Stratified action negation, a logic about travel

Xin Sun[1] and Huimin Dong[2]

[1] Faculty of Science, Technology and Communication, University of Luxembourg
xin.sun@uni.lu
[2] Philosophy and Economics, University of Bayreuth
huimin.dong@uni-bayreuth.de

Abstract. This paper presents a new operator of action negation called stratified action negation. We give a complete axiomatization to stratified action negation via combinatory propositional dynamic logic.

Key words: action negation, logic of action, dynamic logic

1 Introduction

The central question of dynamic deontic logic is: what is the logical structure of deontic statement in terms of the structure of complex action. Therefore dynamic deontic logic study the logic of action over action operators (choice: $\cup$, concurrent execution: $\cap$, sequence: ;, iteration: $*$, action negation: $^-$), rather than standard logical connectives ($\vee, \wedge, \neg$). Among all those action operators, action negation is of special interest to deontic logicians [4, 1, 2, 7].

In the dynamic logic literature [3, 5], action negation is usually interpreted as set theoretical complement with respect to the universal relation. But such treatment is not a smart choice when dynamic logic is applied to the deontic setting. Jan Broersen [2] argues that a good notion of action negation for deontic logic should satisfy the following requirement:

- It should have an intuitive interpretation as an action forming operator.
- It should not impose restrictions on the use of other relevant action operator.
- It should have a meaningful interpretation in the normative context.

Presumably the only up to date interpretation of action negation that satisfies the above requirement is Broersen's relativized action negation [1, 2]. However, in this paper we argue that in addition to Broersen's requirement, the ideal notion of action negation should further satisfy the following requirement:

For two different actions α and β, in general we should **not** have $\alpha \cup \overline{\alpha} = \beta \cup \overline{\beta}$.

The relativized action negation does not satisfy this requirement. In this paper we develop a new interpretation of action negation such that it is intuitively acceptable and satisfies the above requirement.

2 Dynamic logic

Let $\mathbb{P}$ be a countable set of propositional letters and $\mathbb{A}$ a countable set of symbols of action generators. The language of dynamic logic can be defined by the following BNF:

Definition 1 (Language of dynamic logic). *For $a \in \mathbb{A}$ and $p \in \mathbb{P}$,*

- $\alpha := \mathbf{1} \mid a \mid \alpha \cup \alpha \mid \alpha \cap \alpha \mid \alpha;\alpha \mid \alpha^* \mid \phi? \mid \overline{\alpha}$
- $\phi := p \mid \top \mid \neg\phi \mid \phi \wedge \phi \mid [\alpha]\phi$

Here symbols of the form α are action terms and ϕ are formulas. We use $\langle\alpha\rangle\varphi$ as an abbreviation of $\neg[\alpha]\neg\varphi$. Formulas are interpreted by the relational model, which is defined as follows.

Definition 2 (Relational model). *A relational model $\mathbb{M} = (W, R^{\mathbb{A}}, V)$ is a triple:*

- *W is a nonempty set of possible states.*
- *$R^{\mathbb{A}} : \mathbb{A} \to 2^{W\times W}$ is an action interpretation function, assigning a binary relation over W to each action generator $a \in \mathbb{A}$.*
- *$V : \mathbb{P} \to 2^W$ is the valuation function for propositional letters.*

The action interpretation function $R^{\mathbb{A}}$ can be extend to a new function R to interpret arbitrary actions as follows:

- $R(\mathbf{1}) = W \times W$,
- $R(a) = R^{\mathbb{A}}(a)$ for $a \in \mathbb{A}$,
- $R(\alpha \cup \beta) = R(\alpha) \cup R(\beta)$,
- $R(\alpha \cap \beta) = R(\alpha) \cap R(\beta)$,
- $R(\alpha;\beta) = R(\alpha) \circ R(\beta)$,
- $R(\phi?) = \{(w, w) \mid w \in V(\varphi)\}$,
- $R(\alpha^*) = (R(\alpha))^*$.

