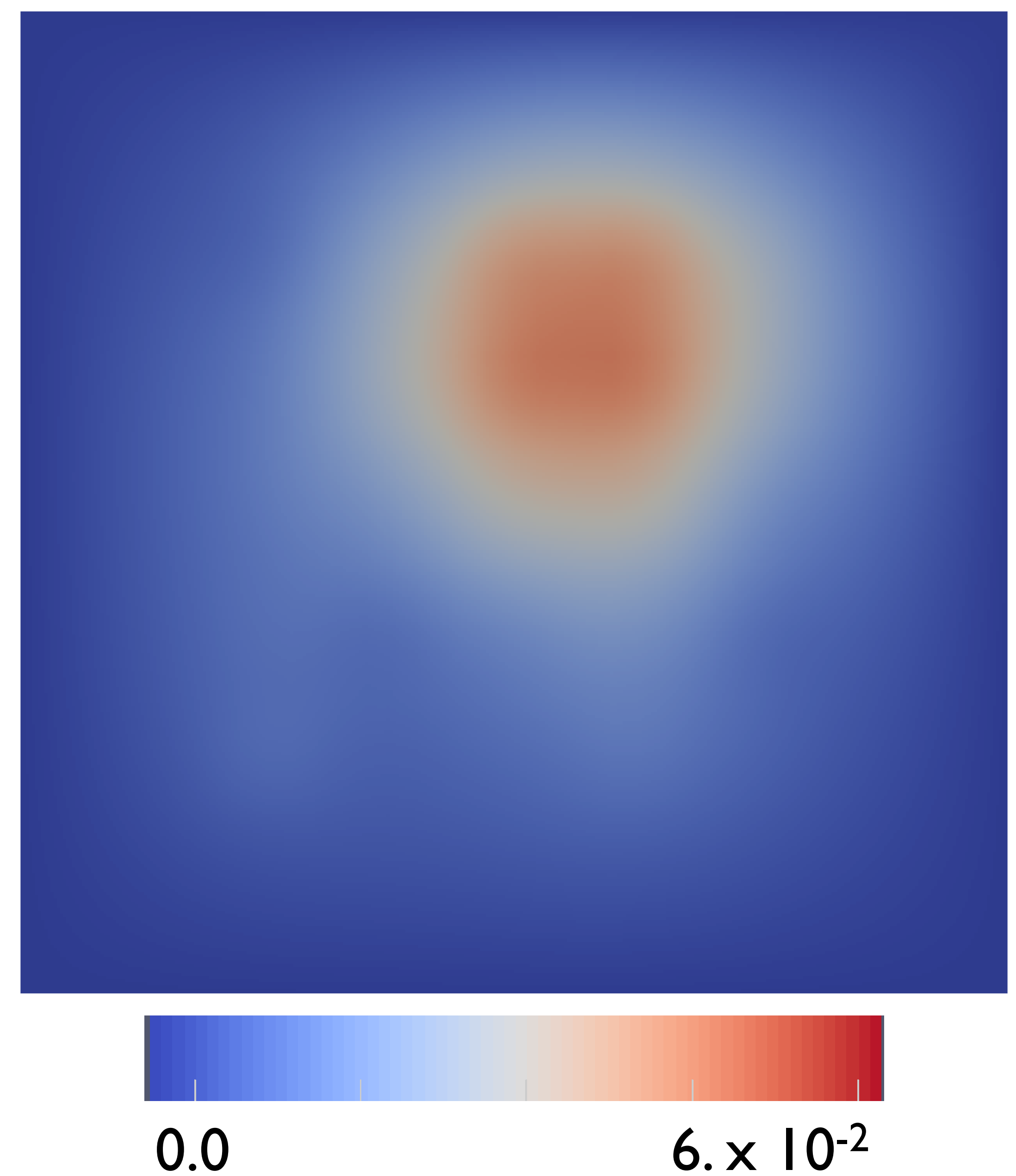


POD (N=10)



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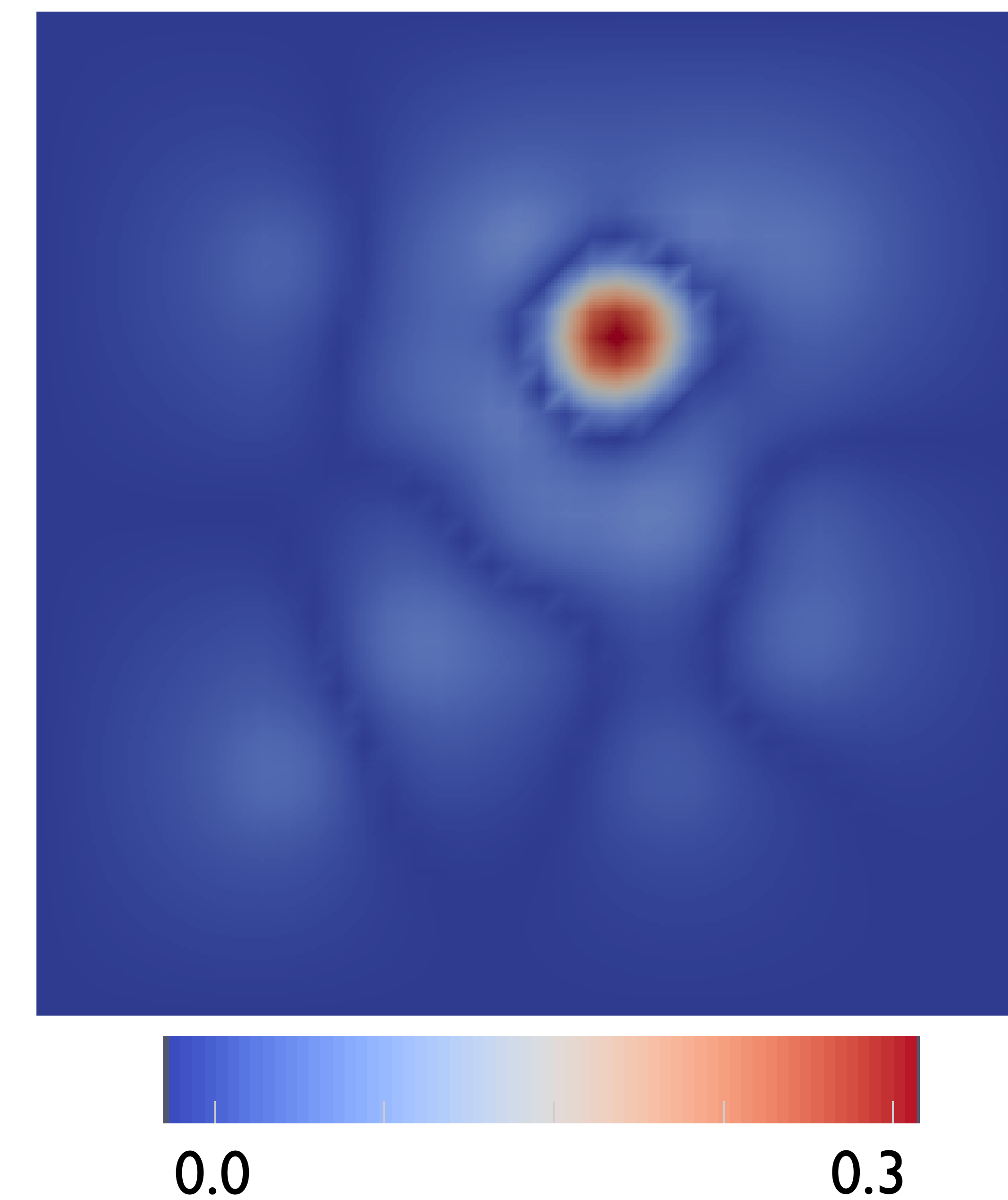

