ℓ-INDEPENDENCE FOR COMPATIBLE SYSTEMS OF (MOD ℓ) REPRESENTATIONS

CHUN YIN HUI

ABSTRACT. Let K be a number field. For any system of semisimple mod ℓ Galois representations $\{\phi_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}/K) \to \mathrm{GL}_N(\mathbb{F}_\ell)\}_\ell$ arising from étale cohomology (Definition 1), there exists a finite normal extension L of K such that if we denote $\phi_\ell(\mathrm{Gal}(\bar{\mathbb{Q}}/K))$ and $\phi_\ell(\mathrm{Gal}(\bar{\mathbb{Q}}/L))$ by respectively $\bar{\Gamma}_\ell$ and $\bar{\gamma}_\ell$ for all ℓ, and let $\bar{\mathbf{S}}_\ell$ be the $\mathbb{F}_\ell$-semisimple subgroup of $\mathrm{GL}_{N,\mathbb{F}_\ell}$ associated to $\bar{\gamma}_\ell$ (or $\bar{\Gamma}_\ell$) by Nori [No87] for all sufficiently large ℓ, then the following statements hold for all sufficiently large ℓ:

A(i) The formal character of $\bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}$ (Definition 3) is independent of ℓ and is equal to the formal character of $(\mathbf{G}_\ell^\circ)^{\mathrm{der}} \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}_\ell}$, where $(\mathbf{G}_\ell^\circ)^{\mathrm{der}}$ is the derived group of the identity component of $\mathbf{G}_\ell$, the monodromy group of the corresponding semi-simplified ℓ-adic Galois representation Φ_ℓ^{ss}.

A(ii) The non-cyclic composition factors of $\bar{\gamma}_\ell$ and $\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)$ are identical. Therefore, the composition factors of $\bar{\gamma}_\ell$ are finite simple groups of Lie type of characteristic ℓ and cyclic groups.

B(i) The total ℓ-rank $\mathrm{rk}_\ell \bar{\Gamma}_\ell$ of $\bar{\Gamma}_\ell$ (Definition 14) is equal to the rank of $\bar{\mathbf{S}}_\ell$ and is therefore independent of ℓ.

B(ii) The A_n-type ℓ-rank $\mathrm{rk}_\ell^{A_n} \bar{\Gamma}_\ell$ of $\bar{\Gamma}_\ell$ (Definition 14) for $n \in \mathbb{N}\backslash\{1,2,3,4,5,7,8\}$ and the parity of $(\mathrm{rk}_\ell^{A_4} \bar{\Gamma}_\ell)/4$ are independent of ℓ.

CONTENTS

1991 *Mathematics Subject Classification.* 11F80, 14F20, 20D05.

Key words and phrases. Galois representations, ℓ-independence, big Galois image, étale cohomology.

Major revisions of the paper were done when I was a postdoctoral fellow at The Hebrew University of Jerusalem supported by Aner Shalev's ERC Advanced Grant no. 247034.

1. Introduction

Let K be a number field, $\mathscr{P} \subset \mathbb{N}$ the set of prime numbers, and X a complete non-singular variety defined over K. For any integer i belonging to $[0, 2\mathrm{dim}X]$, the absolute Galois group $\mathrm{Gal}_K := \mathrm{Gal}(\bar{\mathbb{Q}}/K)$ acts on the ith ℓ-adic étale cohomology group $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ for each prime number $\ell \in \mathscr{P}$. The dimension of $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ as a $\mathbb{Q}_\ell$-vector space is independent of ℓ and we denote it by N. We therefore obtain a system of continuous, ℓ-adic Galois representations indexed by $\mathscr{P}$:

$$\{\Phi_\ell : \mathrm{Gal}_K \to \mathrm{GL}_N(\mathbb{Q}_\ell)\}_{\ell\in\mathscr{P}}$$

which satisfies strict compatibility (Deligne [De74]) in the sense of Serre [Se98, Chapter 1]. There is a conjectural ℓ-independence [Se94] on the images of $\{\Phi_\ell\}$ which has been studied by many people. When X is an elliptic curve without complex multiplication, Serre has proved that the Galois action on the ℓ-adic Tate module $T_\ell(X)$ is the whole $\mathrm{GL}(T_\ell(X))$ when ℓ is sufficiently large by showing that the Galois action ϕ_ℓ on ℓ-torsion points $X[\ell] \cong T_\ell(X)/\ell T_\ell(X)$:

$$\phi_\ell : \mathrm{Gal}_K \to \mathrm{GL}(X[\ell]) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$$

is surjective for $\ell \gg 1$ [Se72]. This paper is motivated by the idea that the largeness of the ℓ-adic Galois image $\Gamma_\ell := \Phi_\ell(\mathrm{Gal}_K)$ can be studied via *taking mod ℓ reduction*. More precisely, given any continuous, ℓ-adic representation $\Phi_\ell : \mathrm{Gal}_K \to \mathrm{GL}_N(\mathbb{Q}_\ell)$, one can find a Galois stable $\mathbb{Z}_\ell$-lattice of $\mathbb{Q}_\ell^N$ so that up to some change of coordinates, we may assume $\Phi_\ell(\mathrm{Gal}_K) \subset \mathrm{GL}_N(\mathbb{Z}_\ell)$ since Gal_K is compact. Then by taking mod ℓ reduction $\mathrm{GL}_N(\mathbb{Z}_\ell) \to \mathrm{GL}_N(\mathbb{F}_\ell)$ and semi-simplification, we obtain a continuous, semisimple, mod ℓ Galois representation

$$\phi_\ell : \mathrm{Gal}_K \to \mathrm{GL}_N(\mathbb{F}_\ell)$$

which is independent of the choice of the $\mathbb{Z}_\ell$-lattice by Brauer-Nesbitt [CR88, Theorem 30.16]. Denote the mod ℓ Galois image $\phi_\ell(\mathrm{Gal}_K)$ by $\bar{\Gamma}_\ell$.

Definition 1. A system of mod ℓ Galois representations

$$\{\phi_\ell : \mathrm{Gal}_K \to \mathrm{GL}_N(\mathbb{F}_\ell)\}_{\ell\in\mathscr{P}}$$

is said to be *arising from étale cohomology* if it is the semi-simplification of a mod ℓ reduction of the ℓ-adic system or its dual system:

$$\{\Phi_\ell : \mathrm{Gal}_K \to \mathrm{GL}(H^i_{\text{ét}}(X_{\bar K}, \mathbb{Q}_\ell))\}_{\ell\in\mathscr{P}},$$

$$\{\Phi_\ell : \mathrm{Gal}_K \to \mathrm{GL}(H^i_{\text{ét}}(X_{\bar K}, \mathbb{Q}_\ell)^\vee)\}_{\ell\in\mathscr{P}}$$

for a complete non-singular variety X defined over K and some i, where $H^i_{\text{ét}}(X_{\bar K}, \mathbb{Q}_\ell)^\vee := \mathrm{Hom}_{\mathbb{Q}_\ell}(H^i_{\text{ét}}(X_{\bar K}, \mathbb{Q}_\ell), \mathbb{Q}_\ell)$.

Let ρ^{ss} denote the semi-simplification for any finite dimensional representation ρ over a perfect field (well defined by Brauer-Nesbitt [CR88, Theorem 30.16]). Let $\{\Phi_\ell\}$ be a compatible system of ℓ-adic representations of Gal_K in Definition 1, the algebraic monodromy group at ℓ of the semi-simplified system $\{\Phi_\ell^{\mathrm{ss}}\}$, denoted by $\mathbf{G}_\ell$, is the Zariski closure of $\Phi_\ell^{\mathrm{ss}}(\mathrm{Gal}_K)$ in $\mathrm{GL}_{N,\mathbb{Q}_\ell}$. Then $\mathbf{G}_\ell$ is reductive. Denote the set of non-Archimedean valuations of K and $\bar K$ by respectively Σ_K and $\Sigma_{\bar K}$. The strict compatibility of $\{\Phi_\ell\}$ implies $\{\phi_\ell\}$ is strictly compatible in the following sense.

Definition 2. A system of mod ℓ Galois representations

$$\{\phi_\ell : \mathrm{Gal}_K \to \mathrm{GL}_N(\mathbb{F}_\ell)\}_{\ell\in\mathscr{P}}$$

is said to be *strictly compatible* if $\{\phi_\ell\}$ is continuous, semisimple, and satisfies the following conditions:

(i) There is a finite subset $S \subset \Sigma_K$ such that ϕ_ℓ is *unramified* outside $S_\ell := S \cup \{v \in \Sigma_K : v|\ell\}$ for all ℓ,

(ii) For any $\ell_1, \ell_2 \in \mathscr{P}$ and any $\bar v \in \Sigma_{\bar K}$ extending some $v \in \Sigma_K \backslash (S_{\ell_1} \cup S_{\ell_2})$, the characteristic polynomials of $\phi_{\ell_1}(\mathrm{Frob}_{\bar v})$ and $\phi_{\ell_2}(\mathrm{Frob}_{\bar v})$ are the reductions mod ℓ_1 and mod ℓ_2 of some polynomial $P_v(x) \in \mathbb{Q}[x]$ depending only on v.

Let $\rho : \mathbf{G} \to \mathrm{GL}_{N,F}$ be a faithful representation of a rank r reductive algebraic group $\mathbf{G}$ defined over field F. We define in the beginning of §2 *the formal character* of ρ as an element of quotient set $\mathrm{GL}_r(\mathbb{Z})\backslash\mathbb{Z}[\mathbb{Z}^r]$. Here $\mathbb{Z}[\mathbb{Z}^r]$ is the free abelian group generated by $\mathbb{Z}^r$ and $\mathrm{GL}_r(\mathbb{Z})$ acts naturally on $\mathbb{Z}[\mathbb{Z}^r]$. This allows us to define what is meant by two representations having *the same formal character* (see Definition 3') and the formal character is *bounded by a constant* $C > 0$ (see Definition 4,4'). Let $\{\phi_\ell\}$ be a strictly compatible system of mod ℓ representations arising from étale cohomology (Definition 1,2), this paper studies ℓ-independence of mod ℓ Galois images $\bar{\Gamma}_\ell$ for all sufficiently large ℓ. Let $\mathfrak{g}$ be a Lie type. We define *total ℓ-rank* $\mathrm{rk}_\ell\,\bar{\Gamma}$ and $\mathfrak{g}$*-type ℓ-rank* $\mathrm{rk}_\ell^{\mathfrak{g}}\,\bar{\Gamma}$ of a finite group $\bar{\Gamma}$ in §3.3 Definition 14. The main results are as follows.

Theorem A. (Main theorem) *Let K be a number field and $\{\phi_\ell : \mathrm{Gal}_K \to \mathrm{GL}_N(\mathbb{F}_\ell)\}_{\ell\in\mathscr{P}}$ a strictly compatible system of mod ℓ Galois representations arising from étale cohomology (Definition 1,2). There exists a finite normal extension L of K such that if we denote $\phi_\ell(\mathrm{Gal}_K)$ and $\phi_\ell(\mathrm{Gal}_L)$ by respectively $\bar{\Gamma}_\ell$ and $\bar{\gamma}_\ell$ for all ℓ, and let $\bar{\mathbf{S}}_\ell \subset \mathrm{GL}_{N,\mathbb{F}_\ell}$ be the connected $\mathbb{F}_\ell$-semisimple subgroup associated to $\bar{\gamma}_\ell$ (or $\bar{\Gamma}_\ell$) by Nori's theory for $\ell \gg 1$, then the following hold for $\ell \gg 1$:*

- (i) *The formal character of $\bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}$ is independent of ℓ (Definition 3') and is equal to the formal character of $(\mathbf{G}_\ell^\circ)^{\mathrm{der}} \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}_\ell}$, where $(\mathbf{G}_\ell^\circ)^{\mathrm{der}}$ is the derived group of the identity component of $\mathbf{G}_\ell$, the algebraic monodromy group of the semi-simplified representation Φ_ℓ^{ss}.*
- (ii) *The composition factors of $\bar{\gamma}_\ell$ and $\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)$ are identical modulo cyclic groups. Therefore, the composition factors of $\bar{\gamma}_\ell$ are finite simple groups of Lie type of characteristic ℓ and cyclic groups.*

Corollary B. *Let $\bar{\mathbf{S}}_\ell$ be defined as above, then the following hold for $\ell \gg 1$:*

- (i) *The total ℓ-rank $\mathrm{rk}_\ell\,\bar{\Gamma}_\ell$ of $\bar{\Gamma}_\ell$ (Definition 14) is equal to the rank of $\bar{\mathbf{S}}_\ell$ and is therefore independent of ℓ.*
- (ii) *The A_n-type ℓ-rank $\mathrm{rk}_\ell^{A_n}\,\bar{\Gamma}_\ell$ of $\bar{\Gamma}_\ell$ (Definition 14) for $n \in \mathbb{N}\backslash\{1,2,3,4,5,7,8\}$ and the parity of $(\mathrm{rk}_\ell^{A_4}\,\bar{\Gamma}_\ell)/4$ are independent of ℓ.*

Remark 1.1. As an application of the main results, we prove in [HL14] that $\Phi_\ell(\mathrm{Gal}_K)$, the ℓ-adic Galois image arising from étale cohomology has certain maximality inside the algebraic monodromy group $\mathbf{G}_\ell$ if ℓ is sufficiently large and $\mathbf{G}_\ell$ is of type A. This generalizes Serre's open image theorem on non-CM elliptic curves [Se72].

Remark 1.2. For any field F, define ι to be the involution of $\mathrm{GL}_{N,F}$ that sends A to $(A^t)^{-1}$. If Γ is a subgroup of $\mathrm{GL}_N(F)$, then Γ is semisimple on F^N if and only if $\iota(\Gamma)$ is semisimple on F^N. If ϕ_ℓ is the mod ℓ Galois representation arising from the dual representation $H^i_{\text{ét}}(X_{\bar{K}},\mathbb{Q}_\ell)^\vee$ (Definition 1), then the mod ℓ representation arising from $H^i_{\text{ét}}(X_{\bar{K}},\mathbb{Q}_\ell)$ is $\iota\circ\phi_\ell$ under suitable basis by Brauer-Nesbitt [CR88, Theorem 30.16]. Since ι is an automorphism of GL_N, it suffices to prove Theorem A by considering only the dual mod ℓ system $\{\phi_\ell\}$ and Galois images $\{\bar{\Gamma}_\ell\}$. Let $\phi_{\bar{v}}$ be the restriction of ϕ_ℓ to inertia subgroup $I_{\bar{v}}$ such that $\bar{v} \in \Sigma_{\bar{K}}$ divides ℓ. The reason for choosing the dual system is that the characters of $\phi_{\bar{v}}^{\text{ss}}$ have bounded exponents in the sense of Definition 8 for $\ell \gg 1$ by Serre's tame inertia conjecture proved by Caruso [Ca08] (see Theorem 2.3.1). Such boundedness makes our arguments simpler.

This paper can be considered as "mod ℓ" version of [Hu13] in which we studied ℓ-independence of monodromy groups of any compatible system of ℓ-adic representations by the theory of abelian ℓ-adic representation [Se98] and the representation theory of complex semisimple Lie algebra. The main difference between [Hu13] and this paper is that you get nothing new for considering monodromy groups of mod ℓ Galois images because they are just finite groups. The strategy in this paper is to first construct for each $\ell \gg 1$ a connected $\mathbb{F}_\ell$-reductive subgroup $\bar{\mathbf{G}}_\ell \subset \mathrm{GL}_{N,\mathbb{F}_\ell}$ with bounded formal characters (Definition 4,4') such that $[\bar{\Gamma}_\ell : \bar{\Gamma}_\ell \cap \bar{\mathbf{G}}_\ell(\mathbb{F}_\ell)]$ and $[\bar{\mathbf{G}}_\ell(\mathbb{F}_\ell) : \bar{\Gamma}_\ell \cap \bar{\mathbf{G}}_\ell(\mathbb{F}_\ell)]$ are both uniformly bounded (Theorem 2.0.5). The idea to construct such $\bar{\mathbf{G}}_\ell$ was due to Serre [Se86] where he considered the Galois action on the ℓ-torsion points of abelian varieties A without complex multiplication. In Serre's case, the semisimple part $\bar{\mathbf{S}}_\ell$ of $\bar{\mathbf{G}}_\ell$ is constructed by Nori's theory [No87] and the center $\bar{\mathbf{C}}_\ell$ of $\bar{\mathbf{G}}_\ell$ is the mod ℓ reduction of some $\mathbb{Q}$-diagonalizable group $\mathbf{C}_\mathbb{Q}$ which is a subgroup of the centralizer of monodromy $\mathbf{G}_\ell$ in $\mathrm{GL}_{N,\mathbb{Q}_\ell}$. Hence, $\{\bar{\mathbf{G}}_\ell \subset \mathrm{GL}_{N,\mathbb{F}_\ell}\}_\ell$ has bounded formal characters. The construction of $\mathbf{C}_\mathbb{Q}$ uses the abelian theory of ℓ-adic representations [Se98] and the Tate conjecture for abelian variety (proved by Faltings [Fa83]) which relates the endomorphism ring of A and the commutant of Galois image Γ_ℓ in $\mathrm{End}_N(\mathbb{Q}_\ell)$. Although we don't have the luxury of the Tate conjecture for étale cohomology in general, it is still possible to construct reductive $\bar{\mathbf{G}}_\ell \subset \mathrm{GL}_{N,\mathbb{F}_\ell}$ with the above conditions for $\ell \gg 1$ by Nori's theory, tame inertia tori [Se86], and Serre's tame inertia conjecture (proved by Caruso [Ca08]). The constructions of these algebraic envelopes $\bar{\mathbf{G}}_\ell$ of $\bar{\Gamma}_\ell$ (see Definition 5) are accomplished in §2. Once these nice envelopes are ready, we can use the techniques in [Hu13, §3] to prove that the formal character (Definition 3) of the semisimple part $\bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}$ is independent of $\ell \gg 1$ (Theorem A). The number of A_n factors of $\bar{\mathbf{S}}_\ell$ for large n are then independent of ℓ for all $\ell \gg 1$ by [Hu13, Theorem 2.19]. Since the group of $\mathbb{F}_\ell$-rational points of $\bar{\mathbf{G}}_\ell$ is commensurate to the Galois image $\bar{\Gamma}_\ell$, one deduces ℓ-independence results on the number of Lie type composition factors of characteristic ℓ of $\bar{\Gamma}_\ell$ for $\ell \gg 1$ (Corollary B). Section 3 is devoted to the proofs of Theorem A and Corollary B. The following summarizes the symbols that are frequently used within the text. Groups inside $\mathrm{GL}_{N,F}$ with $\mathrm{char}F > 0$ have their symbols over-lined and should not be confused with base change to an algebraic closure.

