The splitting problem for complex homogeneous supermanifolds [1]

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Abstract

It is a classical result that any complex analytic Lie supergroup $\mathcal{G}$ is split [5], that is its structure sheaf is isomorphic to the structure sheaf of a certain vector bundle. However, there do exist non-split complex analytic homogeneous supermanifolds.

We study the question how to find out whether a complex analytic homogeneous supermanifold is split or non-split. Our main result is a description of left invariant gradings on a complex analytic homogeneous supermanifold $\mathcal{G}/\mathcal{H}$ in the terms of $\mathcal{H}$-invariants. As a corollary to our investigations we get some simple sufficient conditions for a complex analytic homogeneous supermanifold to be split in terms of Lie algebras.

1 Introduction

A supermanifold is called split if its structure sheaf is isomorphic to the exterior power of a certain vector bundle. By Batchelor's Theorem any real supermanifold is non-canonically split. However, this is false in the complex analytic case. The property of a supermanifold to be split is very important for several reasons. For instance, in [2] it was shown that the moduli space of super Riemann surfaces is not projected (and in particular is not split) for genus $g \geqslant 5$. The physical meaning of this result is that [2]: "certain approaches to superstring perturbation theory that are very powerful in low orders have no close analog in higher orders". Another problem, when the property of a supermanifold to be split is very important, is the calculation of the cohomology group with values in a vector bundle over a supermanifold. In the split case we may use the well understood tools of complex analytic geometry. In the general case, several methods were suggested by Onishchik's school: spectral sequences, see e.g. [12]. All these methods connect the cohomology group with values in a vector bundle with the cohomology group with values in the corresponding split vector bundle.

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How do we determine whether a complex analytic supermanifold is split or non-split? Let me describe here some results in this direction that were obtained by Green, Koszul, Onishchik and Serov. In [3] Green described a moduli space with a marked point such that any non-marked point corresponds to a non-split supermanifold while the marked point corresponds to a split one. His idea was used for instance in [2]. The calculation of the Green moduli space is a difficult problem itself, and in many cases the method is difficult to apply. Furthermore, Onishchik and Serov [9, 10, 11] considered grading derivations, which correspond to $\mathbb{Z}$-gradings of the structure sheaf of a supermanifold. For example, it was shown that almost all supergrassmannians do not possess such derivations, i.e. their structure sheaves do not possess any $\mathbb{Z}$-gradings. Hence, in particular, they are non-split. The idea of grading derivations was independently used by Koszul. In [4] the following statement was proved: if the tangent bundle of a supermanifold $\mathcal{M}$ possesses a (holomorphic) connection then $\mathcal{M}$ is split. (Koszul's proof works in real and complex analytic cases.) In fact, it was shown that we can assign a grading derivation to any supermanifold with a connection and that this grading derivation is induced by a $\mathbb{Z}$-grading of a vector bundle.

Assume that a complex analytic supermanifold $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$ is split. By definition this means that its structure sheaf $\mathcal{O}_{\mathcal{M}}$ is isomorphic to $\bigwedge \mathcal{E}$, where $\mathcal{E}$ is a locally free sheaf on the complex analytic manifold $\mathcal{M}_0$. The sheaf $\bigwedge \mathcal{E}$ is naturally $\mathbb{Z}$-graded and the isomorphism $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$ induces the $\mathbb{Z}$-grading in $\mathcal{O}_{\mathcal{M}}$. We call such gradings *split*. The main result of our paper is a description of those left invariant split gradings on a homogeneous superspace $\mathcal{G}/\mathcal{H}$ which are compatible with split gradings on $\mathcal{G}$. We also give sufficient conditions for pairs $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g} = \mathrm{Lie}\,\mathcal{G}$ and $\mathfrak{h} = \mathrm{Lie}\,\mathcal{H}$, such that $\mathcal{G}/\mathcal{H}$ is split.

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2 Complex analytic supermanifolds. Main definitions.

We will use the word "supermanifold" in the sense of Berezin and Leites, see [1], [7] and [8] for details. Throughout, we will be interested in the complex analytic version of the theory. Recall that a *complex analytic superdomain* *of dimension* $n|m$ is a $\mathbb{Z}_2$-graded ringed space

$$\mathcal{U} = \Big(U, \mathcal{F}_U \otimes \bigwedge(m)\Big),$$

where $\mathcal{F}_U$ is the sheaf of holomorphic functions on an open set $U \subset \mathbb{C}^n$ and $\bigwedge(m)$ is the exterior (or Grassmann) algebra with m generators. A *complex analytic supermanifold* of dimension $n|m$ is a $\mathbb{Z}_2$-graded ringed space that is locally isomorphic to a complex superdomain of dimension $n|m$.

Let $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$ be a complex analytic supermanifold and

$$\mathcal{J}_{\mathcal{M}} = (\mathcal{O}_{\mathcal{M}})_{\bar{1}} + (\mathcal{O}_{\mathcal{M}})_{\bar{1}}^2$$

be the subsheaf of ideals generated by odd elements in $\mathcal{O}_{\mathcal{M}}$. We put $\mathcal{F}_{\mathcal{M}} := \mathcal{O}_{\mathcal{M}}/\mathcal{J}_{\mathcal{M}}$. Then $(\mathcal{M}_0, \mathcal{F}_{\mathcal{M}})$ is a usual complex analytic manifold. It is called the *reduction* or *underlying space* of $\mathcal{M}$. We will write $\mathcal{M}_0$ instead of $(\mathcal{M}_0, \mathcal{F}_{\mathcal{M}})$ for simplicity of notation. Morphisms of supermanifolds are just morphisms of the corresponding $\mathbb{Z}_2$-graded ringed spaces. If $f : \mathcal{M} \to \mathcal{N}$ is a morphism of supermanifolds, then we denote by f_0 the morphism of the underlying spaces $\mathcal{M}_0 \to \mathcal{N}_0$ and by f^* the morphism of the structure sheaves $\mathcal{O}_{\mathcal{N}} \to (f_0)_*(\mathcal{O}_{\mathcal{M}})$. If $x \in \mathcal{M}_0$ and $\mathfrak{m}_x$ is the maximal ideal of the local superalgebra $(\mathcal{O}_{\mathcal{M}})_x$, then the vector superspace $T_x(\mathcal{M}) := (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is the tangent space of $\mathcal{M}$ at $x \in \mathcal{M}_0$.

Denote by $\mathcal{T}_{\mathcal{M}}$ the *tangent sheaf* or the *sheaf of vector fields* of $\mathcal{M}$. In other words, $\mathcal{T}_{\mathcal{M}}$ is the sheaf of derivations of the structure sheaf $\mathcal{O}_{\mathcal{M}}$. Since the sheaf $\mathcal{O}_{\mathcal{M}}$ is $\mathbb{Z}_2$-graded, the tangent sheaf $\mathcal{T}_{\mathcal{M}}$ is also $\mathbb{Z}_2$-graded, i.e. there is the natural decomposition $\mathcal{T}_{\mathcal{M}} = (\mathcal{T}_{\mathcal{M}})_{\bar{0}} \oplus (\mathcal{T}_{\mathcal{M}})_{\bar{1}}$, where

$$(\mathcal{T}_{\mathcal{M}})_{\bar{i}} := \Big\{ v \in \mathcal{T}_{\mathcal{M}} \mid v\big((\mathcal{O}_{\mathcal{M}})_{\bar{j}}\big) \subset (\mathcal{O}_{\mathcal{M}})_{\bar{j}+\bar{i}} \Big\}.$$

Let $\mathcal{M}_0$ be a complex analytic manifold and let $\mathcal{E}$ be the sheaf of holomorphic sections of a vector bundle over $\mathcal{M}_0$. Then the ringed space $(\mathcal{M}_0, \bigwedge \mathcal{E})$ is a supermanifold. In this case dim $\mathcal{M} = n|m$, where $n = \dim \mathcal{M}_0$ and m is the rank of the locally free sheaf $\mathcal{E}$.

Definition 1. A supermanifold $(\mathcal{M}_0, \mathcal{O}_{\mathcal{M}})$ is called *split* if $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$ for a locally free sheaf $\mathcal{E}$ on $\mathcal{M}_0$. The grading of $\mathcal{O}_{\mathcal{M}}$ induces by an isomorphism $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$ and the natural $\mathbb{Z}$-grading of $\bigwedge \mathcal{E} = \bigoplus_p \bigwedge^p \mathcal{E}$ is called *split grading.*

For example, all smooth supermanifolds are split by Batchelor's Theorem. In [4] it was shown that all complex analytic Lie supergroups are split too. In this paper we study the splitting problem for complex analytic homogeneous supermanifolds.