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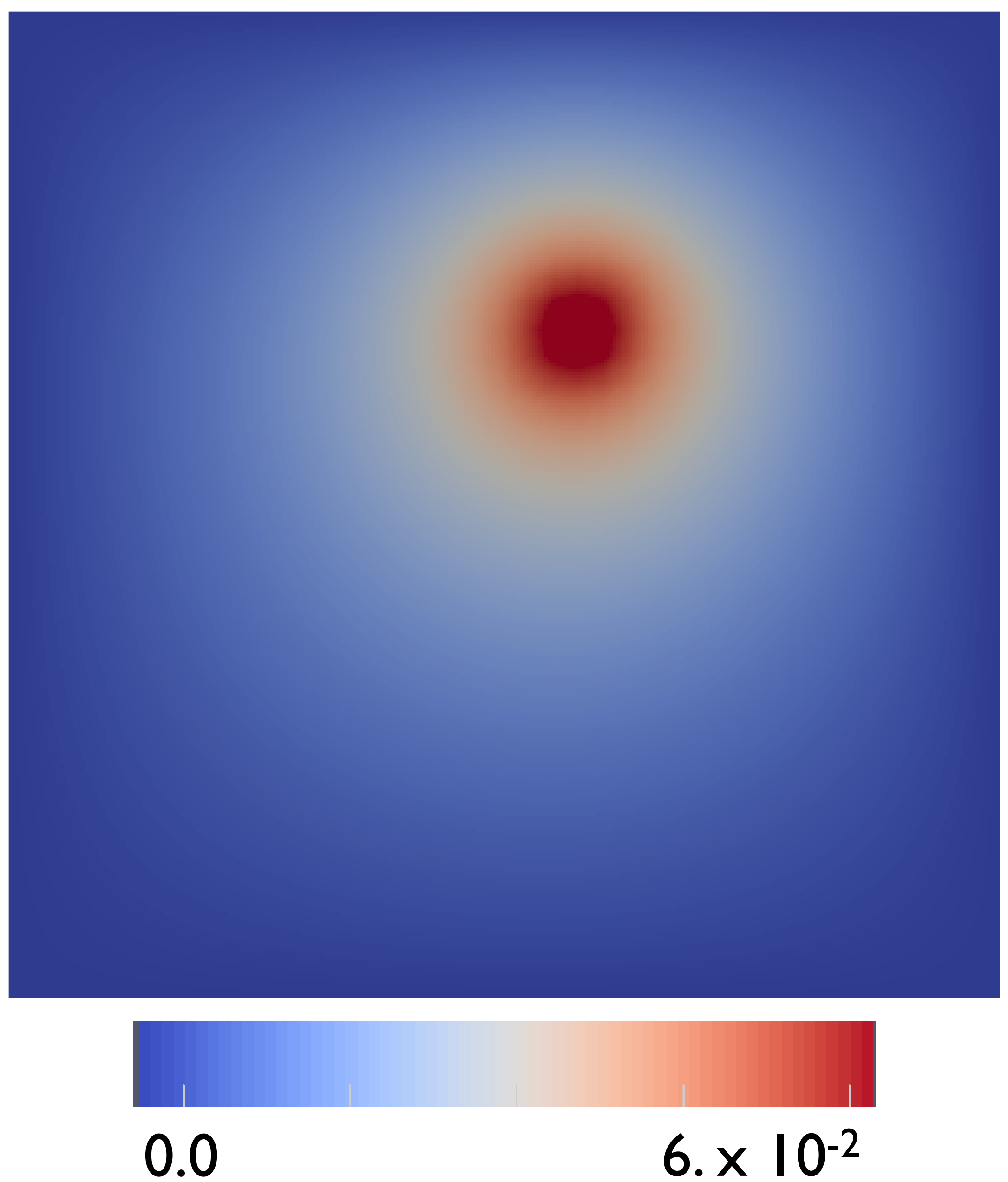
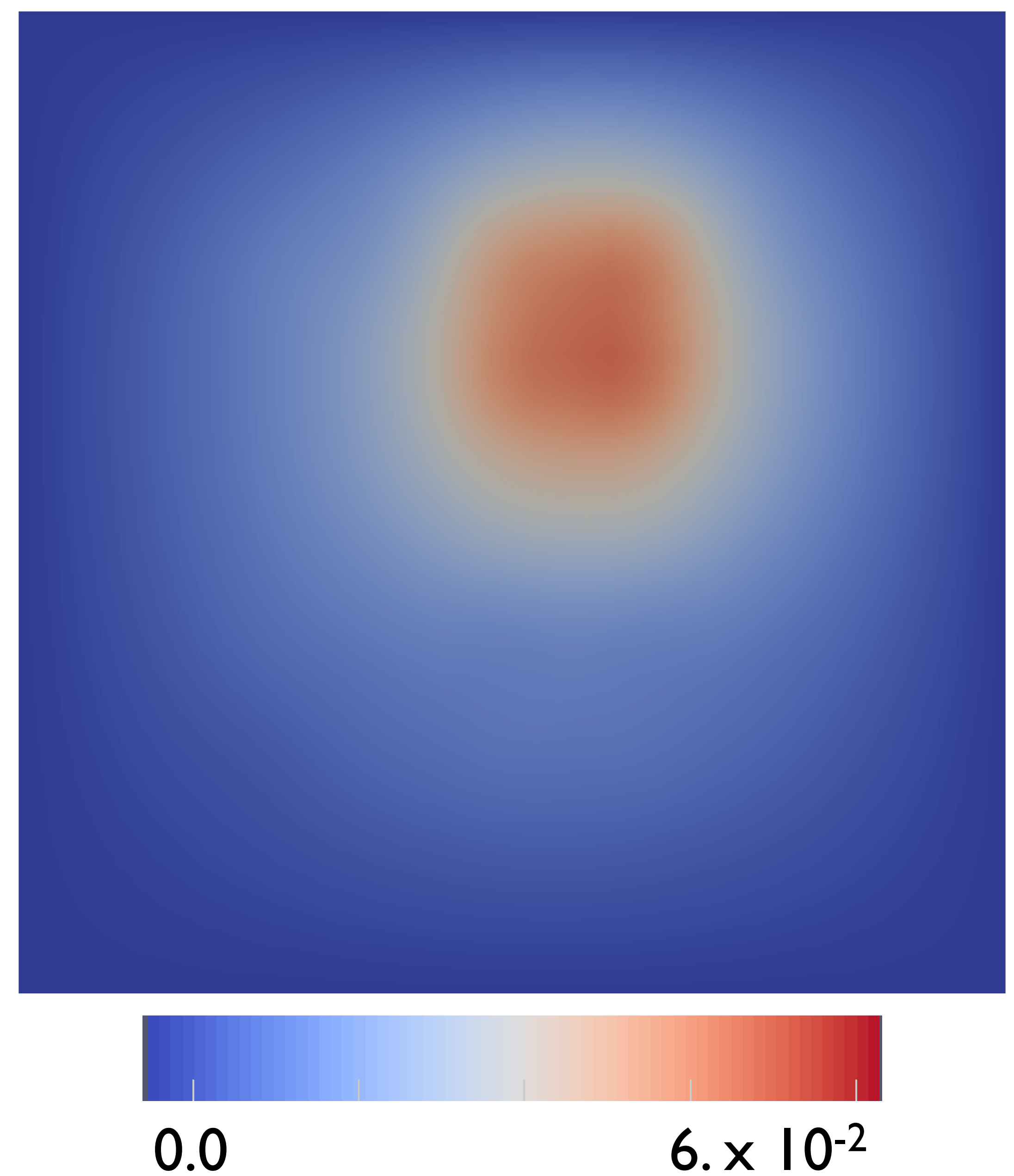


POD (N=15)



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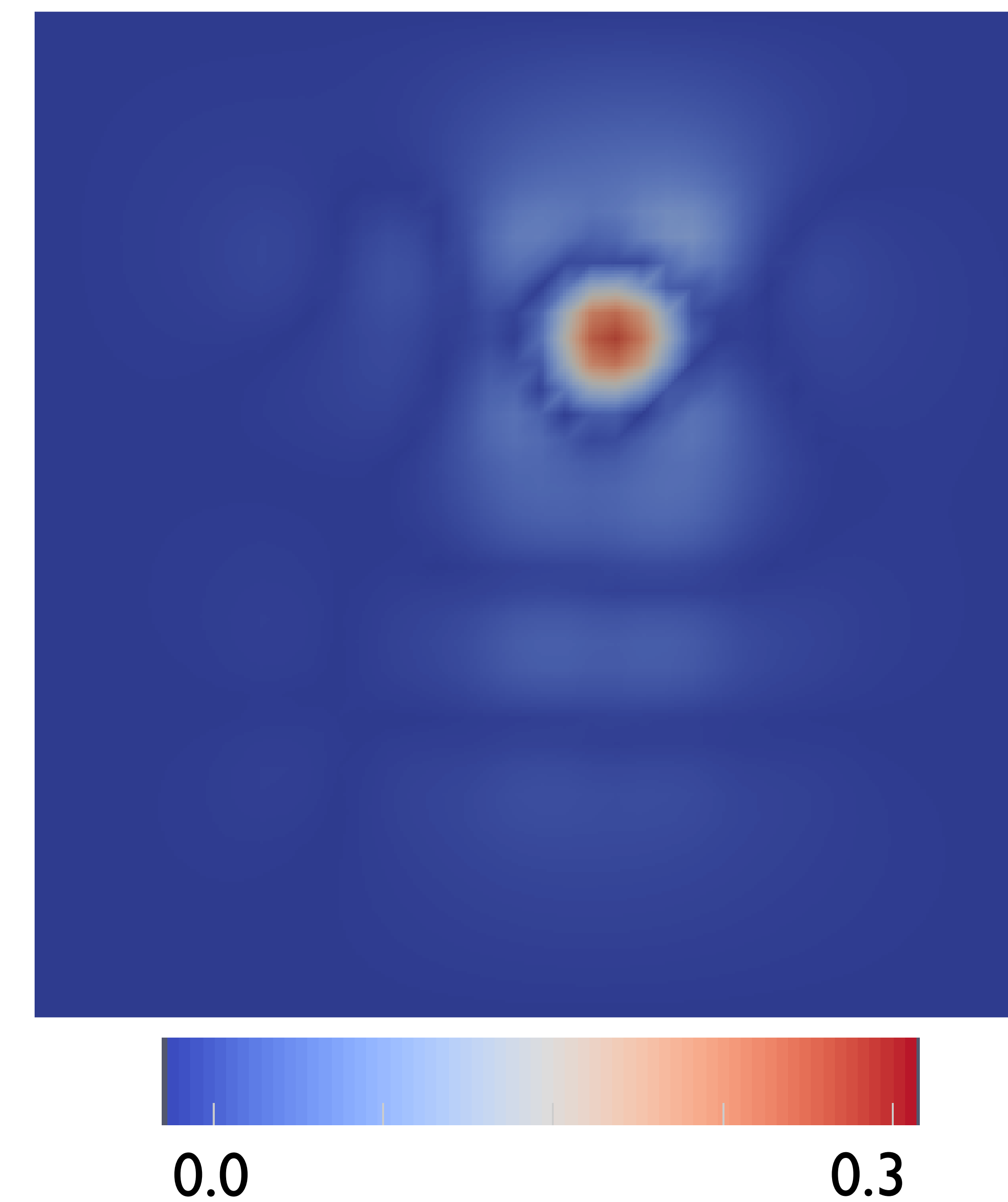