Here $\circ$ is the composition operator for relations and * is the reflexive transitive closure operator of relations. We leave the case for $R(\overline{\alpha})$ to the next section because that is the theme of this paper. With the function R in hand, we can define the semantics for formulas of dynamic logic use relational model as following:

Definition 3 (Semantics of dynamic logic). *Let $M = (W, R^{\mathbb{A}}, V)$ be a relational model. Let $w \in W$.*

- *$M, w \models p$ iff $w \in V(p)$*
- *$M, w \models \neg\phi$ iff not $M, w \models \phi$*
- *$M, w \models \phi \wedge \phi$ iff $M, w \models \phi$ and $M, w \models \phi$*
- *$M, w \models [\alpha]\phi$ iff for all v, if $(w, v) \in R(\alpha)$ then $M, v \models \phi$*

3 Action Negation

Broersen [2] suggests the negation of action α should be different from α, an alternative of α, and can be considered as the act of refraining from α. We accept such intuition. In this section we review the known treatment for dynamic logic on action negation, then we define a new, and better, treatment to implement such intuition.

3.1 Action negation in the literature

The traditional interpretation of action negation [3] is to let $R(\overline{\alpha}) = (W \times W) - R(\alpha)$, *i.e.* the set theoretical complement with respect to the universal relation. Broersen [1, 2] argues that the universal relation is not the ideal background for complement when dynamic logic is applied to deontic logic. The reason is, given such interpretation, $\alpha \cup \overline{\alpha}$ represents the universal relation. Therefore the modality $[\alpha \cup \overline{\alpha}]$ has universal power and can reach every state, including those states which are not reachable by any action. Apparently our arbitrary action, such as "write a paper or not write a paper", don't have such strong power. Therefore the traditional interpretation of action negation is not appropriate for deontic logic, or even the logic of action.

To avoid such predicament, Broersen [2] restricts the universal relation such that those worlds which are unreachable by any action are out of concern. With this motivation Broersen replaces the universal relation $W \times W$ in the interpretation of $R(\overline{\alpha})$ by relations like $\bigcup_{\alpha \in \mathbb{A}} R(\alpha)$, $(\bigcup_{\alpha \in \mathbb{A}} R(\alpha))^+$, $(\bigcup_{\alpha \in \mathbb{A}} R(\alpha))^*$ etc.

Another proposal for action negation is developed by Wansing [7]. Wansing suggests that an operator "$-$" for action is an operation of action negation as long as it satisfies the following: $R(-(\alpha;\beta)) = R(-\alpha;-\beta)$, $R(-(\alpha^*)) = R((-\alpha)^*)$ and $R(-(\varphi?)) = R((\neg\varphi)?)$. Broersen [2] points out that this approach is not ideal because it it might be too liberal. For instance we can define for all action α, $\alpha = -\alpha$, assuming we have no operator '?' in our language. Such definition still satisfies $R(-(\alpha;\beta)) = R(-\alpha;-\beta)$ and $R(-(\alpha^*)) = R((-\alpha)^*)$, but contradicts to our intuition.

3.2 A drawback of reletavized negation

Broersen's approach is more natural than its traditional counterpart in the deontic setting. But there is a drawback. For an illustration, note that for any two action α and β, the action $\alpha \cup \overline{\alpha}$ and $\beta \cup \overline{\beta}$ are identical in Broersen's approach. Now consider the following:

- suppose Hamlet receives the following authorization:
  (1) "You are permitted either to be or not to be."
- and James Bond receives the following authorization:
  (2) "You are permitted either to kill or not to kill. "

Abstract away the factor of agents, (1) offers the agent a free choice between to live and to be dead while (2) offers the agent the license to kill or not. These two permissions convey very different information and should be distinguished. But they are identical in both traditional and Broersen's approach. More generally, any interpretation of action negation validate the following is problematic: for all action α and β, $R(\alpha \cup \overline{\alpha}) = R(\beta \cup \overline{\beta})$. In the following section we develop a new interpretation of action negation such that$R(\alpha \cup \overline{\alpha}) \neq R(\beta \cup \overline{\beta})$ in general.

