Commutative n-ary superalgebras with an invariant skew-symmetric form[1]

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Abstract

We study n-ary symmetric superalgebras and L_∞-algebras that possess skew-symmetric invariant forms, using the derived bracket formalism. This class of superalgebras includes for instance Lie algebras and their n-ary generalizations, commutative associative and Jordan algebras with invariant forms. We give a classification of m-dimensional $(m-3)$-ary algebras with invariant form, and a classification of real simple m-dimensional Lie $(m-3)$-algebras with positive definite invariant form up to isometry. We develop the Hodge Theory for L_∞-algebras with symmetric invariant forms, and we describe quasi-Frobenius structures on skew-symmetric n-ary algebras.

1 Introduction

`Derived bracket formalism.` The derived bracket approach was successfully used in different areas of mathematics: in Poisson geometry, in the theory of Lie algebroids and Courant algebroids, BRST formalism, in the theory of Loday algebras and different types of Drinfeld Doubles. For detailed introduction we recommend a beautiful survey of Y. Kosmann-Schwarzbach [KoSch1].

The idea of the formalism is the following. *One fixes an algebra L, usually a Lie superalgebra, and constructs another multiplication on the same vector space (or some subspace) using derivations of L and the (iterated) multiplication in L. We obtain a class of new algebras, which properties can be studied using original algebra L.* For example, using this formalism we can obtain all Poisson structures on a manifold M from the canonical Poisson algebra on T^*M as was shown by Th. Voronov in [Vor3]. Voronov's idea allows A. Cattaneo and M. Zambon [CZ] to introduce a unified approach to the reduction of Poisson manifolds. Another example was suggested in [Vor1] and [Vor2], where a series of strongly homotopy algebras was obtained from a given Lie superalgebra.

We use this formalism to study n-ary symmetric superalgebras with invariant skew-symmetric forms. More precisely, consider a vector superspace V with a

[1] Supported by AFR-grant, University of Luxembourg.

non-degenerate even skew-symmetric form $(\,,)$. In this case there exists a natural Lie superalgebra structure on $S^*(V)$, where $S^*(V)$ is the symmetric power of V. The main observation is that *we get all symmetric n-ary and strongly homotopy superalgebras on V with invariant skew-symmetric form $(\,,)$. In other words, the property of these n-ary superalgebras having an invariant skew-symmetric form is encoded by the Lie superalgebra $S^*(V)$.* The observation that using the superalgebra $S^*(V)$ we can obtain all Lie algebras with invariant symmetric forms was made by B. Kostant and S. Sternberg in [KS]. The superalgebra $S^*(V)$ was also used in Poisson Geometry to study for instance Lie bialgebras and Drinfeld Doubles, see [KoSch1, KoSch2], [LR] and others.

`Multiple generalizations of Lie algebras.` Using the derived bracket formalism we can study all n-ary symmetric superalgebras with skew-symmetric invariant forms. This class of superalgebras includes for instance different n-ary generalizations of Lie algebras with symmetric invariant form. First of all let us give a short review of such generalizations.

Multiple generalizations arise usually from different readings of the Jacobi identity. For example, the Jacobi identity for a Lie algebra is equivalent to the statement that all adjoint operators are derivations of this Lie algebra. If we use this point of view for the n-ary case we come to the notion of a *Filippov n-algebra* [Fil]. V.T. Filippov considered alternating n-ary algebras A satisfying the following Jacobi identity:

$$\{a_1,\ldots,a_{n-1},\{b_1,\ldots,b_n\}\} = \sum\{b_1,\ldots,b_{i-1}\{a_1,\ldots,a_{n-1},b_i\},\ldots,b_n\}, \qquad (1)$$

where a_i, $b_j \in A$. In other words, the operators $\{a_1,\ldots,a_{n-1},-\}$ are derivations of the n-ary bracket $\{b_1,\ldots,b_n\}$. Such algebras appear naturally in Nambu mechanics [Nam] in the contecst of Nambu-Poisson manifolds, in supersymmetric gravity theory and in supersymmetric gauge theories, the Bagger-Lambert-Gustavsson Theory, see [AI].

Another natural n-ary generalization of the Jacobi identity has the following form:

$$\sum(-1)^{(I,J)}\{\{a_{i_1},\ldots,a_{i_n}\},a_{j_1},\ldots,a_{j_{n-1}}\} = 0, \qquad (2)$$

where the sum is taken over all ordered unshuffle multi-indexes $I=(i_1,\ldots,i_n)$ and $J=(j_1,\ldots,j_{n-1})$ such that (I,J) is a permutation of $(1,\ldots,2n-1)$. We will call such algebras *Lie n-algebras*. This type of n-ary algebras was considered for instance by P. Michor and A. Vinogradov in [MV] and by P. Hanlon and M.L. Wachs [HW]. The homotopy case was studied in [SS] in context of the Schlesinger-Stasheff homotopy algebras and L_∞-algebras. Such algebras are related to the Batalin-Fradkin-Vilkovisky theory and to the string field theory, see [LSt]. In [VV1] A.M. Vinogradov and M.M. Vinogradov proposed a three-parameter family of n-ary algebras such that for some n the above discussed structures appear as particular cases.

The theory of Filippov n-ary algebras is relatively well-developed. For instance, there is a classification of simple real and complex Filippov n-ary algebras and an analog of the Levi decomposition [Ling]. W.X. Ling in [Ling] proved that there exists only one simple finite-dimensional n-ary Filippov algebra over an algebraically closed field of characteristic 0 for any $n > 2$. The simple Filippov n-ary superalgebras in the finite and infinite dimensional case were studied in [CK]. It was shown there that there are no simple linearly compact n-ary Filippov superalgebras which are not n-ary Filippov algebras, if $n > 2$, and a classification of linearly compact n-ary Filippov algebras was given.

In this paper we give a classification of $(m-3)$-ary algebras with symmetric invariant forms, where $\dim V = m$, satisfying the Jacobi identity (2) over $\mathbb{C}$ and $\mathbb{R}$ up to an isomorphism preserving the invariant form in terms of coadjoint orbits of the Lie group $\mathrm{SO}(V)$. In the real case we give a classification of simple algebras of this type. Our result can be formulated as follows: *almost all real* $(m-3)$*-ary algebras with symmetric invariant forms are simple. The exceptional cases are: the trivial* $(m-3)$*-ary algebra and all* $(m-3)$*-ary algebras that corresponds to decomposible elements.*

`Hodge decomposition for real strongly homotopy algebras.` A definition of a strongly homotopy Lie algebras (or L_∞-algebras or sh-algebras) was given by Lada and Stasheff in [LSt]. For more about strongly homotopy algebras see also [LM], [Vor1], [Vor2]. Another result of our paper is a Hodge Decomposition for real metric homogeneous strongly homotopy algebras. This result is expected, but a remarkable fact is that we can obtain easily such kind of decomposition using derived bracket formalism.

We can also use this formalism to define the Hodge operator on a Riemannian compact oriented manifold M. Indeed, in this case there exists the metric on cotangent space T^*M that is induced by Riemannian metric on the tangent space TM. Then we can define a Poisson bracket on $\bigwedge T^*M$, see [Roy], and repeat the construction of the Hodge operator given in the present paper.

`Quasi-Frobenius structures.` We conclude our paper with a description of quasi-Frobenius structures on skew-symmetric n-ary algebras. Our result is as follows. *There is a one-to-one correspondence between quasi-Frobenius structures on a skew-symmetric* n*-ary algebra and maximal isotropic subalgebras in* T_0^**-extension on this algebra.*

2 Commutative n-ary superalgebras with an invariant skew-symmetric form

2.1 Main definitions

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a finite dimensional $\mathbb{Z}_2$-graded vector space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If $a \in V$ is a homogeneous element, we denote by $\bar{a} \in \mathbb{Z}_2$ the parity of a. As usual we assume that elements in $\mathbb{K}$ are even. Recall that a bilinear form $(\,,)$ on V is called *even* (or *odd*) if the corresponding linear map $V \otimes V \to \mathbb{K}$ is even (or odd). A bilinear form is called *skew-symmetric* if $(a, b) = -(-1)^{\bar{a}\bar{b}}(b, a)$ for any homogeneous elements $a, b \in V$.

Definition 1. • An *n-ary superalgebra structure* on V is an n-linear map

$$V \times \cdots \times V \longrightarrow V,$$
$$(a_1, \ldots, a_n) \mapsto \{a_1, \ldots, a_n\}.$$

• An n-ary superalgebra structure is called *commutative* if

$$\{a_1, \ldots, a_i, a_{i+1}, \ldots, a_n\} = (-1)^{\bar{a}_i \bar{a}_{i+1}} \{a_1, \ldots, a_{i+1}, a_i, \ldots, a_n\} \tag{3}$$

for any homogeneous $a_i, a_{i+1} \in V$.