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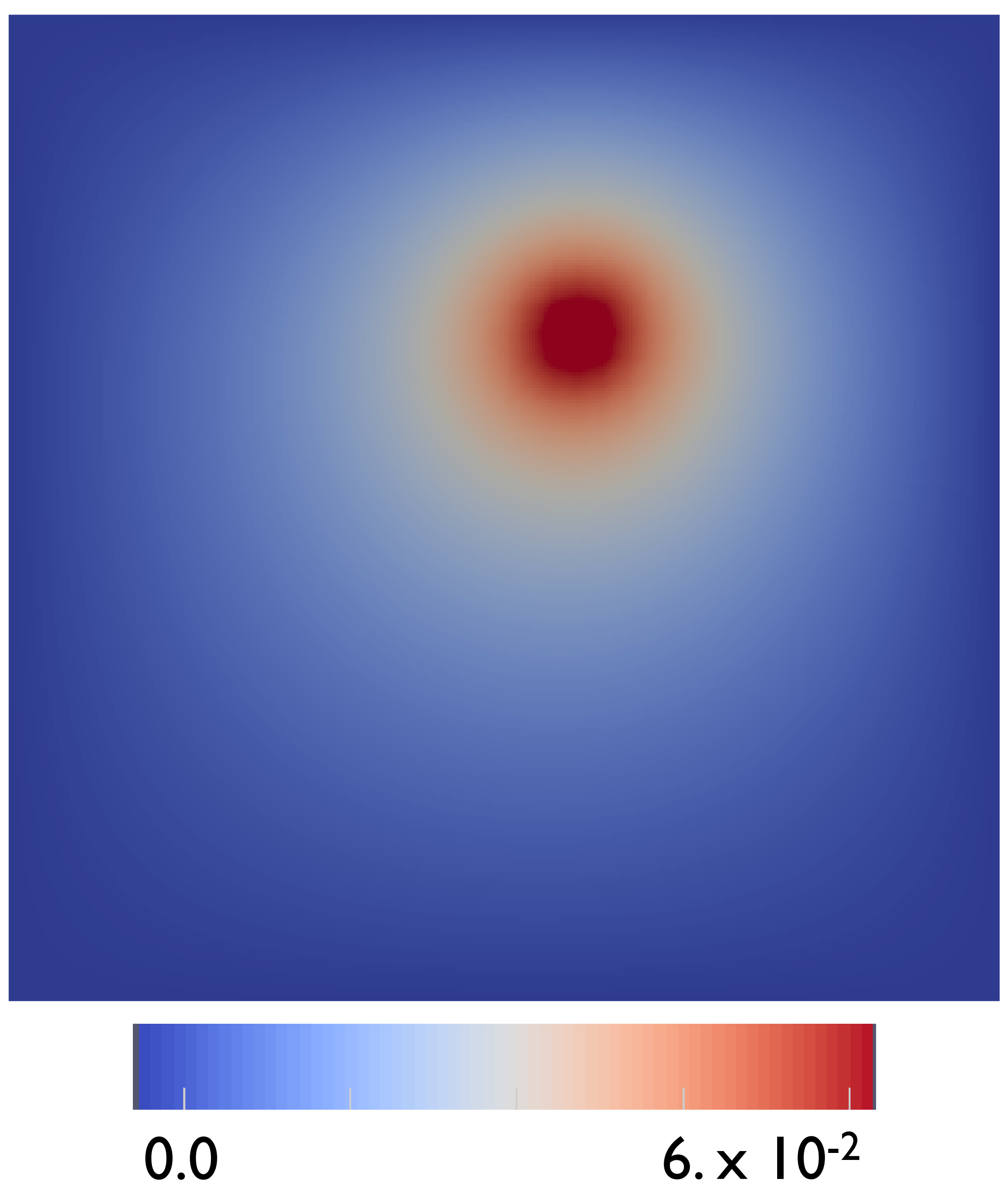
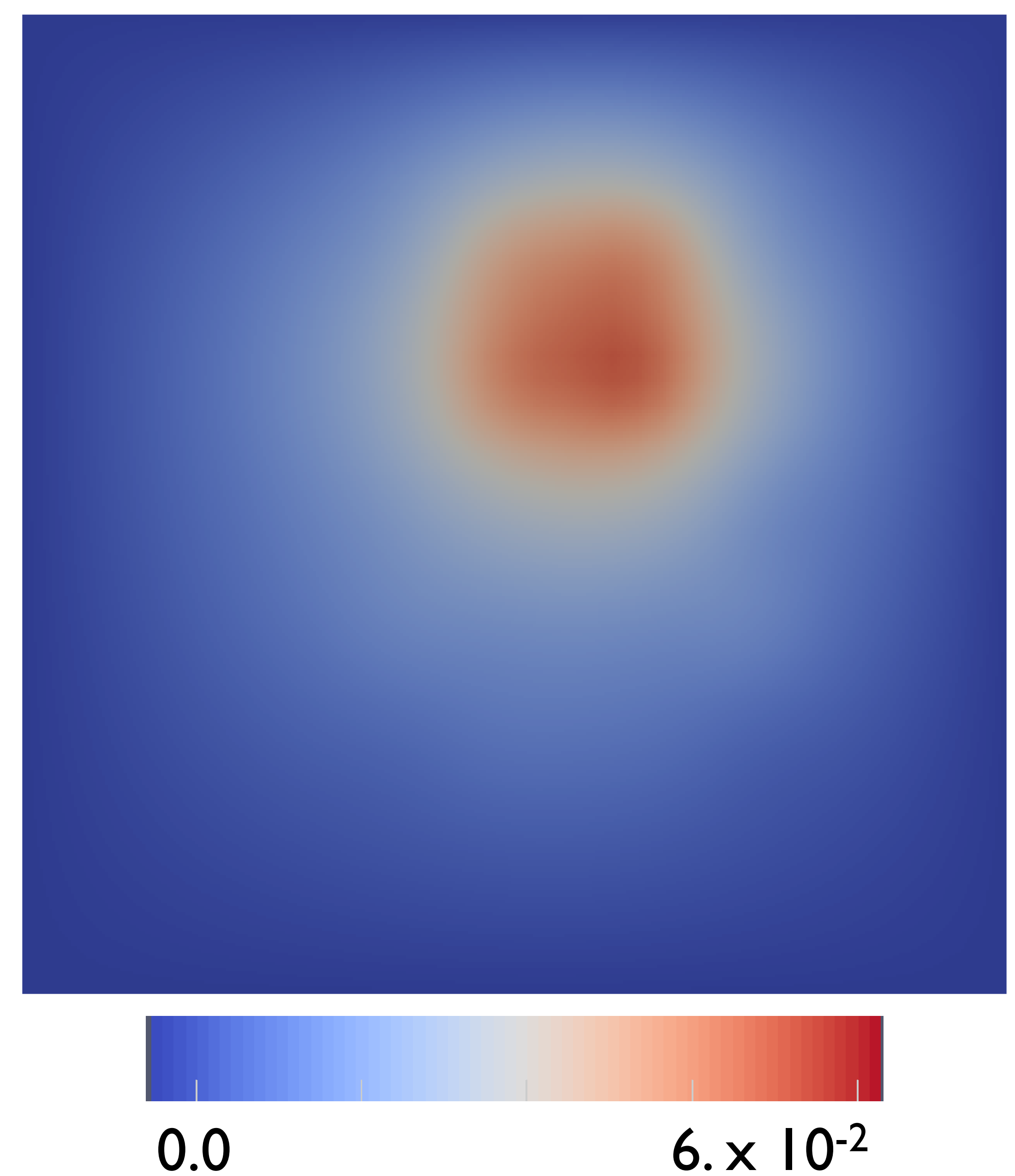


POD (N=20)



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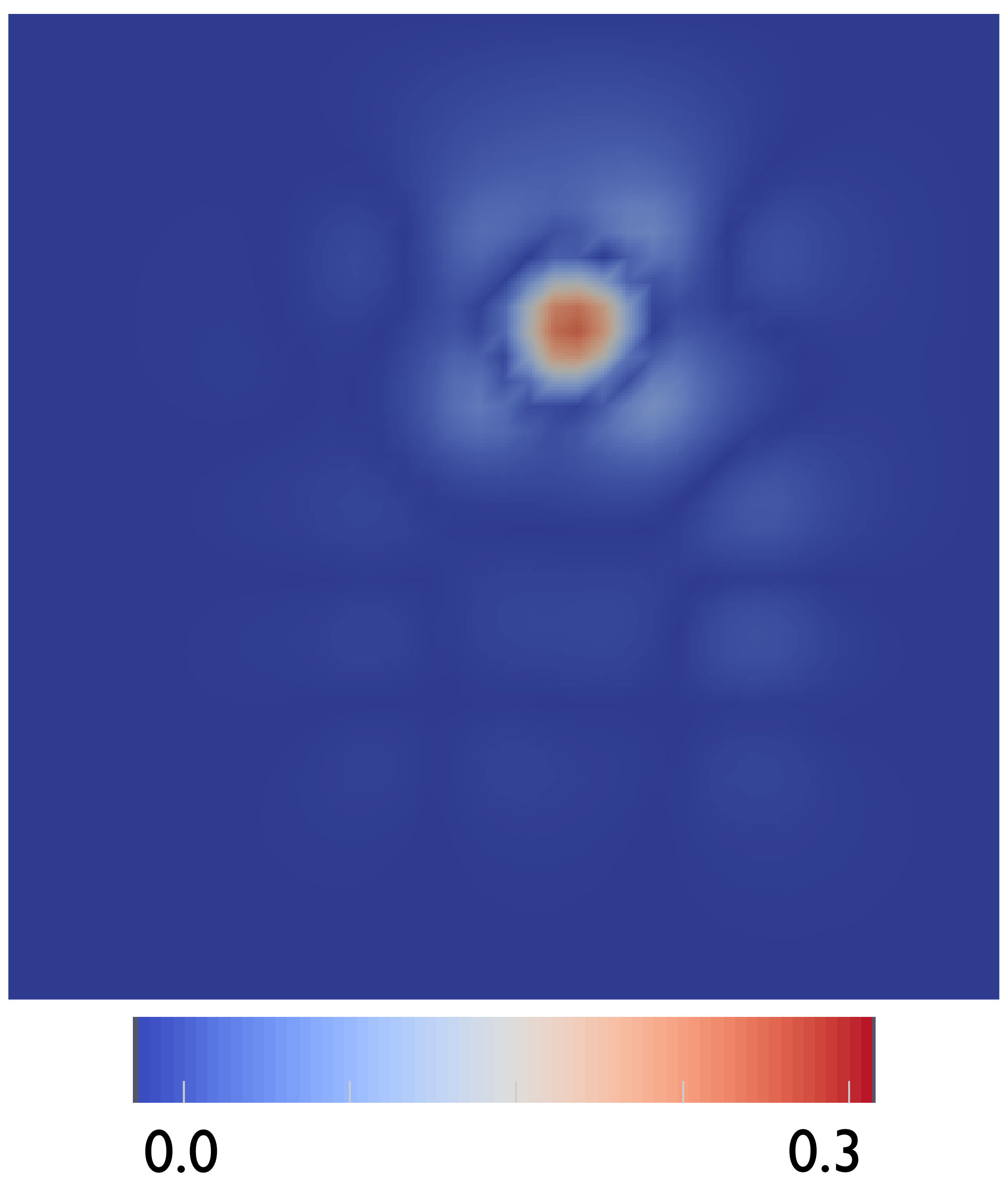