3 Lie supergroups and their homogeneous spaces

3.1 Lie supergroups and super Harish-Chandra pairs.

A *Lie supergroup* is a group object in the category of supermanifolds, i.e. it is a supermanifold $\mathcal{G}$ with three morphisms: the multiplication morphism, the inversion morphism and the identity morphism, which satisfy the usual conditions, modeling the group axioms. In this case the underlying space $\mathcal{G}_0$ is a Lie group. The structure sheaf of a (complex analytic) Lie supergroup can be explicitly described in terms of the corresponding Lie superalgebra and underlying Lie group using super Harish-Chandra pairs (see [5] and [14] for more details). Let us describe this construction briefly.

Definition 2. A *super Harish-Chandra pair* is a pair $(\mathcal{G}_0, \mathfrak{g})$ that consists of a Lie group $\mathcal{G}_0$ and a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that $\mathfrak{g}_{\bar{0}} = \operatorname{Lie} \mathcal{G}_0$ provided with a representation $\operatorname{Ad} : \mathcal{G}_0 \to \operatorname{Aut} \mathfrak{g}$ of $\mathcal{G}_0$ in $\mathfrak{g}$ such that:

- Ad preserves the parity and induces the adjoint representation of $\mathcal{G}_0$ on $\mathfrak{g}_{\bar{0}}$;
- the differential $(\mathrm{d} \operatorname{Ad})_e$ at the identity $e \in \mathcal{G}_0$ coincides with the adjoint representation ad of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}$.

If a super Harish-Chandra pair $(\mathcal{G}_0, \mathfrak{g})$ is given, it determines the Lie supergroup $\mathcal{G}$ in the following way, see [5]. Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping superalgebra of $\mathfrak{g}$. It is clear that $\mathfrak{U}(\mathfrak{g})$ is a $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$-module, where $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$ is the universal enveloping algebra of $\mathfrak{g}_{\bar{0}}$. Recall that we denote by $\mathcal{F}_{\mathcal{G}_0}$ the structure sheaf of the manifold $\mathcal{G}_0$. The natural action of $\mathfrak{g}_{\bar{0}}$ on the sheaf $\mathcal{F}_{\mathcal{G}_0}$ gives rise to a structure of $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$-module on $\mathcal{F}_{\mathcal{G}_0}(U)$ for any open set $U \subset \mathcal{G}_0$. Putting

$$\mathcal{O}_{\mathcal{G}}(U) = \operatorname{Hom}_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}\big(\mathfrak{U}(\mathfrak{g}), \mathcal{F}_{\mathcal{G}_0}(U)\big)$$

for every open $U \subset \mathcal{G}_0$, we get a sheaf $\mathcal{O}_{\mathcal{G}}$ of $\mathbb{Z}_2$-graded vector spaces. (Here we assume that the functions in $\mathcal{F}_{\mathcal{G}_0}(U)$ are even.) The enveloping superalgebra $\mathfrak{U}(\mathfrak{g})$ has a Hopf superalgebra structure. Using this structure we can define the product of elements from $\mathcal{O}_{\mathcal{G}}$ such that $\mathcal{O}_{\mathcal{G}}$ becomes a sheaf of superalgebras, see [5] and [14] for details. A supermanifold structure on $\mathcal{O}_{\mathcal{G}}$ is determined by the isomorphism $\Phi_{\mathfrak{g}} : \mathcal{O}_{\mathcal{G}} \to \operatorname{Hom}\big(\bigwedge(\mathfrak{g}_{\bar{1}}), \mathcal{F}_{\mathcal{G}_0}\big)$, $f \mapsto f \circ \gamma_{\mathfrak{g}}$, where

$$\gamma_{\mathfrak{g}} : \bigwedge(\mathfrak{g}_{\bar{1}}) \to \mathfrak{U}(\mathfrak{g}), \quad X_1 \wedge \cdots \wedge X_r \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} (-1)^{|\sigma|} X_{\sigma(1)} \cdots X_{\sigma(r)}. \tag{1}$$

The following formulas define the multiplication morphism, the inversion morphism and the identity morphism respectively:

$$\begin{aligned} \mu^*(f)\,(X\otimes Y)\,(g,h) &= f\big(\,\mathrm{Ad}(h^{-1})(X)\cdot Y\big)\,(gh)\,; \\ \iota^*(f)(X)(g) &= f\big(\mathrm{Ad}(g)(S(X))\big)\,\big(g^{-1}\big)\,; \\ \varepsilon^*(f) &= f(1)(e). \end{aligned} \tag{2}$$

Here $X, Y \in \mathfrak{U}(\mathfrak{g})$, $f \in \mathcal{O}_{\mathcal{G}}$, $g, h \in \mathcal{G}_0$ and S is the antipode map of the Hopf superalgebra $\mathfrak{U}(\mathfrak{g})$. Here we identify the enveloping superalgebra $\mathfrak{U}(\mathfrak{g}\oplus\mathfrak{g})$ with the tensor product $\mathfrak{U}(\mathfrak{g})\otimes\mathfrak{U}(\mathfrak{g})$.

Sometimes we will identify the Lie superalgebra $\mathfrak{g}$ of a Lie supergroup $\mathcal{G}$ with the tangent space $T_e(\mathcal{G})$ at $e \in \mathcal{G}_0$. The corresponding to $T \in T_e(\mathcal{G})$ left invariant vector field on $\mathcal{G}$ is given by

$$(\mathrm{id}\otimes T)\circ\mu^*, \tag{3}$$

where μ is the multiplication morphism of $\mathcal{G}$. (Recall that a vector field Y on $\mathcal{G}$ is called *left invariant* if $(\mathrm{id}\otimes Y)\circ\mu^* = \mu^*\circ Y$.) Denote by l_g and by r_g the left and right translations with respect to $g \in \mathcal{G}_0$, respectively. The morphisms l_g and r_g are given by the following formulas:

$$l_g^*(f)(X)(h) = f(X)(gh); \quad r_g^*(f)(X)(h) = f\big(\mathrm{Ad}(g^{-1})X\big)(hg), \tag{4}$$

where $f \in \mathcal{O}_{\mathcal{G}}$, $X \in \mathfrak{U}(\mathfrak{g})$ and $g, h \in \mathcal{G}_0$.

3.2 Homogeneous supermanifolds.

An *action of a Lie supergroup $\mathcal{G}$ on a supermanifold $\mathcal{M}$* is a morphism $\nu : \mathcal{G}\times\mathcal{M}\to\mathcal{M}$ such that the usual conditions modeling group action axioms hold. Any vector $X \in T_e(\mathcal{G})$ defines the vector field on $\mathcal{M}$ by the following formula:

$$X \mapsto (X\otimes\,\mathrm{id})\circ\nu^*. \tag{5}$$

Definition 3. An action ν is called *transitive* if ν_0 is a transitive action of the Lie group $\mathcal{G}_0$ on $\mathcal{M}_0$ and the vector fields (5) generates the tangent space $T_x(\mathcal{M})$ at any point $x \in \mathcal{M}_0$. In this case the supermanifold $\mathcal{M}$ is called *$\mathcal{G}$-homogeneous*. A supermanifold $\mathcal{M}$ is called *homogeneous*, if it possesses a transitive action of a Lie supergroup.

If a supermanifold $\mathcal{M}$ is $\mathcal{G}$-homogeneous and $\nu : \mathcal{G}\times\mathcal{M}\to\mathcal{M}$ is the corresponding transitive action, then $\mathcal{M}$ is isomorphic to the supermanifold $\mathcal{G}/\mathcal{H}$, where $\mathcal{H}$ is the isotropy subsupergroup of a certain point (see [16] for details). Recall that the underlying space of $\mathcal{G}/\mathcal{H}$ is the complex analytic manifold $\mathcal{G}_0/\mathcal{H}_0$ and the structure sheaf $\mathcal{O}_{\mathcal{G}/\mathcal{H}}$ of $\mathcal{G}/\mathcal{H}$ is given by

$$\mathcal{O}_{\mathcal{G}/\mathcal{H}} = \Big\{ f \in (\pi_0)_*(\mathcal{O}_{\mathcal{G}}) \,\Big|\, \mu^*_{\mathcal{G}\times\mathcal{H}}(f) = \mathrm{pr}^*(f) \Big\}, \tag{6}$$

where $\pi_0 : \mathcal{G}_0 \to \mathcal{G}_0/\mathcal{H}_0$ is the natural map, $\mu_{\mathcal{G}\times\mathcal{H}}$ is the restriction of the multiplication map on $\mathcal{G}\times\mathcal{H}$ and $\mathrm{pr} : \mathcal{G}\times\mathcal{H} \to \mathcal{G}$ is the natural projection. Using (2) we can rewrite the condition $\mu^*_{\mathcal{G}\times\mathcal{H}}(f) = \mathrm{pr}^*(f)$ in the following way:

$$f\Big(\mathrm{Ad}\,\big(h^{-1}\big)(X)Y\Big)(gh) = \begin{cases} f(X)(g), & Y \in \mathbb{C}; \\ 0, & Y \notin \mathbb{C}; \end{cases} \tag{7}$$

where $X \in \mathfrak{U}(\mathfrak{g})$, $Y \in \mathfrak{U}(\mathfrak{h})$, $\mathfrak{h} = \mathrm{Lie}\,\mathcal{H}$, $g \in \mathcal{G}_0$ and $h \in \mathcal{H}_0$.