• A commutative n-ary superalgebra structure is called *invariant with respect to the form* $(\,,)$ if the following holds:

$$(a_0, \{a_1, \ldots, a_n\}) = (-1)^{\bar{a}_0 \bar{a}_1} (a_1, \{a_0, a_2, \ldots, a_n\}) \tag{4}$$

for any homogeneous $a_i \in V$.

We will write *a commutative invariant n-ary superalgebra structure* or *a commutative invariant n-ary superalgebra* as a shorthand for *a commutative n-ary superalgebra structure on V that is invariant with respect to the form* $(\,,)$.

Example 1. The class of commutative invariant n-ary superalgebras includes for instance the following algebras.

• *Anti-commutative algebras on $V = V_{\bar{1}}$ with an invariant symmetric form.* Indeed, in this case the conditions (3) and (4) are equivalent to the following conditions:

$$\{a, b\} = -\{b, a\}, \quad (\{a, b\}, c) = (a, \{b, c\}). \tag{5}$$

In particular, all Lie algebras with an invariant symmetric form are of this type.

• *Commutative algebras on $V = V_{\bar{0}}$ with an invariant skew-symmetric form.* In this case from (3) and (4) it follows:

$$\{a, b\} = \{b, a\}, \quad (\{a, b\}, c) = -(a, \{b, c\}). \tag{6}$$

In particular, commutative associative and Jordan algebras with an invariant skew-symmetric form are of this type.

• *Anti-commutative n-ary algebras on $V = V_{\bar{1}}$ with an invariant symmetric form.* In this case the condition (4) is equivalent to the following condition:

$$(y, \{x_1, \ldots, x_{n-1}, z\}) = (-1)^n(\{y, x_1, \ldots, x_{n-1}\}, z)$$

that is more familiar for physicists. In particular, anti-commutative n-ary algebras satisfying (1) with an invariant symmetric form are of this type. Such algebras are used in the Bagger-Lambert-Gustavsson model (BLG-model), see [AI] for details.

Remark. For a commutative algebra usually one considers the following invariance condition: $(\{a, b\}, c) = (a, \{b, c\})$. If in addition we assume that the form $(\,,)$ is skew-symmetric and non-degenerate, we obtain $2(ab, c) = 0$ for all $a, b, c \in V$, therefore $ab = 0$. In our case we do not have such additional restrictive relations.

2.2 Derived bracket and commutative invariant n-ary superalgebras

Let V be as above. We denote by S^nV the n-th symmetric power of V and we put $S^*V = \bigoplus_n S^nV$. The superspace S^*V possesses a natural structure $[\,,]$ of a Poisson superalgebra. It is defined by the following formulas:

$$[x, y] := (x, y), \quad x, y \in V;$$

$$[v, w_1 \cdot w_2] := [v, w_1] \cdot w_2 + (-1)^{vw_1} w_1 \cdot [v, w_2],$$
$$[v, w] = -(-1)^{vw}[w, v],$$

where v, w, w_i are homogeneous elements in S^*V. One can show that the multiplication $[\,,]$ satisfies the graded Jacobi identity:

$$[v, [w_1, w_2]] = [[v, w_1], w_2] + (-1)^{\bar{v}\bar{w}_1}[w_1, [v, w_2]].$$

This Poisson superalgebra is well-defined. Indeed, we can repeat the argument from [KS, Page 65] for vector superspaces. The idea is to show that this superalgebra is induced by the Clifford superalgebra corresponding to V and $(\,,)$.

Let us take any element $\mu \in S^{n+1}V$. Then we can define an n-ary superalgebra structure on V in the following way:

$$\{a_1, \ldots, a_n\} := [a_1, [\ldots, [a_n, \mu] \ldots]], \;\; a_i \in V. \tag{7}$$

We will denote the corresponding superalgebra by (V, μ) and we will call the element μ the *derived potential* of (V, μ). The n-ary superalgebras of type (V, μ) have the following two properties:

- The multiplication (7) is *commutative*. (This was noticed in [Vor1].) Indeed, using Jacobi identity for S^*V we have:

$$[a_1,[a_2,\dots,[a_n,\mu]\dots]] = [[a_1,a_2],[\dots,[a_n,\mu]\dots]] + (-1)^{\bar{a}_1\bar{a}_2}[a_2,[a_1,\dots,[a_n,\mu]\dots]] = (-1)^{\bar{a}_1\bar{a}_2}[a_2,[a_1,\dots,[a_n,\mu]\dots]].$$

  We used the fact that $[[a_1,a_2],[\dots,[a_n,\mu]\dots]] = 0$, because $[a_1,a_2] \in \mathbb{K}$. Similarly we can prove the commutativity relation for other a_i.

- The n-ary superalgebra structure (7) is invariant. Indeed,

$$(a_0,\{a_1,\dots,a_n\}) = [a_0,[a_1,[a_2,\dots,[a_n,\mu]\dots]]] = (-1)^{\bar{a}_0\bar{a}_1}[a_1,[a_0,[a_2,\dots,[a_n,\mu]\dots]]] = (-1)^{\bar{a}_0\bar{a}_1}(a_1,\{a_0,a_2,\dots,a_n\}).$$

We conclude this section with the following observation.

Proposition 1. *Assume that V is finite dimensional and $(\,,\,)$ is non-degenerate. Any commutative invariant n-ary superalgebra structures can be obtained by construction (7).*

Proof. Denote by $\mathcal{A}_n$ the vector space of commutative invariant n-ary superalgebra structures on V and by $\mathcal{L}_{n+1}$ the vector space of symmetric $(n+1)$-linear maps from V to $\mathbb{K}$. Clearly, $\dim \mathcal{L}_{n+1} = \dim S^{n+1}V$. Since $(\,,\,)$ is non-degenerate, Formula (7) defines an injective linear map $S^{n+1}V \to \mathcal{A}_n$. We can also define an injective linear map $\mathcal{A}_n \to \mathcal{L}_{n+1}$ in the following way:

$$\mathcal{A}_n \ni \mu \longmapsto L_\mu \in \mathcal{L}_{n+1}, \quad L_\mu(a_1,\dots,a_{n+1}) = (a_1,\mu(a_2,\dots,a_{n+1})).$$

Note that L_μ is symmetric since μ defines an invariant superalgebra structure. Summing up, we have the following sequence of injective maps or isomorphisms:

$$S^{n+1}V \hookrightarrow \mathcal{A}_n \hookrightarrow \mathcal{L}_{n+1} \simeq S^{n+1}V.$$

Since V is finite dimensional, we get $S^{n+1}V \simeq \mathcal{A}_n$.□

3 Examples of commutative invariant n-ary superalgebras

Usually one studies superalgebras with an invariant form in the following way. One considers for example a Lie algebra or a Jordan algebra and assumes that the multiplication in the algebra satisfies the following additional condition: it is invariant with respect to a non-degenerate (skew)-symmetric form. The derived bracket formalism permits to express for instance Jacobi, Filippov and Jordan identities in terms of derived potentials and the Poisson bracket on S^*V. In this case the additional invariance condition is fulfilled automatically.

3.1 Strongly homotopy Lie algebras with an invariant skew-symmetric form

We follow Th. Voronov [Vor1] in conventions concerning L_∞-algebras. We set $I^k := (i_1, \dots, i_k)$ and $J^l := (j_1, \dots, j_l)$, where $i_1 < \dots < i_k$ and $j_1 < \dots < j_l$. We denote $a_{I^k} := (a_{i_1}, \dots, a_{i_k})$, $a_{J^l} := (a_{j_1}, \dots, a_{j_l})$ and $a^s := (a_1, \dots, a_s)$, where $a_i \in V$. We put $[a_{I^k}, \mu] := [a_{i_1}, \dots [a_{i_k}, \mu]]$ and $[a^s, \mu] := [a_1, \dots [a_s, \mu]]$, where $\mu \in S^*V$.

Definition 2. A vector superspace V with a sequence of odd n-linear maps μ_n, where $n \geq 0$, is called an L_∞-*algebra* if

- the maps μ_n are commutative in the sense of Definition 1;
- the following generalized Jacobi identities hold:

$$\sum_{k+l=n} \sum_{(I^k, J^l)} (-1)^{(I^k, J^l)} \mu_{l+1}(a_{I^l}, \mu_k(a_{J^k})) = 0, \quad n \geq 0. \tag{8}$$

  Here (I^k, J^l) is a unshuffle permutation of $(1, \dots, n)$ and $(-1)^{(I^k, J^l)}$ is the sign obtained using the sign rule for the permutation (I^k, J^l) of homogeneous elements $a_1, \dots, a_n \in V$.