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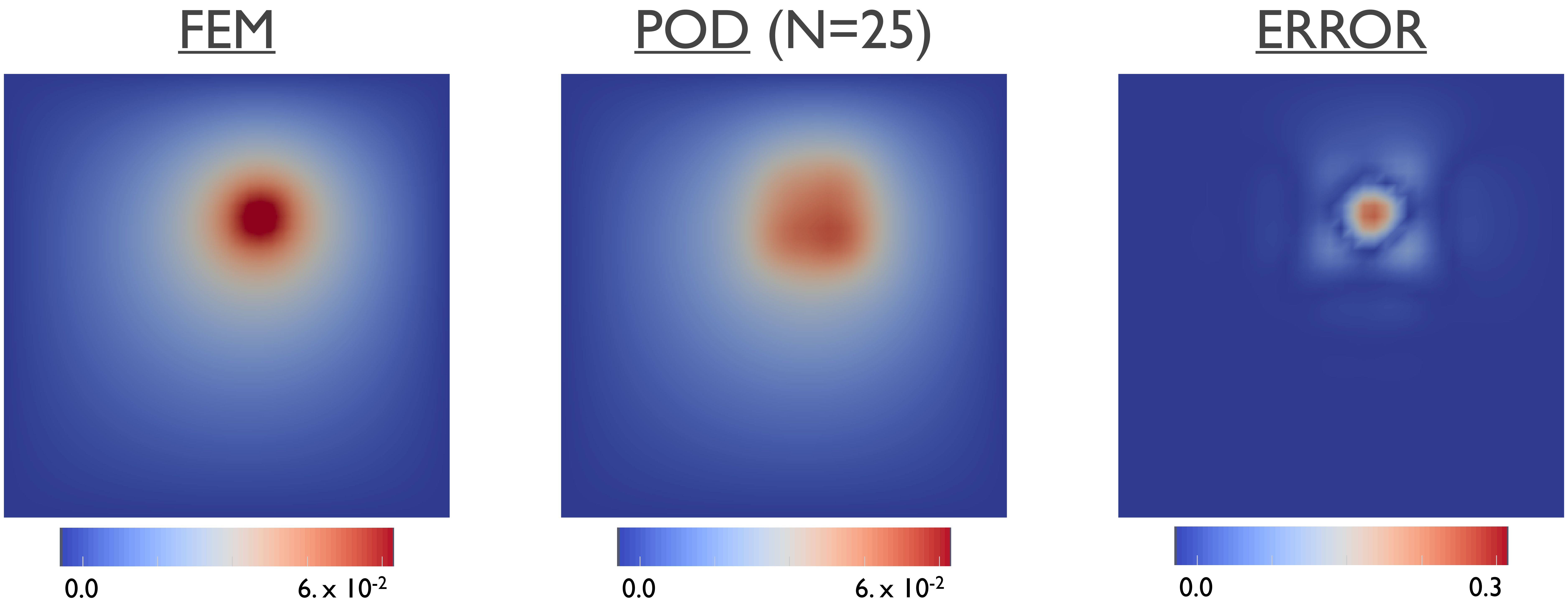

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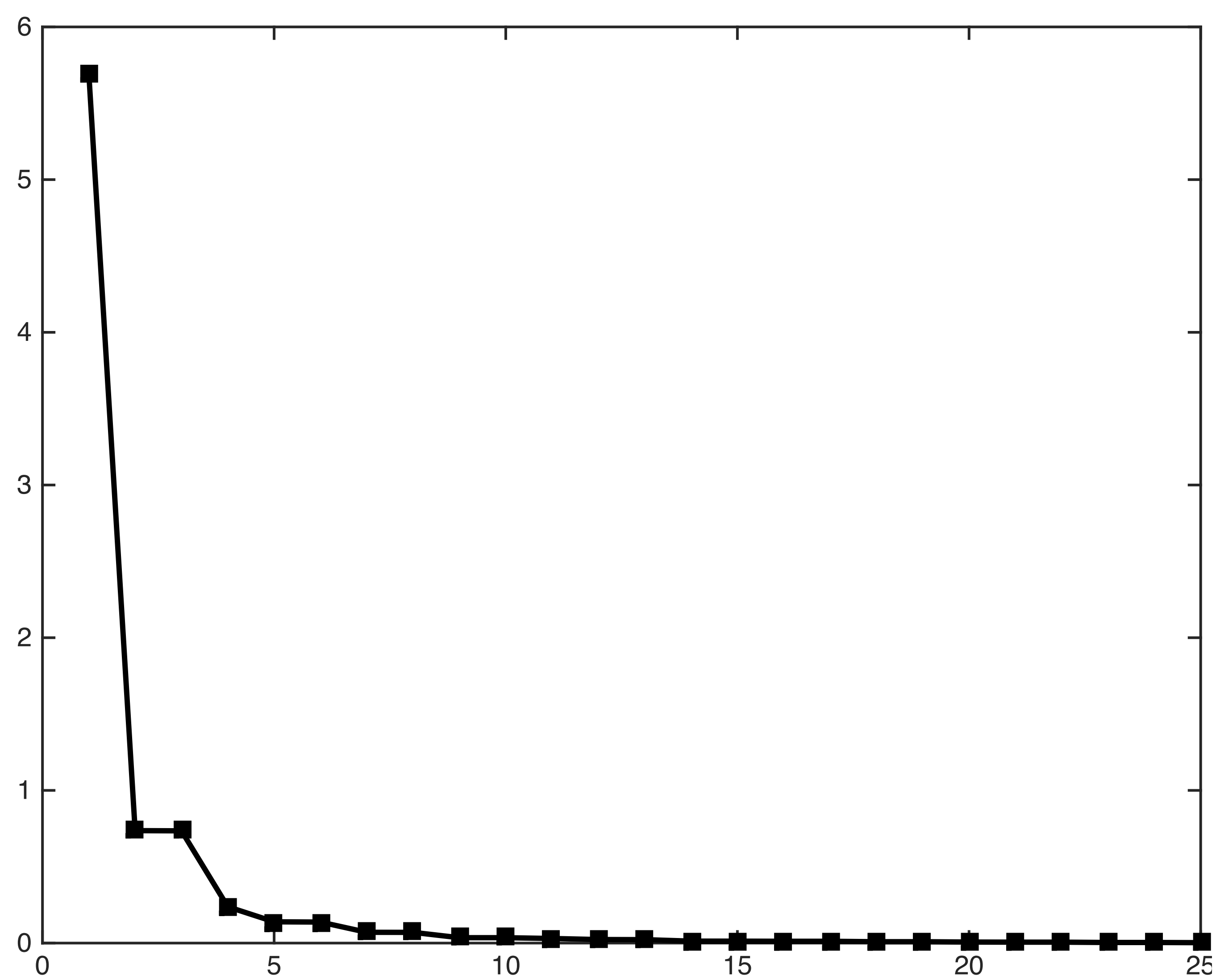
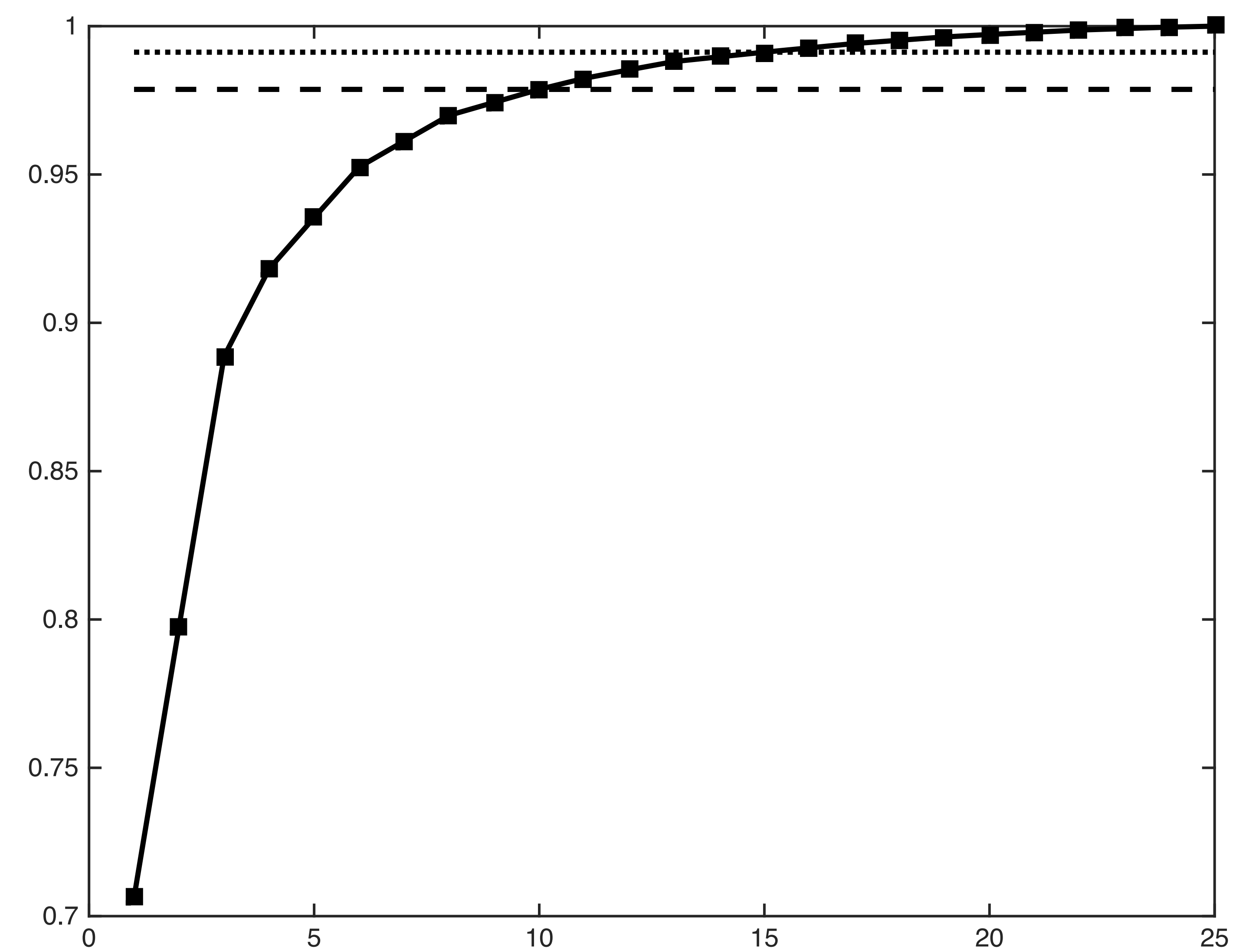
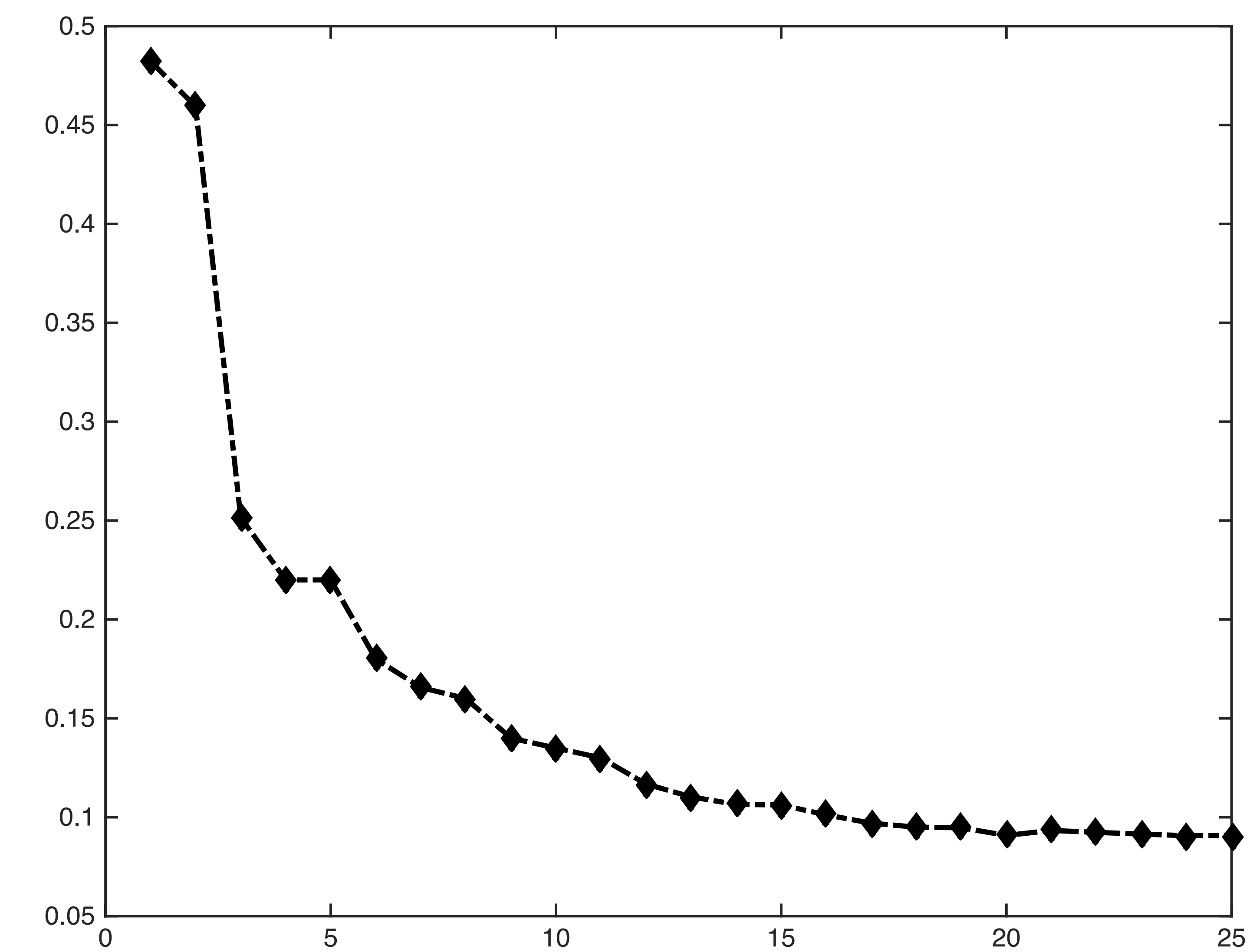
EIGENVALUES