Gal_F	the absolute Galois group of field F
K, L	number fields
$\bar{v}$	a valuation of $\bar{K}$ that divides prime ℓ
$I_{\bar{v}}$	the inertia subgroup of Gal_K at valuation $\bar{v}$
U_ℓ, V_ℓ, W_ℓ ($\bar{U}_\ell$, $\bar{V}_\ell$, $\bar{W}_\ell$), ...	vector spaces defined over $\mathbb{F}_\ell$ (over $\bar{\mathbb{F}}_\ell$)
$\bar{\Gamma}_\ell$, $\bar{\gamma}_\ell$, $\bar{\Omega}_\ell$, $\bar{\Omega}_{\bar{v}}$, ...	finite subgroups of $\mathrm{GL}_N(\mathbb{F}_\ell)$
$\mathbf{G}_\ell$, $\mathbf{T}_\ell$, ...	algebraic subgroups of $\mathrm{GL}_{N,\mathbb{Q}_\ell}$
$\bar{\mathbf{G}}_\ell$, $\bar{\mathbf{S}}_\ell$, $\bar{\mathbf{N}}_\ell$, $\bar{\mathbf{I}}_\ell$, $\bar{\mathbf{I}}_{\bar{v}}$, ...	algebraic subgroups of $\mathrm{GL}_{N,\mathbb{F}_\ell}$
Φ_ℓ, Ψ_ℓ, Θ_ℓ, ...	representations over $\mathbb{Q}_\ell$
ϕ_ℓ, ψ_ℓ, μ_ℓ, t_ℓ, $\rho_{\bar{v}}$, $f_{\bar{v}}$, $w_{\bar{v}}$, ...	representations over $\mathbb{F}_\ell$
ρ^{ss}	the semi-simplification of representation ρ
$\rho^\vee$	the dual representation of representation ρ

2. Algebraic envelope $\bar{\mathbf{G}}_\ell$

We define *formal character* and prove some related propositions before stating the main result (Theorem 2.0.5) of this section. Let $\rho : \mathbf{G} \to \mathrm{GL}_{N,F}$ be a faithful representation of a rank r reductive algebraic group $\mathbf{G}$ defined over field F. Choose a maximal torus $\mathbf{T}$ of $\mathbf{G}$ and denote the character group of $\mathbf{T}$ by $\mathbb{X}$. Let $\{w_1, w_2, ..., w_N\} \subset \mathbb{X}$ be the *multiset* of weights of $\rho|_{\mathbf{T}}$ over $\bar{F}$ and choose an isomorphism $\mathbb{X} \cong \mathbb{Z}^r$. Then the image of $w_1 + w_2 + \cdots + w_N \in \mathbb{Z}[\mathbb{X}] \cong \mathbb{Z}[\mathbb{Z}^r]$ in the quotient set $\mathrm{GL}(\mathbb{X})\backslash\mathbb{Z}[\mathbb{X}] \cong \mathrm{GL}_r(\mathbb{Z})\backslash\mathbb{Z}[\mathbb{Z}^r]$ is independent of the choices of maximal torus $\mathbf{T}$ and isomorphism $\mathbb{X} \cong \mathbb{Z}^r$.

Definition 3. Let ρ be as above. *The formal character* of ρ is defined to be the image of $w_1 + w_2 + \cdots + w_N \in \mathbb{Z}[\mathbb{Z}^r]$ in $\mathrm{GL}_r(\mathbb{Z})\backslash\mathbb{Z}[\mathbb{Z}^r]$.

This definition of formal character is more general than the one in [Hu13, §2.1]. It allows us to compare formal characters of two N-dimensional faithful representations $\rho_1 : \mathbf{G}_1 \to \mathrm{GL}_{N,F_1}$ and $\rho_2 : \mathbf{G}_2 \to \mathrm{GL}_{N,F_2}$ over different fields whenever $\mathbf{G}_1$ and $\mathbf{G}_2$ have the same rank. Let $\mathbb{G}_m^N$ be the diagonal subgroup of GL_N. Every character χ of $\mathbb{G}_m^N$ can be expressed uniquely as $x_1^{m_1}x_2^{m_2}\cdots x_N^{m_N}$, a product of powers of *standard characters* $\{x_1, x_2, ..., x_N\}$, where x_i maps $(a_1, ..., a_N) \in \mathbb{G}_m^N$ to a_i for all i. The following proposition (definition) is particularly useful.

Proposition 2.0.1. *(Definition 3') Let ρ_1 and ρ_2 be as above. If $\mathbf{T}_1 \subset \mathbf{G}_1$ and $\mathbf{T}_2 \subset \mathbf{G}_2$ are maximal tori. The following conditions are equivalent:*

(i) *Representations ρ_1 and ρ_2 have the same formal character.*
(ii) *Tori $\rho_1(\mathbf{T}_1)$ and $\rho_2(\mathbf{T}_2)$ are respectively conjugate (in $\mathrm{GL}_{N,\bar{F}_1}$ and $\mathrm{GL}_{N,\bar{F}_2}$) to some subtori $\mathbf{D}_1$ and $\mathbf{D}_2$ of the diagonal subgroup $\mathbb{G}_m^N \subset \mathrm{GL}_N$ such that $\mathbf{D}_1$ and $\mathbf{D}_2$ are annihilated by the same set of characters of $\mathbb{G}_m^N$.*

Hence, formal characters of N-dimensional faithful representations are in bijective correspondence with subtori in $\mathbb{G}_m^N$ up to natural action of permutation group $\mathrm{Perm}(N)$ *of N letters on $\mathbb{G}_m^N$.*

Proof. Assume $\mathbf{T}_j = \mathbb{G}^r_{m,\bar{F}_j}$ and $\rho_j(\mathbf{T}_j) \subset \mathbb{G}^N_{m,\bar{F}_j} \subset \mathrm{GL}_{N,\bar{F}_j}$ from now on by base change to algebraic closure of F_j and diagonalizations for $j = 1, 2$. Suppose (i) holds, then by taking an automorphism of the character group of $\mathbf{T}_1$ and a permutation of coordinates of $\mathbb{G}_m^N$ we obtain

$$x_i \circ \rho_1 = x_i \circ \rho_2$$

for all standard character x_i of $\mathbb{G}_m^N$ if we identify the character groups of $\mathbb{G}^r_{m,\bar{F}_1}$ and $\mathbb{G}^r_{m,\bar{F}_2}$ naturally. This implies the set of characters of $\mathbb{G}_m^N$ that annihilate $\mathbf{D}_j := \rho_j(\mathbf{T}_j)$ for $j = 1, 2$ are equal which is (ii). Suppose (ii) holds, it suffices to consider the case that ρ_1 and ρ_2 are standard representations (inclusions) since $\rho : \mathbf{G} \to \mathrm{GL}_{N,F}$ and $\rho(\mathbf{G}) \subset \mathrm{GL}_{N,F}$ always have the same formal character. Condition (ii) implies that there exists an automorphism of $\mathbb{G}_m^N$ such that

$$\mathbf{D}_j = \{(a_1, ..., a_N) \in \mathbb{G}_m^N : \ a_1 = a_2 = \cdots = a_{N-r} = 1\}$$

for $j = 1, 2$ because $\mathbf{D}_1$ and $\mathbf{D}_2$ are connected. Then (i) follows easily.

Let $\rho : \mathbf{T} \to \mathrm{GL}_{N,\bar{F}}$ be a representation of a torus $\mathbf{T}$. Since the set of weights of ρ is obtained by diagonalizing $\rho(\mathbf{T})$ and is independent of diagonalizations, subtori of $\mathbb{G}_m^N$ that are conjugate to $\rho(\mathbf{T})$ only differ by a permutation of N coordinates. Therefore, the map from formal characters of N-dimensional faithful representations to subtori of $\mathbb{G}_m^N$ modulo action of $\mathrm{Perm}(N)$ is well defined. Since the equivalence of (i) and (ii) implies injectivity and any subtorus of $\mathbb{G}_m^N$ is the formal character of the standard representation of the subtorus, the map is a bijective correspondence. $\square$

Examples: Denote standard representation and faithful representation by respectively Std and ρ. Below are some pair of representations that have the same formal character:

(i) $(\mathrm{GL}_{2,\mathbb{Q}_\ell}, \mathrm{Std})$ and $(\mathrm{GL}_{2,\mathbb{F}_\ell}, \mathrm{Std})$;
(ii) $(\mathbf{G}, \rho)$ and $(\mathbf{H}, \rho|_{\mathbf{H}})$ if $\mathbf{H}$ is a reductive subgroup of $\mathbf{G}$ of same rank;
(iii) $(\mathbf{G}, \rho)$ and $(\mathbf{G}, \rho^\vee)$;

(iv) $(\mathbf{G}, \rho)$ and $(\rho(\mathbf{G}), \mathrm{Std})$.

Definition 4. The formal character of ρ is said to be *bounded by a constant* $C > 0$ if there exists an isomorphism $\mathbb{X} \cong \mathbb{Z}^r$ such that the coefficients of the images of weights $w_1, w_2, ..., w_N \in \mathbb{X}$ in $\mathbb{Z}^r$ have absolute values bounded by C.
Let N be a fixed integer and $\{\rho_i : \mathbf{G}_i \to \mathrm{GL}_{N_i,F_i}\}_{i\in I}$ a family of faithful representations of reductive groups such that $N_i \leq N$ for all $i \in I$. The family is said to have *bounded formal characters* if the formal character of ρ_i is bounded by some constant $C > 0$ for all $i \in I$.

Remark 2.0.2. Let $\{\rho_i\}_{i\in I}$ be a family of representations in Definition 4 having bounded formal characters. Then the number of distinct formal characters arising from the family is finite.

Let $\chi = x_1^{m_1} x_2^{m_2} \cdots x_N^{m_N}$ be a character of $\mathbb{G}_m^N$ expressed as products of standard characters. We call multiset $\{m_1, ..., m_N\}$ *the exponents* of χ and say *the exponents are bounded by* $C > 0$ if $|m_i| < C$ for all $1 \leq i \leq N$. The following characterization of Definition 4 is very useful in this paper.

Proposition 2.0.3. *(Definition 4') Let* $\{\rho_i\}_{i\in I}$ *be a family of faithful representations of reductive* $\mathbf{G}_i$ *such that* ρ_i *is* N_i*-dimensional and* $N_i \leq N$ *for all* $i \in I$*. Choose a maximal torus* $\mathbf{T}_i$ *of* $\mathbf{G}_i$ *for each* $i \in I$*. The following conditions are equivalent:*

(i) *The family has bounded formal characters.*
(ii) *For any* $i \in I$ *and any subtorus* $\mathbf{D}_i$ *of the diagonal subgroup* $\mathbb{G}_m^{N_i} \subset \mathrm{GL}_{N_i}$ *that is conjugate (in* $\mathrm{GL}_{N_i,\bar{F}_i}$*) to* $\rho_i(\mathbf{T}_i)$*, one can choose a set* R_i *of characters of* $\mathbb{G}_m^{N_i}$ *such that the common kernel of* R_i *is* $\mathbf{D}_i$ *and the exponents of characters in* R_i *are bounded by a constant independent of* $i \in I$*.*

Proof. It follows easily from Definition 4, the bijective correspondence in Proposition 2.0.1, and Remark 2.0.2. □

Proposition 2.0.4. *Let* $\{\rho_i\}_{i\in I}$ *and* $\{\phi_i\}_{i\in I}$ *be two families of faithful representations of reductive* $\mathbf{G}_i$ *and* $\mathbf{H}_i$ *over field* F_i *with bounded formal characters such that the target of* ρ_i *and* ϕ_i *are both equal to* GL_{N_i,F_i} *and* $\rho_i(\mathbf{G}_i)$ *commutes with* $\phi_i(\mathbf{H}_i)$ *for all* $i \in I$*. Then the family of standard representations*

$$\{\rho_i(\mathbf{G}_i) \cdot \phi_i(\mathbf{H}_i) \subset \mathrm{GL}_{N_i,F_i}\}_{i\in I}$$

also has bounded formal characters.

Proof. It follows easily from Remark 2.0.2, Proposition 2.0.3, and the fact (by the commutativity hypothesis) that any maximal torus of $\rho_i(\mathbf{G}_i)\cdot\phi_i(\mathbf{H}_i)$ is generated by some maximal torus of $\rho_i(\mathbf{G}_i)$ and some maximal torus of $\phi_i(\mathbf{H}_i)$. □

Let $\{\phi_\ell\}$ be the strictly compatible system of mod ℓ Galois representations arising from (Definition 1,2) the dual system of ℓ-adic representations $\{\Phi_\ell\}$. Denote the image of ϕ_ℓ by $\bar{\Gamma}_\ell$ and the ambient space of the representation by $V_\ell\cong\mathbb{F}_\ell^N$ for each ℓ. Each $\bar{\Gamma}_\ell:=\phi_\ell(\mathrm{Gal}_K)$ is a subgroup of $\mathrm{GL}_N(\mathbb{F}_\ell)$ for a fixed N. Suppose K' is a finite normal extension of K. Since $[\phi_\ell(\mathrm{Gal}_K):\phi_\ell(\mathrm{Gal}_{K'})]\leq[K':K]$ for all ℓ and the restriction of $\{\phi_\ell\}$ to $\mathrm{Gal}_{K'}$ is semisimple [CR88, Theorem 49.2] and satisfies the compatibility conditions (Definition 2), we are free to replace K by K' in the course of proving the main theorem. The main result of this section states that for $\ell\gg1$, $\bar{\Gamma}_\ell$ can be approximated by some connected, reductive subgroup $\bar{\mathbf{G}}_\ell\subset\mathrm{GL}_{N,\mathbb{F}_\ell}$ with bounded formal characters (Definition 4').

Theorem 2.0.5. *Let $\{\phi_\ell\}_{\ell\in\mathscr{P}}$ be a system of mod ℓ Galois representations as above. There exist a finite normal extension L of K and a connected, $\mathbb{F}_\ell$-reductive subgroup $\bar{\mathbf{G}}_\ell$ of $\mathrm{GL}_{N,\mathbb{F}_\ell}$ for each $\ell\gg1$ such that*

(i) *$\bar{\gamma}_\ell:=\phi_\ell(\mathrm{Gal}_L)$ is a subgroup of $\bar{\mathbf{G}}_\ell(\mathbb{F}_\ell)$ of uniformly bounded index,*
(ii) *the action of $\bar{\mathbf{G}}_\ell$ on $\bar{V}_\ell:=V_\ell\otimes\bar{\mathbb{F}}_\ell$ is semisimple,*
(iii) *the representations $\{\bar{\mathbf{G}}_\ell\hookrightarrow\mathrm{GL}_{N,\mathbb{F}_\ell}\}_{\ell\gg1}$ have bounded formal characters in the sense of Definition 4'.*

Definition 5. A system of connected reductive groups $\{\bar{\mathbf{G}}_\ell\}_{\ell\gg1}$ satisfying the conditions in the above theorem is called *a system of algebraic envelopes* of $\{\bar{\Gamma}_\ell\}_{\ell\gg1}$. We say $\bar{\mathbf{G}}_\ell$ is the *algebraic envelope* of $\bar{\Gamma}_\ell$ when a system of algebraic envelopes is given.

We first establish in §2.1 − 2.4 essential ingredients of the proof of Theorem 2.0.5. Then the proof is presented in §2.5.

2.1. **Nori's theory.** The material in this subsection is due to Nori [No87]. Suppose $\ell>N-1$. Given a subgroup $\bar{\Gamma}$ of $\mathrm{GL}_N(\mathbb{F}_\ell)$, Nori's theory gives us a connected algebraic group $\bar{\mathbf{S}}_\ell$ that captures all the order ℓ elements of $\bar{\Gamma}$ if ℓ is bigger than a constant that only depends on N.

Let $\bar{\Gamma}[\ell]=\{x\in\bar{\Gamma}\mid x^\ell=1\}$. The normal subgroup of $\bar{\Gamma}$ generated by $\bar{\Gamma}[\ell]$ is denoted by $\bar{\Gamma}^+$. Define $\exp(x)$ and $\log(x)$ by

$$\exp(x)=\sum_{i=0}^{\ell-1}\frac{x^i}{i!}\ \text{ and }\ \log(x)=-\sum_{i=1}^{\ell-1}\frac{(1-x)^i}{i}.$$

Denote by $\bar{\mathbf{S}}$ the (connected) algebraic subgroup of $\mathrm{GL}_{N,\mathbb{F}_\ell}$, defined over $\mathbb{F}_\ell$, generated by the one-parameter subgroups

$$t \mapsto x^t := \exp(t \cdot \log(x))$$

for all $x \in \bar{\Gamma}[\ell]$. Algebraic subgroups with the above property are said to be *exponentially generated*. The theorem we need is stated below.

Theorem 2.1.1. [No87, Theorem B(1), 3.6(v)] *There is a constant $c_0 = c_0(N)$ such that if $\ell > c_0$ and $\bar{\Gamma}$ is a subgroup of $\mathrm{GL}_N(\mathbb{F}_\ell)$, then*

(i) $\bar{\Gamma}^+ = \bar{\mathbf{S}}(\mathbb{F}_\ell)^+$,

(ii) $\bar{\mathbf{S}}(\mathbb{F}_\ell)/\bar{\mathbf{S}}(\mathbb{F}_\ell)^+$ *is a commutative group of order* $\leq 2^{N-1}$.

Proposition 2.1.2. *Let $\bar{\mathbf{S}}_\ell$ be the algebraic group associated to $\bar{\Gamma}_\ell$ by Nori's theory for all $\ell > N-1$. There is a constant $c_1 = c_1(N) > c_0(N)$ that depends only on N such that if $\ell > c_1$, then the following hold:*

(i) $\bar{\mathbf{S}}_\ell$ *is a connected, exponentially generated, semisimple $\mathbb{F}_\ell$-subgroup of* $\mathrm{GL}_{N,\mathbb{F}_\ell}$.