Definition 3. An L_∞-algebra structure $(\mu_n)_{n\geq 0}$ on V is called *invariant* if all μ_n are invariant in the sense of Definition 1.

The following statement follows from Theorem 1 in [Vor1] and Proposition 1. For completeness we give here a proof in our notations and agreements.

Proposition 2. *Invariant L_∞-algebra structures on V are in one-to-one correspondence with odd elements $\mu \in S^*(V)$ such that $[\mu, \mu] = 0$.*

Proof. Our objective is to show that $[\mu, \mu] = 0$ is equivalent to (8) together with the invariance condition. Let us take any odd element $\mu = \sum_k \mu_k \in S^*V$, where $\mu_k \in S^{k+1}V$. The equation $[\mu, \mu] = 0$ is equivalent to the following equations

$$\sum_{k+l=n} [a^{n-1}, [\mu_l, \mu_k]] = 0$$

for all $n \geq 0$ and all $a_i \in V$. Furthermore, we have:

$$
\begin{aligned}
[a^{n-1}, [\mu_l, \mu_k]] = & \sum_{(I^l, J^{k-1})} (-1)^{(I^l, J^{k-1}) + \bar{a}_{J^{k-1}}} [[a_{I^l}, \mu_l], [a_{J^{k-1}}, \mu_k]] + \\
& \sum_{(I^{l-1}, J^k)} (-1)^{(I^{l-1}, J^k) + \bar{a}_{J^k}} [[a_{I^{l-1}}, \mu_l], [a_{J^k}, \mu_k]] = \\
& \sum_{(I^l, J^{k-1})} (-1)^{(I^l, J^{k-1}) + \bar{a}_{J^{k-1}}} \mu_k(\mu_l(a_{I^l}), a_{J^{k-1}}) + \\
& \sum_{(I^k, J^{l-1})} (-1)^{(J^k, I^{l-1}) + \bar{a}_{J^{l-1}}} \mu_l(\mu_k(a_{I^k}), a_{J^{l-1}}) = \\
& \sum_{(J^{k-1}, I^l)} (-1)^{(J^{k-1}, I^l)} \mu_k(a_{J^{k-1}}, \mu_l(a_{I^l})) + \\
& \sum_{(I^{l-1}, J^k)} (-1)^{(I^{l-1}, J^k)} \mu_l(a_{I^{l-1}}, \mu_k(a_{J^k})).
\end{aligned}
\tag{9}
$$

Therefore, $[a^{n-1}, \sum_{k+l=n} [\mu_l, \mu_k]] = 0$ is equivalent to the generalized Jacobi identity for $k+l = n$. In other words, the equation $[\mu, \mu] = 0$ is equivalent to the generalized Jacobi identities together with the invariance conditions.□

Corollary. *Assume that* $V = V_{\bar{1}}$ *and* n *is even. Anti-commutative invariant* n*-ary algebra structures on* V *satisfying Jacobi (2) are in one-to-one correspondence with elements* $\mu \in S^{n+1}(V)$ *such that* $[\mu, \mu] = 0$.

Proof. In this case the equation 9 has the form:

$$[a^{2n-1}, [\mu, \mu]] = 2 \sum_{(I,J)} (-1)^{(I,J)} \mu(a^I, \mu(a^J)).$$

Here $I = (i_1, \ldots, i_{n-1})$, $J = (j_1, \ldots, j_n)$ are unshuffels and $I \cup J = \{1, \ldots, 2n-1\}$. Since n is even we have:

$$\sum_{(I,J)} (-1)^{(I,J)} \mu(a^I, \mu(a^J)) = \sum_{(I,J)} (-1)^{(J,I)} \mu(\mu(a^J), a^I).$$

The proof is complete.□

3.2 Filippov algebras with invariant symmetric forms

Definition 4. A skew-symmetric n-ary algebra is called a *Filippov algebra* if its multiplication satisfies (1). We say that a Filippov algebra has an *invariant form*

(,) if its multiplication is invariant with respect to (,) in the sense of Definition 1.

Filippov algebras with an invariant form are described in the following proposition. The idea of the proof we borrow in [VV1].

Proposition 3. *Assume that* $V = V_{\bar{1}}$ *and* $\mu \in S^{n+1}V$ *satisfies*

$$[\mu_{a^{n-1}}, \mu] = 0$$

for all $a^{n-1} = (a_1, \ldots, a_{n-1})$*. Then* (V, μ) *is a Filippov (or Nambu-Poisson)* n*-ary algebra with an invariant form.*

Conversely, any Filippov n*-ary algebra with an invariant form can be obtained by this construction.*

Proof. We need to show that $[\mu_{a^{n-1}}, \mu] = 0$ is equivalent to 1, where $\mu_{a^{n-1}} = [a_1, \ldots, [a_{n-1}, \mu]]$ and $a_i \in V$. Let us take $b_1 \ldots, b_n \in V$. We have:

$$[\mu_{a^{n-1}}, [b_1, \ldots [b_n, \mu]]] = \sum_{i=1}^{n} [b_1, \ldots [[\mu_{a^{n-1}}, b_i], \ldots [b_n, \mu]]] + [b_1, \ldots [b_n, [\mu_{a^{n-1}}, \mu]]]]$$

Further,

$$\begin{aligned}[\mu_{a^{n-1}}, [b_1, \ldots [b_n, \mu]]] = \ & -\{\{b_1, \ldots, b_n\}, a_1 \ldots, a_{n-1}\} = \\ & (-1)^n \{a_1, \ldots, a_{n-1}, \{b_1, \ldots, b_n\}\};\end{aligned}$$

$$\begin{aligned}[b_1, \ldots [[\mu_{a^{n-1}}, b_i], \ldots [b_n, \mu]]] = \ & -\{b_1 \ldots, b_{i-1}, \{b_i, a_1 \ldots, a_{n-1}\}, b_{i+1}, \ldots, b_n\} = \\ & (-1)^n \{b_1 \ldots, b_{i-1}, \{a_1 \ldots, a_{n-1}, b_i\}, b_{i+1}, \ldots, b_n\};\end{aligned}$$

Hence, we have:

$$\begin{aligned}\{a_1, \ldots, a_{n-1}, \{b_1, \ldots, b_n\}\} = \ & \sum_{i=1}^{n} \{b_1 \ldots, b_{i-1}, \{a_1 \ldots, a_{n-1}, b_i\}, b_{i+1}, \ldots, b_n\} + \\ & (-1)^n [b_1, \ldots [b_n, [\mu_{a^{n-1}}, \mu]]].\end{aligned}$$

We see that 1 holds if and only if $[b_1, \ldots [b_n, [\mu_{a^{n-1}}, \mu]]] = 0$. By Proposition 1 all such algebras are invariant with respect to (,). The proof is complete.□

3.3 Jordan algebras with symplectic invariant forms

First of all let us recall the definition of a Jordan algebra.

Definition 5. A *Jordan algebra* is a commutative algebra over $\mathbb{K}$ such that the multiplication satisfies the following axiom:

$$(xy)(xx) = x(y(xx)).$$

We call a Jordan algebra *symplectic* if it possesses a non-degenerate skew-symmetric invariant form.

Proposition 4. *Let V be a pure even vector space with a non-degenerate skew-symmetric form. Assume that $A \in S^3V$ satisfies the following identity:*

$$[A_x, A_{[A_x,x]}] = 0,$$

where $A_x = [x, A]$. Then (V, A) is a symplectic Jordan algebra. Conversely, any symplectic Jordan algebra can be obtained by this construction.

Proof. By Proposition 1 any commutative algebra V with a non-degenerate skew-symmetric form can be obtained by the derived bracket construction. Denote by A the derived potential of a commutative algebra V with a non-degenerate skew-symmetric form $(\,,)$. In other words, the multiplication in V is given by

$$xy = [x, [y, A]].$$

We have:

$$(xy)(xx) = [[y, A_x], [[x, A_x], A]]; \quad x(y(xx)) = -[A_x, [y, [[x, A_x], A]]].$$

Further,

$$[A_x, [y, [[x, A_x], A]]] = [[A_x, y], [[x, A_x], A]] + [y, [A_x, [[x, A_x], A]]].$$

We see that this equation is equivalent to

$$-x(y(xx)) = -(xy)(xx) + [y, [A_x, [[x, A_x], A]]].$$

Hence, the algebra V is Jordan if and only if

$$[y, [A_x, [[x, A_x], A]]] = 0$$

for all $x, y \in V$. The last condition is equivalent to

$$[A_x, [[x, A_x], A]] = 0$$

for all $x \in V$.□

3.4 Associative algebras with symplectic invariant forms

Proposition 5. *Assume that $V = V_{\bar{0}}$ and $\mu \in S^3V$ satisfies the following identity:*

$$[\mu_a, \mu_b] = 0$$

for all $a, b \in V$. Here $\mu_x = [x, \mu]$. Then (V, μ) is a commutative associative algebra with a non-degenerate skew-symmetric invariant form.