3.3 A new approach: stratified action negation

We first make a classification about actions. Since actions are interpreted by relations and the simplest relation is a set that contains an ordered pair of states, we can naturally call an action α **particle** if $R(\alpha)$ contains exactly one ordered pair. For a particle action α, we call the first component of the ordered pair in $R(\alpha)$ the pre-state of α. Formally, if $R(\alpha) = \{(s_1, s_2)\}$, then $pre(\alpha) = \{s_1\}$. And we call the second component the post-state of α, formally $post(\alpha) = \{s_2\}$. Intuitively, a particle action is simply a travel from one state to another.

Based on particle action, we build **atomic action** as a union of particle actions which share the same pre-state. For instance, for two particle actions α_1 and α_2 with $R(\alpha_1) = \{(s_1, s_2)\}$, $R(\alpha_2) = \{(s_1, s_3)\}$, the action α_3 such that $R(\alpha_3) = \{(s_1, s_2), (s_1, s_3)\}$ is an atomic action. See Figure 1 .

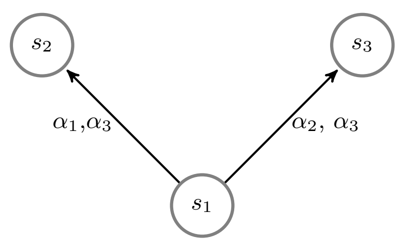

Fig. 1. atomic action

For an atomic action α, its pre-state is the same as its consisting particle actions. The post condition of α is the union of the post-state of its consisting particle actions. Therefore $post(\alpha_3) = \{s_2, s_3\}$. Intuitively, an atomic action is a nondeterministic travel from a specific state to other states.

For an arbitrary action α, we defined its pre-state as the union of the pre-state of its consisting particle actions. Formally, $pre(\alpha) = \{s \in S | (s, t) \in R(\alpha)\}$.

Now we have a classification of actions and the pre/post state of an action has been defined. It is the time to grasp what the negation of an action is. For an atomic action α, say $R(\alpha) = \{(t, t), (t, u)\}$ and $W = \{t, u, v\}$, we tend to define $R(\overline{\alpha}) = \{(t, v)\}$. See Figure 2.

The intuition is, we understand α as a plan of travel from t to either u or t itself, then the negation of α can be understood as "go to those states other

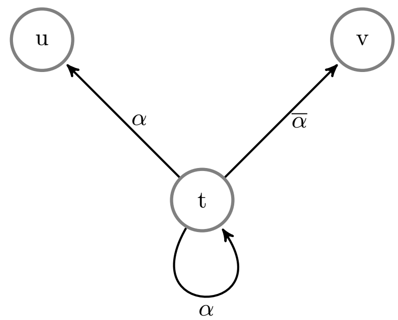

Fig. 2. atomic action negation

than the states α goes". Suppose t, u, v represent Thailand, UK and Vietnam respectively, the action α in Figure 2 is understood as "stay in Thailand, or go to UK from Thailand". The negation of α is "go to Vietnam from Thailand". More formally, we define $R(\overline{\alpha}) = pre(\alpha) \times (W - post(\alpha))$ for an atomic action α.

For an arbitrary action α, we calculate $R(\overline{\alpha})$ via the following steps:

1. We first decompose α to atomic actions $\alpha_1, \ldots, \alpha_n$ such that $R(\alpha) = R(\alpha_1) \cup \ldots \cup R(\alpha_n)$ and for every $i \neq j$, $pre(\alpha_i) \neq pre(\alpha_j)$. It can be verified that such decomposition is unique and each α_i is a maximal atomic sub-action of α in the sense that for every atomic action β, if $R(\beta) \subseteq R(\alpha)$ then there exist a unique α_i in the decomposition such that $R(\beta) \subseteq R(\alpha_i)$
2. For each $i \in \{1, \ldots, n\}$, we then calculate $R(\overline{\alpha_i})$. Since α_i is an atomic action, we have $R(\overline{\alpha_i}) = pre(\alpha_i) \times (W - post(\alpha_i))$.
3. Finally we take the union of these $R(\overline{\alpha_i})$ to form $R(\overline{\alpha}) = R(\overline{\alpha_1}) \cup \ldots \cup R(\overline{\alpha_n})$.