(ii) $\bar{\mathbf{S}}_\ell$ *acts semi-simply on the ambient space* $\bar{V}_\ell \cong \bar{\mathbb{F}}_\ell^N$.

(iii) $[\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell) : \bar{\mathbf{S}}_\ell(\mathbb{F}_\ell) \cap \bar{\Gamma}_\ell] \leq 2^{N-1}$.

Proof. Since $\bar{\Gamma}_\ell$ acts semi-simply on $\bar{V}_\ell$, so does $\bar{\Gamma}_\ell^+$ [CR88, Theorem 49.2]. Part (ii) then follows from [EHK12, Theorem 24] for some sufficiently large constant $c_1(N)$ ($> c_0(N)$) depending only on N, see also [Se86]. Since $\ell > c_0(N)$, $\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)^+ = \bar{\Gamma}_\ell^+$ (Theorem 2.1.1) also acts semi-simply on $\bar{V}_\ell$. This implies $\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)^+$ cannot have normal ℓ-subgroup. If $\bar{\mathbf{S}}_\ell$ has a non-trivial unipotent radical $\bar{\mathbf{U}}_\ell$, then $\bar{\mathbf{U}}_\ell$ is defined over $\mathbb{F}_\ell$ [Sp08, Proposition 14.4.5(v)] and $\bar{\mathbf{U}}_\ell(\mathbb{F}_\ell)$ is then a non-trivial normal ℓ-group of $\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)^+$ which is a contradiction. Therefore $\bar{\mathbf{S}}_\ell$ is reductive. $\bar{\mathbf{S}}_\ell$ is actually semisimple since it is generated by unipotent elements $\bar{\Gamma}_\ell^+$. This proves (i). Since $\ell > c_0(N)$, (iii) is proved by Theorem 2.1.1. ☐

Definition 6. Define the *semisimple envelope* of $\bar{\Gamma}_\ell$ for all sufficiently large ℓ as the connected, semisimple $\mathbb{F}_\ell$-algebraic group $\bar{\mathbf{S}}_\ell$ in Proposition 2.1.2.

Remark 2.1.3. If K' is a finite extension of K, then the semisimple envelopes of $\phi_\ell(\mathrm{Gal}_{K'})$ and $\phi_\ell(\mathrm{Gal}_K)$ are identical for $\ell \gg 1$ because the order ℓ elements of the two finite groups are the same when ℓ is large.

2.2. **Characters of tame inertia group.** Let $\rho_\ell : \mathrm{Gal}_K \to \mathrm{GL}_N(\mathbb{F}_\ell)$ be a continuous representation and $I_{\bar{v}}$ the inertia subgroup of Gal_K at $\bar{v} \in \Sigma_{\bar{K}}$ that divides ℓ. Let $I_{\bar{v}}^{\mathrm{w}}$ be the wild inertia (normal) subgroup of $I_{\bar{v}}$ and $\rho_{\bar{v}}^{\mathrm{ss}}$ the semi-simplification of the restriction of ρ_ℓ to $I_{\bar{v}}$. Since $\rho_\ell^{\mathrm{ss}}(I_{\bar{v}}^{\mathrm{w}})$ is an ℓ-group and semisimple on $\bar{\mathbb{F}}_\ell^N$, $\rho_{\bar{v}}^{\mathrm{ss}}(I_{\bar{v}}^{\mathrm{w}}) = \{1\}$ and $\rho_{\bar{v}}^{\mathrm{ss}}$

factors through a representation of the tame inertia group $I_{\bar{v}}^{\mathrm{t}} := I_{\bar{v}}/I_{\bar{v}}^{\mathrm{w}}$ (still denoted by $\rho_{\bar{v}}^{\mathrm{ss}}$):

$$\rho_{\bar{v}}^{\mathrm{ss}} : I_{\bar{v}}^{\mathrm{t}} \to \mathrm{GL}_N(\mathbb{F}_\ell).$$

The tame inertia group $I_{\bar{v}}^{\mathrm{t}}$ is a projective limit of cyclic groups of order prime to ℓ [Se72, Proposition 2]

$$\theta_{\bar{v}} : I_{\bar{v}}^{\mathrm{t}} \xrightarrow{\cong} \varprojlim_k \mathbb{F}_{\ell^k}^*$$

where the projective system is given by norm maps of finite fields of characteristic ℓ. The isomorphism is unique up to action of $\mathrm{Gal}_{\mathbb{F}_\ell}$ on the target.

Definition 7. The *fundamental characters* of $I_{\bar{v}}^{\mathrm{t}}$ of level d [Se72, §1.7] are defined as

$$\theta_d^{\ell^j},\ j = 0, 1, ..., d-1$$

where $\theta_d : I_{\bar{v}}^{\mathrm{t}} \xrightarrow{\theta_{\bar{v}}} \varprojlim_k \mathbb{F}_{\ell^k}^* \twoheadrightarrow \mathbb{F}_{\ell^d}^* \hookrightarrow \bar{\mathbb{F}}_\ell^*$.

Any continuous character $\chi : I_{\bar{v}}^{\mathrm{t}} \to \bar{\mathbb{F}}_\ell^*$ of $\rho_{\bar{v}}^{\mathrm{ss}}$ factors through a power of some θ_d. Character theory says that $\mathrm{Hom}(\mathbb{F}_{\ell^d}^*, \bar{\mathbb{F}}_\ell^*) \cong \mathrm{Hom}(\mathbb{F}_{\ell^d}^*, \mathbb{C}^*)$ is cyclic generated by θ_d of order $\ell^d - 1$. Therefore, χ can always be expressed as a product of fundamental characters of level d

$$\chi = (\theta_d)^{m_0} \cdot (\theta_d^{\ell})^{m_1} \cdots (\theta_d^{\ell^{d-1}})^{m_{d-1}}$$

Definition 8. Let $\chi : I_{\bar{v}}^{\mathrm{t}} \to \bar{\mathbb{F}}_\ell^*$ be a character of $\rho_{\bar{v}}^{\mathrm{ss}}$ and express χ as a product of fundamental characters of level d as above.

(i) The product is said to be *ℓ-restricted* if $0 \le m_i \le \ell - 1$ for all i and not all m_i equal to $\ell - 1$. It is easy to see that ℓ-restricted expression of χ is unique.

(ii) The *exponents* of χ are defined to be the multiset of powers $\{m_0, m_1, ..., m_{d-1}\}$ in the ℓ-restricted product. Note that the multiset is independent of the action of $\mathrm{Gal}_{\mathbb{F}_\ell}$ on the target.

Lemma 2.2.1. *Let $V \cong \mathbb{F}_\ell^n$ be a continuous, irreducible subrepresentation of $\rho_{\bar{v}}$, then the characters of the representation can be written as a product of fundamental characters of level n.*

Proof. For simplicity, assume $\rho_{\bar{v}}$ is irreducible. The image of $I_{\bar{v}}^{\mathrm{t}}$ in $\mathrm{GL}(V)$ is a cyclic group of order prime to ℓ, therefore V is a $\mathbb{F}_\ell[x]/(f(x))$-module where x corresponds to a generator of the cyclic image and the minimal polynomial $f(x)$ is separable. Irreducibility of V implies $f(x)$ is irreducible over $\mathbb{F}_\ell$. Thus $\rho_{\bar{v}}(I_{\bar{v}}^{\mathrm{t}})$ is contained in a maximal subfield F of $\mathrm{End}(V)$ and $\rho_{\bar{v}} : I_{\bar{v}}^{\mathrm{t}} \to F^* \subset \mathrm{GL}(V)$ can be written as a product of fundamental characters of level n as above. On the other hand, V has a structure of F-vector space of dimension 1 such that the action of $\rho_{\bar{v}}(I_{\bar{v}}^{\mathrm{t}}) \subset F^*$ is through field multiplication. By tensoring F with F (on the right) over $\mathbb{F}_\ell$, we obtain an F-isomorphism

$$F \otimes F \to F \oplus F \oplus \cdots \oplus F$$
$$x \otimes y \mapsto (xy, x^\ell y, ..., x^{\ell^{n-1}} y)$$

where $x, x^\ell, ..., x^{\ell^{n-1}}$ are just conjugate of x over $\mathbb{F}_\ell$. If $x \in \rho_{\bar{v}}(I_{\bar{v}}^{\mathrm{t}}) \subset F^*$, then we see the action of $I_{\bar{v}}^{\mathrm{t}}$ on $V \otimes_{\mathbb{F}_\ell} F$ is a direct sum of products of fundamental characters of level n. □

2.3. **Exponents of characters arising from étale cohomology.** Every character χ of $\rho_{\bar{v}}^{\mathrm{ss}} : I_{\bar{v}}^{\mathrm{t}} \to \mathrm{GL}_N(\mathbb{F}_\ell)$ can be written as

$$\chi = (\theta_n)^{m_0} \cdot (\theta_n^\ell)^{m_1} \cdots (\theta_n^{\ell^{n-1}})^{m_{n-1}},$$

a product of fundamental characters of level $n \leq N$ by Lemma 2.2.1. One would like to study the exponents $m_0, ..., m_{n-1}$ (Definition 8) and in the case of étale cohomology we have the following theorem proved by Caruso [Ca08].

Theorem 2.3.1. *(Serre's tame inertia conjecture) Let X be a proper and smooth variety over a local field K (a finite extension of $\mathbb{Q}_\ell$) with semi-stable reduction over $\mathscr{O}_K$, the ring of integers of K and i an integer. The Galois group Gal_K acts on $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/\ell\mathbb{Z})^\vee$, the $\mathbb{F}_\ell$-dual of the ith cohomology group with $\mathbb{Z}/\ell\mathbb{Z}$ coefficients. If we restrict the representation to the inertia group of Gal_K, then the exponents of the characters of the tame inertia group on any Jordan-Holder quotient of $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/\ell\mathbb{Z})^\vee$ are between 0 and ei where e is the ramification index of $K/\mathbb{Q}_\ell$.*

We now relate our mod ℓ Galois representation ϕ_ℓ to representation $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/\ell\mathbb{Z})^\vee$ in Theorem 2.3.1. Cohomology group $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_\ell)$ is a finitely generated, free $\mathbb{Z}_\ell$-module [Ga83] for $\ell \gg 1$:

$$H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell \oplus \cdots \oplus \mathbb{Z}_\ell.$$

Reduction mod ℓ gives

$$H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_\ell) \otimes \mathbb{F}_\ell = \mathbb{Z}/\ell\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\ell\mathbb{Z}$$

and the semi-simplification of $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_\ell) \otimes \mathbb{F}_\ell$ is then isomorphic to the semi-simplification of a mod ℓ reduction of ℓ-adic representation $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ by Brauer-Nesbitt [CR88, Theorem 30.16]. Since the sequence

$$H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_\ell) \xrightarrow{\ell} H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_\ell) \to H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/\ell\mathbb{Z})$$

is exact [Mi13, Theorem 19.2], $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_\ell)\otimes\mathbb{F}_\ell$ is isomorphic to $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/\ell\mathbb{Z})$. Recall V_ℓ is the semi-simplification of a mod ℓ reduction of $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)^\vee$. Thus, we conclude that

Proposition 2.3.2. *For all sufficiently large ℓ, $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_\ell)\otimes\mathbb{F}_\ell$ is isomorphic to $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/\ell\mathbb{Z})$ and the semi-simplification of $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/\ell\mathbb{Z})$ is $V_\ell^\vee$.*

The following theorem is the main result of this subsection.

Theorem 2.3.3. *Let K be a number field. Let $\phi_\ell : \text{Gal}_K \to \text{GL}(V_\ell) \cong \text{GL}_N(\mathbb{F}_\ell)$ be the mod ℓ Galois representation arising from étale cohomology group $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)^\vee$ for sufficiently large ℓ. If we restrict ϕ_ℓ to the inertia group $I_{\bar{v}}$ of a valuation $\bar{v}|\ell$ of $\bar{K}$ and semi-simplify the representation, then every character χ of the representation can be written as*

$$\chi = (\theta_{N!})^{m_0} \cdot (\theta_{N!}^{\ell})^{m_1} \cdots (\theta_{N!}^{\ell^{N!-1}})^{m_{N!-1}}$$

a product of fundamental characters of level $N!$ with exponents (Definition 8) $m_0, ..., m_{N!-1}$ (depending on ℓ) belonging to $[0, ei]$ where e is the ramification index of $K_v/\mathbb{Q}_\ell$, $v = \bar{v}|_K$, and K_v is the completion of K with respect to v.

Proof. Proposition 2.3.2 implies that if ℓ is sufficiently large, then Galois representations $V_\ell = (V_\ell^\vee)^\vee$ and $(H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/\ell\mathbb{Z})^\vee)^{\text{ss}}$ are isomorphic. Let χ be a character of $I^{\text{t}}_{\bar{v}}$ given by the semi-simplification of the restriction of V_ℓ to inertia subgroup $I_{\bar{v}}$. By Theorem 2.3.1, χ can be written as

$$\chi = (\theta_d)^{m_0} \cdot (\theta_d^{\ell})^{m_1} \cdots (\theta_d^{\ell^{d-1}})^{m_{d-1}},$$

a product of fundamental characters of level d ($\leq N$ by Lemma 2.2.1) with exponents $m_0, ..., m_{d-1}$ belonging to $[0, ei]$ where e is the ramification index of $K_v/\mathbb{Q}_\ell$. Since d divides $N!$, $\theta_{N!}$ factors through χ. Consider the norm map $\text{Nm} : \mathbb{F}^*_{\ell^{N!}} \to \mathbb{F}^*_{\ell^d}$

$$x \mapsto x \cdot x^{\ell^d} \cdot x^{\ell^{2d}} \cdots x^{\ell^{(N!-d)}}.$$

Then we obtain a product of fundamental characters of level $N!$

$$\chi = (\text{Nm} \circ \theta_{N!})^{m_0 + m_1\ell + \cdots + m_{d-1}\ell^{d-1}}$$

$$= (\theta_{N!})^{s_0} \cdot (\theta_{N!}^{\ell})^{s_1} \cdots (\theta_{N!}^{\ell^{N!-1}})^{s_{N!-1}}$$

with exponents $s_0, ..., s_{N!-1}$ belonging to $[0, ei]$. □

2.4. **Tame inertia tori and rigidity.** Tame inertia tori were considered by Serre when he studied Galois action on ℓ-torsion points of abelian varieties without complex multiplication [Se86]. He observed that these tori have certain rigidity which will be explained in this subsection.

Assume $\ell > N-1$ as in §2.1. Since every non-trivial element of every ℓ-Sylow subgroup of $\bar{\Gamma}_\ell$ is of order ℓ and $\bar{\Gamma}_\ell^+$ is contained in $\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)$ by Theorem 2.1.1(i), index $[\bar{\Gamma}_\ell : \bar{\Gamma}_\ell \cap \bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)]$ is prime to ℓ. Let $\bar{\mathbf{N}}_\ell$ be the normalizer of $\bar{\mathbf{S}}_\ell$ in $\mathrm{GL}_{N,\mathbb{F}_\ell}$; clearly $\bar{\Gamma}_\ell \subset \bar{\mathbf{N}}_\ell$.

Theorem 2.4.1. [Se86, §1 Theorem] *There are constants $c_2 = c_2(N)$ and $c_3 = c_3(N)$ such that if $\ell > c_2$, $\bar{\mathbf{S}}_\ell \subset \mathrm{GL}_{N,\mathbb{F}_\ell}$ is an exponentially generated semisimple algebraic group defined over $\mathbb{F}_\ell$, and the action on $\bar{V}_\ell \cong \bar{\mathbb{F}}_\ell^N$ is semisimple. If W_ℓ is the $\mathbb{F}_\ell$-subspace of*

$$U_\ell := \bigoplus_{i=1}^{c_3} (\otimes^i V_\ell)$$

fixed by $\bar{\mathbf{S}}_\ell$, then $t_\ell : \bar{\mathbf{N}}_\ell/\bar{\mathbf{S}}_\ell \to \mathrm{GL}_{W_\ell}$ is an $\mathbb{F}_\ell$-embedding. Moreover, if $x \notin \bar{\mathbf{S}}_\ell$, then there is an element of $\bar{W}_\ell$ that is not fixed by x.

By Theorem 2.4.1, $\bar{\Gamma}_\ell/(\bar{\Gamma}_\ell \cap \bar{\mathbf{S}}_\ell(\mathbb{F}_\ell))$ embeds in $\mathrm{GL}(W_\ell)$ with $\dim(W_\ell) \leq c_4 = c_4(N)$ uniformly for some integer c_4. Theorem 2.4.2 below is the main result of this subsection.

Definition 9. Define $\mu_\ell : \mathrm{Gal}_K \to \mathrm{GL}(W_\ell)$ to be the composition $t_\ell \circ \phi_\ell$ for each ℓ and $\bar{\Omega}_\ell$ to be the image μ_ℓ, where t_ℓ is defined in Theorem 2.4.1.