Conversely, any commutative associative algebra with a non-degenerate skew-symmetric invariant form can be obtained by this construction.

Proof. Let us use the notation:

$$a \circ b := [a, [b, \mu]].$$

We have to show that the associativity relation for $\circ$ is equivalent to $[\mu_a, \mu_b] = 0$ for all $a, b \in V$. Indeed,

$$a \circ (b \circ c) = [a, [[b, [c, \mu]]\mu]] = -[a, [\mu, [b, [c, \mu]]] = \\ -[\mu_a, [b, [c, \mu]]] = -[[\mu_a, b], [c, \mu]] - [b, [\mu_a, \mu_c]] = [[b, \mu_a], [c, \mu]] - [b, [\mu_a, \mu_c]] = \\ (b \circ a) \circ c - [b, [\mu_a, \mu_c]].$$

Therefore, the equality $[\mu_a, \mu_c] = 0$ for all $a, c \in V$ and the associativity law are equivalent. □

Remark. We see that the associativity law for commutative algebras is equivalent to commutativity of all operators μ_a, $a \in V$, where $\mu_a(b) = a \circ b$.

4 Hodge operator and its applications

4.1 $*$-operator and n-ary algebras

Let V be a pure odd vector space of dimension m with a non-degenerate skew-symmetric even bilinear form $(,)$. Recall that means that $(a, b) = (b, a)$ for all $a, b \in V$. Let us choose a normalized orthogonal basis (e_i) of V. Denote by $L := e_1 \dots e_m$ the top form corresponding to the chosen basis. We define the operator $* : S^pV \to S^{m-p}V$ by the following formula:

$$* (x_1 \dots x_p) = [x_1, [\dots [x_p, L]]]. \tag{10}$$

In particular, we have:

$$*(e_{i_1} \dots e_{i_p}) = [e_{i_1}, [\dots [e_{i_p}, L]]] = (-1)^{\sigma} e_{j_1} \dots e_{j_{m-p}},$$

where $\sigma(1, \dots, m) = (i_p, \dots, i_1, j_1, \dots, j_{m-p})$. Clearly, this definition depends only on orientation of V and on the bilinear form $(,)$. Note that $* : S^pV \to S^{m-p}V$ is an isomorphism for all p. This follows for example from the following formula:

$$* * (e_{i_1} \dots e_{i_p}) = (-1)^{\frac{m(m-1)}{2}} e_{i_1} \dots e_{i_p}.$$

The following well-known result we can easily prove using derived bracket formalism:

Proposition 6. *The vector space* $\mathfrak{so}(V)$ *of linear operators preserving the form* $(\,,)$ *is isomorphic to* $S^2(V)$.

Proof. The isomorphism is given by the formula $w \longmapsto \operatorname{ad} w$, where $w \in S^2(V)$ and $\operatorname{ad} w(v) := [w, v]$ for $v \in V$. Indeed, for all $v_1, v_2 \in V$ we have:

$$0 = \operatorname{ad} w([v_1, v_2]) = [[w, v_1], v_2] + [v_1, [w, v_2]] = ([w, v_1], v_2) + (v_1, [w, v_2]).$$

Obviously, this map is injective. We complete the proof observing that the dimensions of $\mathfrak{so}(V)$ and $S^2(V)$ are equal.□

We have seen in previous sections that elements from $S^{(n+1)}V$ corresponds to n-ary algebras with an invariant form. The existence of the $*$-operator for $V = V_{\bar{1}}$ leads to the idea that n-ary and $(m-n)$-ary algebras can have some common properties. In particular such algebras have the same algebra of orthogonal derivations.

Definition 6. A *derivation* of an n-ary algebra (V, μ) is a linear map $D : V \to V$ such that

$$D(\{v_1, \ldots, v_n\}) = \sum_j \{v_1, \ldots, D(v_j), \ldots, v_n\}.$$

We denote by $\operatorname{IDer}(\mu)$ the vector space of all derivations of the algebra (V, μ) preserving the form $(\,,)$.

Proposition 7. *Let us take any* $w \in S^2(V)$ *and* $\mu \in S^{n+1}(V)$.
a. *We have:*

$$\operatorname{IDer}(\mu) \simeq \operatorname{lin}\{w \in S^2(V) \mid \operatorname{ad} w(\mu) = 0\}.$$

b. *The isomorphism* $* : S^p(V) \to S^{m-p}(V)$ *is equivariant with respect to the natural action of* $\mathfrak{so}(V)$ *on* $S^*(V)$. *In particular,*

$$\operatorname{IDer}(\mu) = \operatorname{IDer}(*\mu).$$

Proof. **a.** First of all by the standard argument we obtain:

$$\operatorname{ad} w(\{v_1, \ldots, v_p\}) = [w, [v_1, \ldots, [v_n, \mu] \ldots]] = \sum_i^n [v_1, \ldots, [[w, v_i] \ldots, [v_n, \mu]] \ldots] + [v_1, \ldots, [v_n, [w, \mu]] \ldots] = \sum_j \{v_1, \ldots, [w, v_j], \ldots, v_n\} + [v_1, \ldots, [v_n, [w, \mu]] \ldots].$$

We see that $\operatorname{ad} w$ is a derivation if and only if $[w, \mu] = 0$.

b. Let $L = e_1 \ldots e_m$ be as above and $w \in S^2(V)$. We have,

$$*([w, e_{i_1} \ldots e_{i_p}]) = *(\sum_{j=1}^p e_{i_1} \ldots [w, e_{i_j}] \ldots e_{i_p}) = \sum_{j=1}^p [e_{i_1}, \ldots, [[w, e_{i_j}] \ldots, [e_{i_p}, L]] \ldots].$$

On the other side,

$$[w, *(e_{i_1} \dots e_{i_p})] = [w, [e_{i_1}, \dots [e_{i_p}, L]]] = \sum_{j=1}^{p} [e_{i_1}, \dots, [[w, e_{i_j}] \dots, [e_{i_p}, L]] \dots].$$

We use here the fact that $[w, L] = 0$. Therefore, the $*$-operator is $\mathfrak{so}(V)$-equivariant.

Furthermore, assume that $w \in \mathrm{IDer}(\mu)$ or equivalently that $[w, \mu] = 0$. Therefore,

$$[w, *\mu] = *([w, \mu]) = *(0) = 0.$$

Hence, $w \in \mathrm{IDer}(*\mu)$. Conversely, if $w \in \mathrm{IDer}(*\mu)$ then

$$*([w, \mu]) = [w, *\mu] = 0.$$

This finishes the proof. □

4.2 Hodge decomposition for real metric strongly homotopy algebras

4.2.1 Hodge decomposition for a vector space.

In this Subsection we follow Kostant's approach [Kost, Page 332 - 333]. Let W be a finite dimensional vector space with two linear operators d and δ such that $\mathrm{d}^2 = \delta^2 = 0$.

Definition 7. [Kostant] Linear maps d and δ are called *disjoint* if the following holds:

1. $\mathrm{d} \circ \delta(x) = 0$ implies $\delta(x) = 0$;
2. $\delta \circ \mathrm{d}(x) = 0$ implies $\mathrm{d}(x) = 0$.

Denote $\mathcal{L} = \delta \circ \mathrm{d} + \mathrm{d} \circ \delta$.

Proposition 8. [Kostant] *Assume that* d *and* δ *are disjoint. Then we have:*

$$\mathrm{Ker}(\mathcal{L}) = \mathrm{Ker}(\mathrm{d}) \cap \mathrm{Ker}(\delta)$$

and a direct sum (an analog of a Hodge Decomposition):

$$W = \mathrm{Im}(\mathrm{d}) \oplus \mathrm{Im}(\delta) \oplus \mathrm{Ker}(\mathcal{L}).$$

In this case the restriction $\pi|_{\mathrm{Ker}(\mathcal{L})}$ *of the canonical mapping*

$$\pi : \mathrm{Ker}(\mathrm{d}) \to \mathrm{Ker}(\mathrm{d}) / \mathrm{Im}(\mathrm{d}) =: H(W, \mathrm{d})$$

is a bijection. In other words $\mathrm{Ker}(\mathcal{L}) \simeq H(W, \mathrm{d})$.□

We will use this Proposition to obtain a Hodge decomposition for metric L_∞-algebras.