Let $Y \in \mathfrak{g}$ and $f \in \mathcal{O}_{\mathcal{G}}$. Then the operator defined by the formula

$$Y(f)(X) = (-1)^{p(Y)} f(XY), \tag{8}$$

where $p(Y)$ is the parity of Y, is a left invariant vector field on $\mathcal{G}$. From (4), (7) and (8) it follows that

$f \in \mathcal{O}_{\mathcal{G}/\mathcal{H}}$ if and only if f is $\mathcal{H}_0$-right invariant, i.e. $r^*_h(f) = f$ for any $h \in \mathcal{H}_0$, and $Y(f) = 0$ for all $Y \in \mathfrak{h}_{\bar 1}$, where $\mathfrak{h} = \mathfrak{h}_{\bar 0} \oplus \mathfrak{h}_{\bar 1}$.

Sometimes we will consider also the left action $\mathcal{H}\times\mathcal{G} \to \mathcal{G}$ of a subsupergroup $\mathcal{H}$ on a Lie supergroup $\mathcal{G}$. The corresponding quotient supermanifold we will denote by $\mathcal{H}\backslash\mathcal{G}$.

3.3 More about split supermanifolds.

Recall that a supermanifold $\mathcal{M}$ is called split if its structure sheaf $\mathcal{O}_{\mathcal{M}}$ is isomorphic to $\bigwedge \mathcal{E}$, where $\mathcal{E}$ is a locally free sheaf on $\mathcal{M}_0$. In this case, $\mathcal{O}_{\mathcal{M}}$ possesses the $\mathbb{Z}$-grading induced by the natural $\mathbb{Z}$-grading of $\bigwedge \mathcal{E} = \bigoplus_p \bigwedge^p \mathcal{E}$ and by isomorphism $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge \mathcal{E}$. Such gradings of $\mathcal{O}_{\mathcal{M}}$ we call split.

Proposition 1. *Any Lie supergroup $\mathcal{G}$ is split.*

This statement follows from the fact that any Lie supergroup is determined by its super Harish-Chandra pair. A different proof of this result (probably the first one) was given in [4]. For completeness we give here another proof.

Proof. The underlying space $\mathcal{G}_0$ is a closed Lie subsupergroup of $\mathcal{G}$. Hence, there exists the homogeneous space $\mathcal{G}/\mathcal{G}_0$, which is isomorphic to the supermanifold $\mathcal{N}$ such that $\mathcal{N}_0$ is a point $\mathrm{pt} = \mathcal{G}_0/\mathcal{G}_0$ and $\mathcal{O}_{\mathcal{N}} \simeq \bigwedge(m)$, where $m = \dim \mathfrak{g}_{\bar 1}$. By definition, the structure sheaf $\mathcal{O}_{\mathcal{N}}$ consists of all

r_g-invariant functions, $g \in \mathcal{G}_0$. We have the natural map $\varphi : \mathcal{G} \to \mathcal{G}/\mathcal{G}_0$, where $\varphi_0 : \mathcal{G}_0 \to \mathrm{pt}$ and $\varphi^* : \mathcal{O}_{\mathcal{N}} \to (\varphi_0)_*(\mathcal{O}_{\mathcal{G}})$ is the inclusion. It is known that $\varphi : \mathcal{G} \to \mathcal{G}/\mathcal{G}_0$ is a principal bundle (see [16]). Using the fact that the underlying space of $\mathcal{G}/\mathcal{G}_0$ is a point we get $\mathcal{G} \simeq \mathcal{N} \times \mathcal{G}_0$. Note that this is an isomorphism of supermanifolds but not of Lie supergroups.□

Example 1. As an example of a homogeneous non-split supermanifold we can cite the super-grassmannian $\mathbf{Gr}_{m|n,r|s}$ for $0 < r < m$ and $0 < s < n$. Super-grassmannians of other types are split (see Example 3).

Denote by `SSM` the category of split supermanifolds. Objects Ob `SSM` in this category are all split supermanifolds $\mathcal{M}$ with fixed split gradings. Further if $X, Y \in$ Ob `SSM`, we put

$$\mathrm{Hom}(X,Y) = \left\{ \begin{array}{l} \text{morphisms from } X \text{ to } Y \\ \text{preserving the split gradings} \end{array} \right\}$$

As in the category of supermanifolds, we can define in `SSM` a group object (*split Lie supergroup*), an action of a split Lie supergroup on a split supermanifold (*split action*) and a *split homogeneous supermanifold.*

There is a functor gr from the category of supermanifolds to the category of split supermanifolds. Let us briefly describe this construction. Let $\mathcal{M}$ be a supermanifold. Denote by $\mathcal{J}_{\mathcal{M}} \subset \mathcal{O}_{\mathcal{M}}$ the subsheaf of ideals generated by odd elements of $\mathcal{O}_{\mathcal{M}}$. Then by definition $\mathrm{gr}\, \mathcal{M} = (\mathcal{M}_0, \mathrm{gr}\, \mathcal{O}_{\mathcal{M}})$ is the split supermanifold with the structure sheaf

$$\mathrm{gr}\, \mathcal{O}_{\mathcal{M}} = \bigoplus_{p \geq 0} (\mathrm{gr}\, \mathcal{O}_{\mathcal{M}})_p, \quad \mathcal{J}^0_{\mathcal{M}} := \mathcal{O}_{\mathcal{M}}, \quad (\mathrm{gr}\, \mathcal{O}_{\mathcal{M}})_p := \mathcal{J}^p_{\mathcal{M}} / \mathcal{J}^{p+1}_{\mathcal{M}}.$$

In this case $(\mathrm{gr}\, \mathcal{O}_{\mathcal{M}})_1$ is a locally free sheaf and there is a natural isomorphism of $\mathrm{gr}\, \mathcal{O}_{\mathcal{M}}$ onto $\bigwedge (\mathrm{gr}\, \mathcal{O}_{\mathcal{M}})_1$. If $\psi = (\psi_0, \psi^*) : \mathcal{M} \to \mathcal{N}$ is a morphism, then $\mathrm{gr}(\psi) = (\psi_0, \mathrm{gr}(\psi^*)) : \mathrm{gr}\, \mathcal{M} \to \mathrm{gr}\, \mathcal{N}$ is defined by

$$\mathrm{gr}(\psi^*)(f + \mathcal{J}^p_{\mathcal{N}}) := \psi^*(f) + \mathcal{J}^p_{\mathcal{M}} \text{ for } f \in (\mathcal{J}_{\mathcal{N}})^{p-1}.$$

Recall that by definition every morphism of supermanifolds is even and as a consequence sends $\mathcal{J}^p_{\mathcal{N}}$ into $\mathcal{J}^p_{\mathcal{M}}$.

3.4 Split Lie supergroups.

Let $\mathcal{G}$ be a Lie supergroup with the supergroup morphisms μ, ι and ε: the multiplication, the inversion and the identity morphism, respectively. In this section we assign three split Lie supergroups $\mathcal{G}^1$, $\mathcal{G}^2$ and $\mathcal{G}^3$ to $\mathcal{G}$ and we show that these split Lie supergroups are pairwise isomorphic.

(1) The construction of $\mathcal{G}^1$ is very simple: we just apply functor gr to $\mathcal{G}$. Clearly, $\mathcal{G}^1 := \operatorname{gr}\mathcal{G}$ is a split Lie supergroup with the supergroup morphisms $\operatorname{gr}(\mu)$, $\operatorname{gr}(\iota)$ and $\operatorname{gr}(\varepsilon)$.

(2) Consider the super Harish-Chandra pair $(\mathcal{G}_0, \mathfrak{g}^2)$, where $\mathfrak{g}^2$ is the following Lie superalgebra: $\mathfrak{g}^2$ and $\mathfrak{g}$ are isomorphic as vector superspaces and the Lie bracket in $\mathfrak{g}^2$ is defined by the following formula:

$$[X,Y]_{\mathfrak{g}^2} = \begin{cases} [X,Y]_{\mathfrak{g}}, & \text{if } X,Y \in \mathfrak{g}_{\bar 0} \text{ or } X \in \mathfrak{g}_{\bar 0} \text{ and } Y \in \mathfrak{g}_{\bar 1}; \\ 0, & \text{if } X,Y \in \mathfrak{g}_{\bar 1}. \end{cases} \tag{9}$$

Denote by $\mathcal{G}^2$ the Lie supergroup corresponding to $(\mathcal{G}_0, \mathfrak{g}^2)$.