Theorem 2.4.2. *Let $\bar{\mathbf{I}}_\ell$ be the algebraic group generated by a set of tame inertia tori $\bar{\mathbf{I}}_{\bar{v}}$ (Definition 10) for $\ell \gg 1$. There exist constant $c_8 = c_8(N)$ and a finite normal field extension L/K such that if $\ell \gg 1$, then $\bar{\mathbf{I}}_\ell$ is a torus, called the inertia torus at ℓ, and $\mu_\ell(\mathrm{Gal}_L) \subset \bar{\Omega}_\ell$ is a subgroup of $\bar{\mathbf{I}}_\ell(\mathbb{F}_\ell)$ such that*

(i) *$\{\bar{\mathbf{I}}_\ell \hookrightarrow \mathrm{GL}_{W_\ell}\}_{\ell \gg 1}$ have bounded formal characters (Definition 4'),*
(ii) *$[\bar{\mathbf{I}}_\ell(\mathbb{F}_\ell) : \mu_\ell(\mathrm{Gal}_L)]$ is bounded by c_8.*

Theorem 2.4.3. [Jo78, Jordan's theorem on finite linear groups] *For every n there exists a constant $J(n)$ such that any finite subgroup of GL_n over a field of characteristic zero possesses an abelian normal subgroup of index $\leq J(n)$.*

The order of $\bar{\Omega}_\ell$ is prime to ℓ. $\bar{\Omega}_\ell$ can thus be lifted to a subgroup of $\mathrm{GL}_{N'}(\mathbb{C})$ such that N' only depends on N. Theorem 2.4.3 (Jordan) then says that $\bar{\Omega}_\ell$ has a abelian normal subgroup $\bar{J}_\ell$ of index less than a constant $c_5 = c_5(N) := J(N')$ depends on N'. Since N' depends on N, we have $[\bar{\Omega}_\ell : \bar{J}_\ell] \leq c_5$. If $\bar{v}$ divides ℓ, then the action of the inertia group $I_{\bar{v}}$ on W_ℓ is semisimple because $|\bar{\Omega}_\ell|$ is prime to ℓ. Since $\dim(W_\ell)|c_4!$ We obtain

$$\mu_\ell : I_{\bar{v}}^{\mathrm{t}} \overset{\theta_{c_4!}}{\twoheadrightarrow} \mathbb{F}^*_{\ell^{c_4!}} \to \mathrm{GL}(W_\ell).$$

By Theorem 2.3.3 and W_ℓ in Theorem 2.4.1, there exist $c_6 = c_6(N) \geq 0$ such that if χ is a character, then χ can be written as a product of fundamental characters of level $c_4!$

$$\chi = (\theta_{c_4!})^{m_0} \cdot (\theta_{c_4!}^{\ell})^{m_1} \cdots (\theta_{c_4!}^{\ell^{c_4!-1}})^{m_{c_4!-1}}$$

with exponents $m_0, ..., m_{c_4!-1}$ belonging to $[0, c_6]$ for all $\ell \gg 1$. Therefore, we make the following definition.

Definition 10. Denote field $\mathbb{F}_{\ell^{c_4!}}$ by $\mathbb{E}_\ell$ for all ℓ. This gives a homomorphism

$$f_{\bar{v}} : \mathbb{E}_\ell^* \to \mathrm{GL}(W_\ell)$$

if $\ell > c_6(N) + 1$. Let $\bar{\mathbf{E}}_\ell$ denote $\mathrm{Res}_{\mathbb{E}_\ell/\mathbb{F}_\ell}(\mathbb{G}_m)$ (Weil restriction of scalars) for all ℓ. We have $\bar{\mathbf{E}}_\ell(\mathbb{F}_\ell) = \mathbb{E}_\ell^*$. Then $f_{\bar{v}}$ extends uniquely [Ha11, §3] to an ℓ-restricted $\mathbb{F}_\ell$-morphism below:

$$w_{\bar{v}} : \bar{\mathbf{E}}_\ell := \mathrm{Res}_{\mathbb{E}_\ell/\mathbb{F}_\ell}(\mathbb{G}_m) \to \mathrm{GL}_{W_\ell}.$$

Denote the image of $w_{\bar{v}}$ by $\bar{\mathbf{I}}_{\bar{v}}$ for $\bar{v}|\ell \gg 1$. It is called the *tame inertia torus at* $\bar{v} \in \Sigma_{\bar{K}}$.

Lemma 2.4.4. *There exists a constant $c_7 = c_7(N)$ such that for any $\bar{v}|\ell > c_6(N) + 1$, we have*

(i) $\{\bar{\mathbf{I}}_{\bar{v}} \hookrightarrow \mathrm{GL}_{W_\ell}\}_{\bar{v}}$ *have bounded formal characters (Definition 4');*
(ii) $[\bar{\mathbf{I}}_{\bar{v}}(\mathbb{F}_\ell) : f_{\bar{v}}(\mathbb{E}_\ell^*)] \leq c_7$.

Proof. Since $\dim(W_\ell)$ and $\dim(\bar{\mathbf{E}}_\ell)$ are bounded by a constant independent of ℓ and the exponents of the characters of $w_{\bar{v}}$ in terms of the fundamental characters [Ha11, §3] belong to $[0, c_6]$, we find by Proposition 2.0.3 a set of characters $R_{\bar{v}}$ of uniformly bounded exponents of the diagonal subgroup of GL_{W_ℓ} by diagonalizing $\bar{\mathbf{I}}_{\bar{v}}$ and then obtain assertion (i). For assertion (ii), uniform boundedness of exponents of characters and $\dim(\bar{\mathbf{E}}_\ell) = c_4!$ (for all ℓ) imply the number of connected components of $\mathrm{Ker}(w_{\bar{v}})$ is uniformly bounded by c_7. On the other hand, the number of $\mathbb{F}_\ell$-rational points of any $\mathbb{F}_\ell$-torus of dimension k is between $(\ell - 1)^k$ and $(\ell + 1)^k$ by [No87, Lemma 3.5]. Therefore,

$\mu_\ell(I_{\bar v}^{\mathrm{t}})=f_{\bar v}(\mathbb{E}_\ell^*)$ has at least

$$\frac{|\mathbb{E}_\ell^*|}{c_7(\ell+1)^{\dim(\mathrm{Ker}(w_{\bar v}))}}=\frac{\ell^{c_4!}-1}{c_7(\ell+1)^{\dim(\mathrm{Ker}(w_{\bar v}))}}$$

points and $[\bar{\mathbf{I}}_{\bar v}(\mathbb{F}_\ell):\mu_\ell(I_{\bar v}^{\mathrm{t}})]$ is bounded by

$$\frac{c_7(\ell+1)^{\dim(\mathrm{Ker}(w_{\bar v}))+\dim(\mathrm{Im}(w_{\bar v}))}}{\ell^{c_4!}-1}=\frac{c_7(\ell+1)^{c_4!}}{\ell^{c_4!}-1}\to c_7$$

when ℓ is big. This proves (ii). □

Lemma 2.4.5. *(Rigidity)* [Ha11, §3],[Se86, §3] *Let $s\in\mathrm{GL}(W_\ell)$ be a semisimple element and $f_{\bar v}:\mathbb{E}_\ell^*\to\mathrm{GL}(W_\ell)$ a representation such that the exponents of characters of $f_{\bar v}$ belong to $[0,c]$ for some $c>0$. If $H\subset\mathbb{E}_\ell^*$ is a subgroup such that $f_{\bar v}(H)$ commutes with s in $\mathrm{GL}(W_\ell)$ and $c\cdot[\mathbb{E}_\ell^*:H]\le\ell-1$, then $\bar{\mathbf{I}}_{\bar v}$ commutes with s, and hence so does $f_{\bar v}(\mathbb{E}_\ell^*)$.*

Recall from Definition 2 that there is a finite subset $S\subset\Sigma_K$ such that ϕ_ℓ is unramified outside $S_\ell:=S\cup\{v\in\Sigma_K:v|\ell\}$ for all ℓ.

Proof of Theorem 2.4.2. The following arguments are influenced by the arguments Serre gave for [Se86, Theorem 1].

Proof. Denote the image of $\mu_\ell(I_{\bar v}^{\mathrm{t}})$ under the map $\bar\Gamma_\ell/(\bar\Gamma_\ell\cap\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell))\hookrightarrow\mathrm{GL}(W_\ell)$ by $\bar\Omega_{\bar v}$ whenever $\bar v|\ell$. Let $\bar J_\ell$ be a maximal abelian normal subgroup of $\bar\Omega_\ell:=\mu_\ell(\mathrm{Gal}_K)$. We first prove that $\bar\Omega_{\bar v}$ commutes with $\bar J_\ell$ if ℓ is large. Since $\bar\Omega_{\bar v}$ and $\bar J_\ell$ are abelian and

$$[\bar\Omega_{\bar v}:\bar\Omega_{\bar v}\cap\bar J_\ell]\le c_5$$

by Theorem 2.4.3 (Jordan), the tame inertia torus $\bar{\mathbf{I}}_{\bar v}$ at $\bar v$ (Definition 10) and hence $f_{\bar v}(\mathbb{E}_\ell^*)=\bar\Omega_{\bar v}$ commute with $\bar J_\ell$ if $\ell>c_5c_6+1$ by rigidity (Lemma 2.4.5). For any $\bar v_1,\bar v_2|\ell$, since $\bar\Omega_{\bar v_1}\cap\bar J_\ell$ commutes with $\bar\Omega_{\bar v_2}\cap\bar J_\ell$ which are of bounded index in $\bar\Omega_{\bar v_1}$ and $\bar\Omega_{\bar v_2}$ respectively, we obtain $\bar{\mathbf{I}}_{\bar v_1}$ commutes with $\bar{\mathbf{I}}_{\bar v_2}$ if $\ell\gg1$ by rigidity (Lemma 2.4.5). The subgroup $\bar H_\ell$ of $\bar\Omega_\ell$ generated by the inertia subgroups $\bar\Omega_{\bar v}$ for all $\bar v|\ell$ is abelian and normal for $\ell\gg1$. As $\bar J_\ell$ is maximal normal abelian in $\bar\Omega_\ell$, $\bar H_\ell\subset\bar J_\ell$ for all $\ell\gg1$. Therefore, $\bar\Omega_\ell/\bar J_\ell$ corresponds to a field extension of K of degree bounded by c_5 that only ramifies in S (Definition 2) for $\ell\gg1$. By Hermite's Theorem [La94, p.122], the composite of these fields is still a finite field extension K' of K. Therefore, $\mu_\ell(\mathrm{Gal}_{K'})\subset\bar J_\ell$ for $\ell\gg1$.

Since the representations $\{\phi_\ell\}$ come from étale cohomology and $I_{\bar v}\cap\mathrm{Gal}_{K'}$ is the inertia subgroup of $\mathrm{Gal}_{K'}$ at $\bar v$ [Ne99, Proposition 9.5], they are potentially semi-stable which means there exists a finite extension K'' of K' such that $\phi_\ell(I_{\bar v}\cap \mathrm{Gal}_{K''})$ is unipotent for any $\bar v$ not dividing ℓ [deJ96, §1]. Therefore, for each $\ell \gg 1$ we have a finite abelian extension of K'' with Galois group $\mu_\ell(\mathrm{Gal}_{K''})$ contained in $\bar J_\ell$ that only ramifies at $v \in \Sigma_{K''}$ dividing ℓ. Since $\mu_\ell(\mathrm{Gal}_{K''})$ is an abelian Galois group over K'', each ramified prime $v\in\Sigma_{K''}$ dividing large ℓ corresponds to an inertia subgroup $\bar I''_v \subset \mu_\ell(\mathrm{Gal}_{K''})$ and there are at most $[K'':\mathbb{Q}]$ of them. For each inertia subgroup $\bar I''_v$, choose a tame inertia torus $\bar{\mathbf{I}}_{\bar v}$ such that $\bar I''_v \subset \bar{\mathbf{I}}_{\bar v}(\mathbb{F}_\ell)$. Since these tame inertia tori commute with each other, the algebraic group $\bar{\mathbf{I}}_\ell$ generated by them is an $\mathbb{F}_\ell$-torus, called *the inertia torus at* ℓ. Since $\{\bar{\mathbf{I}}_{\bar v}\to \mathrm{GL}_{W_\ell}\}_{\bar v|\ell\gg1}$ have bounded formal characters (Lemma 2.4.4(i)) and each $\bar{\mathbf{I}}_\ell$ is generated by at most $[K'':\mathbb{Q}]$ tame inertia tori, $\{\bar{\mathbf{I}}_\ell\hookrightarrow \mathrm{GL}_{W_\ell}\}_{\ell\gg1}$ have bounded formal characters by Proposition 2.0.4. This proves (i).

Let $\bar I''_\ell$ be the subgroup of $\mu_\ell(\mathrm{Gal}_{K''})$ generated by $\bar I''_v$ for all $v|\ell$. Then, for $\ell\gg1$ we have

$$\mu_\ell(\mathrm{Gal}_{K''})/\bar I''_\ell$$

is the Galois group of a finite abelian extension of K'' that is unramified at every non-Archimedean valuation. By abelian class field theory, these fields generate a finite extension K''' of K''. Choose L normal over K such that $K'''\subset L$. Then, we obtain

$$(*):\ \mu_\ell(\mathrm{Gal}_L)\subset \bar I''_\ell \subset \bar{\mathbf{I}}_\ell(\mathbb{F}_\ell).$$

It remains to prove (ii). Suppose $\bar{\mathbf{I}}_\ell$ is generated by tame inertia tori $\bar{\mathbf{I}}_{\bar v_i}$ for $1\le i\le k$ for some fixed $k\le[K'':\mathbb{Q}]$. We have

$$\begin{aligned}[\bar{\mathbf{I}}_\ell(\mathbb{F}_\ell):\mu_\ell(\mathrm{Gal}_L)] &= [\bar{\mathbf{I}}_\ell(\mathbb{F}_\ell):\bar{\mathbf{I}}_\ell(\mathbb{F}_\ell)\cap\bar\Omega_\ell]\cdot[\bar{\mathbf{I}}_\ell(\mathbb{F}_\ell)\cap\bar\Omega_\ell:\mu_\ell(\mathrm{Gal}_L)]\\ &\le [\bar{\mathbf{I}}_\ell(\mathbb{F}_\ell):f_{\bar v_1}(\mathbb{E}_\ell^*)\cdots f_{\bar v_k}(\mathbb{E}_\ell^*)]\cdot[L:K].\end{aligned}$$

It suffices to show $[\bar{\mathbf{I}}_\ell(\mathbb{F}_\ell):f_{\bar v_1}(\mathbb{E}_\ell^*)\cdots f_{\bar v_k}(\mathbb{E}_\ell^*)]$ is bounded independent of ℓ. The proof is identical to Lemma 2.4.4(ii) since $f_{\bar v_1}(\mathbb{E}_\ell^*)\cdots f_{\bar v_k}(\mathbb{E}_\ell^*)$ is the image of

$$f_{\bar v_1}\times\cdots\times f_{\bar v_k}:(\mathbb{E}_\ell^*)^k\to \mathrm{GL}(W_\ell),$$

$\bar{\mathbf{I}}_\ell$ is the image of

$$w_{\bar v_1}\times\cdots\times w_{\bar v_k}:(\bar{\mathbf{E}}_\ell)^k\to \mathrm{GL}_{W_\ell},$$

k (depending on ℓ) is always less than $[K'':\mathbb{Q}]$, and the exponents of characters (ℓ-restricted 10) of $w_{\bar v_1}\times\cdots\times w_{\bar v_k}$ are uniformly bounded. Therefore, there exists $c_8=c_8(N)$ such that $[\bar{\mathbf{I}}_\ell(\mathbb{F}_\ell):\mu_\ell(\mathrm{Gal}_L)]\le c_8$ for $\ell\gg1$. $\square$

2.5. **Construction of $\bar{\mathbf{G}}_\ell$.** An $\mathbb{F}_\ell$-torus $\bar{\mathbf{I}}_\ell \subset \mathrm{GL}_{W_\ell}$ is constructed in §2.4 for $\ell \gg 1$ and we have the following map defined in Theorem 2.4.1

$$t_\ell : \bar{\mathbf{N}}_\ell \twoheadrightarrow \bar{\mathbf{N}}_\ell/\bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{W_\ell}.$$

One has to show that $\bar{\mathbf{I}}_\ell \subset t_\ell(\bar{\mathbf{N}}_\ell)$ so that $t_\ell^{-1}(\bar{\mathbf{I}}_\ell)$ is connected. It suffices to consider tame inertia tori $\bar{\mathbf{I}}_{\bar{v}}$. Recall vector space U_ℓ from Theorem 2.4.1.

Lemma 2.5.1. *Let $\bar{\mathbf{H}}_\ell$ be an algebraic subgroup of $\mathrm{GL}_{\bar{V}_\ell}$. Then $\bar{\mathbf{H}}_\ell$ acts on $\bar{U}_\ell$. If $\bar{\mathbf{H}}_\ell$ is invariant on the subspace*

$$\bar{W}_\ell \subset \bar{U}_\ell$$

fixed by $\bar{\mathbf{S}}_\ell$, then $\bar{\mathbf{H}}_\ell$ is contained in $\bar{\mathbf{N}}_\ell$.

Proof. Let $x \in \bar{\mathbf{H}}_\ell \backslash \bar{\mathbf{N}}_\ell$. Then there exists $s \in \bar{\mathbf{S}}_\ell$ such that $xsx^{-1} \notin \bar{\mathbf{S}}_\ell$. There exists $w \in \bar{W}_\ell$ such that

$$xsx^{-1}w \neq w$$

by the last statement of Theorem 2.4.1. Therefore,

$$sx^{-1}w \neq x^{-1}w$$

implies $x^{-1}w \notin \bar{W}_\ell$, a contradiction. Hence, $\bar{\mathbf{H}}_\ell$ is contained in $\bar{\mathbf{N}}_\ell$. $\square$

Proposition 2.5.2. *The $\mathbb{F}_\ell$-torus $\bar{\mathbf{I}}_\ell$ in GL_{W_ℓ} is a subgroup of the image of*

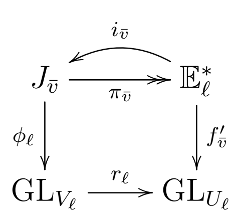
$$t_\ell : \bar{\mathbf{N}}_\ell \twoheadrightarrow \bar{\mathbf{N}}_\ell/\bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{W_\ell}$$

defined in Theorem 2.4.1.

Proof. Let $\bar{v}|\ell$ be a valuation of $\bar{K}$ and $I_{\bar{v}}$ the inertia subgroup of Gal_K at $\bar{v}$. The restriction $\phi_\ell : I_{\bar{v}} \to \mathrm{GL}(V_\ell)$ factors through a finite quotient $\pi_{\bar{v}} : I_{\bar{v}} \twoheadrightarrow J_{\bar{v}}$ such that $|J_{\bar{v}}| = \ell^k \cdot (\ell^{c_4!} - 1)$. Recall vector spaces $W_\ell \subset U_\ell$ from Theorem 2.4.1 and $f_{\bar{v}} : \mathbb{E}_\ell^* \to \mathrm{GL}(W_\ell)$ from Definition 10. Consider the following diagram so that

$$r_\ell \circ \phi_\ell \circ i_{\bar{v}} = f'_{\bar{v}}$$

and the actions of $\mathbb{E}_\ell^*$ on W_ℓ via $f'_{\bar{v}}$ and $f_{\bar{v}}$ are the same. Here r_ℓ is the obvious map and $i_{\bar{v}}$ is a splitting of $\pi_{\bar{v}}$. This is possible because $\mathbb{E}_\ell^*$ defined in §2.4 is cyclic of order $(\ell^{c_4!} - 1)$ prime to ℓ.