Since we calculate the action negation by decomposing complex actions to simple actions, we call such an approach **stratified action negation**. For an example, if $W = \{u, v, w\}$, $R(\alpha) = \{(w, w), (w, u), (v, w)\}$, then $R(\overline{\alpha}) = \{(w, v), (v, v), (v, u)\}$. See Figure 3.

It will then be nice to have a uniform formal representation of the above procedure. To achieve this we give the following definition:

Definition 4 (Stratified action negation). *For an arbitrary action α, we define*

$R(\overline{\alpha}) = Pre(\alpha) \times W - R(\alpha)$, *if* $R(\alpha \neq \emptyset)$. *Otherwise we let* $R(\overline{\alpha}) = W \times W$.

The above definition makes the negation of an action be sensitive to its preconditions. We believe such treatments is intuitively acceptable. According to Definition 4, we have for two action α and β, as long as $pre(\alpha) \neq pre(\beta)$, we have $R(\alpha \cup \overline{\alpha}) \neq R(\beta \cup \overline{\beta})$.

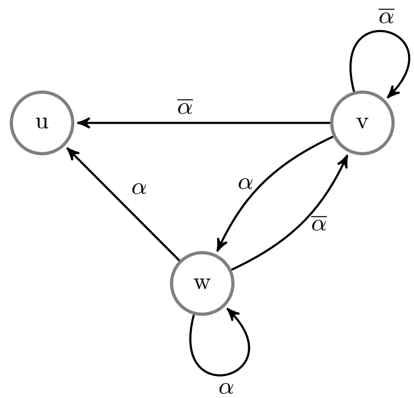

Fig. 3. arbitrary action negation

3.4 Axiomatization via combinatory propositional dynamic logic

Our stratified action negation can be expressed in language of combinatory propositional dynamic logic (CPDL) [6]. In Passy and Tinchev [6], CPDL is an extension of PDL with nomials to name every state.

Let $\mathbb{P}, \Sigma$ be countable sets of propositional letters and nominals respectively, $\mathbb{A}$ a countable set of symbols of action generators. We require $\mathbb{P}, \Sigma$ and $\mathbb{A}$ to be disjoint. The language of CPDL can be defined by the following BNF:

Definition 5 (Language of CPDL [6]). *For $a \in \mathbb{A}$, $i \in \Sigma$ and $p \in \mathbb{P}$,*

- $\alpha := \mathbf{1} \mid a \mid \alpha \cup \alpha \mid \alpha \cap \alpha \mid \alpha;\alpha \mid \alpha^* \mid \phi? \mid \widetilde{\alpha} \mid \alpha^{-1}$
- $\phi := p \mid i \mid \top \mid \neg\phi \mid \phi \wedge \phi \mid \langle\alpha\rangle\phi \mid \alpha = \alpha \mid \alpha \subset \alpha$

Definition 6 (Relational model with nominal [6]). *A relational model with nominal is a quadruple $\mathbb{M} = (W, R^{\mathbb{A}}, \chi, V)$:*

- *W is a nonempty set of possible states.*
- *$R^{\mathbb{A}} : \mathbb{A} \to 2^{W \times W}$ is an action interpretation function, assigning a binary relation over W to each action generator $a \in \mathbb{A}$.*
- *$\chi : \sigma \to W$ is a surjective function.*
- *$V : \mathbb{P} \to 2^W$ is the valuation function for propositional letters.*

The action interpretation function $R^{\mathbb{A}}$ is extend to a new function R to interpret arbitrary actions as follows:

- $R(\mathbf{1}) = W \times W$,
- $R(a) = R^{\mathbb{A}}(a)$ for $a \in \mathbb{A}$,
- $R(\alpha \cup \beta) = R(\alpha) \cup R(\beta)$,
- $R(\alpha \cap \beta) = R(\alpha) \cap R(\beta)$,
- $R(\alpha; \beta) = R(\alpha) \circ R(\beta)$,

- $R(\alpha^*) = (R(\alpha))^*$,
- $R(\phi?) = \{(w, w) \mid w \in V(\varphi)\}$,
- $R(\widetilde{\alpha}) = W \times W - R(\alpha)$,
- $R(\alpha^{-1}) = R(\alpha)^{-1}$.

Definition 7 (Semantics of CPDL [6]). *Let $M = (W, R^{\mathbb{A}}, \chi, V)$ be a relational model with nominal. Let $w \in W$,*

- *$M, w \models i$ iff $w = \chi(i)$,*
- *$M, w \models \alpha = \beta$ iff for all v, $(w, v) \in R(\alpha)$ iff $(w, v) \in R(\beta)$,*
- *$M, w \models \alpha \subset \beta$ iff for all v, if $(w, v) \in R(\alpha)$ then $(w, v) \in R(\beta)$.*

Other cases are the same as PDL.

The following is the proof-system of CPDL [6].

1. Axiom Schemes:
   - All proposition tautologies.
   - $\langle \mathbf{1} \rangle i$
   - $\langle \mathbf{1} \rangle (i \wedge \varphi) \to [\mathbf{1}](i \to \varphi)$
   - $\varphi \to \langle \mathbf{1} \rangle \varphi$
   - $\langle \mathbf{1} \rangle \langle \mathbf{1} \rangle \varphi \to \langle \mathbf{1} \rangle \varphi$
   - $\varphi \to [\mathbf{1}] \langle \mathbf{1} \rangle \varphi$
   - $\langle \alpha \rangle \varphi \to \langle \mathbf{1} \rangle \varphi$
   - $\langle \alpha; \beta \rangle \varphi \leftrightarrow \langle \alpha \rangle \langle \beta \rangle \varphi$
   - $\langle \alpha \cup \beta \rangle i \leftrightarrow \langle \alpha \rangle i \vee \langle \beta \rangle i$
   - $\langle \alpha \cap \beta \rangle i \leftrightarrow \langle \alpha \rangle i \wedge \langle \beta \rangle i$
   - $\langle \widetilde{\alpha} \rangle i \leftrightarrow [\alpha] \neg i$
   - $\alpha \subset \beta \leftrightarrow [\alpha \cap \widetilde{\beta}] \bot$
   - $\langle \mathbf{1} \rangle (i \wedge \langle \alpha^{-1} \rangle j) \leftrightarrow \langle 1 \rangle (j \wedge \langle \alpha \rangle i)$
   - $\langle \varphi? \rangle \psi \leftrightarrow \varphi \wedge \psi$
   - $\langle \alpha^* \rangle \varphi \leftrightarrow \varphi \vee \langle \alpha \rangle \langle \alpha^* \rangle \varphi$
   - $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$
2. Rules:
   - If $\vdash [\alpha] \neg i$ for all $i \in \Sigma$, then $\vdash [\alpha] \bot$
   - If $\vdash [\beta][\alpha^i]\varphi$ for all $i \in \mathbb{N}$, then $\vdash [\beta][\alpha^*]\varphi$
   - If $\vdash \varphi$, then $\vdash [\mathbf{1}]\varphi$
   - If $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$

It is proved by Passy and Tinchev [6] that the above proof-system is sound and complete with the class of relational models with nominal. Our stratified action negation can be expressed in the language of CPDL. To achieve this, first note that the notion of pre-condition and stratified action negation can both semantically be define using notions of CPDL.