$$\frac{\sum^{N} \lambda_i}{\sum^{N_T} \lambda_i}$$

ERROR



$$\frac{\|u_{\mathrm{FEM}} - u_{\mathrm{POD}}\|_2}{\|u_{\mathrm{FEM}}\|_2}$$

POD application in Cardiac Electrophysiology

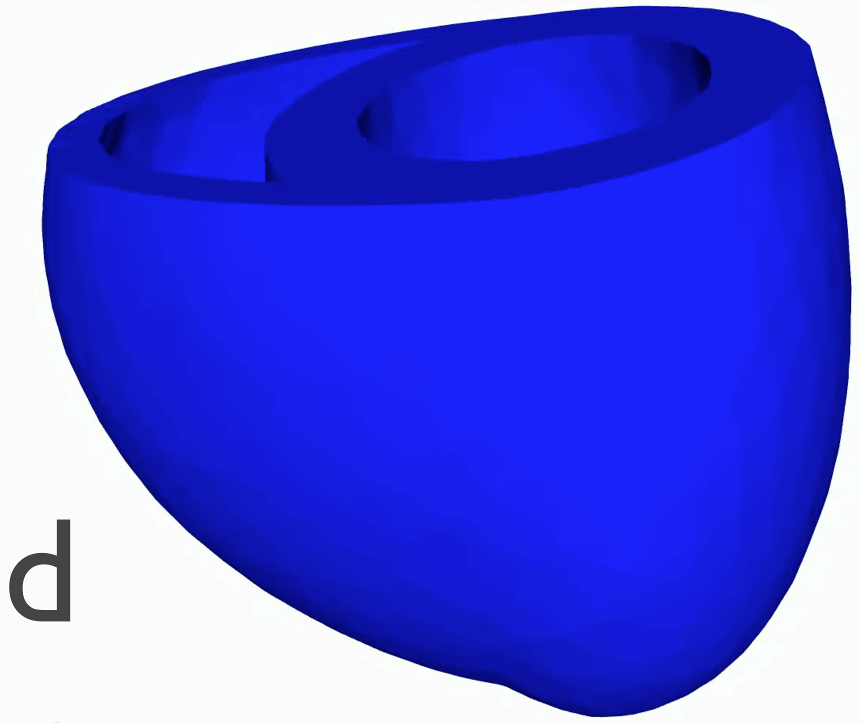
Simulation of a myocardial infarction

- Infarcted tissue: damaged area which cannot be activated
  ➡ to build the POD basis, we use a FEM solution with NO infarction

[Boulakia, Schenone, Gerbeau - 2012]

POD application in Cardiac Electrophysiology

Simulation of a myocardial infarction



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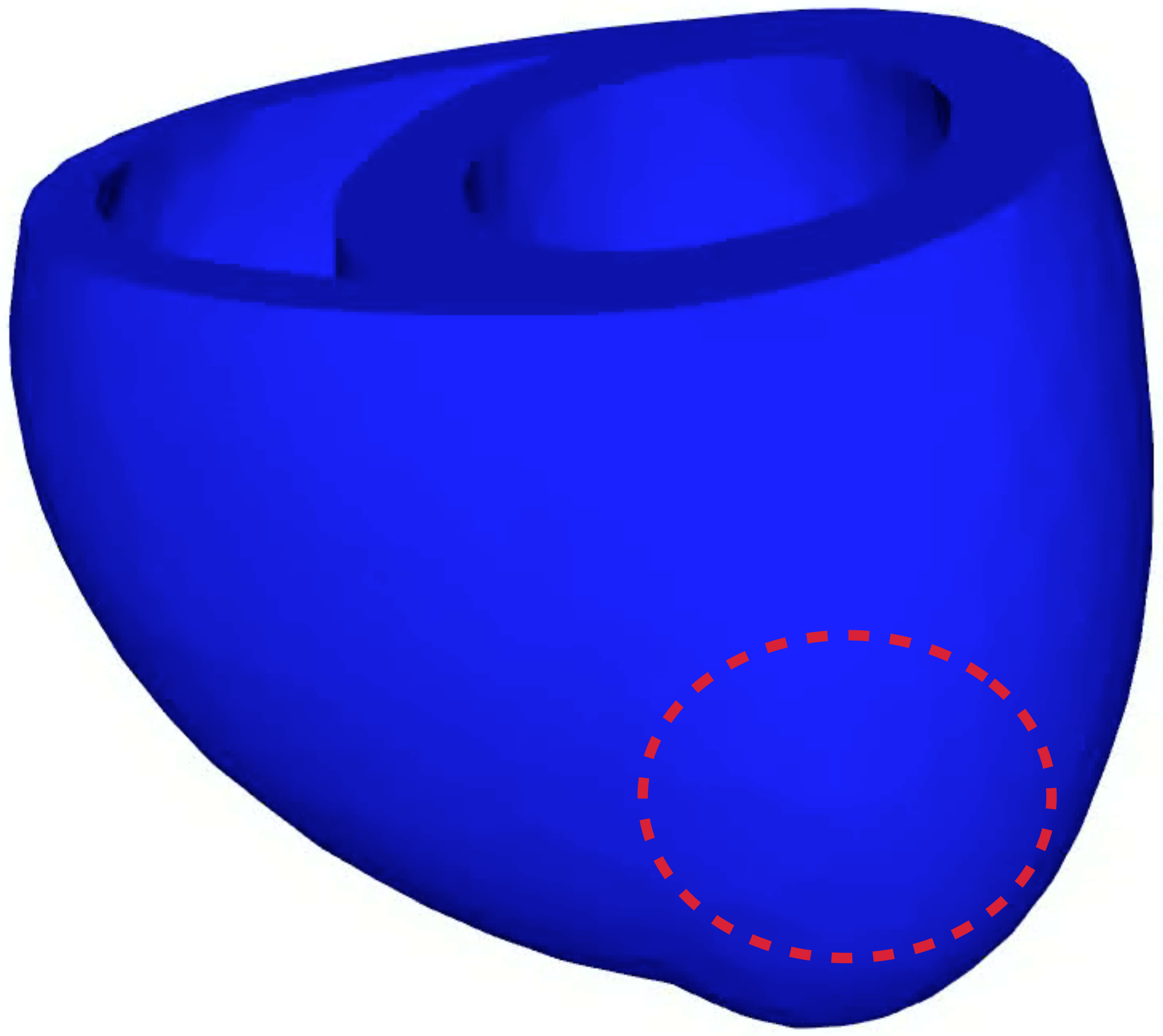