If ℓ is sufficiently large, then the exponents of the characters (ℓ-restricted) of representations $\phi_\ell \circ i_{\bar{v}}$ and $r_\ell \circ \phi_\ell \circ i_{\bar{v}}$ belong to $[0, i]$ and $[0, ic_3]$ respectively by Theorem 2.3.3 and the construction of U_ℓ. Recall $\bar{\mathbf{E}}_\ell$ from definition 10. By Weil restriction of scalars, we obtain two $\mathbb{F}_\ell$-morphisms

$$\alpha_\ell : \bar{\mathbf{E}}_\ell \to \mathrm{GL}_{V_\ell}$$
$$\beta_\ell : \bar{\mathbf{E}}_\ell \to \mathrm{GL}_{U_\ell}.$$

Since $r_\ell \circ \alpha_\ell$ and β_ℓ are both ℓ-restricted [Ha11, §3] and equal to $r_\ell \circ \phi_\ell \circ i_{\bar{v}}$ when restricting to $\mathbb{E}_\ell^*$, by uniqueness [Ha11, §3] we have

$$r_\ell \circ \alpha_\ell = \beta_\ell.$$

The image $r_\ell \circ \phi_\ell \circ i_{\bar{v}}(\mathbb{E}_\ell^*) = f'_{\bar{v}}(\mathbb{E}_\ell^*)$ maps W_ℓ and hence $\bar{W}_\ell$ to itself, so $\beta_\ell(\bar{\mathbf{E}}_\ell)$ also maps $\bar{W}_\ell$ to itself. Since $r_\ell \circ \alpha_\ell(\bar{\mathbf{E}}_\ell) = \beta_\ell(\bar{\mathbf{E}}_\ell)$, we conclude that $\alpha_\ell(\bar{\mathbf{E}}_\ell) \subset \bar{\mathbf{N}}_\ell$ by Lemma 2.5.1. One also observes that the following morphism

$$t_\ell : \bar{\mathbf{N}}_\ell \twoheadrightarrow \bar{\mathbf{N}}_\ell / \bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{W_\ell}$$

maps $\alpha_\ell(\bar{\mathbf{E}}_\ell)$ to $\bar{\mathbf{I}}_{\bar{v}} := w_{\bar{v}}(\bar{\mathbf{E}}_\ell)$. Therefore, tame inertia torus $\bar{\mathbf{I}}_{\bar{v}}$ and thus $\bar{\mathbf{I}}_\ell$ is a subgroup of $t_\ell(\bar{\mathbf{N}}_\ell)$. $\square$

Definition 11. Let L be the normal extension of K in Theorem 2.4.2. Denote $\phi_\ell(\mathrm{Gal}_L)$ by $\bar{\gamma}_\ell$ for all ℓ. Then $[\bar{\Gamma}_\ell : \bar{\gamma}_\ell] \leq [L : K]$ for all ℓ.

Proof of Theorem 2.0.5(i),(ii).

Proof. Since $\bar{\mathbf{S}}_\ell$ is a connected normal subgroup of $\bar{\mathbf{N}}_\ell$, $\bar{\mathbf{I}}_\ell$ is a torus, and t_ℓ is an $\mathbb{F}_\ell$-morphism, Proposition 2.5.2 implies $t_\ell^{-1}(\bar{\mathbf{I}}_\ell)$, the preimage of the $\mathbb{F}_\ell$-torus $\bar{\mathbf{I}}_\ell$ is a connected $\mathbb{F}_\ell$-reductive group $\bar{\mathbf{G}}_\ell$. Moreover, $\bar{\gamma}_\ell \subset \bar{\mathbf{G}}_\ell(\mathbb{F}_\ell)$ by construction of $\bar{\mathbf{G}}_\ell$ for $\ell \gg 1$. We obtain an exact sequences of $\mathbb{F}_\ell$ algebraic groups for $\ell \gg 1$

$$1 \to \bar{\mathbf{S}}_\ell \to \bar{\mathbf{G}}_\ell \to \bar{\mathbf{I}}_\ell \to 1.$$

and hence

$$1 \to \bar{\mathbf{S}}_\ell(\mathbb{F}_\ell) \to \bar{\mathbf{G}}_\ell(\mathbb{F}_\ell) \to \bar{\mathbf{I}}_\ell(\mathbb{F}_\ell).$$

Recall $\mu_\ell(\mathrm{Gal}_L) = t_\ell(\bar{\gamma}_\ell)$ from Theorem 2.4.2. Since the semisimple envelopes (Definition 6) of $\bar{\Gamma}_\ell$ and $\bar{\gamma}_\ell$ are identical for $\ell \gg 1$ by Remark 2.1.3, the above exact sequence implies

$$[\bar{\mathbf{G}}_\ell(\mathbb{F}_\ell) : \bar{\gamma}_\ell] \leq [\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell) : \bar{\gamma}_\ell \cap \bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)][\bar{\mathbf{I}}_\ell(\mathbb{F}_\ell) : \mu_\ell(\mathrm{Gal}_L)] \leq 2^{N-1} c_8$$

by Proposition 2.1.2(iii) and Theorem 2.4.2 for $\ell \gg 1$. Since the derived group of $\bar{\mathbf{G}}_\ell$ is $\bar{\mathbf{S}}_\ell$, the action of $\bar{\mathbf{G}}_\ell$ on the ambient space is semisimple if $\ell \gg 1$ by Proposition 2.1.2(ii). Therefore, we have proved Theorem 2.0.5 (i) and (ii). $\square$

Proof of Theorem 2.0.5(iii).

Proof. Let $\bar{\mathbf{S}}_\ell^{\mathrm{sc}} \to \bar{\mathbf{S}}_\ell$ be the simply connected cover of $\bar{\mathbf{S}}_\ell$. The representation $(\bar{\mathbf{S}}_\ell^{\mathrm{sc}} \to \bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}) \times \bar{\mathbb{F}}_\ell$ is semisimple and has a $\mathbb{Z}$-form which belongs to a finite set of $\mathbb{Z}$-representations of simply-connected Chevalley schemes [EHK12, Theorem 24] if $\ell \gg 1$. Thus, $\{\bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}\}_{\ell \gg 1}$ have bounded formal characters (Definition 4'). Let $\bar{\mathbf{C}}_\ell$ be the center of $\bar{\mathbf{G}}_\ell$. Since $\bar{\mathbf{S}}_\ell$ acts semi-simply on $\bar{V}_\ell$ by Proposition 2.1.2(ii) for $\ell \gg 1$, we decompose the representation $\bar{\mathbf{S}}_\ell \to \mathrm{GL}(\bar{V}_\ell)$ into a sum of absolutely irreducible representations $\bar{M}_i$

$$\bar{V}_\ell = (\bigoplus_1^{m_1} \bar{M}_1) \oplus (\bigoplus_1^{m_2} \bar{M}_2) \oplus \cdots \oplus (\bigoplus_1^{m_k} \bar{M}_k)$$

such that $\bar{M}_i \not\cong \bar{M}_j$ if $i \neq j$. If $c \in \bar{\mathbf{C}}_\ell$, then $\bar{M}_i$ and $c(\bar{M}_i)$ are isomorphic representations of $\bar{\mathbf{S}}_\ell$ for all i. Hence, c is invariant on $\oplus_1^{m_i} \bar{M}_i$ and $\oplus_1^{m_i} \bar{M}_i$ is a subrepresentation of $\bar{\mathbf{G}}_\ell$ on $\bar{V}_\ell$ for all i. Let n_i be the dimension of $\bar{M}_i$. Denote the representation of $\bar{\mathbf{S}}_\ell$ on $\bar{M}_i$ under some coordinates by

$$u_i : \bar{\mathbf{S}}_\ell \to \mathrm{GL}_{n_i}(\bar{\mathbb{F}}_\ell).$$

Then, the representation of $\bar{\mathbf{G}}_\ell$ on $\bigoplus_1^{m_i} \bar{M}_i$ is given by:

$$q_i : \bar{\mathbf{G}}_\ell \to \mathrm{GL}_{n_i m_i}(\bar{\mathbb{F}}_\ell)$$

so that when restricting to $\bar{\mathbf{S}}_\ell$, the action is "diagonal"

$$q_i : \bar{\mathbf{S}}_\ell \stackrel{u_i}{\to} \mathrm{GL}_{n_i}(\bar{\mathbb{F}}_\ell) \to \bigoplus_1^{m_i} \mathrm{GL}_{n_i}(\bar{\mathbb{F}}_\ell) \subset \mathrm{GL}_{n_i m_i}(\bar{\mathbb{F}}_\ell)$$

$$x \mapsto u_i(x) \mapsto (u_i(x), ..., u_i(x)).$$

Since u_i is a irreducible representation and $q_i(c)$ commutes with $q_i(\bar{\mathbf{S}}_\ell)$, $q_i(c)$ is contained in the subgroup

$$\bar{\mathbf{H}}_i = \begin{pmatrix} \bar{\mathbf{D}}_{11} & \bar{\mathbf{D}}_{12} & ... & \bar{\mathbf{D}}_{1m_i} \\ \bar{\mathbf{D}}_{21} & \bar{\mathbf{D}}_{22} & ... & \bar{\mathbf{D}}_{2m_i} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\mathbf{D}}_{m_i 1} & \bar{\mathbf{D}}_{m_i 2} & ... & \bar{\mathbf{D}}_{m_i m_i} \end{pmatrix},$$

where $\bar{\mathbf{D}}_{jk}$ is the subgroup of scalars of $\mathrm{GL}_{n_i}(\bar{\mathbb{F}}_\ell)$ for all $1 \leq j \leq m_i$, $1 \leq k \leq m_i$. We see that $\bar{\mathbf{H}}_i$ is isomorphic to $\mathrm{GL}_{m_i}(\bar{\mathbb{F}}_\ell)$. Since $q_i(\bar{\mathbf{C}}_\ell)$ is a diagonalizable group which commutes with $q_i(\bar{\mathbf{S}}_\ell)$ and $q_i|_{\bar{\mathbf{S}}_\ell}$ is "diagonal", we may assume $q_i(\bar{\mathbf{C}}_\ell)$ is contained in the following torus

$\bar{\mathbf{D}}_i$ for all i

$$\bar{\mathbf{D}}_i = \begin{pmatrix} \bar{\mathbf{D}}_{11} & 0 & ... & 0 \\ 0 & \bar{\mathbf{D}}_{22} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \bar{\mathbf{D}}_{m_i m_i} \end{pmatrix}$$

after a change of coordinates by some element in $\bar{\mathbf{H}}_i \cong \mathrm{GL}_{m_i}(\bar{\mathbb{F}}_\ell)$. Therefore, we may assume that $\bar{\mathbf{C}}_\ell$ is a subgroup of

$$\bar{\mathbf{B}}_\ell := \bar{\mathbf{D}}_1 \times \bar{\mathbf{D}}_2 \times \cdots \times \bar{\mathbf{D}}_k \subset \mathrm{GL}_N(\bar{\mathbb{F}}_\ell).$$

in suitable coordinates. Torus $\bar{\mathbf{B}}_\ell$ centralizes $\bar{\mathbf{S}}_\ell$ implies $\bar{\mathbf{B}}_\ell \subset \bar{\mathbf{N}}_\ell$. Denote the restriction $t_\ell|_{\bar{\mathbf{B}}_\ell}$ by s_ℓ. Since $\bar{\mathbf{N}}_\ell$ acts on $\bar{W}_\ell$, we have

$$s_\ell : \bar{\mathbf{B}}_\ell \to \mathrm{GL}_{W_\ell}.$$

We obtain $(s_\ell^{-1}(\bar{\mathbf{I}}_\ell))^\circ = \bar{\mathbf{C}}_\ell^\circ$ because $\mathrm{Ker}(s_\ell)$ is discrete. Consider the construction of U_ℓ from Theorem 2.4.1. This implies the exponents of characters of s_ℓ on $\bar{\mathbf{D}}_i \cong \prod_1^{m_i} \bar{\mathbb{F}}_\ell^*$ are between 0 and c_3 for all i. By Theorem 2.4.2(i) and above, the diagonalizable groups $\{s_\ell^{-1}(\bar{\mathbf{I}}_\ell)\}_{\ell\gg1}$ satisfies the bounded exponents condition in Definition 4'. Hence, $\{\bar{\mathbf{C}}_\ell^\circ = (s_\ell^{-1}(\bar{\mathbf{I}}_\ell))^\circ \hookrightarrow \bar{\mathbf{B}}_\ell \hookrightarrow \mathrm{GL}_{V_\ell}\}_{\ell\gg1}$ have bounded formal characters. Since $\{\bar{\mathbf{C}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}\}_{\ell\gg1}$ and $\{\bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}\}_{\ell\gg1}$ both have bounded formal characters and $\bar{\mathbf{C}}_\ell^\circ$ commutes with $\bar{\mathbf{S}}_\ell$ for $\ell \gg 1$, $\{\bar{\mathbf{G}}_\ell = \bar{\mathbf{C}}_\ell^\circ \cdot \bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}\}_{\ell\gg1}$ have bounded formal characters by Proposition 2.0.4. This prove Theorem 2.0.5(iii). □

3. ℓ-INDEPENDENCE OF $\bar{\Gamma}_\ell$

3.1. **Formal character of $\bar{\mathbf{G}}_\ell \subset \mathrm{GL}_{N,\mathbb{F}_\ell}$.** A system of algebraic envelopes $\{\bar{\mathbf{G}}_\ell\}_{\ell\gg1}$ of $\{\bar{\Gamma}_\ell\}_{\ell\gg1}$ (Definition 5) are constructed in §2.5. Let $\mathbf{G}_\ell$ be the algebraic monodromy group of Φ_ℓ^{ss} for all ℓ. The compatibility (Definition 2) of the system $\{\phi_\ell\}$ implies that the formal characters of $\{\bar{\mathbf{G}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}\}_{\ell\gg1} \cup \{\mathbf{G}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}_\ell}\}_{\ell\gg1}$ are the same in the sense of Definition 3':

Theorem 3.1.1. *Let $\{\bar{\mathbf{G}}_\ell\}_{\ell\gg1}$ be a system of algebraic envelopes of $\{\bar{\Gamma}_\ell\}_{\ell\gg1}$ (Definition 5).*

- (i) *The formal characters of $\bar{\mathbf{G}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}$ and $\mathbf{G}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}_\ell}$ are the same for $\ell \gg 1$.*
- (ii) *The formal characters of $\{\bar{\mathbf{G}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}\}_{\ell\gg1}$ are the same.*

Proof. The mod ℓ system $\{\phi_\ell : \mathrm{Gal}_K \to \mathrm{GL}_N(\mathbb{F}_\ell)\}$ comes from the ℓ-adic system $\{\Phi_\ell^{\mathrm{ss}} : \mathrm{Gal}_K \to \mathrm{GL}_N(\mathbb{Q}_\ell)\}$ (Definition 1). The algebraic monodromy group $\mathbf{G}_\ell$ is reductive for all ℓ. By taking a finite extension K^{conn} of K [Se81], we may assume $\mathbf{G}_\ell$ is connected for all ℓ. This does not change the formal character of $\mathbf{G}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}_\ell}$. It is well known that these algebraic monodromy groups have same reductive rank r. Define

$$Char\colon \mathrm{GL}_N \to \mathbb{G}_a^{N-1}\times\mathbb{G}_m$$

that maps a matrix to the coefficients of its characteristic polynomial. We know that $Char(\mathbf{G}_\ell)$ is a $\mathbb{Q}$-variety of dimension r that is independent of ℓ (by the compatibility conditions) and can be defined over $\mathbb{Z}[\frac{1}{N'}]$ for some positive integer N' that is sufficiently divisible. Let $\mathbf{P}_{\mathbb{Z}[\frac{1}{N'}]}$ be the Zariski closure of $Char(\mathbf{G}_\ell)$ in the projective $\mathbb{P}^N_{\mathbb{Z}[\frac{1}{N'}]}$. Since ϕ_ℓ is continuous, every element of $\bar{\Gamma}_\ell$ is the image of a Frobenius element. Therefore, $Char(\bar{\Gamma}_\ell)$ is a subset of the $\mathbb{F}_\ell$-rational points of $\mathbf{P}_{\mathbb{F}_\ell} := \mathbf{P}_{\mathbb{Z}[\frac{1}{N'}]}\times_{\mathbb{Z}}\mathbb{F}_\ell$ for $\ell \gg 1$.

Generic flatness [DG65, Theorem 6.9.1] implies $\mathbf{P}_{\mathbb{Z}[\frac{1}{N'}]}$ is flat over $\mathbb{Z}[\frac{1}{N'}]$ for sufficiently divisible N', so the dimension of every irreducible component of $\mathbf{P}_{\mathbb{Z}[\frac{1}{N'}]}$ is $r+1$ [Ha77, Chapter 3 Proposition 9.5] and hence the dimension of every irreducible component of $\mathbf{P}_{\mathbb{F}_\ell}$ is r [Ha77, Chapter 3 Corollary 9.6] for $\ell \gg 1$. Also, the Hilbert polynomial of $\mathbf{P}_{\mathbb{F}_\ell}$ and in particular the *degree* (let it be d) of $\mathbf{P}_{\mathbb{F}_\ell}\subset\mathbb{P}^N_{\mathbb{F}_\ell}$ is independent of ℓ for $\ell\gg 1$ [Ha77, Chapter 3 Theorem 9.9]. Since d is a positive integer, we conclude that the number and degrees of irreducible components of $\mathbf{P}_{\mathbb{F}_\ell}$ are bounded by d [Ha77, Chapter 1 Proposition 7.6(a),(b)]. By [LW54, Theorem 1] and above, we have

$$|\mathbf{P}_{\mathbb{F}_\ell}(\mathbb{F}_\ell)| \le 3d\cdot \ell^r$$

for $\ell\gg 1$. Let $\bar{\mathbf{T}}_\ell$ be a $\mathbb{F}_\ell$-maximal torus of $\bar{\mathbf{G}}_\ell$. [No87, Lemma 3.5] implies $\bar{\mathbf{T}}_\ell$ has at least $(\ell-1)^{\dim(\bar{\mathbf{T}}_\ell)}$ $\mathbb{F}_\ell$-rational points. By Theorem 2.0.5 (i), there is an integer $n>0$ such that the nth power of $\bar{\mathbf{T}}_\ell(\mathbb{F}_\ell)$ is contained in $\bar{\gamma}_\ell$ for $\ell\gg 1$. One sees by diagonalizing $\bar{\mathbf{T}}_\ell$ in $\mathrm{GL}_{N,\bar{F}_\ell}$ that the order of the kernel of this nth power homomorphism is less than or equal to n^N. Hence, we obtain

$$|\bar{\mathbf{T}}_\ell(\mathbb{F}_\ell)\cap\bar{\gamma}_\ell| \ge \frac{(\ell-1)^{\dim(\bar{\mathbf{T}}_\ell)}}{n^N}.$$

Also, morphism *Char* restricted to any maximal torus of GL_N is finite morphism of degree $N!$. Therefore, there is a constant $c>0$ such that

$$c\cdot\ell^{\dim(\bar{\mathbf{T}}_\ell)} \le |Char(\bar{\mathbf{T}}_\ell(\mathbb{F}_\ell)\cap\bar{\gamma}_\ell)| \le |Char(\bar{\gamma}_\ell)| \le |\mathbf{P}_{\mathbb{F}_\ell}(\mathbb{F}_\ell)| \le 3d\cdot\ell^r$$

for $\ell\gg 1$. This implies $\dim(\bar{\mathbf{T}}_\ell)\le r$ for $\ell\gg 1$.