4.2.2 Hodge decomposition for real metric L_∞-algebras.

Let V be a pure odd real vector space with a non-degenerate skew-symmetric positive defined form $(\, ,)$ and $\mu \in S^*V$ be a homogeneous element such that $[\mu, [\mu, -]] = 0$. If μ is an odd element, the condition $[\mu, [\mu, -]] = 0$ is equivalent to $[\mu, \mu] = 0$. Denote by $\mathrm{d} : V \to V$ the linear operator $v \mapsto [\mu, v]$. Obviously, $\mathrm{d} \circ \mathrm{d} = 0$. Using Hodge $*$-operator we can define the following operator

$$\delta = * \, \mathrm{d} \, *.$$

Again we have $\delta \circ \delta = 0$. We can also define a bilinear product $\langle \, , \rangle$ in S^*V by the following formula:

$$\langle v_1, v_2 \rangle L = \begin{cases} (-1)^{\frac{p(p-1)}{2}} v_1 \cdot (*v_2), & \text{if } v_1, v_2 \in S^pV; \\ 0, & \text{if } v_1 \in S^pV, \, v_2 \in S^qV \text{ and } p \neq q. \end{cases}$$

This bilinear product has the following properties:

Proposition 9. *We have*

$$\langle e_I, e_J \rangle = \begin{cases} 0, & \text{if } I \neq J, \\ 1, & \text{if } I = J. \end{cases}$$

Here $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_p)$ such that $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_p$. In particular, the pairing $\langle \, , \rangle$ is symmetric and positive definite.

Proof. A straightforward computation.□

Proposition 10. *Assume that $\mu \in S^*(V)$ is a homogeneous element and* d *and δ are as above. Then we have*

$$\langle \mathrm{d}(v), w \rangle = -(-1)^{\bar{\mu}\bar{v} + \frac{m(m-1)}{2}} \langle v, \delta(w) \rangle$$

for $v, w \in S^*V$, and the operators d *and δ are disjoint.*

Proof. **Step A.** Let us take $\mu_k \in S^{k+2}V$, $v \in S^{p-k}V$ and $w \in S^pV$. (We assume that $S^rV = \{0\}$ for $r < 0$ and $r > m$, where $m = \dim V$.) Then, $v \cdot *w \in S^{m-k}V$ and we have:

$$[\mu_k, v \cdot *w] \subset [S^{k+2}V, S^{m-k}(V)] = 0.$$

Furthermore,

$$0 = [\mu_k, v \cdot *w] = [\mu_k, v] \cdot *w + (-1)^{\bar{\mu}_k \bar{v}} v \cdot [\mu_k, *w] =$$
$$[\mu_k, v] \cdot *w + (-1)^{\bar{\mu}_k \bar{v} + \frac{m(m-1)}{2}} v \cdot ** \, [\mu_k, *w] =$$
$$\mathrm{d}_k(v) \cdot *w + (-1)^{\bar{\mu}_k \bar{v} + \frac{m(m-1)}{2}} v \cdot *\delta_k(w),$$

where $\mathrm{d}_k(v) = [\mu_k, v]$ and $\delta_k(w) = *[\mu_k, *w]$. Therefore,

$$\langle \mathrm{d}_k(v), w\rangle = -(-1)^{\bar{\mu}_k\bar{v}+\frac{m(m-1)}{2}}\langle v, \delta_k(w)\rangle$$

for all $v \in S^{p-k}V$ and $w \in S^pV$. Note that this equation holds trivially for $v \in S^pV$ and $w \in S^qV$, where $q - p \neq k$. Therefore, we have

$$\langle \mathrm{d}_k(v), w\rangle = -(-1)^{\bar{\mu}_k\bar{v}+\frac{m(m-1)}{2}}\langle v, \delta_k(w)\rangle \tag{11}$$

for all $v, w \in S^*V$, where v is homogeneous, and $\mu_k \in S^{k+2}V$.

Let us take any homogeneous $\mu \in S^*(V)$. Then $\mu = \sum_k \mu_k$, where $\mu_k \in S^k(V)$ and k are all odd or all even. Therefore, d and δ also possess corresponding decomposition: $\mathrm{d} = \sum_k \mathrm{d}_k$ and $\delta = \sum_k \delta_k$, where $\mathrm{d}_k = [\mu_k, -]$ and $\delta_k = * \mathrm{d}_k *$. Using 11, we get for homogeneous $v, w \in S^*V$:

$$\langle \mathrm{d}(v), w\rangle = \sum_k \langle \mathrm{d}_k(v), w\rangle = -\sum_k (-1)^{\bar{\mu}_k\bar{v}+\frac{m(m-1)}{2}}\langle v, \delta_k(w)\rangle =$$
$$-(-1)^{\bar{\mu}\bar{v}+\frac{m(m-1)}{2}}\sum_k \langle v, \delta_k(w)\rangle = -(-1)^{\bar{\mu}\bar{v}+\frac{m(m-1)}{2}}\langle v, \delta(w)\rangle.$$

The first statement is proven.

Step B. Let us show that $\mathrm{d}\circ\delta(v) = 0$ implies $\delta(v) = 0$, i.e. the operators d and δ are disjoint. (This argument we borrow from [Kost].) Indeed,

$$0 = \langle \mathrm{d}\circ\delta(v), v\rangle = -(-1)^{\bar{\mathrm{d}}(\bar{\delta}+\bar{v})+\frac{m(m-1)}{2}}\langle \delta(v), \delta(v)\rangle.$$

The pairing $\langle\, ,\rangle$ is positive definite, hence $\delta(v) = 0$. Analogously we can show that $\delta \circ \mathrm{d}(v) = 0$ implies $\mathrm{d}(v) = 0$.□

Denote by $H(V, \mu)$ the cohomology space of the L_∞-algebra (V, μ), where $\mu \in S^*V$ is an odd element such that $[\mu, \mu] = 0$. By definition $H(V, \mu) := \mathrm{Ker}(\mathrm{d})/\mathrm{Im}(\mathrm{d})$. The main result of this section is the following theorem.

Theorem 1. [Hodge decomposition for real metric L_∞-algebras] *Let $\mu \in S^*(V)$ be a real metric L_∞-algebra structure on V and* d *and δ be as above. Then we have a direct sum decomposition:*

$$V = \mathrm{Im}(\mathrm{d}) \oplus \mathrm{Im}(\delta) \oplus \mathrm{Ker}(\mathcal{L}),$$

where $\mathcal{L} = \delta \circ \mathrm{d} + \mathrm{d}\circ\delta$, and $\mathrm{Ker}(\mathcal{L}) \simeq H(V, \mu)$.

Proof. The statement follows from Propositions 8 and 10 □.

5 m-dimensional Filippov and Lie $(m-3)$-algebras

5.1 $(m-3)$-ary algebras with non-degenerate symmetric forms and coadjoint orbits

Another application of the $*$-operator is the following: we can classify all $(m-3)$-ary symmetric algebras up to orthogonal isomorphism in terms of coadjoint orbits. If we assume in addition that such $(m-3)$-ary algebras are real and that the form $(\,,)$ is positive definite then we can classify all simple algebras. Let again V be a pure odd vector space with an even non-degenerate skew-symmetric form $(\,,)$, i.e. $(a,b) = (b,a)$ for all $a, b \in V$. As usual we denote by $\mathrm{O}(V)$ the Lie group of all invertible linear operators on V that preserve the form $(\,,)$ and by $\mathrm{SO}(V)$ the subgroup of $\mathrm{O}(V)$ that contains all operators with the determinant $+1$. We have $\mathfrak{so}(V) = \mathrm{Lie}\,\mathrm{O}(V) = \mathrm{Lie}\,\mathrm{SO}(V)$.

Definition 8. Two n-ary algebra structures $\mu, \mu' \in S^*V$ on V are called *isomorphic* if there exists $\varphi \in \mathrm{SO}(V)$ such that

$$\varphi(\{v_1, \dots, v_n\}_\mu) = \{\varphi(v_1), \dots, \varphi(v_n)\}_{\mu'}$$

for all $v_i \in V$. Here we denote by $\{\dots\}_\nu$ the multiplication on V corresponding to the algebra structure ν.