(3) Consider the sheaf $\mathcal{O}_{\mathcal{G}^3} := \operatorname{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_{\bar 1}, \mathcal{F}_{\mathcal{G}_0})$. For the ringed space $\mathcal{G}^3 := (\mathcal{G}_0, \mathcal{O}_{\mathcal{G}^3})$ we can repeat the construction from Section 3.1. Indeed, this ringed space is clearly a supermanifold. Futhermore, the exterior algebra $\bigwedge \mathfrak{g}_{\bar 1}$ is also a Hopf algebra. Therefore, we can define on $\mathcal{G}^3$ the multiplication, the inversion and the identity morphisms respectively by the following formulas:

$$\begin{aligned} (\mu^3)^*(f)\big(X \wedge Y\big)(g,h) &= f\big(\operatorname{Ad}(h^{-1})(X) \wedge Y\big)(gh); \\ (\iota^3)^*\big(f\big)(X)(g) &= f\big(\operatorname{Ad}(g)(S'(X))\big)\big(g^{-1}\big); \\ (\varepsilon^3)^*\big(f\big) &= f\big(1\big)(e). \end{aligned} \tag{10}$$

Here $X, Y \in \bigwedge \mathfrak{g}_{\bar 1}$, $f \in \operatorname{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_{\bar 1}, \mathcal{F}_{\mathcal{G}_0})$, g, $h \in \mathcal{G}_0$ and S' is the antipode map of the Hopf superalgebra $\bigwedge \mathfrak{g}_{\bar 1}$. Hence, $\mathcal{G}^3 := (\mathcal{G}_0, \mathcal{O}_{\mathcal{G}^3})$ is a Lie supergroup. Since

$$\operatorname{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_{\bar 1}, \mathcal{F}_{\mathcal{G}_0}) = \bigoplus_{p \geq 0} \operatorname{Hom}_{\mathbb{C}}(\bigwedge^p \mathfrak{g}_{\bar 1}, \mathcal{F}_{\mathcal{G}_0})$$

is $\mathbb{Z}$-graded and the morphisms (10) preserve this $\mathbb{Z}$-grading, we see that $\mathcal{G}^3$ is a split Lie supergroup.

Later on we will need the explicit expression of left and right translations l'_g and r'_g in $\mathcal{G}^3$:

$$(l'_g)^*(f)(X)(h) = f(X)(gh); \quad (r'_g)^*(f)(X)(h) = f\big(\operatorname{Ad}(g^{-1})X\big)(hg), \tag{11}$$

where $f \in \operatorname{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_{\bar 1}, \mathcal{F}_{\mathcal{G}_0})$, $X \in \bigwedge \mathfrak{g}_{\bar 1}$ and $g, h \in \mathcal{G}_0$.

In fact, all these split Lie supergroups are isomorphic. To show this we need the following lemma:

Lemma 1. *Let $\mathfrak{k}$ be a Lie superalgebra and $X_i, Y_j \in \mathfrak{k}_{\bar 1}$, $i = 1, \ldots, r$, $j = 1, \ldots, s$ be any elements. Assume that $[X_i, Y_j] = 0$ for any i, j. Then we have*

$$\gamma_{\mathfrak{k}}(X_1 \wedge \cdots \wedge X_r \wedge Y_1 \wedge \cdots \wedge Y_s) = \gamma_{\mathfrak{k}}(X_1 \wedge \cdots \wedge X_r) \cdot \gamma_{\mathfrak{k}}(Y_1 \wedge \cdots \wedge Y_s),$$

where $\gamma_{\mathfrak{k}}$ is given by (1).

Proof. A direct calculation.$\Box$

Proposition 2. *We have $\mathcal{G}^1 \simeq \mathcal{G}^2 \simeq \mathcal{G}^3$ in the category of Lie supergroups.*

Proof. **(a)** The statement $\mathcal{G}^1 \simeq \mathcal{G}^2$ was proven in [15], Theorem 3.

(b) Let us show that $\mathcal{G}^2 \simeq \mathcal{G}^3$. Applying Lemma 1 to $\mathfrak{g}^2$ and to any elements X_i, $Y_j \in \mathfrak{g}^2_{\bar{1}}$, we see that in this case $\gamma_{\mathfrak{g}^2}$ is not only isomorphism of super coalgebras but of Hopf superalgebras. In other words, the isomorphism

$$\Phi_{\mathfrak{g}^2} : \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}^2_{\bar{0}})}\big(\mathfrak{U}(\mathfrak{g}^2), \mathcal{F}_{\mathcal{G}_0}\big) \to \mathrm{Hom}_{\mathbb{C}}\big(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0}\big)$$

is an isomorphism of Lie supergroups.$\Box$

4 Split grading operators

Let again $\mathcal{M}$ be a supermanifold, $\mathrm{gr}\,\mathcal{M}$ be the corresponding split supermanifold and $\mathcal{J}$ be the sheaf of ideals generated by odd elements of $\mathcal{O}_{\mathcal{M}}$. We denote by $\mathcal{T} = \mathcal{D}er\mathcal{O}_{\mathcal{M}}$ and by $\mathrm{gr}\,\mathcal{T} = \mathcal{D}er(\mathcal{O}_{\mathrm{gr}\,\mathcal{M}})$ the tangent sheaf of $\mathcal{M}$ and of $\mathrm{gr}\,\mathcal{M}$, respectively. The sheaf $\mathcal{T}$ is naturally $\mathbb{Z}_2$-graded and the sheaf $\mathrm{gr}\,\mathcal{T}$ is naturally $\mathbb{Z}$-graded: the gradings are induced by the $\mathbb{Z}_2$ and $\mathbb{Z}$-grading of $\mathcal{O}_{\mathcal{M}}$ and $\mathrm{gr}\,\mathcal{O}_{\mathcal{M}}$, respectively. In other words, we have the decomposition:

$$\mathcal{T} = \mathcal{T}_{\bar{0}} \oplus \mathcal{T}_{\bar{1}}, \quad \mathrm{gr}\,\mathcal{T} = \bigoplus_{p \geq -1} (\mathrm{gr}\,\mathcal{T})_p.$$

The sheaves $\mathcal{T}$ and $\mathrm{gr}\,\mathcal{T}$ are related: this relation can be expressed by the following exact sequence:

$$0 \longrightarrow \mathcal{T}_{(2)\bar{0}} \longrightarrow \mathcal{T}_{\bar{0}} \stackrel{\alpha}{\longrightarrow} (\mathrm{gr}\,\mathcal{T})_0 \longrightarrow 0, \tag{12}$$

where

$$\mathcal{T}_{(2)\bar{0}} = \{v \in \mathcal{T}_{\bar{0}} \mid v(\mathcal{O}_{\mathcal{M}}) \subset \mathcal{J}^2\}.$$

The morphism α in (12) is the composition of the natural morphism $\mathcal{T}_{\bar{0}} \to \mathcal{T}_{\bar{0}}/\mathcal{T}_{(2)\bar{0}}$ and the isomorphism $\mathcal{T}_{\bar{0}}/\mathcal{T}_{(2)\bar{0}} \to (\mathrm{gr}\,\mathcal{T})_0$ that is given by

$$[w] \longmapsto \tilde{w}, \quad \tilde{w}\big(f + \mathcal{J}^{p+1}\big) := w(f) + \mathcal{J}^{p+1},$$

where $w \in \mathcal{T}_{\bar{0}}$, $[w]$ is the image of w in $\mathcal{T}_{\bar{0}}/\mathcal{T}_{(2)\bar{0}}$ and $f \in \mathcal{J}^p$.

Assume that the sheaf $\mathcal{O}_{\mathcal{M}}$ is $\mathbb{Z}$-graded, i.e. $\mathcal{O}_{\mathcal{M}} = \bigoplus\limits_p (\mathcal{O}_{\mathcal{M}})_p$. Then we have the map $w : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$ defined by $w(f) = pf$, where $f \in (\mathcal{O}_{\mathcal{M}})_p$. Such maps are called *grading operators* on $\mathcal{M}$.

Definition 4. We call a grading operator w on $\mathcal{M}$ a *split grading operator* if it corresponds to a split grading of $\mathcal{O}_{\mathcal{M}}$, see Definition 1.

In fact any split grading operator w on $\mathcal{M}$ is an even vector field on $\mathcal{M}$. Indeed, w is linear, it preserves the parity in $\mathcal{O}_{\mathcal{M}}$ and for $f \in (\mathcal{O}_{\mathcal{M}})_p$ and $g \in (\mathcal{O}_{\mathcal{M}})_q$ we have:

$$w(fg) = (p+q)fg = (pf)g + f(qg) = w(f)g + fw(g).$$

Note that $fg \in (\mathcal{O}_{\mathcal{M}})_{p+q}$.