On the other hand, we find for each $\ell\gg 1$ a set R_ℓ of characters of $\mathbb{G}_m^N$ of exponents bounded by $C>0$ such that $\bar{\mathbf{T}}_\ell$ is conjugate in $\mathrm{GL}_{N,\bar{F}_\ell}$ to the kernel of R_ℓ by Theorem 2.0.5(iii) and Definition 4'. Let $\mathscr{L}$ be an infinite subset of prime numbers $\mathscr{P}$ such that for all $\ell,\ell'\in\mathscr{L}$, we have equality $R_\ell = R_{\ell'}$. Denote this common set of characters by R and define $\mathbf{Y}_\mathbb{C} = \{y \in \mathbb{G}_{m,\mathbb{C}}^N : \chi(y) = 1 \ \ \forall \chi \in R\}$ so that $\dim_\mathbb{C}\mathbf{Y}_\mathbb{C} = \dim_{\bar{\mathbb{F}}_\ell}\bar{\mathbf{T}}_\ell$ for all $\ell \in \mathscr{L}$. If $\bar{v}$ divides $v \in \Sigma_K\backslash S_\ell$ (S_ℓ in Definition 2), then the characteristic polynomial of $\phi_\ell(\mathrm{Frob}_{\bar{v}})$ is just the mod ℓ reduction of the characteristic polynomial of $\Phi_\ell^{\mathrm{ss}}(\mathrm{Frob}_{\bar{v}}) = P_v(x) \in \mathbb{Q}[x]$ which depends only on v (Definition 2). Therefore, for each $v \notin S$ (Definition 2), we can put the roots of $P_v(x)$ in some order $\alpha_1, \alpha_2, ..., \alpha_N$ such that the following congruence equation holds:

$$\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_N^{m_N} \equiv 1 \qquad \pmod{\ell'}$$

for any character $x_1^{m_1}x_2^{m_2}\cdots x_N^{m_N} \in R$ and

$$\ell' \in \mathscr{L}_v := \mathscr{L}\backslash\{\ell'' \in \mathscr{P} : \exists v' \in S_\ell \text{ s.t. } v'|\ell''\}$$

if $v|\ell$. Since $\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_N^{m_N}$ is an algebraic number and $\mathscr{L}_v$ consists of infinitely many primes, we obtain equality

$$\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_N^{m_N} = 1$$

for any character $x_1^{m_1}x_2^{m_2}\cdots x_N^{m_N} \in R$. Therefore,

$$(\mathit{Char}|_{\mathbb{G}_m^N})^{-1}(\{P_v(x) : v \in \Sigma_K\backslash S\}) \subset \bigcup_{g\in\mathrm{Perm}(N)} g(\mathbf{Y}_\mathbb{C}),$$

where $\mathrm{Perm}(N)$ is the group of permutations of N letters permuting the coordinates. Since $\{P_v(x) : v \in \Sigma_K\backslash S\}$ is Zariski dense in $\mathit{Char}(\mathbf{G}_\ell)$ of dimension r and $\mathit{Char}|_{\mathbb{G}_m^N}$ is a finite morphism of degree $N!$, the Zariski closure of $(\mathit{Char}|_{\mathbb{G}_m^N})^{-1}(\{P_v(x) : v \in \Sigma_K\backslash S\})$ in $\mathbb{G}_{m,\mathbb{C}}^N$ denoted by $\mathbf{D}_\mathbb{C}$ is also of dimension r. Since we have obtained $\dim(\bar{\mathbf{T}}_\ell) \le r$ at the end of the second paragraph and any maximal torus of the algebraic monodromy group $\mathbf{G}_\ell$ is conjugate in $\mathrm{GL}_{N,\mathbb{C}}$ to an irreducible component of $\mathbf{D}_\mathbb{C}$ [Se81], the inclusion

$$\mathbf{D}_\mathbb{C} \subset \bigcup_{g\in\mathrm{Perm}(N)} g(\mathbf{Y}_\mathbb{C})$$

implies the formal characters of $\bar{\mathbf{G}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}$ and $\mathbf{G}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}_\ell}$ are the same in the sense of Definition 3' for all $\ell \in \mathscr{L}$. There are only finitely many possibilities for R_ℓ by Remark 2.0.2 and Proposition 2.0.3. By excluding the primes ℓ such that R_ℓ appears finitely many times, we conclude that the formal characters of $\bar{\mathbf{G}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}$ and $\mathbf{G}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}_\ell}$ are the same for $\ell \gg 1$. This proves (i) and hence (ii) since formal character of $\mathbf{G}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}_\ell}$ is independent of ℓ [Se81]. $\square$

3.2. **Formal character of $\bar{\mathbf{S}}_\ell \subset \mathrm{GL}_{N,\mathbb{F}_\ell}$.** We make the following assumptions for this subsection.

Assumptions: By taking a field extension of K, we may assume

(i) $\mathbf{G}_\ell$, the algebraic monodromy group of Φ_ℓ^{ss} is connected for all ℓ (see [Se81]),

(ii) $\bar{\Omega}_\ell := \mu_\ell(\bar{\Gamma}_\ell)$ corresponds to an abelian extension of K that is unramified at all primes not dividing ℓ for all ℓ (see the first paragraph of the proof of Theorem 2.4.2).

Theorem 3.2.1 below is the main result in this subsection. Denote a finite extension of K by K'. Since $\bar{\mathbf{S}}_\ell$ is independent of K' over K for $\ell \gg 1$ by Remark 2.1.3, the assumptions above remain valid for K', and $\{\bar{\mathbf{G}}_\ell\}_{\ell\gg 1}$ constructed in §2.5 are still algebraic envelopes of $\{\phi_\ell(\mathrm{Gal}_{K'})\}_{\ell\gg 1}$, we are free to replace K by K' in this subsection.

Theorem 3.2.1. *Let $\bar{\mathbf{S}}_\ell \subset \mathrm{GL}_{N,\mathbb{F}_\ell}$ be the semisimple envelope of $\bar{\Gamma}_\ell$ (Definition 6) for all $\ell \gg 1$.*

(i) *The formal character of $\bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}$ is equal to the formal character of $\mathbf{G}_\ell^{\mathrm{der}} \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}_\ell}$ for $\ell \gg 1$, where $\mathbf{G}_\ell^{\mathrm{der}}$ is the derived group of the algebraic monodromy group $\mathbf{G}_\ell$ of Φ_ℓ^{ss}.*

(ii) *The formal character of $\bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}$ is independent of ℓ if $\ell \gg 1$.*

In [Hu13, §3], we used mainly abelian ℓ-adic representations to prove that the formal character of $\mathbf{G}_\ell^{\mathrm{der}} \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}_\ell}$ is independent of ℓ. To prove Theorem 3.2.1, we adopt this strategy in a mod ℓ fashion. The key point is to prove that the inertia characters of μ_ℓ (Definition 9) for $\ell \gg 1$ are in some sense the mod ℓ reduction of inertia characters of some Serre group $\mathbf{S}_{\mathfrak{m}}$ [Se98, Chapter 2] (Proposition 3.2.4).

Definition 12. For each prime $\ell \in \mathscr{P}$, choose a valuation $\bar{v}_\ell$ of $\bar{\mathbb{Q}}$ that extends the ℓ-adic valuation of $\mathbb{Q}$. This valuation on $\bar{\mathbb{Q}}$ is equal to the restriction of the unique non-Archimedean valuation on $\bar{\mathbb{Q}}_\ell$ (extending the ℓ-adic valuation on $\mathbb{Q}_\ell$) to $\bar{\mathbb{Q}}$ with respect to some embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$. Denote also this valuation on $\bar{\mathbb{Q}}_\ell$ by $\bar{v}_\ell$. Define the following notation.

- $\mathrm{Gal}_K^{\mathrm{ab}}$: the Galois group of the maximal abelian extension of K,
- I_K: the group of idéles of K,
- $(x_v)_{v\in\Sigma_K}$: a representation of a finite idéle,
- K_v: the completion of K with respect to $v \in \Sigma_K$,
- U_v: the unit group of K_v^*,
- k_v: the residue field of K_v,
- $\mathfrak{m}_0$: the modulus of empty support,

- $U_{\mathfrak{m}_0} := \prod_v U_v$,
- $K_\ell := \prod_{v|\ell} K_v = K \otimes \mathbb{Q}_\ell$,
- $\bar{\mathbb{Z}}_\ell$: the valuation ring of $\bar{v}_\ell$,
- $\mathfrak{p}_\ell$: the maximal ideal of $\bar{v}_\ell$,
- k_ℓ: the residue field of $\bar{v}_\ell$,
- $x_\ell := (x_v)_{v|\ell}$.

Let $\sigma : K \to \bar{\mathbb{Q}}$ be an embedding of K in $\bar{\mathbb{Q}}$. The composition of σ with $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$ extends to a $\mathbb{Q}_\ell$-algebra homomorphism $\sigma_\ell : K_\ell \to \bar{\mathbb{Q}}_\ell$.

Remark 3.2.2. The field k_ℓ is an algebraic closure of $\mathbb{F}_\ell$ and homomorphism σ_ℓ is trivial on the components K_v of K_ℓ when v is not equivalent to $\bar{v}_\ell \circ \sigma$.

Recall representation $\mu_\ell : \mathrm{Gal}_K \to \mathrm{GL}(W_\ell)$ (abelian by Assumption (ii)) from Definition 9. Thus, μ_ℓ induces ρ_ℓ below for each ℓ by composing with $I_K \to \mathrm{Gal}_K^{\mathrm{ab}}$:

$$\rho_\ell : I_K \to \mathrm{GL}(W_\ell).$$

Proposition 3.2.3. *If $\chi_\ell : I_K \to \bar{\mathbb{F}}_\ell^*$ is a character of ρ_ℓ for $\ell \gg 1$, then for all finite idéle $x \in U_{\mathfrak{m}_0}$ we have the congruence*

$$\chi_\ell(x) \equiv \prod_{\sigma \in \mathrm{Hom}(K,\bar{\mathbb{Q}})} \sigma_\ell(x_\ell^{-1})^{m(\sigma,\ell)} \qquad (\mathrm{mod}\ \mathfrak{p}_\ell)$$

such that $0 \leq m(\sigma, \ell) \leq c_6$.

Proof. Since $|\bar{\Omega}_\ell|$ is prime to ℓ, the following homomorphism

$$U_v \hookrightarrow K_v^* \to I_K \xrightarrow{\rho_\ell} \mathrm{GL}(W_\ell)$$

factors through $\alpha_v : k_v^* \to \mathrm{GL}(W_\ell)$ for all $v|\ell$. On the other hand, let $\bar{v} \in \Sigma_{\bar{K}}$ divide ℓ. Since $\bar{\Omega}_\ell$ is abelian and of order prime to ℓ, the restriction of $\mu_\ell : \mathrm{Gal}_K \to \mathrm{GL}(W_\ell)$ to $I_{\bar{v}}$ factors through

$$I_{\bar{v}} \to I_{\bar{v}}^{\mathrm{t}} \xrightarrow{\cong} \varprojlim \mathbb{F}_{\ell^k}^* \to k_v^*$$

and induces $\beta_v : k_v^* \to \mathrm{GL}(W_\ell)$ that depends on $v = \bar{v}|_{\bar{K}}$. By [Se72, Proposition 3], α_v and β_v are inverse of each other. Since $f_{\bar{v}}$ (Definition 10) factors through β_v and the exponents of any character of $f_{\bar{v}}$ when expressed as a ℓ-restricted (Definition 8) product of fundamental characters of level $c_4!$ are bounded by c_6 for $\ell \gg 1$ (§2.4), the exponents of χ_ℓ when expressed as a ℓ-restricted product of fundamental characters of level $[k_v : \mathbb{F}_\ell]$ are also bounded by c_6 for $\ell \gg 1$. Since ρ_ℓ is unramified at all v not dividing ℓ, ρ_ℓ is trivial on subgroup $\prod_{v\nmid\ell} U_v$ of $U_{\mathfrak{m}_0} := \prod_v U_v$. Therefore, we conclude the congruence for $\ell \gg 1$. ☐

Definition 13. Let $\mathbf{S}_{\mathfrak{m}}$ be the Serre group of K with modulus $\mathfrak{m}$ [Se98, Chapter 2] and $\Theta : \mathbf{S}_{\mathfrak{m}} \to \mathbb{G}_{m,\bar{\mathbb{Q}}_\ell}$ a character of $\mathbf{S}_{\mathfrak{m}}$ over $\bar{\mathbb{Q}}_\ell$. Since the image of the abelian representation Θ_ℓ attached to Θ [Se98, Chapter 2]

$$\Theta_\ell : \mathrm{Gal}_K^{\mathrm{ab}} \to \mathbf{S}_{\mathfrak{m}}(\mathbb{Q}_\ell) \stackrel{\Theta}{\to} \bar{\mathbb{Q}}_\ell^*$$

is contained in $\bar{\mathbb{Z}}_\ell^*$, define

$$\theta_\ell : I_K \to k_\ell^* \cong \bar{\mathbb{F}}_\ell^*$$

as the mod $\mathfrak{p}_\ell$ reduction of the composition of $I_K \to \mathrm{Gal}_K^{\mathrm{ab}}$ with Θ_ℓ.

Proposition 3.2.4. *Let χ_ℓ be a character of ρ_ℓ as above. If ℓ is sufficiently large, then there is a character Θ of $\mathbf{S}_{\mathfrak{m}_0}$ such that*

$$\chi_\ell(x) = \theta_\ell(x)$$

for all $x \in U_{\mathfrak{m}_0}$, where θ_ℓ is defined in Definition 13.

Proof. Since $0 \leq m(\sigma,\ell) \leq c_6$ for all $\sigma \in \mathrm{Hom}(K,\bar{\mathbb{Q}})$ and $\ell \gg 1$ by Proposition 3.2.3, the proposition follows by the proof of [Se72, Proposition 20]. □

Let $\Psi : \mathbf{S}_{\mathfrak{m}_0} \to \mathrm{GL}_{n,\mathbb{Q}}$ be a $\mathbb{Q}$-morphism of the Serre group $\mathbf{S}_{\mathfrak{m}_0}$ with finite kernel. Then Ψ induces a strictly compatible system $\{\Psi_\ell\}_{\ell\in\mathscr{P}}$ of abelian ℓ-adic representations of Gal_K [Se98, Chapter 2] with $S = \emptyset$ (Definition 2):

$$\Psi_\ell : \mathrm{Gal}_K \to \mathrm{Gal}_K^{\mathrm{ab}} \to \mathrm{GL}_n(\mathbb{Q}_\ell).$$

We may assume $\{\Psi_\ell\}$ is integral [Se98, Chapter 2 §3.4] by twisting $\{\Psi_\ell\}$ with suitable big power of the system of cyclotomic characters.

Proposition 3.2.5. *Given Ψ and $\{\Psi_\ell\}_{\ell\in\mathscr{P}}$ as above.*

(i) *The subgroup generated by the characters of Ψ is of finite index in the character group of $\mathbf{S}_{\mathfrak{m}_0}$. Denote this index by k.*

(ii) *For any ℓ and character θ_ℓ of I_K induced from a character Θ of $\mathbf{S}_{\mathfrak{m}_0}$ in Definition 13, we obtain the following congruence for all $x \in U_{\mathfrak{m}_0} \subset I_K$*

$$\theta_\ell(x) \equiv \prod_{\sigma\in\mathrm{Hom}(K,\bar{\mathbb{Q}})} \sigma_\ell(x_\ell^{-1})^{m(\sigma)} \pmod{\mathfrak{p}_\ell}.$$

such that $m(\sigma) \geq 0$ for all σ.

Proof. Part (i) follows by Ψ is an isogeny from $\mathbf{S}_{\mathfrak{m}_0}$ onto $\Psi(\mathbf{S}_{\mathfrak{m}_0})$. Part (ii) follows by the integrality of the system $\{\Psi_\ell\}$ and the theory of abelian ℓ-adic representations [Se98, Chapter 2,3]. □

Denote the semi-simplification of some mod ℓ reduction of Ψ_ℓ by ψ_ℓ for all ℓ. Consider the following strictly compatible system of ℓ-adic representations

$$\{\Phi_\ell \times \Psi_\ell : \mathrm{Gal}_K \to \mathrm{GL}_N(\mathbb{Q}_\ell) \times \mathrm{GL}_n(\mathbb{Q}_\ell)\}_{\ell\in\mathscr{P}}.$$

The semi-simplification of some mod ℓ reduction of $\{\Phi_\ell \times \Psi_\ell\}_{\ell\in\mathscr{P}}$:

$$\{\phi_\ell \times \psi_\ell : \mathrm{Gal}_K \to \mathrm{GL}_N(\mathbb{F}_\ell) \times \mathrm{GL}_n(\mathbb{F}_\ell)\}_{\ell\in\mathscr{P}}$$

is then a strictly compatible system of mod ℓ representations (Definition 2). Denote the image of $\phi_\ell \times \psi_\ell$ by $\bar{\Gamma}'_\ell$. Let $\bar{v} \in \Sigma_{\bar{K}}$ divide ℓ. When we restrict $\phi_\ell \times \psi_\ell$ to inertia subgroup $I_{\bar{v}}$ of Gal_K and then semi-simplify, the exponents of characters of tame inertia quotient $I_{\bar{v}}^{\mathrm{t}}$ for some level are bounded independent of ℓ by §2.3, Proposition 3.2.5(ii), and [Se72, Proposition 3]. Therefore, we can construct as in §2 semisimple envelopes $\{\bar{\mathbf{S}}'_\ell\}_{\ell\gg1}$ (Definition 6), inertia tori $\{\bar{\mathbf{I}}'_\ell\}_{\ell\gg1}$ (Theorem 2.4.2), and algebraic envelopes $\{\bar{\mathbf{G}}'_\ell\}_{\ell\gg1}$ (Definition 5) of $\{\bar{\Gamma}'_\ell\}_{\ell\gg1}$.