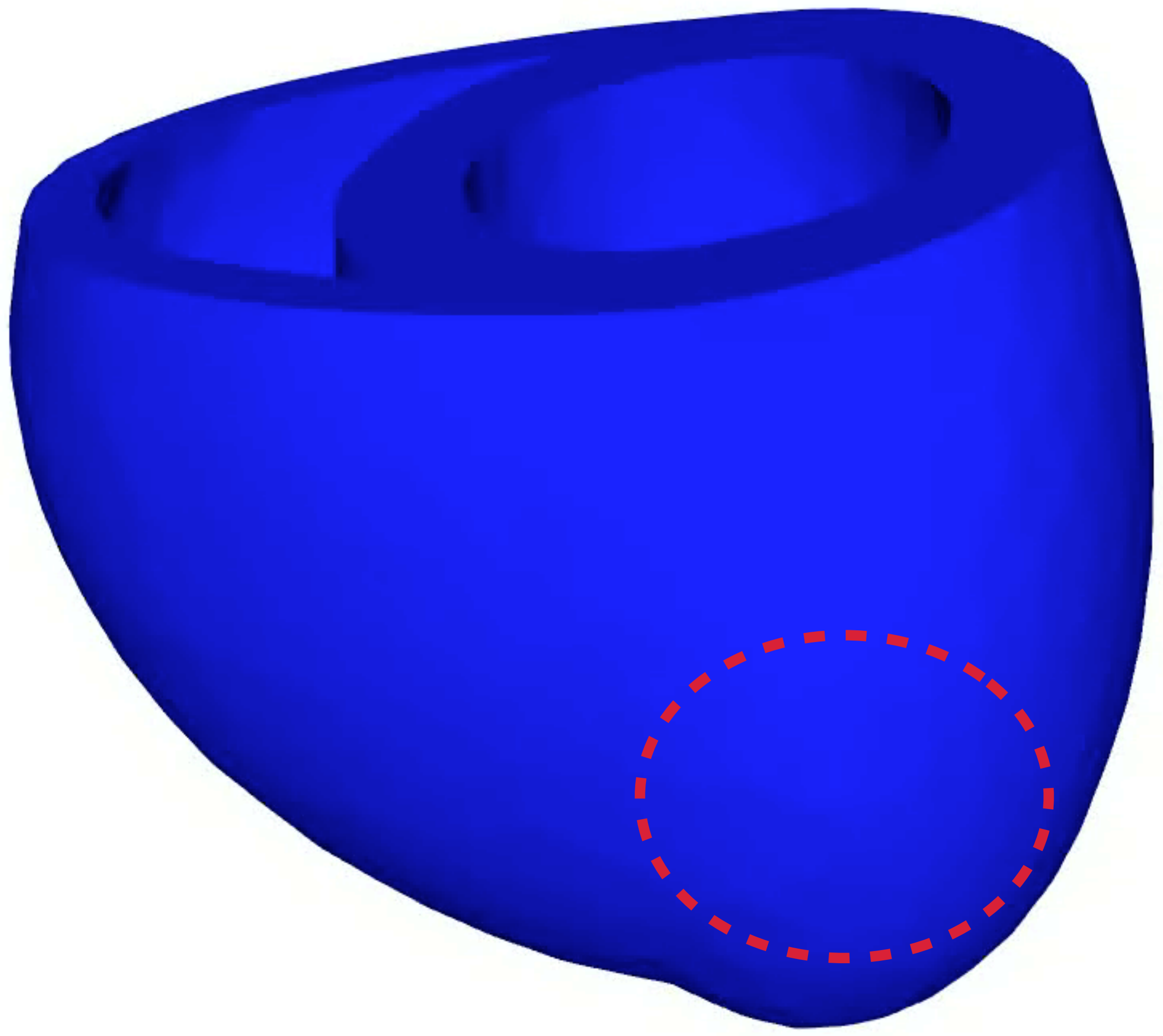

FEM (79,537 basis) POD (100 basis)

[Boulakia, Schenone, Gerbeau - 2012]

POD application in Cardiac Electrophysiology

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- Infarcted tissue: damaged area which cannot be activated
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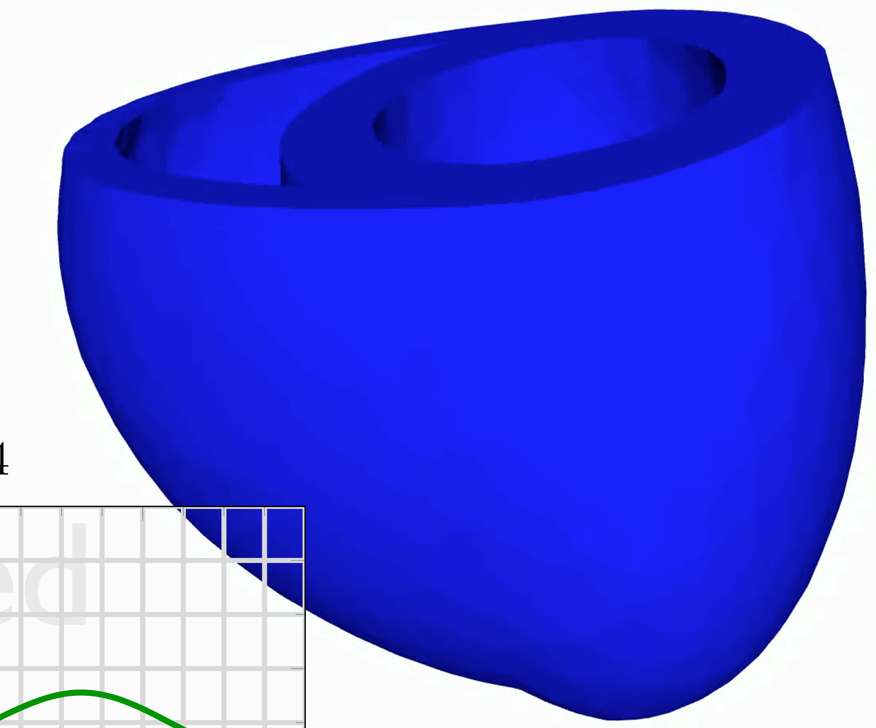
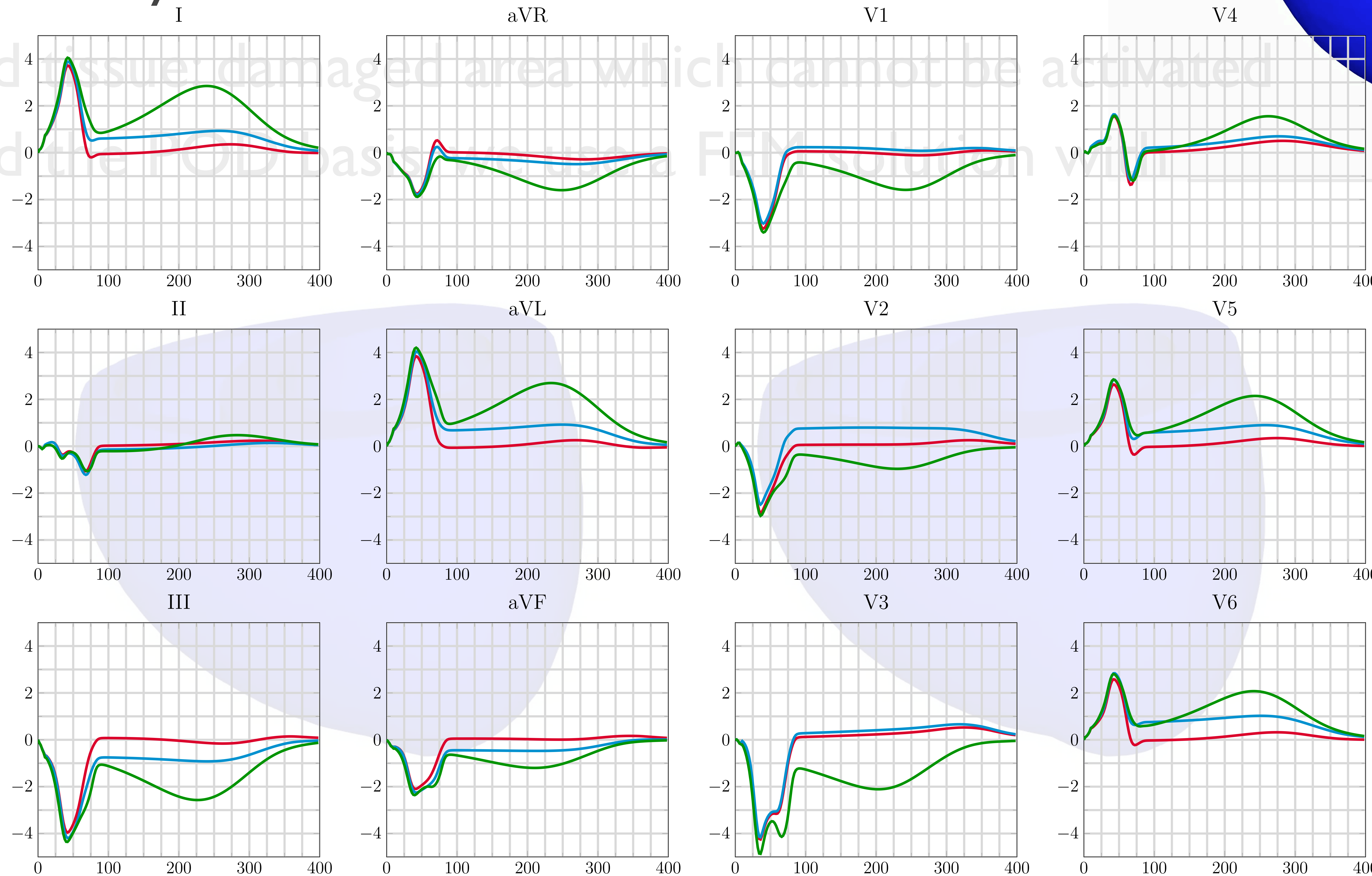