By definition the sheaf $\operatorname{gr} \mathcal{O}_{\mathcal{M}}$ is $\mathbb{Z}$-graded. Denote by a the corresponding split grading operator.

Lemma 2. *1. A supermanifold $\mathcal{M}$ is split if and only if the vector field a is contained in* $\operatorname{Im} H^0(\alpha)$*, where*

$$H^0(\alpha) : H^0(\mathcal{M}_0, \mathcal{T}_{\bar{0}}) \to H^0(\mathcal{M}_0, (\operatorname{gr} \mathcal{T})_0).$$

(We applied the functor $H^0(\mathcal{M}_0, -)$ to the sequence (12). We write $H^0(\alpha)$ instead of $H^0(\mathcal{M}_0, \alpha)$ for notational simplicity.)

2. If w is a split grading operator on $\mathcal{M}$, then any other split grading operator on $\mathcal{M}$ has the form $w + \chi$, where $\chi \in H^0(\mathcal{M}_0, \mathcal{T}_{(2)\bar{0}})$.

Proof. 1. The statement of the lemma can be deduced from the following observation made by Koszul in [4, Lemma 1.1 and Section 3]. Let A be a commutative superalgebra over $\mathbb{C}$ and $\mathfrak{m}$ be a nilpotent ideal in A. An even derivation w of A is called *adapted to the filtration*

$$A \supset \mathfrak{m} \supset \mathfrak{m}^2 \dots$$

if

$$(w - r\,\mathrm{id})(\mathfrak{m}^r) \subset \mathfrak{m}^{r+1} \text{ for any } r \geq 0.$$

Denote by $D^{ad}_{\mathfrak{m}}$ the set of all derivations adapted to $\mathfrak{m}$. In [4, Lemma 1.1] it was shown that $D^{ad}_{\mathfrak{m}}$ is not empty if and only if the filtration of A is splittable. Moreover, if $w \in D^{ad}_{\mathfrak{m}}$, then the corresponding splitting of A is given by eigenspaces of the derivation w: $A = \bigoplus_i A_i$, where A_i is the eigenspace of w with the eigenvalue i, and $\mathfrak{m}^r = A_r \oplus \mathfrak{m}^{r+1}$ for all $r \geq 0$.

We apply Koszul's observation to the sheaf of superalgebras $\mathcal{O}_{\mathcal{M}}$ and its subsheaf of ideals $\mathcal{J}$. The set $D^{ad}_{\mathcal{J}}$ is in this case the set of global derivations of $\mathcal{O}_{\mathcal{M}}$ adapted to the filtration

$$\mathcal{O}_{\mathcal{M}} \supset \mathcal{J} \supset \mathcal{J}^2 \supset \dots . \tag{13}$$

Clearly, $D^{ad}_{\mathcal{J}}$ is not empty if and only if a is contained in $\operatorname{Im} H^0(\alpha)$. (Actually, $H^0(\alpha)(D^{ad}_{\mathcal{J}}) = a$.) Furthermore, if the supermanifold $\mathcal{M}$ is split, i.e.

we have a split grading $\mathcal{O}_{\mathcal{M}} = \bigoplus_{p\geq 0}(\mathcal{O}_{\mathcal{M}})_p$, then $\mathcal{J}^q = \bigoplus_{p\geq q}(\mathcal{O}_{\mathcal{M}})_p$ and $\mathcal{J}^q = (\mathcal{O}_{\mathcal{M}})_q \oplus \mathcal{J}^{q+1}$. Hence, the split grading determine the splitting of the filtration (13) and the corresponding split grading operator belongs to $D^{ad}_{\mathcal{J}}$.

Conversely, if there exists $w \in D^{ad}_{\mathcal{J}}$, then we can decompose the sheaf $\mathcal{O}_{\mathcal{M}}$ into eigenspaces

$$(\mathcal{O}_{\mathcal{M}})_q := \{f \in \mathcal{O}_{\mathcal{M}} | w(f) = qf\}.$$

In this case the sheaves $\bigoplus\limits_p(\mathcal{O}_{\mathcal{M}})_p$ and $\operatorname{gr}\mathcal{O}_{\mathcal{M}}$ are isomorphic as $\mathbb{Z}$-graded sheaves of superalgebras since $\mathcal{J}^q = (\mathcal{O}_{\mathcal{M}})_q \oplus \mathcal{J}^{q+1}$. Hence, the supermanifold is split.

2. Applying the left-exact functor $H^0(\mathcal{M}_0, -)$ to (12), we get the following exact sequence:

$$0 \longrightarrow H^0(\mathcal{M}_0, \mathcal{T}_{(2)\bar{0}}) \longrightarrow H^0(\mathcal{M}_0, \mathcal{T}_{\bar{0}}) \xrightarrow{H^0(\alpha)} H^0(\mathcal{M}_0, (\operatorname{gr}\mathcal{T})_0).$$

If w_1, w_2 are two split grading operators on $\mathcal{M}$, then

$$H^0(\alpha)(w_1) = H^0(\alpha)(w_2) = a,$$

according to the part 1. Therefore, $w_1 - w_2 \in H^0(\mathcal{M}_0, \mathcal{T}_{(2)\bar{0}})$. The result follows.□

Example 2. Consider the supermanifold $\mathcal{G}_0\backslash\mathcal{G}$. Its structure sheaf is isomorphic to $\bigwedge(\mathfrak{g}_{\bar{1}})$ (compare with Example 1). Denote by (ε_i) the system of odd (global) coordinates on $\mathcal{G}_0\backslash\mathcal{G}$. An example of a split grading operator on the Lie supergroup $\mathcal{G}$ is $\sum \varepsilon^i X_i$. Here (X_i) is a basis of odd left invariant vector fields on $\mathcal{G}$ such that $X_i(\varepsilon^j)(e) = \delta_i^j$. We may produce other examples if we use right invariant vector fields or odd (global) coordinates on $\mathcal{G}/\mathcal{G}_0$.

By Lemma 2, any split grading operator on a Lie supergroup $\mathcal{G}$ is given by $\sum \varepsilon^i X_i + \chi$, where $\chi \in H^0(\mathcal{G}_0, \mathcal{T}_{(2)\bar{0}})$ is any vector field on $\mathcal{G}$.

5 Compatible split gradings on $\mathcal{G}/\mathcal{H}$

5.1 Compatible gradings on $\mathcal{G}/\mathcal{H}$.

Let $\mathcal{G}$ be a Lie supergroup and $\mathcal{M} = \mathcal{G}/\mathcal{H}$ be a homogeneous supermanifold. As above we denote by $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{H}$ the natural projection.

Definition 5. A split grading of the sheaf $\mathcal{O}_{\mathcal{G}} = \bigoplus\limits_p(\mathcal{O}_{\mathcal{G}})_p$ is called *compatible* with the inclusion $\mathcal{O}_{\mathcal{M}} \subset (\pi_0)_*(\mathcal{O}_{\mathcal{G}})$ if the following holds:

$$f \in \mathcal{O}_{\mathcal{M}} \ \Rightarrow f_p \in \mathcal{O}_{\mathcal{M}} \text{ for all } p,$$

where $f = \sum f_p$ and $f_p \in (\pi_0)_*((\mathcal{O}_\mathcal{G})_p)$.

Let us take any split grading operator w on $\mathcal{G}$. Clearly, the corresponding split grading of $\mathcal{O}_\mathcal{G}$ is compatible with $\mathcal{O}_\mathcal{M}$ if and only if $w(\mathcal{O}_\mathcal{M}) \subset \mathcal{O}_\mathcal{M}$. It is not clear from Definition 5 that the compatible grading

$$(\mathcal{O}_\mathcal{M})_p = \mathcal{O}_\mathcal{M} \cap (\pi_0)_*((\mathcal{O}_\mathcal{G})_p) \tag{14}$$

of $\mathcal{O}_\mathcal{M}$, if it exists, is a split grading of $\mathcal{O}_\mathcal{M}$. However, the following proposition holds:

Proposition 3. *Assume that we have the $\mathbb{Z}$-grading:*

$$\mathcal{O}_\mathcal{M} = \bigoplus_{p \geq 0} (\mathcal{O}_\mathcal{M})_p,$$

where $(\mathcal{O}_\mathcal{M})_p$ are as in (14). Then this grading is a split grading.