Since ψ_ℓ is semisimple and abelian, we see that Nori's construction gives $\bar{\mathbf{S}}'_\ell = \bar{\mathbf{S}}_\ell \times \{1\} \subset \mathrm{GL}_{N,\mathbb{F}_\ell} \times \mathrm{GL}_{n,\mathbb{F}_\ell}$. The normalizer of $\bar{\mathbf{S}}_\ell \times \{1\}$ in $\mathrm{GL}_{N,\mathbb{F}_\ell} \times \mathrm{GL}_{n,\mathbb{F}_\ell}$ is $\bar{\mathbf{N}}_\ell \times \mathrm{GL}_{n,\mathbb{F}_\ell}$. We have

$$t_\ell \times \mathrm{id} : \bar{\mathbf{N}}_\ell \times \mathrm{GL}_{n,\mathbb{F}_\ell} \to \mathrm{GL}_{W_\ell} \times \mathrm{GL}_{n,\mathbb{F}_\ell}$$

with kernel $\bar{\mathbf{S}}_\ell \times \{1\}$. Therefore, we obtain a map

$$\mu_\ell \times \psi_\ell : \mathrm{Gal}_K^{\mathrm{ab}} \to \mathrm{GL}(W_\ell) \times \mathrm{GL}_n(\mathbb{F}_\ell)$$

with image denoted by $\bar{\Omega}'_\ell$. As $\bar{\Omega}'_\ell$ is abelian, denote the composition of μ_ℓ and ψ_ℓ with $I_K \to \mathrm{Gal}_K^{\mathrm{ab}}$ by $\widetilde{\mu}_\ell$ and $\widetilde{\psi}_\ell$ for all ℓ. By $(*)$ in the proof of Theorem 2.4.2 and [Ne99, Proposition 9.5], we assume by taking a finite extension of K that

$$(**): \ (\widetilde{\mu}_\ell \times \widetilde{\psi}_\ell)(\prod_{v|\ell} U_v) = \bar{\Omega}'_\ell \quad \forall \ell \gg 1.$$

Proposition 3.2.6. *Let $p_2 : \mathrm{GL}_{W_\ell} \times \mathrm{GL}_{n,\mathbb{F}_\ell}$ be the projection to the second factor. Then p_2 is an isogeny from $\bar{\mathbf{I}}'_\ell$ onto $p_2(\bar{\mathbf{I}}'_\ell)$ for $\ell \gg 1$.*

Proof. Let $(x,1) \in \mathrm{GL}_{W_\ell} \times \mathrm{GL}_{n,\mathbb{F}_\ell}$ be an element of $\bar{\Omega}'_\ell \cap \mathrm{Ker}(p_2)$, where $(x,1) = (\widetilde{\mu}_\ell \times \widetilde{\psi}_\ell)(x_\ell)$ for some $x_\ell \in \prod_{v|\ell} U_v$ (Definition 12) by $(**)$ above. Since $\Psi : \mathbf{S}_{\mathfrak{m}_0} \to \mathrm{GL}_{n,\mathbb{Q}}$ has finite kernel and $\widetilde{\mu}_\ell \times \widetilde{\psi}_\ell$ is abelian and semisimple, we have $x^k = 1$ for $\ell \gg 1$ by $1 = \widetilde{\psi}_\ell(x_\ell)$, Proposition 3.2.4, and Proposition 3.2.5(i). Since $\bar{\Omega}'_\ell$ is abelian of order prime to ℓ, $x^k = 1$ implies x has at most $k^{\dim(W_\ell)}$ possibilities (by diagonalizing the image of $\widetilde{\mu}_\ell$) which implies

$$|\bar{\Omega}'_\ell \cap \mathrm{Ker}(p_2)| \le k^{\dim(W_\ell)}.$$

Therefore, the $\mathbb{F}_\ell$-diagonalizable group $\mathrm{Ker}(p_2)\cap\bar{\mathbf{I}}'_\ell$ cannot have positive dimension for $\ell \gg 1$ because $[\bar{\mathbf{I}}'_\ell(\mathbb{F}_\ell) : \bar{\Omega}'_\ell \cap \bar{\mathbf{I}}'_\ell(\mathbb{F}_\ell)]$ is also uniformly bounded by Theorem 2.4.2(ii). Thus, p_2 is an isogeny from $\bar{\mathbf{I}}'_\ell$ onto $p_2(\bar{\mathbf{I}}'_\ell)$. □

Proof of Theorem 3.2.1.

Proof. The mod ℓ system

$$\{\phi_\ell \times \psi_\ell : \mathrm{Gal}_K \to \mathrm{GL}_N(\mathbb{F}_\ell) \times \mathrm{GL}_n(\mathbb{F}_\ell)\}$$

comes from the ℓ-adic system (i.e., the semi-simplification of a mod ℓ reduction)

$$\{\Phi_\ell^{\mathrm{ss}} \times \Psi_\ell : \mathrm{Gal}_K \to \mathrm{GL}_N(\mathbb{Q}_\ell) \times \mathrm{GL}_n(\mathbb{Q}_\ell)\}.$$

Let $\mathbf{G}'_\ell$ be the algebraic monodromy group of semisimple representation $\Phi_\ell^{\mathrm{ss}} \times \Psi_\ell$ for all ℓ. Thus, $\mathbf{G}'_\ell$ is reductive and we may assume $\mathbf{G}'_\ell$ is connected for all ℓ by taking a finite extension of K. Denote the projection to the first and second factor of $\mathrm{GL}_N \times \mathrm{GL}_n$ by respectively p_1 and p_2. Consider the map

$$\mathit{Char}_1 \times \mathit{Char}_2 : \mathrm{GL}_N \times \mathrm{GL}_n \to (\mathbb{G}_a^{N-1} \times \mathbb{G}_m) \times (\mathbb{G}_a^{n-1} \times \mathbb{G}_m)$$

where $\mathit{Char}_i = \mathit{Char} \circ p_i$, $i = 1, 2$. Note that the restriction of $\mathit{Char}_1 \times \mathit{Char}_2$ to $\mathbb{G}_m^N \times \mathbb{G}_m^n$ is a finite morphism. Let $\mathbf{T}'_\ell$ be a maximal torus of monodromy group $\mathbf{G}'_\ell$ and $\bar{\mathbf{T}}'_\ell$ a maximal torus of $\bar{\mathbf{G}}'_\ell$, the algebraic envelope of the mod ℓ representation $\phi_\ell \times \psi_\ell$. Up to conjugation by $\mathrm{GL}_N \times \mathrm{GL}_n$ (over algebraically closed fields), we may assume $\mathbf{T}'_\ell$ and $\bar{\mathbf{T}}'_\ell$ are diagonal (i.e., inside $\mathbb{G}_m^{N+n}$). We claim that up to permutation of coordinates by $\mathrm{Perm}(N) \times \mathrm{Perm}(n)$, $\mathbf{T}'_\ell$ and $\bar{\mathbf{T}}'_\ell$ are annihilated by the same set of characters of $\mathbb{G}_m^{N+n}$ for all sufficiently large ℓ. The proof of the claim goes exactly the same as the proof of Theorem 3.1.1(i) with the following replacements:

- $\mathrm{GL}_N \longrightarrow \mathrm{GL}_N \times \mathrm{GL}_n$
- Morphism $\mathit{Char} \longrightarrow$ morphism $\mathit{Char}_1 \times \mathit{Char}_2$
- $\mathbb{Q}$-variety $\mathit{Char}(\mathbf{G}_\ell) \longrightarrow \mathbb{Q}$-variety $\mathit{Char}_1 \times \mathit{Char}_2(\mathbf{G}'_\ell)$
- $\mathrm{Perm}(N) \longrightarrow \mathrm{Perm}(N) \times \mathrm{Perm}(n)$

Therefore, $\mathbf{T}''_\ell := \mathrm{Ker}(p_2 : \mathbf{T}'_\ell \to p_2(\mathbf{T}'_\ell))^\circ$ and $\bar{\mathbf{T}}''_\ell := \mathrm{Ker}(p_2 : \bar{\mathbf{T}}'_\ell \to p_2(\bar{\mathbf{T}}'_\ell))^\circ$ as subtori of $\mathbb{G}_m^N$ are annihilated by the same set of characters for $\ell \gg 1$. Torus $\mathbf{T}''_\ell$ is the formal character of $\mathbf{G}_\ell^{\mathrm{der}} \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}_\ell}$ [Hu13, proof of Theorem 3.19]. It suffices to show $\bar{\mathbf{T}}''_\ell$ is a maximal torus of $\bar{\mathbf{S}}_\ell$ for $\ell \gg 1$. Since the dimension of torus $\bar{\mathbf{I}}'_\ell$ is equal to the dimension of the center of algebraic envelope $\bar{\mathbf{G}}'_\ell$ for $\ell \gg 1$ (see §2.5) and p_2 is an isogeny from $\bar{\mathbf{I}}'_\ell$ onto $p_2(\bar{\mathbf{I}}'_\ell)$ by Proposition 3.2.6 for $\ell \gg 1$, the identity component of the kernel of

$$p_2 : \bar{\mathbf{G}}'_\ell \to p_2(\bar{\mathbf{G}}'_\ell) = p_2(\bar{\mathbf{I}}'_\ell)$$

is $\bar{\mathbf{S}}'_\ell$ (the semisimple part of $\bar{\mathbf{G}}'_\ell$) for $\ell \gg 1$. Since $p_2(\bar{\mathbf{T}}'_\ell) = p_2(\bar{\mathbf{G}}'_\ell) = p_2(\bar{\mathbf{I}}'_\ell)$ for $\ell \gg 1$, $\bar{\mathbf{T}}''_\ell$ by construction is a maximal torus of $\bar{\mathbf{S}}'_\ell = \bar{\mathbf{S}}_\ell \times \{1\}$ for $\ell \gg 1$. Hence, the formal character of $\bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}$ and $\mathbf{G}_\ell^{\mathrm{der}} \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}_\ell}$ are the same for $\ell \gg 1$. This proves (i). Since the formal character of $\mathbf{G}_\ell^{\mathrm{der}} \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}_\ell}$ is independent of ℓ [Hu13, Theorem 3.19], we obtain (ii) by (i). □

3.3. **Proofs of Theorem A and Corollary B.** The following purely representation theoretic result is crucial to the study of Galois images $\bar{\Gamma}_\ell$ for $\ell \gg 1$.

Theorem 3.3.1. [Hu13, Theorem 2.19] *Let V be a finite dimensional $\mathbb{C}$-vector space and $\rho_1 : \mathfrak{g} \to \mathrm{End}(V)$ and $\rho_2 : \mathfrak{h} \to \mathrm{End}(V)$ are two faithful representations of complex semisimple Lie algebras. If the formal characters of ρ_1 and ρ_2 are equal, then the number of A_n factors for $n \in \mathbb{N}\backslash\{1,2,3,4,5,7,8\}$ and the parity of A_4 factors of $\mathfrak{g}$ and $\mathfrak{h}$ are equal.*

Theorem 3.3.2. *The number of $A_n = \mathfrak{sl}_{n+1}$ factors for $n \in \mathbb{N}\backslash\{1,2,3,4,5,7,8\}$ and the parity of A_4 factors of $\bar{\mathbf{S}}_\ell \times_{\mathbb{F}_\ell} \bar{\mathbb{F}}_\ell$ are independent of ℓ if $\ell \gg 1$.*

Proof. Let $\bar{\mathbf{S}}_\ell^{\mathrm{sc}} \to \bar{\mathbf{S}}_\ell$ be the simply connected cover of the semisimple $\bar{\mathbf{S}}_\ell$ for $\ell \gg 1$. Then the representation $(\bar{\mathbf{S}}_\ell^{\mathrm{sc}} \to \bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}) \times \bar{\mathbb{F}}_\ell$ can be lifted to a representation of a simply connected Chevalley scheme $\mathbf{H}_{\ell,\mathbb{Z}}$ defined over $\mathbb{Z}$ for $\ell \gg 1$ [EHK12, Theorem 24]

$$\pi_{\ell,\mathbb{Z}} : \mathbf{H}_{\ell,\mathbb{Z}} \to \mathrm{GL}_{N,\mathbb{Z}}$$

which is also a $\mathbb{Z}$-form of a representation of simply connected $\mathbb{C}$-semisimple group $\mathbf{H}_{\ell,\mathbb{C}}$ [St68a]

$$\pi_{\ell,\mathbb{C}} : \mathbf{H}_{\ell,\mathbb{C}} \to \mathrm{GL}_{N,\mathbb{C}}.$$

Hence, $\bar{\mathbf{S}}_\ell \subset \mathrm{GL}_{N,\mathbb{F}_\ell}$ and $\pi_{\ell,\mathbb{C}}(\mathbf{H}_{\ell,\mathbb{C}}) \subset \mathrm{GL}_{N,\mathbb{C}}$ have the same formal character for $\ell \gg 1$. This and Theorem 3.2.1 imply the formal character of $\pi_{\ell,\mathbb{C}}(\mathbf{H}_{\ell,\mathbb{C}}) \subset \mathrm{GL}_{N,\mathbb{C}}$ is independent of ℓ when ℓ is sufficiently large. This in turn implies the formal character of $\mathrm{Lie}(\pi_{\ell,\mathbb{C}}(\mathbf{H}_{\ell,\mathbb{C}})) \hookrightarrow \mathrm{End}(\mathbb{C}^N)$ (see [Hu13, §2.1]) is independent of ℓ when ℓ is sufficiently large. Therefore, the number of A_n factors for $n \in \mathbb{N}\backslash\{1,2,3,4,5,7,8\}$ and the parity of A_4 factors of $\pi_{\ell,\mathbb{C}}(\mathbf{H}_{\ell,\mathbb{C}})$ and hence $\mathbf{H}_{\ell,\mathbb{C}}$ (the homomorphism $\mathbf{H}_{\ell,\mathbb{C}} \to \pi_{\ell,\mathbb{C}}(\mathbf{H}_{\ell,\mathbb{C}})$ is an isogeny since $\bar{\mathbf{S}}_\ell^{\mathrm{sc}} \to \bar{\mathbf{S}}_\ell$ is an isogeny) are independent of ℓ for $\ell \gg 1$ by Theorem 3.3.1. Since the number of simple factors of each type of $\bar{\mathbf{S}}_\ell^{\mathrm{sc}} \times \bar{\mathbb{F}}_\ell$ and $\mathbf{H}_{\ell,\mathbb{C}}$ are equal, we are done. □

Let $\mathfrak{g}$ be a simple Lie type (e.g., $A_n, B_n, C_n, D_n, ...$) and $\bar{\Gamma}$ a finite group. Suppose $\ell \geq 5$. We measure the number of $\mathfrak{g}$-type simple factors of characteristic ℓ and the total number of Lie type simple factors of characteristic ℓ in the set of composition factors of $\bar{\Gamma}$ in the following sense: Let $\mathbb{F}_q$ be a finite field of characteristic ℓ, σ the Frobenius automorphism of $\bar{\mathbb{F}}_q/\mathbb{F}_q$, and $\bar{\mathbf{G}}$ a connected $\mathbb{F}_q$-group which is almost simple over $\bar{\mathbb{F}}_q$. The identification of $\bar{\mathbf{G}}_\sigma := \bar{\mathbf{G}}(\mathbb{F}_q)$ is related to $\mathfrak{g}$, the simple type of $\bar{\mathbf{G}} \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ [St68b, 11.6]:

Type of $\bar{\mathbf{G}}$	Composition factors of $\bar{\mathbf{G}}(\mathbb{F}_q)$
A_1	$A_1(q) = \mathrm{PSL}_2(q)$ + cyclic groups
A_n $(n \geq 2)$	$A_n(q)$ or ${}^2A_n(q^2)$ + cyclic groups
B_n $(n \geq 2)$	$B_n(q)$ + cyclic groups
C_n $(n \geq 3)$	$C_n(q)$ + cyclic groups
D_4	$D_4(q)$ or ${}^2D_4(q^2)$ or ${}^3D_4(q^3)$ + cyclic groups
D_n $(n \geq 5)$	$D_n(q)$ or ${}^2D_n(q^2)$ + cyclic groups
E_6	$E_6(q)$ or ${}^2E_6(q^2)$ + cyclic groups
E_7	$E_7(q)$ + cyclic groups
E_8	$E_8(q)$ + cyclic groups
F_4	$F_4(q)$ + cyclic groups
G_2	$G_2(q)$ + cyclic groups

$\bar{\mathbf{G}}(\mathbb{F}_q)$ has only one non-cyclic composition factor which is either a Chevalley group or a Steinberg group of type $\mathfrak{g}$. For example, the non-cyclic composition factor is $A_n(q)$ or ${}^2A_n(q^2)$ if $\mathfrak{g} = A_n$ and $n \geq 2$. For any semisimple algebraic group $\mathbf{H}/F$ and complex semisimple Lie algebra $\mathfrak{h}$, denote by $\mathrm{rk}\,\mathbf{H}$ and $\mathrm{rk}\,\mathfrak{h}$ respectively the rank of $\mathbf{H}/\bar{F}$ and the rank of $\mathfrak{h}$.