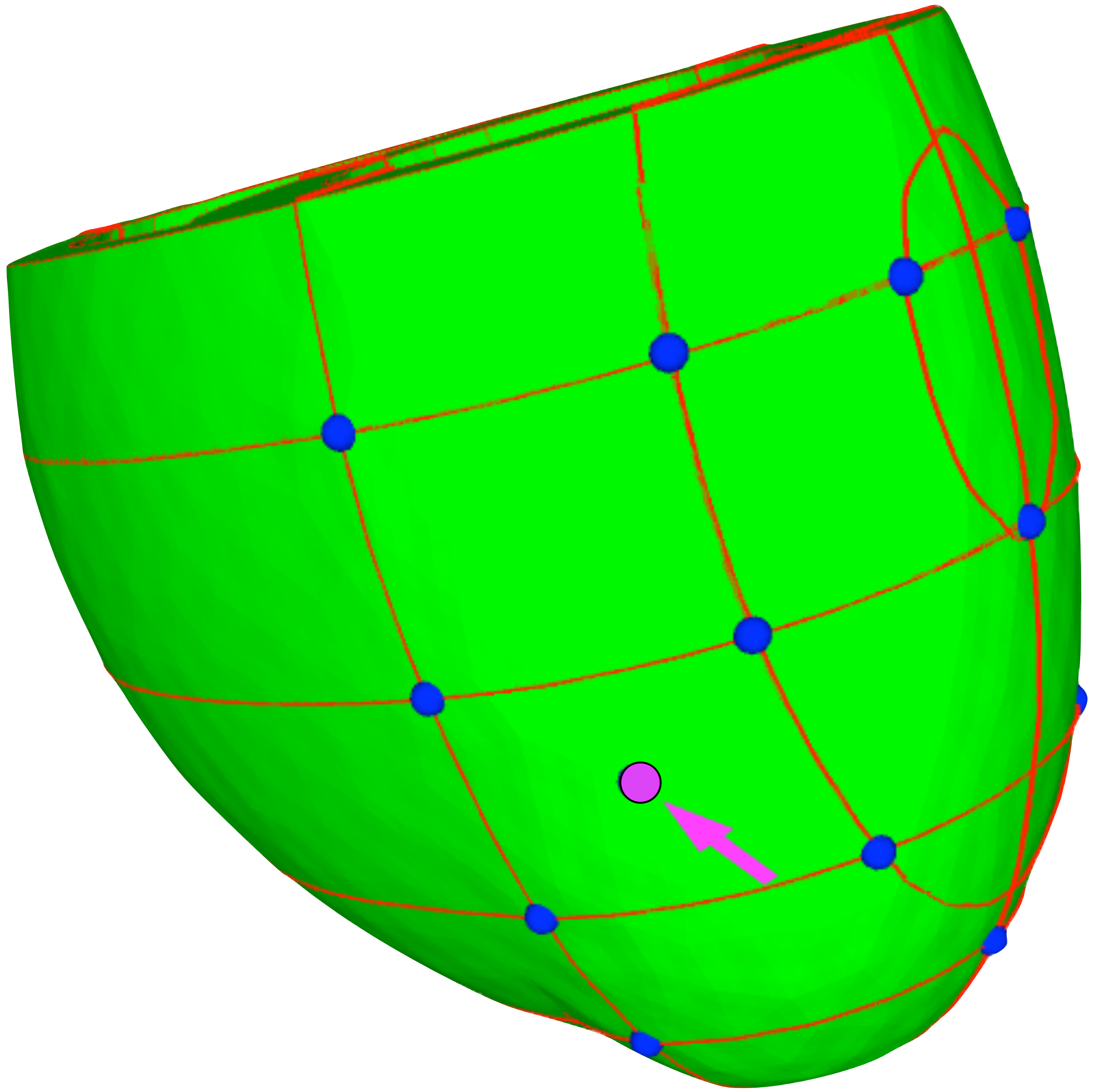
FEM (79,537 basis) POD (100 basis)

[Boulakia, Schenone, Gerbeau - 2012]

POD application in Cardiac Electrophysiology

Simulation of a myocardial infarction

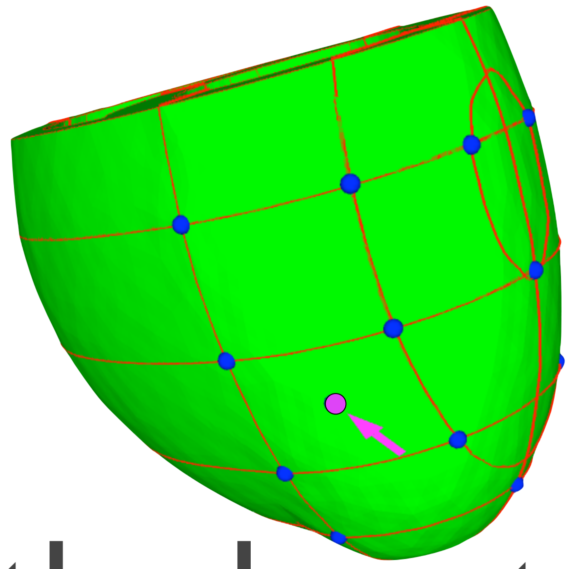
- An efficient POD to simulate an infarction in any point of the heart
  ➡ to build the POD basis, we use <u>many</u> FEM solutions <u>with infarction</u>



[Boulakia, Schenone, Gerbeau - 2012]

POD application in Cardiac Electrophysiology

Simulation of a myocardial infarction



- An efficient POD to simulate an infarction in any point of the heart
  - ➡ to build the POD basis, we use <u>many</u> FEM solutions <u>with infarction</u>

- healthy FEM solution $S_h = \left[\mathbf{u}_h^1 | \mathbf{u}_h^2 | \dots | \mathbf{u}_h^{N_T}\right] \in \mathbb{R}^{\mathcal{N} \times N_T}$
- infarct 1 FEM solution $S_{I_1} = \left[\mathbf{u}_{I_1}^1 | \mathbf{u}_{I_1}^2 | \dots | \mathbf{u}_{I_1}^{N_T}\right] \in \mathbb{R}^{\mathcal{N} \times N_T}$
- infarct 2 FEM solution $S_{I_2} = \left[\mathbf{u}_{I_2}^1 | \mathbf{u}_{I_2}^2 | \dots | \mathbf{u}_{I_2}^{N_T}\right] \in \mathbb{R}^{\mathcal{N} \times N_T}$

$$\dots \quad \dots \quad \dots$$

- infarct m FEM solution $S_{I_m} = \left[\mathbf{u}_{I_m}^1 | \mathbf{u}_{I_m}^2 | \dots | \mathbf{u}_{I_m}^{N_T}\right] \in \mathbb{R}^{\mathcal{N} \times N_T}$

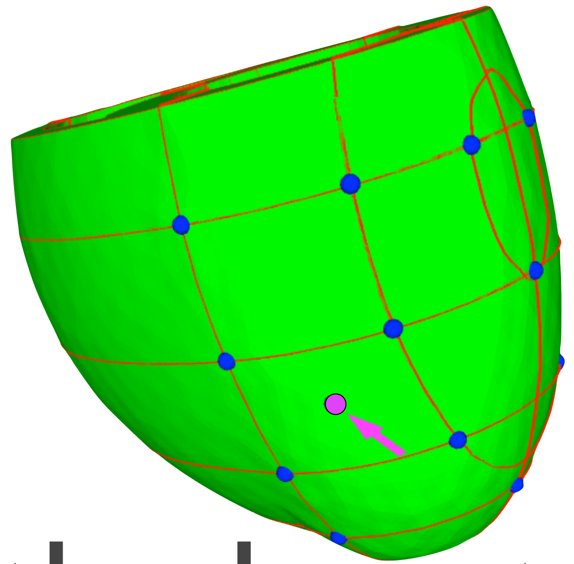
➡ compute the SVD on an enlarged matrix

$$S = \left[S_h | S_{I_1} | S_{I_2} | \dots | S_{I_m}\right] \in \mathbb{R}^{\mathcal{N} \times (m+1) N_T}$$

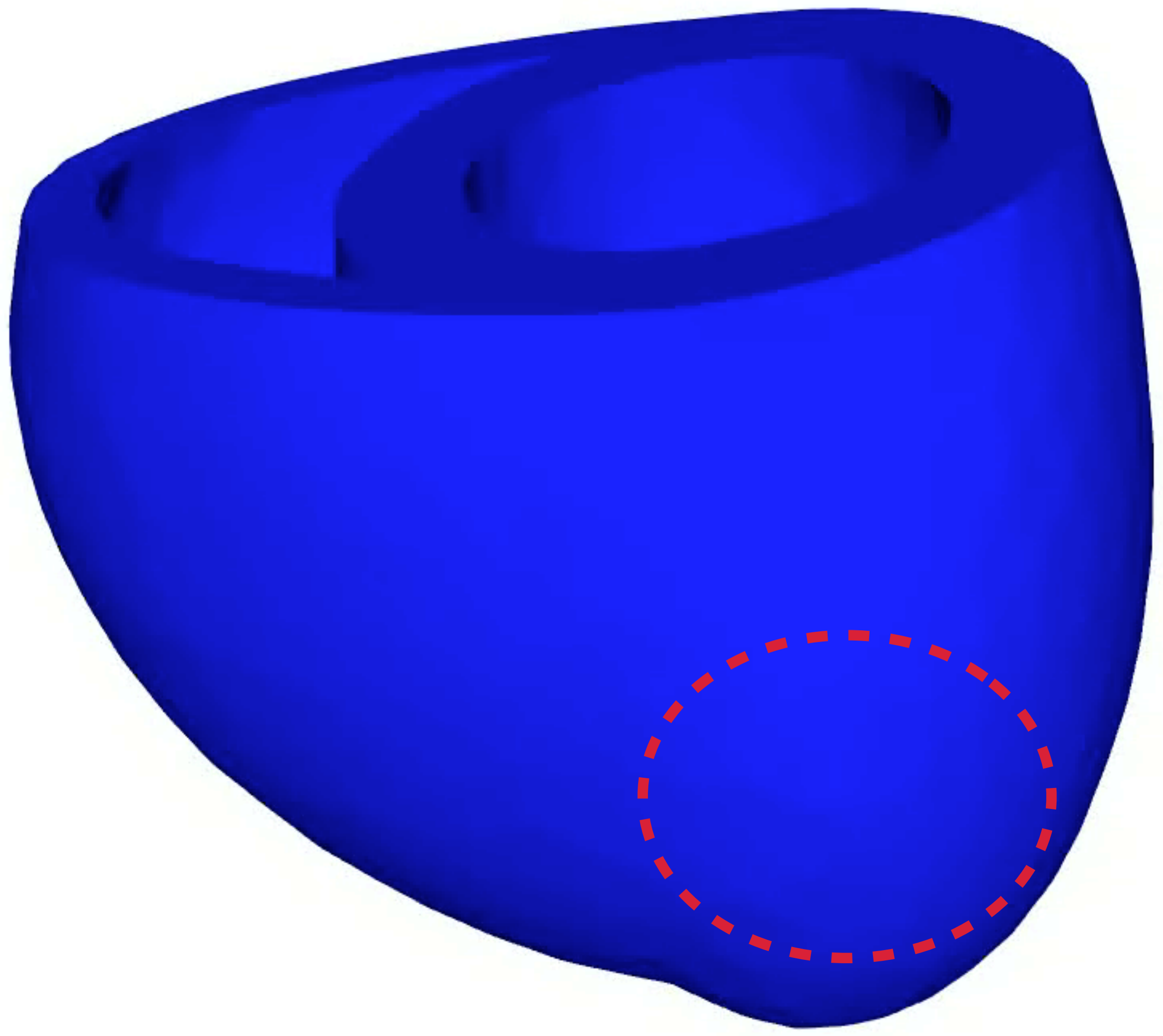
[Boulakia, Schenone, Gerbeau - 2012]