Proof. The idea of the proof is to apply Lemma 2 to the grading operator $w' := w|_{\mathcal{O}_\mathcal{M}}$ on $\mathcal{M}$. Denote by $\mathcal{J}_\mathcal{M}$ and by $\mathcal{J}_\mathcal{G}$ the sheaves of ideals generated by odd elements of $\mathcal{O}_\mathcal{M}$ and $\mathcal{O}_\mathcal{G}$, respectively. Our aim is to show that

$$w'(f) + \mathcal{J}_\mathcal{M}^{p+1} = pf + \mathcal{J}_\mathcal{M}^{p+1},$$

where $f \in \mathcal{J}_\mathcal{M}^p$. In other words, we want to show that $H^0(\alpha)(w')$ is a split grading operator for the grading of $\operatorname{gr} \mathcal{O}_\mathcal{M}$. (We use notations of Lemma 2.) We have:

$$(\operatorname{gr} \pi)^*(w'(f) + \mathcal{J}_\mathcal{M}^{p+1}) = w(f) + \mathcal{J}_\mathcal{G}^{p+1} = pf + \mathcal{J}_\mathcal{G}^{p+1};$$
$$(\operatorname{gr} \pi)^*(pf + \mathcal{J}_\mathcal{M}^{p+1}) = pf + \mathcal{J}_\mathcal{G}^{p+1}.$$

Since the map $(\operatorname{gr} \pi)^*$ is injective, we get, $w'(f) + \mathcal{J}_\mathcal{M}^{p+1} = pf + \mathcal{J}_\mathcal{M}^{p+1}$. □

5.2 $\mathcal{H}$-invariant split grading operators.

First of all let us consider the situation when a split grading operator w on $\mathcal{G}$ is invariant with respect to a Lie subsupergroup $\mathcal{H}$. In terms of super Harish-Chandra pairs this means:

$$\begin{array}{ll} r_h^* \circ w = w \circ r_h^*, & \text{for all } \ h \in \mathcal{H}_0; \\ [Y, w] = 0, & \text{for all } Y \in \mathfrak{h}_{\bar{1}}. \end{array} \tag{15}$$

Here $(\mathcal{H}_0, \mathfrak{h})$ is the super Harish-Chandra pair of $\mathcal{H}$, r_h is the right translation and Y is an odd left invariant vector field.

Proposition 4. *Assume that w is an $\mathcal{H}$-invariant split grading operator on $\mathcal{G}$, i.e. equations (15) hold. Then $\mathcal{H}$ is an ordinary Lie group.*

Proof. The idea of the proof is to show that the Lie superalgebra $\mathfrak{h}$ of $\mathcal{H}$ has the trivial odd part: $\mathfrak{h}_{\bar{1}} = \{0\}$.

In Example 2 we saw that any split grading operator on $\mathcal{G}$ is given by $w = \sum \varepsilon^i X_i + \chi$. If Z is a vector field on $\mathcal{G}$, denote by $Z_e \in T_e(\mathcal{G})$ the corresponding tangent vector at the identity $e \in \mathcal{G}_0$. Consider the second equation in (15). At the point e, we have

$$[Y, w]_e = \big(\sum_i Y(\varepsilon^i)X_i - \sum_i \varepsilon^i Y \circ X_i - \sum_i \varepsilon^i X_i \circ Y + [Y, \chi]\big)_e = 0$$

for any $Y \in \mathfrak{h}_{\bar{1}}$. Furthermore,

$$(\sum_i \varepsilon^i Y \circ X_i - \sum_i \varepsilon^i X_i \circ Y)_e = 0 \quad \text{and} \quad [Y, \chi]_e = 0,$$

because $\varepsilon^i(e) = 0$ and because $\chi \in H^0(\mathcal{M}_0, \mathcal{T}_{(2)\bar{0}})$. Therefore,

$$[Y, w]_e = \sum_i Y(\varepsilon^i)(e)(X_i)_e = 0$$

The tangent vectors $(X_i)_e$ form a basis in $T_e(\mathcal{G})_{\bar{1}}$, hence $Y(\varepsilon^i)(e) = 0$ for all i. The last statement is equivalent to $Y_e = 0$. Since Y is a left invariant vector field, we get $Y = 0$. The proof is complete.□

Remark. It is well known that the supermanifold $\mathcal{G}/\mathcal{H}$, where $\mathcal{H}$ is an ordinary Lie group, is split (see [5] or [14]). Therefore, the case of $\mathcal{H}$-invariant split grading operators does not lead to new examples of homogeneous split supermanifolds.

5.3 $\mathcal{G}_0$-left invariant split grading operators.

Consider now a more general situation, when a split grading operator w leaves $\mathcal{O}_\mathcal{M}$ invariant. Let $f \in \mathcal{O}_\mathcal{M}$. Then $w(f) \in \mathcal{O}_\mathcal{M}$ if and only if

$$r_h^*(w(f)) = w(f) \quad \text{and} \quad Y(w(f)) = 0$$

for $h \in \mathcal{H}_0$ and $Y \in \mathfrak{h}_{\bar{1}}$. These conditions are equivalent to the following ones:

$$(r_h^* \circ w \circ (r_h^{-1})^* - w)|_{\mathcal{O}_\mathcal{M}} = 0; \quad [Y, w]|_{\mathcal{O}_\mathcal{M}} = 0. \qquad (16)$$

Recall that $r_h^{-1} = r_{h^{-1}}$.

It seems to us that the system (16) is hard to solve in general. Consider now a special type of split grading operators, called $\mathcal{G}_0$-left invariant grading operators.

Definition 6. A split grading of $\mathcal{O}_\mathcal{G}$ is called $\mathcal{G}_0$*-left invariant* if it is invariant with respect to left translations. In other words, from $f \in (\mathcal{O}_\mathcal{G})_p$ it follows that $l_g^*(f) \in (\mathcal{O}_\mathcal{G})_p$ for all $g \in \mathcal{G}_0$.

It is easy to see that a split grading of $\mathcal{O}_\mathcal{G}$ is $\mathcal{G}_0$-left invariant if and only if the corresponding split grading operator w is invariant with respect to left translations: $l_g^* \circ w = w \circ l_g^*$, $g \in \mathcal{G}_0$. For example, the split grading operator $\sum \varepsilon^i X_i$ constructed in Example 2 is a $\mathcal{G}_0$-left invariant split grading operator, because ε^i are $\mathcal{G}_0$-left invariant functions and X_i are left invariant vector fields. In this section we will describe all such operators.

In Section 3.4 we have seen that the supermanifold $(\mathcal{G}_0, \mathrm{Hom}_\mathbb{C}(\bigwedge \mathfrak{g}_{\bar 1}, \mathcal{F}_{\mathcal{G}_0}))$ is a Lie supergroup isomorphic to $\mathrm{gr}\,\mathcal{G}$. We need the following lemma:

Lemma 3. *The map*

$$\begin{aligned} \Phi_\mathfrak{g} : \mathcal{O}_\mathcal{G} &\to \mathrm{Hom}_\mathbb{C}(\textstyle\bigwedge \mathfrak{g}_{\bar 1}, \mathcal{F}_{\mathcal{G}_0}),\\ f &\mapsto f \circ \gamma_\mathfrak{g} \end{aligned}$$

from Section 3.1 *is invariant with respect to left and right translations.*

Proof. For any $h \in \mathcal{G}_0$, denote by r'_h and l'_h the right and the left translation in the Lie supergroup $\mathcal{G}^3 = (\mathcal{G}_0, \mathrm{Hom}_\mathbb{C}(\bigwedge \mathfrak{g}_{\bar 1}, \mathcal{F}_{\mathcal{G}_0}))$, respectively. (See, (11)) Let us show that

$$(r'_h)^* \circ \Phi_\mathfrak{g} = \Phi_\mathfrak{g} \circ r_h^*. \tag{17}$$

Let us take $Z \in \bigwedge \mathfrak{g}_{\bar 1}$ and $g, h \in \mathcal{G}_0$. Using (4) we have

$$\begin{gathered} \big[(r'_h)^* \circ \Phi_\mathfrak{g}\big](f)(Z)(g) = \Phi_\mathfrak{g}(f)\big(\mathrm{Ad}(h^{-1})(Z)\big)(gh) = \\ f\big(\gamma_\mathfrak{g}(\mathrm{Ad}(h^{-1})(Z))\big)(gh) = f\big(\mathrm{Ad}(h^{-1})(\gamma_\mathfrak{g}(Z))\big)(gh) = \\ r_h^*(f)\big(\gamma_\mathfrak{g}(Z)\big)(g) = [\Phi_\mathfrak{g} \circ r_h^*](f)(Z)(g). \end{gathered}$$

Similarly, we get

$$(l'_h)^* \circ \Phi_\mathfrak{g} = \Phi_\mathfrak{g} \circ l_h^*.$$

□

The following observation is known to experts, but we cannot find it in the literature:

Lemma 4. *The space of* $\mathcal{G}_0$*-left invariant vector fields* $H^0(\mathcal{G}_0, \mathcal{T})^{\mathcal{G}_0}$ *on a Lie supergroup* $\mathcal{G}$ *is isomorphic to* $H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0 \backslash \mathcal{G}}) \otimes \mathfrak{g}$*. The isomorphism is given by:*

$$f \otimes Z \overset{F}{\longmapsto} fZ,$$

where $f \in H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})$ *and* $Z \in \mathfrak{g}$.