Proposition 1. *Let M be an arbitrary relational model with nominal,*

1. *$w \in pre(\alpha)$ iff $M, w \models \langle \alpha \rangle \top$,*
2. *$M, w \models [\overline{\alpha}]\varphi$ iff $M, w \models \langle \alpha \rangle \top \to [\widetilde{\alpha}]\varphi$.*

Proof. 1. (left-to-right) Assume $w \in pre(\alpha)$, then there exist some $v \in W$ such that $(w, v) \in R(\alpha)$. Therefore $M, w \models \langle\alpha\rangle\top$.
(right-to-left) Assume $M, w \models \langle\alpha\rangle\top$, then there exist some $v \in W$ such that $(w, v) \in R(\alpha)$ and $M, v \models \top$. Therefore $w \in pre(\alpha)$.

2. (left-to-right) Assume $M, w \models [\overline{\alpha}]\varphi$, then $M, v \models \varphi$ for all v such that $(w, v) \in pre(\alpha) \times W - R(\alpha)$. Note that $pre(\alpha) \times W - R(\alpha) = pre(\alpha) \times W \cap R(\widetilde{\alpha})$. Hence for all v such that $(w, v) \in pre(\alpha) \times W \cap R(\widetilde{\alpha})$, $M, v \models \varphi$. Now if $M, w \models \langle\alpha\rangle\top$, then by item 1 we know $w \in pre(\alpha)$.
Let u be an arbitrary state such that $(w, u) \in R(\widetilde{\alpha})$. We know that $(w, u) \in pre(\alpha) \times W$ because $w \in pre(\alpha)$. Therefore $(w, u) \in pre(\alpha) \times W \cap R(\widetilde{\alpha})$. Hence $M, u \models \varphi$, $M, w \models [\widetilde{\alpha}]\varphi$.

(right-to-left) Assume $M, w \models \langle\alpha\rangle\top \to [\widetilde{\alpha}]\varphi$. Let v be an arbitrary state such that $(w, v) \in pre(\alpha) \times W - R(\alpha)$. Then $(w, v) \in pre(\alpha) \times W \cap R(\widetilde{\alpha})$. So we have $pre(\alpha) \neq \emptyset$ and $M, w \models \langle\alpha\rangle\top$. Therefore $M, w \models [\widetilde{\alpha}]\varphi$. From $(w, v) \in R(\widetilde{\alpha})$ we deduce $M, v \models \varphi$. Therefore $M, w \models [\overline{\alpha}]\varphi$. □

Therefore we can syntactically let $[\overline{\alpha}]\varphi$ be an abbreviation of $\langle\alpha\rangle\top \to [\widetilde{\alpha}]\varphi$. Then the axiomatization of CPDL is also the axiomatization of our stratified action negation.

4 Conclusion

This paper develops a logic of stratified action negation. We classify action by particle, atomic action and calculate its negation step by step. We axiomatise this logic via combinatory propositional dynamic logic.

Acknowledgment

The first author thanks Dov Gabbay for his inspiring lectures given in the University of Luxembourg.

References

1. Jan Broersen. Relativized action negation for dynamic logics. In P. Balbiani, N-Y. Suzuki, F. Wolter, and M. Zakharyaschev, editors, *Advances in Modal Logic*, volume 4, pages 51–70, 2003.
2. Jan Broersen. Action negation and alternative reductions dynamic deontic logics. *Journal of applied logic*, 2004.
3. D. Harel, D. Kozen, and J. Tiuryn. *Dynamic logic*. The MIT Press, 2000.
4. John Jule Meyer. A different approach to deontic logic: deontic logic viewed as a variant of dynamic logic. *Notre Dame Journal of Formal Logic*, pages 109–136, 1988.
5. Ewa Orlowska. Dynamic logic with program specifications and its relational proof system. *Journal of Applied Non-Classical Logics*, 3(2):147–171, 1993.
6. Solomon Passy and Tinko Tinchev. Pdl with data constants. *Information Processing Letters*, 20(1):35–41, 1985.
7. Heinrich Wansing. On the negation of action types: Constructive concurrent pdl. *Dresden Preprints in theoretical philosophy and philosophical logic*, 2004.