POD application in Cardiac Electrophysiology

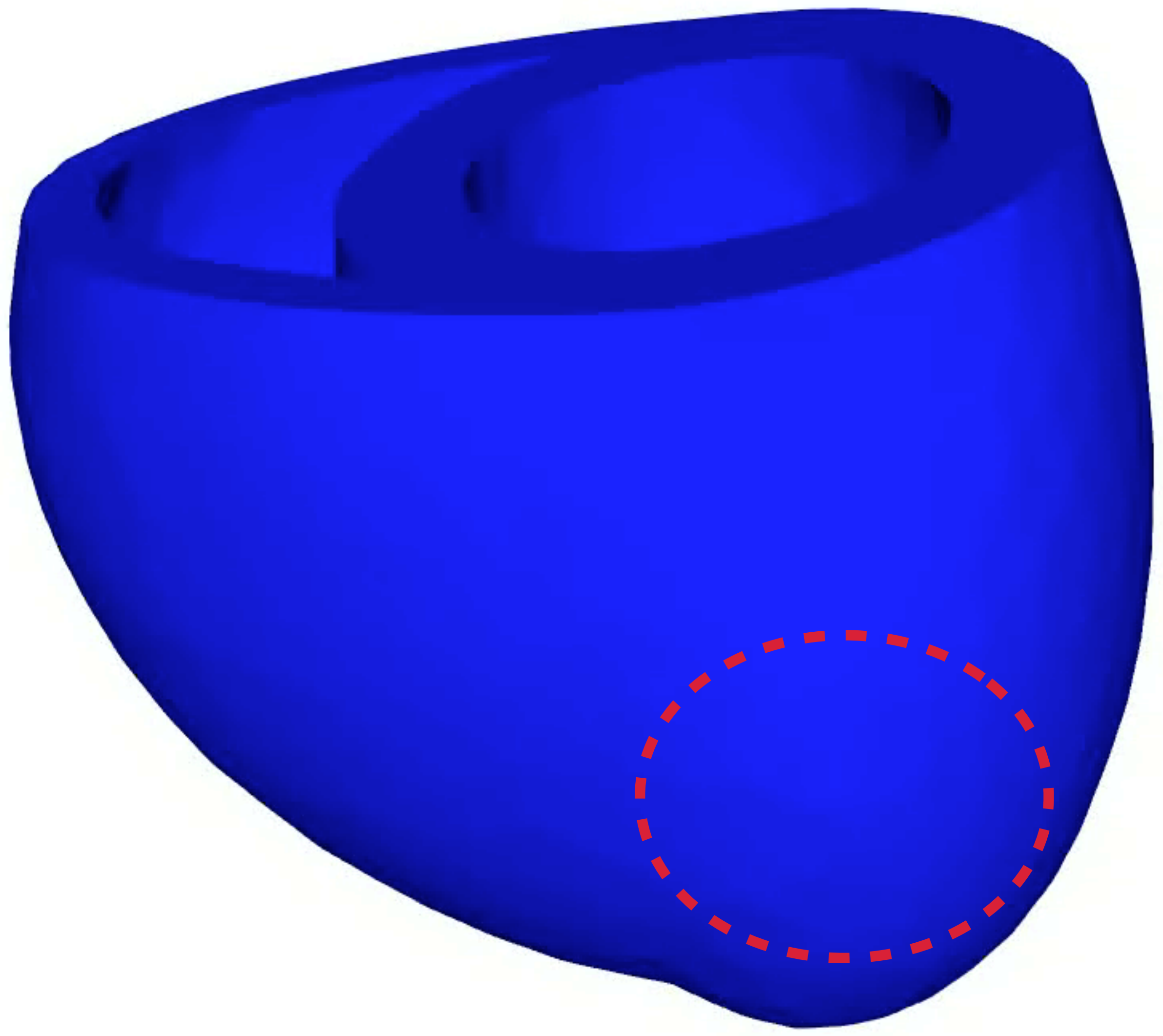
Simulation of a myocardial infarction



- An efficient POD to simulate an infarction in any point of the heart
  - ➡ to build the POD basis, we use <u>many</u> FEM solutions <u>with infarction</u>



FEM (79,537 basis)

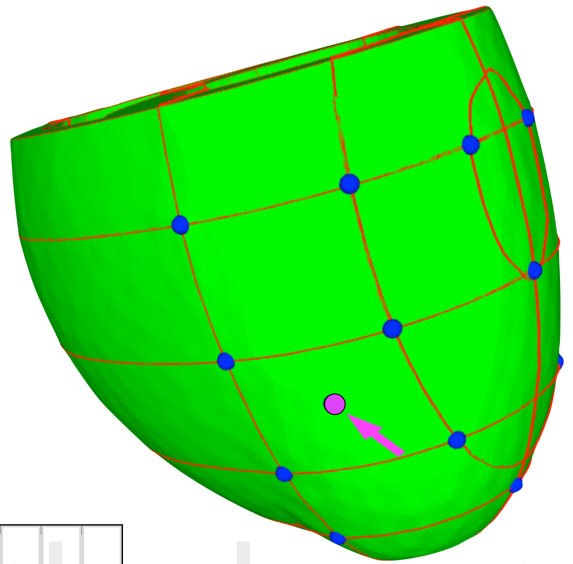


POD (100 basis)

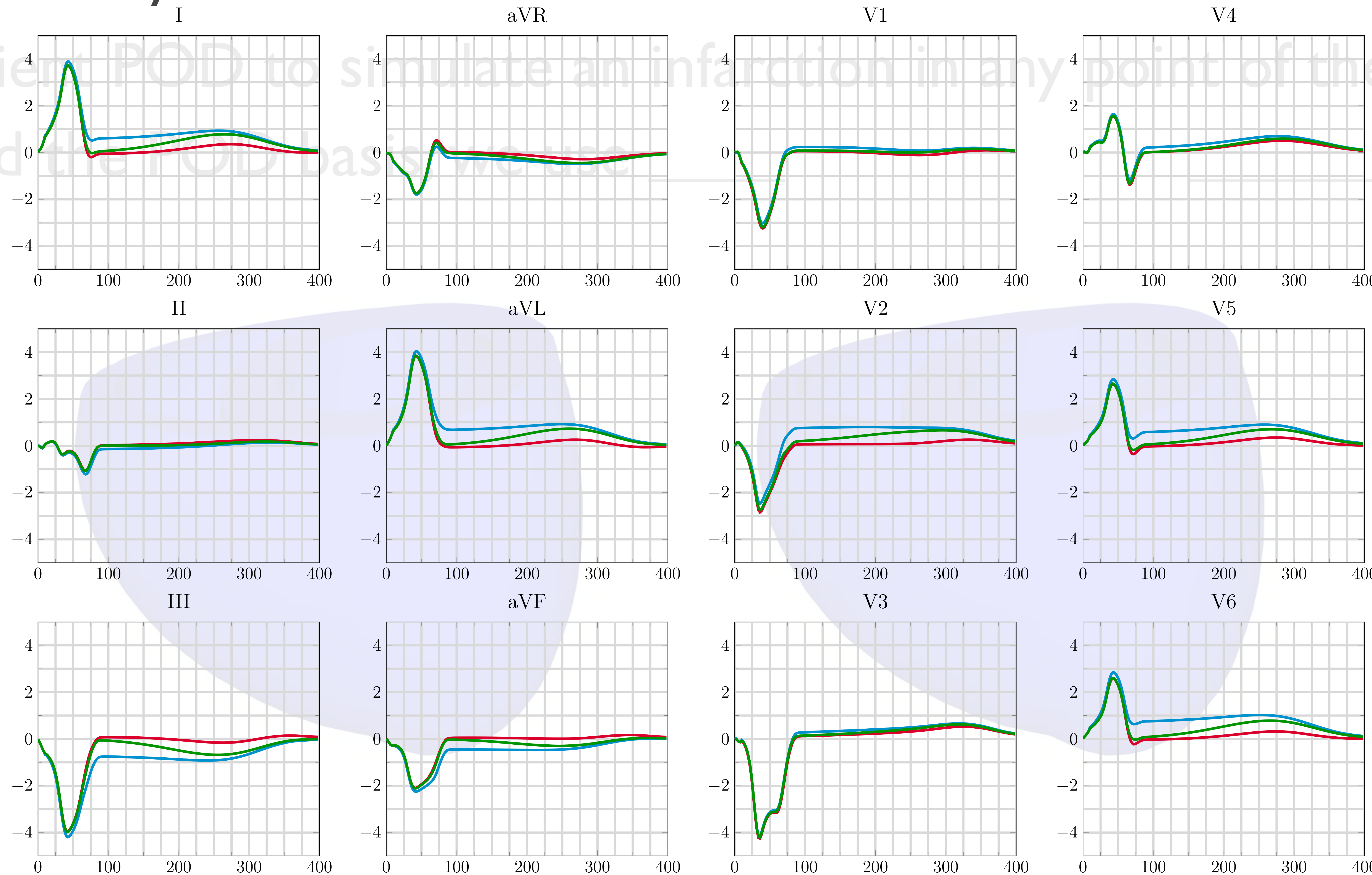
[Boulakia, Schenone, Gerbeau - 2012]

POD application in Cardiac Electrophysiology

Simulation of a myocardial infarction



- An efficient POD to simulate an infarction in any point of the heart
  - ➡ to build the POD basis we use



FEM (79,537 basis) POD (100 basis)

[Boulakia, Schenone, Gerbeau - 2012]

Non-linear problems
Example

$$\begin{cases} -\nabla \cdot (\gamma(u)\nabla u) = f, & \Omega \subset \mathbb{R}^d \\ \text{B.C.} & \partial\Omega \end{cases}$$

Weak formulation

$$\underbrace{\int_\Omega \gamma(u)\nabla u \nabla v}_{K(u)\mathbf{u}} + \text{B.C.} = \int_\Omega f v, \ \forall v \in V$$

possible solution

<u>Empirical Interpolation Methods</u>

<u>Ingredients:</u>

- a reduced- basis approximation space, *e.g.* RB or POD
- an interpolation procedure tho choose some points where the PDE is solved in its strong form

References

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<u>Proper Orthogonal Decomposition</u>

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