Proof. Clearly, the map F is injective and its image is contained in the vector space $H^0(\mathcal{G}_0, \mathcal{T})^{\mathcal{G}_0}$. Let us show that any vector field v in $H^0(\mathcal{G}_0, \mathcal{T})^{\mathcal{G}_0}$ is contained in $\mathrm{Im}(F)$.

Let (X_i) and (Z_j) be a basis of odd and even left invariant (with respect to the supergroup $\mathcal{G}$) vector fields on $\mathcal{G}$, respectively. Assume that

$$v = \sum f^i X_i + \sum g^j Z_j,$$

where $f^i, g^j \in H^0(\mathcal{G}_0, \mathcal{O}_{\mathcal{G}})$, be the decomposition of v with respect to this basis. We have:

$$\begin{aligned} l_g^* \circ v = & \sum l_g^*(f^i) l_g^* \circ X_i + \sum l_g^*(g^j) l_g^* \circ Z_j = \\ & \sum l_g^*(f^i) X_i \circ l_g^* + \sum l_g^*(g^j) Z_j \circ l_g^* = v \circ l_g^*. \end{aligned}$$

Therefore, $l_g^*(f^i) = f^i$ and $l_g^*(g^j) = g^j$ for all $g \in \mathcal{G}_0$. In other words, $f^i, g^j \in H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})$. The proof is complete.□

The Lie supergroup $\mathcal{G}$ acts on the vector superspace $H^0(\mathcal{G}_0, \mathcal{T})^{\mathcal{G}_0}$. This action we can describe in terms of the corresponding super Harish-Chandra pair $(\mathcal{G}_0, \mathfrak{g})$ in the following way:

$$g \mapsto (X \mapsto r_g^* \circ X \circ (r_g^{-1})^*), \quad Y \mapsto (X \mapsto [Y, X]), \tag{18}$$

where $g \in \mathcal{G}_0$, $X \in H^0(\mathcal{G}_0, \mathcal{T})^{\mathcal{G}_0}$ and $Y \in \mathfrak{g}$. Note that this action is well-defined because $\mathcal{G}$-left and right actions on $H^0(\mathcal{G}_0, \mathcal{T})$ commute. The Lie supergroup $\mathcal{G}$ acts also on the vector superspace $H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{g}$. This action is given by right translations r_g^* on $H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})$ and by the formulas (18) on $\mathfrak{g}$ if we assume that $X \in \mathfrak{g}$. Clearly, the isomorphism F from Lemma 4 is equivariant. From now on we will identify $H^0(\mathcal{G}_0, \mathcal{T})^{\mathcal{G}_0}$ and $H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{g}$ via isomorphism F from Lemma 4.

If $\mathcal{H}$ is a Lie subsupergroup of $\mathcal{G}$ and $\mathfrak{h} = \mathrm{Lie}\,\mathcal{H}$ then $\mathfrak{g}/\mathfrak{h}$ is an $\mathcal{H}$-module.

Lemma 5. *Let us take a $\mathcal{G}_0$-left invariant split grading operator w. The vector field w satisfies (16) if and only if*

$$\overline{w} \in \big(H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h}\big)^{\mathcal{H}}, \tag{19}$$

where $\overline{w}$ is the image of w by the natural mapping

$$H^0\big(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}\big) \otimes \mathfrak{g} \to H^0\big(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}\big) \otimes \mathfrak{g}/\mathfrak{h}.$$

Proof. Let $\overline{w} \in (H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h})^{\mathcal{H}}$. It follows that

$$r_h^* \circ w \circ (r_h^{-1})^* - w \in H^0\big(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}\big) \otimes \mathfrak{h},\ h \in \mathcal{H}_0,$$

and

$$[Y, w] \in H^0\big(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}\big) \otimes \mathfrak{h},\ Y \in \mathfrak{h}.$$

Hence, the conditions (16) are satisfied.

On the other hand, if the conditions (16) are satisfied, then the vector fields $r_h^* \circ w \circ (r_h^{-1})^* - w$ and $[Y, w]$ are vertical with respect to the projection $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{H}$. Therefore, $r_h^* \circ w \circ (r_h^{-1})^* - w$ and $[Y, w]$ belong to the superspace $H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{h}$. It is equivalent to conditions (19).□

Now our aim is to describe the space $\big(H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h}\big)^{\mathcal{H}_0}$. We have seen in Proposition 1 that the superspace $H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})$ is isomorphic to $\bigwedge \mathfrak{g}_{\bar{1}}^*$. Actually this isomorphism can be chosen in $\mathcal{G}_0$-equivariant way. More precisely, we need the following lemma.

Proposition 5. **a.** *We have*

$$\begin{aligned} H^0\big(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}\big) \otimes \mathfrak{g} &\simeq \textstyle\bigwedge(\mathfrak{g}_{\bar{1}}^*) \otimes \mathfrak{g} \quad \textit{as } \mathcal{G}_0\textit{-modules},\\ H^0\big(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}\big) \otimes \mathfrak{g}/\mathfrak{h} &\simeq \textstyle\bigwedge(\mathfrak{g}_{\bar{1}}^*) \otimes \mathfrak{g}/\mathfrak{h} \quad \textit{as } \mathcal{H}_0\textit{-modules}, \end{aligned}$$

where the action of $\mathcal{G}_0$ on $\bigwedge(\mathfrak{g}_{\bar{1}}^*)$ is standard.

b. *There exists a $\mathcal{G}_0$-left and right invariant split grading operator on $\mathcal{G}$.*

Proof. **a.** We have to show that there exists an $\mathcal{G}_0$-equivariant isomorphism

$$H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \xrightarrow{\beta} \textstyle\bigwedge \mathfrak{g}_{\bar{1}}^*.$$

Then the map $\beta \otimes \mathrm{id}$ will provide the required isomorphism of $\mathcal{G}_0$-modules. Consider the Lie supergroup

$$\mathcal{G}^3 = (\mathcal{G}_0, \mathrm{Hom}_{\mathbb{C}}(\textstyle\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0}))$$

from Section 3.4. It follows from (4) that

$$H^0(\mathcal{G}_0, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}^3}) = \mathrm{Hom}_{\mathbb{C}}(\textstyle\bigwedge \mathfrak{g}_{\bar{1}}, \mathbb{C}) = (\bigwedge \mathfrak{g}_{\bar{1}})^*.$$

Note that the action of $\mathcal{G}_0$ on $(\bigwedge \mathfrak{g}_{\bar{1}})^*$ by right translations in $\mathcal{G}^3$, denoted by $(r'_g)^*$, coincides with the standard action of $\mathcal{G}_0$ on $(\bigwedge \mathfrak{g}_{\bar{1}})^*$. Indeed, let us take

$$f \in H^0(\mathcal{G}_0, \mathcal{O}_{\mathcal{G}^3})^{\mathcal{G}_0} = (\textstyle\bigwedge \mathfrak{g}_{\bar{1}})^*.$$

By (11), we have:

$$(r'_g)^*(f)(X)(e) = (r'_g)^*(f)(X)(h) = f\big(\mathrm{Ad}(g^{-1})X\big)(hg) = f\big(\mathrm{Ad}(g^{-1})X\big)(e).$$

Here $g, h \in \mathcal{G}_0$, $X \in \bigwedge \mathfrak{g}_{\bar{1}}$ and $e \in \mathcal{G}_0$ is the identity. It remains to note that by Lemma 3, the map $\Phi_{\mathfrak{g}}$ induces the equivariant isomorphism between the superspaces of left invariants $H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})$ and $(\bigwedge \mathfrak{g}_{\bar{1}})^*$.

b. We need to show that in the vector space

$$\left(\bigwedge(\mathfrak{g}_{\bar{1}}^*) \otimes \mathfrak{g}\right)^{\mathcal{G}_0} = \left(\bigwedge(\mathfrak{g}_{\bar{1}}^*) \otimes \mathfrak{g}_{\bar{0}}\right)^{\mathcal{G}_0} \oplus \left(\bigwedge(\mathfrak{g}_{\bar{1}}^*) \otimes \mathfrak{g}_{\bar{1}}\right)^{\mathcal{G}_0}$$

there exists points corresponding to split grading operators. This space always possesses a $\mathcal{G}_0$-invariant, precisely, the identity operator $\mathrm{id} \in \mathfrak{g}_{\bar{1}}^* \otimes \mathfrak{g}_{\bar{1}}$. The pre-image of $\beta^{-1}(\mathrm{id}) \in H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{g}$ has the form $\sum \varepsilon^i X_i$ for some choice of local coordinates such that $X_i(\varepsilon^j)(e) = \delta_i^j$, see Example 2. We have seen that such vector fields correspond to $\mathcal{G}_0$-left invariant split grading operators on $\mathcal{G}$.$\Box$