Sometimes we will consider isomorphism of n-ary algebra structures up to $\varphi \in \mathrm{O}(V)$. We need the following two lemmas:

Lemma 1. *Let us take $\varphi \in \mathrm{O}(V)$ and $w, v \in S^*V$. Then, $\varphi([w,v]) = [\varphi(w), \varphi(v)]$. In other words, φ preserves the Poisson bracket.*

Proof. It follows from the following two facts:

- $(\varphi(w), \varphi(v)) = (w, v),$ if $w, v \in V$;
- $\varphi(w \cdot v) = \varphi(w) \cdot \varphi(v)$ for all $w, v \in S^*V$. □

Lemma 2. *Two n-ary algebras (V, μ) and (V, μ'), where $\mu, \mu' \in S^*V$, are isomorphic if and only if there exists $\varphi \in \mathrm{SO}(V)$ such that $\varphi(\mu) = \mu'$. In other words, two n-ary algebras are isomorphic if and only if the corresponding n-ary algebra structures are in the same orbit of the action* $\mathrm{SO}(V)$ *on* $S^{n+1}V$.

Proof. From Lemma 1 it follows that

$$\varphi(\{v_1, \dots, v_n\}_\mu) = \{\varphi(v_1), \dots, \varphi(v_n)\}_{\varphi(\mu)}.$$

Furthermore, if (V, μ) and (V, μ') are isomorphic and $\varphi \in \mathrm{SO}(V)$ is an isomorphism then from the definition it follows that:

$$\varphi(\{v_1, \dots, v_n\}_\mu) = \{\varphi(v_1), \dots, \varphi(v_n)\}_{\mu'}$$

for all $v_i \in V$. Therefore, $\varphi(\mu) = \mu'$. The converse statement is obvious. $\square$

Assume that $\dim V = m$.

Theorem 2. *Classes of isomorphic real or complex $(m-3)$-ary algebras with the invariant form $(\,,)$ are in one-to-one correspondence with coadjoint orbits of the Lie group* $\mathrm{SO}(V)$.

Proof. It follows from Proposition 7 and Lemma 2. Note that in the case of the Lie group $\mathrm{SO}(V)$ the adjoint and coadjoint action are equivalent. $\square$

It is well-known that any real skew-symmetric matrix A can be written in the following form:

$$A = QA'Q^{-1},$$

where

$$A' = \mathrm{diag}(J_{a_1}, \dots, J_{a_k}, 0, \dots, 0),$$

$$J_{a_j} = \begin{pmatrix} 0 & a_j \\ -a_j & 0 \end{pmatrix}, \quad a_j \in \mathbb{R},$$

and $Q \in \mathrm{SO}(V)$. If we assume in addition that $Q \in \mathrm{O}(V)$ and $0 < a_k \leq \cdots \leq a_1$, then A' is unique. (This follows from the uniqueness of the Jordan normal form of a given matrix up to the order of the Jordan blocks and from the fact that A has the following eigenvalues: $\pm i a_j$, where $j = 1, \dots, k$, and 0.) Furthermore, by Proposition 6 we have an isomorphism $\mathfrak{so}(V) \simeq S^2V$. Let (ξ_i) be an orthogonal basis of V such that the matrix $A \in \mathfrak{so}(V)$ has the form

$$A = \mathrm{diag}(J_{a_1}, \dots, J_{a_k}, 0, \dots, 0).$$

Then the corresponding element in S^2V is

$$v_A = a_1\xi_1\xi_2 + \dots, a_k\xi_{2k-1}\xi_{2k},$$

where $0 < a_k \leq \cdots \leq a_1$ and $a_j \in \mathbb{R}$.

We obtained the following theorem:

Theorem 3. [Classification of real $(m-3)$-ary algebras up to $\mathrm{O}(V)$-isomorphism] *Real $(m-3)$-ary algebras with the invariant positive definite form $(\,,)$ are parametrized by vectors*

$$v = a_1\xi_1\xi_2 + \dots, a_k\xi_{2k-1}\xi_{2k},$$

where $a_i \in \mathbb{R}$, $0 < a_k \leq \cdots \leq a_1$ and $0 \leq k \leq [\frac{m}{2}]$. Explicitly such algebras are given by (V, μ_v), where

$$\mu_v = *(v).$$

5.2 Classification of real simple $(m-3)$-ary algebras with positive definite invariant forms

In this section we give a classification of simple $(m-3)$-ary algebras with invariant forms up to orthogonal isomorphism.

Definition 9. A vector subspace $W \subset V$ is called an *ideal* of a symmetric n-ary algebra (V, μ) if $\mu(V, \dots, V, W) \subset W$.

In other words, the vector space W is an ideal if and only if it is invariant with respect to the set of endomorphisms $\mu_{v_1,\dots,v_{n-1}} : V \to V$, where $v_i \in V$. Clearly, the vector space W is an ideal if and only if it is invariant with respect to the Lie algebra $\mathfrak{g}$ that is generated by all $\mu_{v_1,\dots,v_{n-1}}$.

Definition 10. An n-ary Lie algebra is called *simple* if it is not 1-dimensional and it does not have any proper ideals.

Example 2. The classification of simple complex and real Filippov n-ary algebras was done in [Ling]: there is one series of complex Filippov n-ary algebras A_k, where k is a natural number and several real forms for each A_k. All these algebras have invariant forms and in our terminology they are given by the top form L and formula (7).

Example 3. Let $m = 5$. By Theorem 3 we see that we have three types of 2-ary algebras up to isomorphism:

- $\mu_1 = 0$;
- $\mu_2 = b_1\xi_3\xi_4\xi_5$, where $b_1 \neq 0$;
- $\mu_3 = b_1\xi_3\xi_4\xi_5 + b_2\xi_1\xi_2\xi_5$, where $b_1, b_2 \neq 0$.

Obviously, the zero algebra $\mu_1 = 0$ is not simple. The second algebra structure $\mu_2 = b_1\xi_3\xi_4\xi_5$ has a non-trivial center because

$$(\mu_2)_{\xi_1} = (\mu_2)_{\xi_2} = 0,$$

Therefore it is also not simple. We will see that the algebra μ_3 is simple. It is not a Lie algebra because $[\mu_3, \mu_3] = -2b_1b_2\xi_1\xi_2\xi_3\xi_4 \neq 0$.

Theorem 4. **[Classification of real simple $(m-3)$-ary algebras with invariant forms]** *Assume that $m > 4$. All real $(m-3)$-ary algebras from Theorem 3 are simple except of two cases:*

- $v = 0$*;*
- $v = a_1\xi_1\xi_2$*, where* $a_1 \neq 0$*.*

Proof. Clearly, the trivial derived potential $\mu = 0$ determines a non-simple algebra. The algebra (V, μ), where $\mu = *(v)$ and $v = a_1\xi_1\xi_2$, has a non-trivial center. Indeed, we have $*(v) = \pm a_1\xi_3 \cdots \xi_m$. Therefore,

$$[x_1, \ldots, [\xi_1, \mu]] = 0 \text{ for all } x_i \in V$$

We see that $\operatorname{lin}\{\xi_1\}$ is an ideal. Hence, this algebra is also not simple.

Let as show that the other algebras from Theorem 3 are simple. Consider the Lie algebra $\mathfrak{g}(\mu) \subset S^2V$ generated by linear operators $\mu_{v_1,\ldots,v_{m-4}} : V \to V$, where $v_i \in V$, and the linear space $L(\mu) = \operatorname{lin}\{\mu_{v_1,\ldots,v_{m-3}}\} \subset V$. The idea of the proof is to show by induction that

1. $\mathfrak{g}(\mu) := \operatorname{Lie}\{\mu_{v_1,\ldots,v_{m-4},-}\} = S^2V \simeq \mathfrak{so}(V)$;
2. $L(\mu) = \operatorname{lin}\{\mu_{v_1,\ldots,v_{m-3}}\} = V$.

From the first observation it follows that V is an irreducible module of $\mathfrak{g}(\mu)$ or equivalently that (V, μ) does not contain any non-trivial ideals. The second observation is an auxiliary statement.