Definition 14. Suppose $\ell \geq 5$ is a prime number and $q = \ell^f$. Let $\bar{\Gamma}$ be a finite simple group of Lie type (of characteristic ℓ) in the above table and $\mathfrak{g}$ the simple Lie type of the corresponding $\bar{\mathbf{G}}$. We define *the* $\mathfrak{g}$*-type* ℓ*-rank* of $\bar{\Gamma}$ to be

$$\mathrm{rk}_\ell^{\mathfrak{g}}\bar{\Gamma} := \begin{cases} f \cdot \mathrm{rk}\,\mathfrak{g} & \text{if } \bar{\Gamma} \text{ is associated with } \mathfrak{g} \text{ in the above table,} \\ 0 & \text{otherwise.} \end{cases}$$

For finite simple group $\bar{\Gamma}'$ not in the table, $\mathrm{rk}_\ell^{\mathfrak{g}}\bar{\Gamma}'$ is defined to be 0 for any $\mathfrak{g}$. We extend this definition to arbitrary finite groups by defining the $\mathfrak{g}$-type ℓ-rank of any finite group to be the sum of the $\mathfrak{g}$-type ℓ-ranks of its composition factors. *The total* ℓ*-rank* of a finite group $\bar{\Gamma}$ is defined to be

$$\mathrm{rk}_\ell\bar{\Gamma} := \sum_{\mathfrak{g}} \mathrm{rk}_\ell^{\mathfrak{g}}\bar{\Gamma}.$$

Remark 3.3.3. The definition of $\mathfrak{g}$-type ℓ-rank is equivalent to the following. For any finite simple group $\bar{\Gamma}$ of Lie type of characteristic ℓ, we have

$$\bar{\Gamma} = \bar{\mathbf{G}}(\mathbb{F}_{\ell^{f'}})^{\mathrm{der}}$$

for some adjoint simple group $\bar{\mathbf{G}}/\mathbb{F}_{\ell^{f'}}$ so that

$$\bar{\mathbf{G}} \times_{\mathbb{F}_{\ell^{f'}}} \bar{\mathbb{F}}_\ell = \prod^{m} \bar{\mathbf{H}},$$

where $\bar{\mathbf{H}}$ is an $\bar{\mathbb{F}}_\ell$-adjoint simple group of some Lie type $\mathfrak{h}$. We then set the $\mathfrak{g}$-type ℓ-rank of $\bar{\Gamma}$ to be

$$\mathrm{rk}_\ell^{\mathfrak{g}} \bar{\Gamma} := \begin{cases} f' \cdot \mathrm{rk}\, \bar{\mathbf{G}} & \text{if } \mathfrak{g} = \mathfrak{h}. \\ 0 & \text{otherwise.} \end{cases}$$

We extend this definition to arbitrary finite groups by defining the $\mathfrak{g}$-type ℓ-rank of any finite group to be the sum of the $\mathfrak{g}$-type ℓ-ranks of its composition factors.

Let $\bar{\mathbf{G}}$ be a connected semisimple algebraic group over $\mathbb{F}_q$ and $\pi : \bar{\mathbf{G}}^{\mathrm{sc}} \to \bar{\mathbf{G}}$ the simply-connected cover of $\bar{\mathbf{G}}$. Simply-connected $\bar{\mathbf{G}}^{\mathrm{sc}}$ and isogeny π are defined over $\mathbb{F}_q$ [St68b, 9.16]. Group $\bar{\mathbf{G}}^{\mathrm{sc}}$ is a direct product of $\mathbb{F}_q$-simple, simply-connected semisimple groups $\bar{\mathbf{G}}_i^{\mathrm{sc}}$ [CF65, Chapter 10 §1.3]:

$$\bar{\mathbf{G}}_1^{\mathrm{sc}} \times \bar{\mathbf{G}}_2^{\mathrm{sc}} \times \cdots \times \bar{\mathbf{G}}_k^{\mathrm{sc}} \xrightarrow{\mathbb{F}_q \cong} \bar{\mathbf{G}}^{\mathrm{sc}}.$$

For each $\bar{\mathbf{G}}_i^{\mathrm{sc}}$, there exist an integer m_i and an algebraic group $\bar{\mathbf{H}}_i^{\mathrm{sc}}$ defined over $\mathbb{F}_{q^{m_i}}$ such that $\bar{\mathbf{H}}_i^{\mathrm{sc}} \times_{\mathbb{F}_{q^{m_i}}} \bar{\mathbb{F}}_q$ is almost simple and

$$\bar{\mathbf{G}}_i^{\mathrm{sc}} \times_{\mathbb{F}_q} \mathbb{F}_{q^{m_i}} = \prod^{m_i} \bar{\mathbf{H}}_i^{\mathrm{sc}}.$$

We have [CF65, Chapter 10 §1.3]

$$\bar{\mathbf{G}}_i^{\mathrm{sc}} = \mathrm{Res}_{\mathbb{F}_{q^{m_i}}/\mathbb{F}_q}(\bar{\mathbf{H}}_i^{\mathrm{sc}})$$

so that

$$\bar{\mathbf{G}}_i^{\mathrm{sc}}(\mathbb{F}_q) = \bar{\mathbf{H}}_i^{\mathrm{sc}}(\mathbb{F}_{q^{m_i}}).$$

The following proposition relates $\mathrm{rk}_\ell^{\mathfrak{g}} \bar{\mathbf{G}}(\mathbb{F}_q)$ and $\mathrm{rk}_\ell \bar{\mathbf{G}}(\mathbb{F}_q)$ to $\bar{\mathbf{G}} \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$.

Proposition 3.3.4. *Let $\ell \geq 5$ be a prime and $\bar{\mathbf{G}}$ a connected semisimple algebraic group over $\mathbb{F}_q$, where $q = \ell^f$. The composition factors of $\bar{\mathbf{G}}(\mathbb{F}_q)$ are cyclic groups and finite simple groups of Lie type of characteristic ℓ. Moreover, let m be the number of almost simple factors of $\bar{\mathbf{G}} \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ of simple type $\mathfrak{g}$. Then,*

$$\mathrm{rk}_\ell^{\mathfrak{g}} \bar{\mathbf{G}}(\mathbb{F}_q) = mf \cdot \mathrm{rk}\, \mathfrak{g} \quad \text{and} \quad \mathrm{rk}_\ell \bar{\mathbf{G}}(\mathbb{F}_q) = f \cdot \mathrm{rk}\, \bar{\mathbf{G}}.$$

Proof. Since the kernel and the cokernel of $\pi : \bar{\mathbf{G}}^{\mathrm{sc}}(\mathbb{F}_q) \to \bar{\mathbf{G}}(\mathbb{F}_q)$ are both abelian [St68b, 12.6], the composition factors of $\bar{\mathbf{G}}(\mathbb{F}_q)$ and $\prod_{i=1}^k \bar{\mathbf{H}}_i^{\mathrm{sc}}(\mathbb{F}_{q^{m_i}})$ defined above are identical modulo cyclic groups. Hence, the composition factors of $\bar{\mathbf{G}}(\mathbb{F}_q)$ are cyclic groups and finite simple groups of Lie type of characteristic ℓ by the table. Let

$$\{\bar{\mathbf{H}}_1^{\mathrm{sc}}, \bar{\mathbf{H}}_2^{\mathrm{sc}}, ..., \bar{\mathbf{H}}_j^{\mathrm{sc}}\}$$

be the subset of $\{\bar{\mathbf{H}}_1^{\mathrm{sc}}, ..., \bar{\mathbf{H}}_k^{\mathrm{sc}}\}$ of type $\mathfrak{g}$. The equation

$$m_1 + m_2 + \cdots + m_j = m$$

follows immediately from the fact that each $\bar{\mathbf{G}}_i^{\mathrm{sc}}$ is a direct product of m_i copies of $\bar{\mathbf{H}}_i^{\mathrm{sc}}$ over $\bar{\mathbb{F}}_q$. Since $\bar{\mathbf{H}}_i^{\mathrm{sc}}$ is almost simple over $\bar{\mathbb{F}}_q$, we obtain by Definition 14 that the $\mathfrak{g}$-type ℓ-rank

$$\mathrm{rk}_\ell^{\mathfrak{g}} \bar{\mathbf{G}}(\mathbb{F}_q) = \sum_{i=1}^{k} \mathrm{rk}_\ell^{\mathfrak{g}} \bar{\mathbf{H}}_i^{\mathrm{sc}}(\mathbb{F}_{q^{m_i}}) = \sum_{i=1}^{j} m_i f \cdot \mathrm{rk}\,\mathfrak{g} = mf \cdot \mathrm{rk}\,\mathfrak{g}.$$

and therefore the total ℓ-rank

$$\mathrm{rk}_\ell \bar{\mathbf{G}}(\mathbb{F}_q) = f \cdot \mathrm{rk}\,\bar{\mathbf{G}}.$$

□

We can now prove our main results.

Theorem A. (Main Theorem) *Let K be a number field and $\{\phi_\ell : \mathrm{Gal}_K \to \mathrm{GL}_N(\mathbb{F}_\ell)\}_{\ell\in\mathscr{P}}$ a strictly compatible system of mod ℓ Galois representations arising from étale cohomology (Definition 1,2). There exists a finite normal extension L of K such that if we denote $\phi_\ell(\mathrm{Gal}_K)$ and $\phi_\ell(\mathrm{Gal}_L)$ by respectively $\bar{\Gamma}_\ell$ and $\bar{\gamma}_\ell$ for all ℓ, and let $\bar{\mathbf{S}}_\ell \subset \mathrm{GL}_{N,\mathbb{F}_\ell}$ be the connected $\mathbb{F}_\ell$-semisimple subgroup associated to $\bar{\gamma}_\ell$ (or $\bar{\Gamma}_\ell$) by Nori's theory for $\ell \gg 1$, then the following hold for $\ell \gg 1$:*

(i) *The formal character of $\bar{\mathbf{S}}_\ell \hookrightarrow \mathrm{GL}_{N,\mathbb{F}_\ell}$ is independent of ℓ (Definition 3') and is equal to the formal character of $(\mathbf{G}_\ell^\circ)^{\mathrm{der}} \hookrightarrow \mathrm{GL}_{N,\mathbb{Q}_\ell}$, where $(\mathbf{G}_\ell^\circ)^{\mathrm{der}}$ is the derived group of the identity component of $\mathbf{G}_\ell$, the algebraic monodromy group of the semi-simplified representation Φ_ℓ^{ss}.*

(ii) *The composition factors of $\bar{\gamma}_\ell$ and $\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)$ are identical modulo cyclic groups. Therefore, the composition factors of $\bar{\gamma}_\ell$ are finite simple groups of Lie type of characteristic ℓ and cyclic groups.*

Proof. By Proposition 2.1.2(i), $\bar{\mathbf{S}}_\ell \subset \mathrm{GL}_{N,\mathbb{F}_\ell}$ is a connected $\mathbb{F}_\ell$-semisimple subgroup for $\ell \gg 1$. Part (i) is proved by Theorem 3.2.1. Since there is a finite normal extension L/K such that $\bar{\gamma}_\ell := \phi_\ell(\mathrm{Gal}_L)$ is a subgroup of $\bar{\mathbf{G}}_\ell(\mathbb{F}_\ell)$ of uniform bounded index by Theorem 2.0.5 and $\bar{\mathbf{S}}_\ell$ is the derived group of $\bar{\mathbf{G}}_\ell$, the composition factors of $\bar{\gamma}_\ell$ and $\bar{\gamma}_\ell \cap \bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)$ are identical modulo cyclic groups. Together with $\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)/\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)^+$ abelian and normal series

$$\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)^+ = \bar{\gamma}_\ell^+ \ \triangleleft\ \bar{\gamma}_\ell \cap \bar{\mathbf{S}}_\ell(\mathbb{F}_\ell) \ \triangleleft\ \bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)$$

for $\ell \gg 1$ by Theorem 2.1.1 and Remark 2.1.3, we conclude that the composition factors of $\bar{\gamma}_\ell$ and $\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)$ are identical modulo cyclic groups. Since Proposition 3.3.4 implies the non-cyclic composition factors of $\bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)$ are finite simple groups of Lie type of characteristic ℓ, we obtain (ii). □

Corollary B. *Let $\bar{\mathbf{S}}_\ell$ be defined as above, then the following hold for $\ell \gg 1$:*

(i) *The total ℓ-rank $\mathrm{rk}_\ell\,\bar{\Gamma}_\ell$ of $\bar{\Gamma}_\ell$ (Definition 14) is equal to the rank of $\bar{\mathbf{S}}_\ell$ and is therefore independent of ℓ.*

(ii) *The A_n-type ℓ-rank $\mathrm{rk}_\ell^{A_n}\,\bar{\Gamma}_\ell$ of $\bar{\Gamma}_\ell$ (Definition 14) for $n \in \mathbb{N}\backslash\{1,2,3,4,5,7,8\}$ and the parity of $(\mathrm{rk}_\ell^{A_4}\,\bar{\Gamma}_\ell)/4$ are independent of ℓ.*

Proof. Since $\bar{\gamma}_\ell$ is a normal subgroup of $\bar{\Gamma}_\ell$ of index bounded by $[L:K]$, they have equal total ℓ-rank and $\mathfrak{g}$-type ℓ-rank for all sufficiently large ℓ. It suffices to prove (i) and (ii) for $\bar{\gamma}_\ell$. Since (i) is a direct consequence of Proposition 3.3.4 and Theorem A and (ii) follows easily from Theorem 3.3.2, Proposition 3.3.4, and Theorem A, we are done. □

Acknowledgments

It is my great pleasure to acknowledge my adviser, Michael Larsen, for useful conversations during the course of the work and helpful comments on an earlier preprint of this project. I am also grateful to the anonymous referee for many helpful comments and suggestions, which largely improved the exposition of the paper.

References

[Ca08] X. Caruso: Conjecture de l'inertie modérée de Serre, *Invent. Math.* **171** (2008), no. 3, p. 629-699.

[CF65] Algebraic Number Theory, Proceedings of the instructional conference held at the University of Sussex, Brighton, September 1-17, 1965 (2nd ed.), edited by J. W. S. Cassels and A. Fröhlich, *London Mathematical Society.*

[CR88] C. W. Curtis, I. Reiner: Representation theory of finite groups and associative algebras, reprint of the 1962 original. Wiley Classics Library. A Wiley-Interscience Publication. *John Wiley & Sons, Inc., New York*, 1988.

[deJ96] A. J. de Jong: Smoothness, semi-stability and alterations, *Inst. Hautes tudes Sci. Publ. Math.* no. **83** (1996), 51-93.

[De74] P. Deligne: La conjecture de Weil I, *Publ. Math. I.H.E.S.*, **43** (1974), 273-307.

[DG65] J. Dieudonné, A. Grothendieck: Éléments de géométrie algébrique. IV. Étude locale des schmas et des morphismes de schémas. II, *Inst. Hautes Études Sci. Publ. Math.* no. **24** (1965) 231 pp.

[EHK12] J. Ellenberg, C. Hall, E. Kowalski: Expander graphs, gonality, and variation of Galois representations, *Duke Math. J.* **161** (2012), no. 7, 1233-1275.

[Fa83] G. Faltings: Endlichkeitssätze für abelsche Varietäten über Zahlkörpern, *Invent. Math.* **73** (1983), no. 3, 349-366.

[Ga83] O. Gabber: Sur la torsion dans la cohomologie l-adique d'une variété, *C. R. Acad. Sc. Paris* **297** Série I (1983), 179-182.

[Ha11] C. Hall: An open image theorem for a general class of abelian varieties, *Bull. Lond. Math. Soc.* **43** (2011), no. 4, 703-711.

[Ha77] R. Hartshorne: Algebraic Geometry, Graduate Texts in Mathematics, no. **52**, *Springer-Verlag, New York-Heidelberg*, 1977.

[Hu13] C. Y. Hui: Monodromy of Galois representations and equal-rank subalgebra equivalence, *Math. Res. Lett.* **20** (2013), no. 4, 705-728.

[HL14] C. Y. Hui, M. Larsen: Type A images of Galois representations and maximality, arXiv:1305.1989, preprint 2014.

[Jo78] C. Jordan: Mémoire sur les équations differentielles linéaires á intégrale algébrique, *J. für Math.* **84** (1878), 89-215.

[La94] S. Lang: Algebraic number theory, Second edition. Graduate Texts in Mathematics **110**, *Springer-Verlag, New York*, 1994.

[LW54] S. Lang, A. Wéil: Number of points of varieties in finite fields, *Amer. J. Math.* **76** (1954), 819-827.

[Mi13] J. S. Milne: Lecture on Étale Cohomology, online notes, `http://www.jmilne.org/math/CourseNotes/LEC.pdf`.

[Ne99] J. Neukirch: Algebraic Number Theory, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. **322**. *Springer, Berlin* (1999).

[No87] M. V. Nori: On subgroups of $\mathrm{GL}_n(\mathbb{F}_p)$, *Invent. Math.* **88** (1987), no. 2, 257-275.

[Se72] J.-P. Serre: Propriétés galoisiennes des points d'ordre fini des courbes elliptiques, *Invent. Math.* **15** (1972), no. 4, 259-331.

[Se81] J.-P. Serre: Letter to K. A. Ribet, Jan. 1, 1981, reproduced in *Coll. Papers*, vol. IV, no. 133.

[Se86] J.-P. Serre: Lettre á Marie-France Vignéras du 10/2/1986, reproduced in *Coll. Papers*, vol. IV, no. 137.

[Se94] J.-P. Serre: Propriétés conjecturales des groupes de Galois motiviques et des représentations l-adiques, *Motives (Seattle, WA, 1991)*, pp. 377-400, Proc. Sympos. Pure Math., Vol. 55, Part 1, *Amer. Math. Soc., Providence, RI,* 1994.

[Se98] J.-P. Serre: Abelian l-adic representation and elliptic curves, Research Notes in Mathematics Vol. 7 (2nd ed.), *A K Peters* (1998).

[Sp08] T.A. Springer: Linear algebraic groups, reprint of the 1998 2nd ed., *Birkhaüser* (2008).

[St68a] R. Steinberg: Lectures on Chevalley groups, *Yale University Press* (1968).

[St68b] R. Steinberg: Endomorphisms of Linear Algebraic Groups, *Mem. Amer. Math. Soc.* **80** (1968).

E-mail address: `chunyin.hui@uni.lu, pslnfq@gmail.com`

University of Luxembourg, Mathematics Research Unit, 6 rue Richard Coudenhove-Kalergi, L-1359 Luxembourg