Denote by $\mathcal{T}_{\mathcal{G}}$ the tangent sheaf of a Lie group $\mathcal{G}$ and by $\overline{v}$ is the image of v by the natural mapping

$$H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{g} \to H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h}.$$

The result of our study is:

Theorem 1. *The following conditions are equivalent:*

a. *A homogeneous supermanifold $\mathcal{M} = \mathcal{G}/\mathcal{H}$ admits a $\mathcal{G}_0$-left invariant split grading that is induced by a grading of $\mathcal{O}_{\mathcal{G}}$ and the inclusion $\mathcal{O}_{\mathcal{M}} \subset (\pi_0)_*(\mathcal{O}_{\mathcal{G}})$.*

b. *There exists a $\mathcal{G}_0$-left invariant vector field $\chi \in H^0(\mathcal{G}_0, (\mathcal{T}_{\mathcal{G}})_{(2)\bar{0}})$ such that*

$$\overline{\chi} \in \left(\bigwedge(\mathfrak{g}_{\bar{1}})^* \otimes \mathfrak{g}/\mathfrak{h}\right)^{\mathcal{H}_0}, \tag{20}$$

and such that for $w = \beta^{-1}(\mathrm{id}) + \chi$, where $\beta^{-1}(\mathrm{id}) = \sum \varepsilon^i X_i$ is from the proof of Proposition 5.b, we have

$$[Y, w] \in H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{h}, \;\; Y \in \mathfrak{h}_{\bar{1}}. \tag{21}$$

6 An application

As above let $\mathcal{G}$ be a Lie supergroup and $\mathcal{H}$ be a Lie subsupergroup, $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie superalgebras of $\mathcal{G}$ and $\mathcal{H}$, respectively, and $\mathcal{M} := \mathcal{G}/\mathcal{H}$. Consider the map

$$\rho : \mathfrak{g}_{\bar{0}} \to H^0(\mathrm{pt}, \mathcal{T}_{\mathcal{G}_0\backslash\mathcal{G}})$$

induced by the action of $\mathcal{G}_0$ on $\mathcal{M}$. (Here $\mathcal{T}_{\mathcal{G}_0\backslash\mathcal{G}}$ is the sheaf of vector fields on $\mathcal{G}_0\backslash\mathcal{G}$.) Let us describe its kernel. For $X \in \mathfrak{g}_{\bar{0}}$ and $f \in H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})$, we have:

$$\begin{aligned} X(f)(Y)(e) = & \ \tfrac{\mathrm{d}}{\mathrm{d}\,t}|_{t=0} f\big(\mathrm{Ad}(\exp(-tX))Y\big)\big(\exp(tX)\big) = \\ & \ \tfrac{\mathrm{d}}{\mathrm{d}\,t}|_{t=0} f\big(\mathrm{Ad}(\exp(-tX))Y\big)(e), \end{aligned}$$

where $Y = Y_1 \cdots Y_r$, $Y_i \in \mathfrak{g}_{\bar{1}}$ and t is an even parameter. A vector field X is in $\mathrm{Ker}\,\rho$ if and only if $X(f)(Y)(e) = 0$ for all f and Y. Hence,

$$\mathrm{Ker}\,\rho = \mathrm{Ker}(\mathrm{ad}\,|_{\mathfrak{g}_{\bar{1}}}),$$

where ad is the adjoint representation of $\mathfrak{g}_{\bar{0}}$ in $\mathfrak{g}$.

Furthermore, denote

$$\begin{aligned} A := & \ \mathrm{Ker}\,\big(\mathcal{G}_0 \ni g \mapsto \bar{l}_g : \mathcal{G}/\mathcal{H} \to \mathcal{G}/\mathcal{H}\big); \\ \mathfrak{a} := & \ \mathrm{Ker}\,\big(\mathfrak{g} \ni X \mapsto H^0(\mathcal{G}_0/\mathcal{H}_0, \mathcal{T}_{\mathcal{G}/\mathcal{H}})\big). \end{aligned}$$

Here $\bar{l}_g$ is the automorphism of $\mathcal{G}/\mathcal{H}$ induced by the left translation l_g. The pair $(A, \mathfrak{a})$ is a super Harish-Chandra pair. An action of $\mathcal{G}$ on $\mathcal{M}$ is called *effective* if the corresponding to $(A, \mathfrak{a})$ Lie supergroup is trivial. As in the case of Lie groups any action of a Lie supergroup can be factored to be effective.

Theorem 2. *Assume that the action of $\mathcal{G}$ on $\mathcal{M}$ is effective. If*

$$[\mathfrak{g}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] \subset \mathfrak{h}_{\bar{0}} \cap \mathrm{Ker}(\mathrm{ad}\,|_{\mathfrak{g}_{\bar{1}}}),$$

then $\mathcal{M}$ is split.

Proof. Let us show that in this case the vector field $w = \sum \varepsilon^i X_i + 0 = \sum \varepsilon^i X_i$ from Proposition 5.b is a (left invariant) split grading operator on $\mathcal{M}$ using Theorem 1.

The condition (20) is satisfied trivially, because $\chi = 0$. Let us check the condition (21). We have:

$$[Y, v] = \sum Y(\varepsilon^i) X_i - \sum \varepsilon^i [Y, X_i].$$

Since $[\mathfrak{g}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] \subset \mathfrak{h}_{\bar{0}}$, we get

$$\sum \varepsilon^i [Y, X_i] \in H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{h}.$$

Hence, we have to show that

$$\sum Y(\varepsilon^i) X_i \in H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{h}.$$

Assume that $X_1, \dots X_k$ is a basis of $\mathfrak{h}_{\bar{1}}$, $X_1, \dots X_k, X_{k+1}, \dots X_m$ is a basis of $\mathfrak{g}_{\bar{1}}$ and (ε^i) is the system of global odd $\mathcal{G}_0$-left invariant coordinates corresponding to this basis such that $\sum \varepsilon^i X_i$ is as in Proposition 5.b. In particular, $\varepsilon^i(\gamma_{\mathfrak{g}}(X_j)) = \delta^i_j$, because $\sum(\varepsilon^i \circ \gamma_{\mathfrak{g}}) \otimes X_i$ is the identity operator in $\mathfrak{g}^*_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}}$.

Let us take $Z \in \operatorname{Ker} \rho$. Clearly, $Z(\varepsilon^i) = 0$ and $X_j(\varepsilon^i)$ is again a $\mathcal{G}_0$-left invariant function on $\mathcal{G}$. By (8), we also have:

$$\varepsilon^i(X_{i_1} \cdots Z \cdots X_{i_k}) = 0.$$

Furthermore, by definition of ε^i, we get that $\varepsilon^i \circ \gamma_{\mathfrak{g}} \in \mathfrak{g}^*_{\bar{1}}$. Hence,

$$\varepsilon^i(\gamma_{\mathfrak{g}}(X_{i_1} \wedge \cdots \wedge X_{i_k})) = 0,$$

if $k > 1$. Summing up all these observations we see that

$$\varepsilon^i(\gamma_{\mathfrak{g}}(X) \cdot Y) = \varepsilon^i(\gamma_{\mathfrak{g}}(X \wedge Y)) + 0,$$

where $Y \in \mathfrak{h}$ and $X \in \bigwedge \mathfrak{g}_{\bar{1}}$. Now we can conclude that

$$\sum Y(\varepsilon^i) X_i = -Y \in \mathfrak{h} \subset H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0 \backslash \mathcal{G}}) \otimes \mathfrak{h}.$$

The proof is complete.□

Example 3. Consider the super-grassmannian $\mathbf{Gr}_{m|n,k|l}$. It is a $\mathrm{GL}_{m|n}$-homogeneous space, see [9] for more details. Hence, $\mathbf{Gr}_{m|n,k|l} \simeq \mathrm{GL}_{m|n}/\mathcal{H}$ for a certain $\mathcal{H}$. (See, for example, [15].) It the case $k = 0$ or $k = m$, the following holds $[(\mathfrak{gl}_{m|n})_{\bar{1}}, \mathfrak{h}_{\bar{1}}] = 0$. Therefore, by Theorem 2, the super-grassmannian is split.

In [9] it was shown that the super-grassmannian $\mathrm{GL}_{m|n,k|l}$ is not split if and only if $0 < k < m$ and $0 < l < n$. (This fact also follows from results in [6] and [13] about non-projectivity of super-grassmannian.)

Finally, let us recall a result proved in [14]:

Theorem 3. *If a complex homogeneous supermanifold $\mathcal{M}$ is split, then there is a Lie supergroup $\mathcal{G}$ with $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = 0$, where $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \operatorname{Lie} \mathcal{G}$, such that $\mathcal{G}$ acts on $\mathcal{M}$ transitively.*

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