Base case. Consider the case $\dim V = 5$ and $k = 2$. Then

$$\mu = *(a_1\xi_1\xi_2 + a_2\xi_3\xi_4) = b_1\xi_3\xi_4\xi_5 + b_2\xi_1\xi_2\xi_5,$$

where $b_i = \pm a_i$. A direct computation shows that:

$$[\xi_1, \mu] = b_2\xi_2\xi_5, \quad [\xi_2, \mu] = -b_2\xi_1\xi_5, \quad [\xi_3, \mu] = b_1\xi_4\xi_5, \quad [\xi_4, \mu] = -b_1\xi_3\xi_5$$

Therefore, the Lie algebra $\mathfrak{g}(\mu)$ contains all endomorphisms of the form $w \cdot \xi_5$, where $w \in \operatorname{lin}\{\xi_1, \ldots, \xi_4\}$. Further, let us take $\xi_i\xi_5$ and $\xi_j\xi_5$ in $\mathfrak{g}(\mu)$, where $i \neq j$ and $i, j \in \{1, \ldots, 4\}$. Then

$$[\xi_i\xi_5, \xi_j\xi_5] = \pm\xi_i\xi_j \in \mathfrak{g}(\mu).$$

Therefore, $\mathfrak{g}(\mu) = S^2V$ and we prove the first statement. Again by direct computation we obtain:

$$[\xi_i, \xi_i\xi_5] = \xi_5, \quad [\xi_5, \xi_i\xi_5] = -\xi_i.$$

Hence, $L(\mu) = V$ and the second statement in proven in this case.

Inductive step. Assume that $\dim V = m > 5$ and $\mu = *(v)$, where

$$v = a_1\xi_1\xi_2 + \ldots + a_k\xi_{2k-1}\xi_{2k}, \quad a_i \neq 0, \quad k > 1, \quad m \geqslant 2k.$$

Explicitly the derived potential μ is given by

$$\mu = \sum_{i=1}^{k} b_i\xi_1 \cdots \hat{\xi}_{2i-1}\hat{\xi}_{2i} \cdots \xi_m, \quad b_i = \pm a_i.$$

Here $\hat{\xi}_{2i-1}\hat{\xi}_{2i}$ are omitted. Consider first the cases $m > 2k$. Then, $\mu = \mu'\xi_m$, where μ' is an $(m-4)$-ary algebra on the vector space $V' = \mathrm{lin}\{\xi_1, \ldots, \xi_{m-1}\}$ of dimension $m-1$. By induction, we have

$$\mathfrak{g}(\mu') \simeq \mathfrak{so}(V') \qquad \text{and} \qquad L(\mu') = V'. \tag{12}$$

Using $[\xi_m, \mu] = \pm\mu'$ and (12), we see that $\mathfrak{g}(\mu) \supset \mathfrak{so}(V')$ and $L(\mu) \supset V'$. Again using (12), the second equality, we get that for any $w \in V'$ there exist $x_1, \ldots, x_{m-4} \in V'$ such that

$$[x_1, \ldots, [x_{m-4}, \mu']] = w.$$

Therefore,

$$[x_1, \ldots, [x_{m-4}, \mu]] = w \cdot \xi_m.$$

In other words, $\mathfrak{g}(\mu)$ contains all endomorphisms of the form $w \cdot \xi_m$, where $w \in V'$. Hence,

$$\mathfrak{g}(\mu) \supset \mathfrak{so}(V') \oplus (V' \cdot \xi_m) = \mathfrak{so}(V).$$

Since, $[\xi_i, \xi_i\xi_m] = \xi_m$, where $i \neq m$, we have $L(\mu) = V$.

Now consider the case $m = 2k$. We can rewrite μ in the following form

$$\mu = (\mu')\xi_{m-1}\xi_m + b_k\xi_1\xi_2 \cdots \xi_{2k-2}, \quad b_k \neq 0,$$

where μ' is an $(m-5)$-ary algebra on the vector space $\mathrm{lin}\{\xi_1, \ldots \xi_{m-2}\}$ of dimension $m-2$. Since

$$[\xi_m, \mu] = \pm\mu'\xi_{m-1} \quad \text{and} \quad [\xi_{m-1}, \mu] = \pm\mu'\xi_m,$$

we see as above that $\mathfrak{g}(\mu) \supset \mathfrak{so}(V')$ and $\mathfrak{g}(\mu) \supset \mathfrak{so}(V'')$, where $V' = \mathrm{lin}\{\xi_1, \ldots \xi_{m-1}\}$ and $V'' = \mathrm{lin}\{\xi_1, \ldots \xi_{m-2}, \xi_m\}$. Since,

$$[\xi_i\xi_m, \xi_i\xi_{m-1}] = \pm\xi_{m-1}\xi_m$$

for any $i \in \{1, \ldots, m-2\}$, we get that $\mathfrak{g}(\mu) = \mathfrak{so}(V)$. It is also clear that $L(\mu) \supset V'$ and $L(\mu) \supset V''$, hence $L(\mu) = V$. The proof is complete.□

5.3 Classification of real simple $(m-3)$-ary algebras satisfying Jacobi identity 1 and 2

In this Section we classify real simple n-ary algebras with a positive definite invariant form satisfying Jacobi identity 1 and 2.

Jacobi identity 1. In [Ling] it was proven that there exist only one complex Filippov n-ary algebra for any $n > 2$. This algebra is $(n+1)$-dimensional. In our notations it is given by $*(1) = L$. Another result in [Ling] is the following:

A real simple Filippov n-ary algebra is isomorphic to the realification of a simple complex Filippov n-ary algebra or to a real form of a simple complex Filippov n-ary algebra.

In particular real simple Filippov n-ary algebras are of dimension $n+1$ or $2n+2$. It follows that simple n-ary algebras in Theorem 4 are not of Filippov type. For $n = m-2$ any derived potential has the form $\mu = *(v)$, where $v \in V \setminus \{0\}$. All such algebras have non-trivial centers because $[v, \mu] = 0$. Therefore, they are not simple. Furthermore, such algebras are of Filippov type. Indeed, since L satisfy 1 by Proposition 3 we have $[L_{a_1,\ldots,a_{m-1}}, L] = 0$ for any $a_i \in V$. Hence,

$$[v, [L_{a_1,\ldots,a_{m-2},v}, L]] = [L_{a_1,\ldots,a_{m-2},v}, L_v] = [\mu_{a_1,\ldots,a_{m-2}}, \mu] = 0.$$

By Proposition 3, we see that (V, μ) is a Filippov algebra. By the same argument the derived potential $[v, [w, L]]$ also corresponds to a Filippov algebra.

Theorem 5. *Assume that $m > 4$. Real m-dimensional n-ary Filippov algebras with a symmetric positive definite invariant form, where $n = m-1$, $m-2$ or $m-3$, are given up to isometry by the following derived potentials:*

- $\mu = 0$*, the trivial algebra;*
- $\mu = a\xi_1 \cdots \xi_m$*, where* $a \in \mathbb{R} \setminus \{0\}$*;*
- $\mu = a\xi_1 \cdots \xi_{m-1}$*, where* $a \in \mathbb{R} \setminus \{0\}$*;*
- $\mu = a\xi_1 \cdots \xi_{m-2}$*, where* $a \in \mathbb{R} \setminus \{0\}$*;*

Jacobi identity 2. Assume that $m > 4$ and $(\, , \,)$ is a symmetric positive definite form.

Theorem 6. *All algebras in Theorem 3 satisfy Jacobi identity 2 with the exception of the following cases:*

- $m = 5$*, the algebras with derived potential* $\mu = *(a_1\xi_1\xi_2 + a_2\xi_3\xi_4)$*, where* $a_1, a_2 \neq 0$*;*
- $m = 6$*, the algebras with derived potentials* $\mu = *(a_1\xi_1\xi_2 + a_2\xi_3\xi_4)$ *and* $\mu = *(a_1\xi_1\xi_2 + a_2\xi_3\xi_4 + a_3\xi_5\xi_6)$*, where* $a_i \neq 0$*;*

Proof. Assume that m is odd. By Corollary of Proposition 2 in this case Jacobi identity 2 is equivalent to $[\mu, \mu] = 0$. Assume that $m > 5$, then $[\mu, \mu] \in S^{2m-6}V = \{0\}$. In the case $m = 5$ the result follows from Example 3.

Assume that m is even. First of all consider the case $m = 6$. Let us take

$$\mu = b_1\xi_3\xi_4\xi_5\xi_6 + b_2\xi_1\xi_2\xi_5\xi_6, \quad b_1, b_2 \neq 0.$$

Deanote by LHS the left hand side of 2. Let us calculate LHS for $a_i = \xi_i$, $i = 1 \dots, 5$.

$$LHS = \{\{\xi_1, \xi_2, \xi_5\}, \xi_3, \xi_4\} + \{\{\xi_3, \xi_4, \xi_5\}, \xi_1, \xi_2\} = -2b_1 b_2 \xi_5 \neq 0.$$

The main idea here to use the fact that $\{x, y, z\} = 0$ if $x \in \{\xi_1, \xi_2\}$ and $y \in \{\xi_3, \xi_4\}$. The proof for

$$\mu = b_1\xi_3\xi_4\xi_5\xi_6 + b_2\xi_1\xi_2\xi_5\xi_6 + b_3\xi_1\xi_2\xi_3\xi_4, \quad b_i \neq 0$$

is similar.

Consider the case $m > 6$. Without loss of generality we can assume that between elements a_i, where $i = 1, \dots, 2m-7$, are at least two equal. Let $a_s = a_t = v$. Clearly, $\{a_{i_1}, \dots, v, \dots, v, \dots, a_{i_n}\} = 0$. Therefore,

$$LHS = \sum_{k,l} J_1^{(k,l)} + \sum_{k,l} J_2^{(k,l)},$$

where $J_1^{(k,l)}$ and $J_2^{(k,l)}$ is the sum of all summands of the form

$$\{\{a_{i_1}, \dots, \underset{k}{a_s}, \dots, a_{i_{m-3}}\}, a_{j_1}, \dots, \underset{l}{a_t}, \dots, a_{j_{m-4}}\},$$
$$\{\{a_{i_1}, \dots, \underset{k}{a_t}, \dots, a_{i_{m-3}}\}, a_{j_1}, \dots, \underset{l}{a_s}, \dots, a_{j_{m-4}}\}$$

respectively. Further,

$$\begin{aligned} J_1^{(k,l)} = \; & \pm \sum (-1)^{(I,J)} \{\{a_{i_1}, \dots, \underset{k}{\hat{a}_s}, \dots, a_{i_{m-3}}, a_s\}, a_{j_1}, \dots, \underset{l}{\hat{a}_t}, \dots, a_{j_{m-4}}, a_t\} = \\ & \pm \sum (-1)^{(I',J')} \{\{a_{i_1}, \dots, \underset{k}{\hat{a}_s}, \dots, a_{i_{m-3}}\}_v, a_{j_1}, \dots, \underset{l}{\hat{a}_t}, \dots, a_{j_{m-4}}\}_v, \end{aligned}$$

where $\{\dots\}_v$ is the multiplication corresponding to the derived potential $\mu_v = [v, \mu]$ and $(-1)^{(I',J')}$ is the sign of the permutation

$$(a_1, \dots, \hat{a}_s, \dots, \hat{a}_t, \dots, a_{2m-7}) \longmapsto (a_{i_1}, \dots, \underset{k}{\hat{a}_s}, \dots, a_{i_{m-3}}, a_{j_1}, \dots, \underset{l}{\hat{a}_t}, \dots, a_{j_{m-4}}).$$

Since $\mu_v \in S^{m-3}W$, where $W = \langle v \rangle^\perp$, we see that $[\mu_v, \mu_v] = 0$. Therefore 2 holds for $\{\dots\}_v$ and $J_1^{(k,l)} = 0$. Similarly, $J_2^{(k,l)} = 0$. The proof is complete.□

Corollary. *All simple algebras in Theorem 4 satisfy Jacobi identity 2 for $m > 6$.*

6 Quasi-Frobenius skew-symmetric n-ary algebras

Using "derived bracket" construction it is possible to answer the question when a skew-symmetric n-ary algebra is quasi-Frobenius. Let V be a pure odd vector space and $\mu \in S^n(V^*) \otimes V$ be an n-ary symmetric algebra structure on V.

Definition 11. An n-ary algebra (V, μ) is called *quasi-Frobenius* if it is equipped with a symmetric bilinear form φ such that

$$\sum_{cycl} \varphi(a_1, \mu(a_2, \ldots, a_{n+1})) = 0. \tag{13}$$

If we forget about superlanguage this means that the algebra (V, μ) is skew-symmetric and φ is a skew-symmetric bilinear form on V.

Example 4. Assume that $n = 2$ and (V, μ) is a Lie algebra. Then our definition coincides with the definition of a quasi-Frobenius Lie algebra. Recall that a *quasi-Frobenius Lie algebra* is a Lie algebra $\mathfrak{g}$ equipped with a non-degenerate skew-symmetric bilinear form β such that

$$\beta([x, y], z) + \beta([z, x], y) + \beta([y, z], x) = 0.$$

We may assign an n-ary algebra $(V \oplus V^*, \mu^T)$ to (V, μ), called the T_0^*-extension of (V, μ). (The notion of T_θ^*-extension for algebras was introduced and studied in [Bord]. We will need this notion only for $\theta = 0$.) The construction of $(V \oplus V^*, \mu^T)$ is very simple: the n-ary algebra structure μ^T is just the image of μ by the natural inclusion $S^n(V^*) \otimes V \hookrightarrow S^*(V^* \oplus V)$. Furthermore, the pure odd vector space $V \oplus V^*$ has a skew-symmetric (in supersense) pairing given by

$$(a, \alpha) = (\alpha, a) = \alpha(a),$$

where $\alpha \in V^*$ and $a \in V$. This defines a Poisson bracket on $S^*(V \oplus V^*)$. So $(V \oplus V^*, \mu^T)$ as a quadratic symmetric n-ary algebra, where the multiplication is given by the derived bracket with the derived potential $\mu^T \in S^*(V^* \oplus V)$. More precisely, the new multiplication μ^T in $V \oplus V^*$ is given by:

$$\mu^T|_{S^n(V)} = \mu, \quad \mu^T|_{S^{n-k}(V)\cdot S^k(V^*)} = 0 \text{ if } k > 1, \quad \mu^T(S^{n-1}(V) \cdot S^1(V^*)) \subset V^*$$

and

$$\mu^T(a_1, \ldots, a_{n-1}, b^*)(c) := -b^*(\mu(a_1, \ldots, a_{n-1}, c)).$$

The main observation here is:

Proposition 11. *Let V be a pure odd vector space and n be even. Then an n-ary algebra (V, μ) has a quasi-Frobenius structure with respect to a symmetric form φ if and only if the maximal isotropic subspace $B_\varphi = \{a + \varphi(a, -)\} \subset V \oplus V^*$ is a subalgebra in $(V \oplus V^*, \mu^T)$.*

In other words, there is a one-to-one correspondence between quasi-Frobenius structures on (V, μ) and maximal isotropic subalgebras in $(V \oplus V^*, \mu^T)$ that are transversal to V^*.

Proof. First of all it is well-known that maximal isotropic subspaces in $V \oplus V^*$ that are transversal to V^* are in one-to-one correspondence with $\varphi \in S^2V$. Let us show that φ satisfies (13) if and only if B_φ is a subalgebra. Denote $a^* := \varphi(a, -) \in V^*$. Then we have:

$$\begin{gathered}(\mu^T(a_1 + a_1^*, \ldots, a_n + a_n^*), c + c^*) = \\ c^*(\mu(a_1, \ldots, a_n)) + \sum_k (\mu^T(a_1, \ldots, a_k^*, \ldots, a_n), c) = \\ \varphi(c, \mu(a_1, \ldots, a_n)) - \sum_k a_k^*(\mu(a_1, \ldots, a_{k-1}, c, a_{k+1}, \ldots, a_n)) = \\ \varphi(c, \mu(a_1, \ldots, a_n)) - \sum_k \varphi(a_k, \mu(a_1, \ldots, a_{k-1}, c, a_{k+1} \ldots, a_n)).\end{gathered}$$

Furthermore,

$$\begin{gathered}\varphi(a_k, \mu(a_{k+1}, \ldots, a_n, c, a_1 \ldots, a_{k-1})) = \\ (-1)^{(k-1)(n-k-1)}\varphi(a_k, \mu(a_1, \ldots, a_{k-1}, a_{k+1} \ldots, a_n, c)) = \\ (-1)^{k(n-k-1)+1}\varphi(a_k, \mu(a_1, \ldots, a_{k-1}, c, a_{k+1} \ldots, a_n)).\end{gathered}$$

If n is even, $(-1)^{k(n-k-1)+1} = -1$. Therefore, we have:

$$(\mu^T(a_1 + a_1^*, \ldots, a_n + a_n^*), a_{n+1} + a_{n+1}^*) = \sum_{cycl} \varphi(a_1, \mu(a_2, \ldots, a_{n+1})).$$

This expression is equal to 0 if and only if the algebra (V, μ) is quasi-Frobenius with respect to φ. On other side, $(\mu^T(a_1 + a_1^*, \ldots, a_n + a_n^*), a_{n+1} + a_{n+1}^*)$ is equal to 0 if and only if B_φ is a subalgebra in $(V \oplus V^*, \mu^T)$. The proof is complete.□

Remark. The result of Proposition 11 is well-known for Lie algebras.

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