Ito's and Tanaka's type formulae for the stochastic heat equation: the linear case

Mihai GRADINARU, **Ivan NOURDIN**, **Samy TINDEL**

Université Henri Poincaré, Institut de Mathématiques Élie Cartan, B.P. 239,
F - 54506 Vandœuvre-lès-Nancy Cedex
{Mihai.Gradinaru, Ivan.Nourdin, Samy.Tindel}@iecn.u-nancy.fr

Abstract

In this paper we consider the linear stochastic heat equation with additive noise in dimension one. Then, using the representation of its solution X as a stochastic convolution of the cylindrical Brownian motion with respect to an operator-valued kernel, we derive Itô's and Tanaka's type formulae associated to X.

Keywords: Stochastic heat equation, Malliavin calculus, Itô's formula, Tanaka's formula.

MSC 2000: 60H15, 60H05, 60H07, 60G15.

1 Introduction

The study of stochastic partial differential equations (SPDE in short) has been seen as a challenging topic in the past thirty years for two main reasons. On the one hand, they can be associated to some natural models for a large amount of physical phenomenon in random media (see for instance [4]). On the other hand, from a more analytical point of view, they provide some rich examples of Markov processes in infinite dimension, often associated to a nicely behaved semi-group of operators, for which the study of smoothing and mixing properties give raise to some elegant, and sometimes unexpected results. We refer for instance to [9], [10], [5] for a deep and detailed account on these topics.

It is then a natural idea to try to construct a stochastic calculus with respect to the solution to a SPDE. Indeed, it would certainly give some insight on the properties of such a canonical object, and furthermore, it could give some hints about the relationships between different classes of remarkable equations (this second

1

motivation is further detailed by L. Zambotti in [20], based on some previous results obtained in [19]). However, strangely enough, this aspect of the theory is still poorly developped, and our paper proposes to make one step in that direction.

Before going into details of the results we have obtained so far and of the methodology we have adopted, let us describe briefly the model we will consider, which is nothing but the stochastic heat equation in dimension one. On a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$, let $\{W^n; n \geq 1\}$ be a sequence of independent standard Brownian motions. We denote by $(\mathcal{F}_t)$ the filtration generated by $\{W^n; n \geq 1\}$. Let also H be the Hilbert space $L_0^2([0,1])$ of square integrable functions on $[0,1]$ with Dirichlet boundary conditions, and $\{\mathbf{e}_n; n \geq 1\}$ the trigonometric basis of H, that is

$$\mathbf{e}_n(x) = \sqrt{2}\sin(\pi n x), \qquad x \in [0,1], n \geq 1.$$

The inner product in H will be denoted by $\langle\,,\,\rangle_H$.

The stochastic equation will be driven by a cylindrical noise (see [9] for further details on this object) defined by the formal series

$$W_t = \sum_{n \geq 1} W_t^n \mathbf{e}_n, \qquad t \in [0,T],\, T > 0.$$

Observe that $W_t \notin H$, but for any $\mathbf{y} \in H$, $\langle W_t, \mathbf{y} \rangle_H$ is a well defined Gaussian random variable with variance $|\mathbf{y}|_H^2$. It is also worth observing that W coincides with the space-time white noise (see [9]), in so far that, if $h_1, h_2 \in L^2([0,T]; H)$ for a given $T > 0$, we have

$$\mathrm{E}\left[\left(\int_0^T \langle h_1(t), dW_t \rangle_H\right)\left(\int_0^T \langle h_2(t), dW_t \rangle_H\right)\right] = \int_0^T \langle h_1(t), h_2(t) \rangle_H\, dt. \qquad (1.1)$$

Let now $\Delta = \frac{\partial^2}{\partial x^2}$ be the Laplace operator on $[0,1]$ with Dirichlet boundary conditions. Notice that Δ is an unbounded negative operator that can be diagonalized in the orthonormal basis $\{\mathbf{e}_n; n \geq 1\}$, with $\Delta \mathbf{e}_n = -\lambda_n \mathbf{e}_n$ and $\lambda_n = \pi^2 n^2$. The semi-group generated by Δ on H will be denoted by $\{e^{t\Delta}; t \geq 0\}$. In this context, we will consider the following stochastic heat equation:

$$dX_t = \Delta X_t\, dt + dW_t, \quad t \in (0,T],\, X_0 = 0. \qquad (1.2)$$

Of course, equation (1.2) has to be understood in the so-called *mild* sense, and in this linear additive case, it can be solved explicitly in the form of a stochastic convolution, which takes a particularly simple form in the present case:

$$X_t = \int_0^t e^{(t-s)\Delta} dW_s = \sum_{n \geq 1} X_t^n\, \mathbf{e}_n,\, t \in [0,T], \qquad (1.3)$$

where $\{X^n; n \geq 1\}$ is a sequence of independent one-dimensional Ornstein-Uhlenbeck processes:

$$X_t^n = \int_0^t e^{-\lambda_n(t-s)} dW_s^n,\, n \geq 1,\, t \in [0,T].$$

With all those notations in mind, let us go back to the main motivations of this paper: if one wishes to get, for instance, an Itô's type formula for the process X

defined above, a first natural idea would be to start from a finite-dimensional version (of order $N \geq 1$) of the the representation given by formula (1.3), and then to take limits as $N \to \infty$. Namely, if we set

$$X_t^{(N)} = \sum_{n \leq N} X_t^n \, \mathbf{e}_n, \; t \in [0, T],$$

and if $F_N : \mathbb{R}^N \to \mathbb{R}$ is a C_b^2-function, then $X^{(N)}$ is just a N-dimensional Ornstein-Uhlenbeck process, and the usual semi-martingale representation of this approximation yields, for all $t \in [0, T]$,

$$F_N\left(X_t^{(N)}\right) = F_N(0) + \sum_{n \leq N} \int_0^t \partial_{x_n} F_N(X_s^{(N)}) \, dX_s^n + \frac{1}{2} \int_0^t \mathrm{Tr}\left(F_N''(X_s^{(N)})\right) \, ds, \quad (1.4)$$

where the stochastic integral has to be interpreted in the Itô sense. However, when one tries to take limits in (1.4) as $N \to \infty$, it seems that a first requirement on $F \equiv \lim_{N \to \infty} F_N$ is that $\mathrm{Tr}(F'')$ is a bounded function. This is certainly not the case in infinite dimension, since the typical functional to which we would like to apply Itô's formula is of the type $F : H \to \mathbb{R}$ defined by

$$F(\ell) = \int_0^1 \sigma(\ell(x))\phi(x) \, dx, \quad \text{with} \quad \sigma \in C_b^2(\mathbb{R}), \; \phi \in L^\infty([0,1]),$$

and it is easily seen in this case that, for non degenerate coefficients σ and ϕ, F is a $C_b^2(H)$-functional, but F'' is not trace class. One could imagine another way to make all the terms in (1.4) convergent, but it is also worth mentioning at this point that, even if our process X is the limit of a semi-martingale sequence $X^{(N)}$, it is not a semi-martingale itself, since the mapping $t \in [0, T] \mapsto X_t \in H$ is only Hölder-continuous of order $(1/4)^-$ (see Lemma 2.1 below). This fact also explains why the classical semi-martingale approach fails in the current situation.

In order to get an Itô's formula for the process X, we have then decided to use another natural approach: the representation (1.3) of the solution to (1.2) shows that X is a centered Gaussian process, given by the convolution of W by the operator-valued kernel $e^{(t-s)\Delta}$. Furthermore, this kernel is divergent at $t = 0$: in order to define the stochastic integral $\int_0^t e^{(t-s)\Delta} dW_s$, one has to get some bounds on $\|e^{t\Delta}\|_{\mathrm{HS}}$ (see Theorem 5.2 in [9]), which diverges as $t^{-3/2}$. However, we will see that the important quantity to control for us is $\|\Delta e^{t\Delta}\|_{\mathrm{op}}$, which diverges as t^{-1} only. In any case, in one dimension, the stochastic calculus with respect to Gaussian processes defined by an integral of the form

$$\int_0^t K(t, s) \, dB_s, \qquad t \geq 0,$$

where B is a standard Brownian motion and K is a kernel with a certain divergence on the diagonal, has seen some spectacular advances during the last ten years, mainly motivated by the example of fractional Brownian motion. For this latter process, Itô's formula (see [2]), as well as Tanaka's one (see [7]) and the representation of Bessel type processes (see [11], [12]) are now fairly well understood. Our idea is then to adapt this methodology to the infinite dimensional case.

Of course, this leads to some technical and methodological problems, inherent to this infinite dimensional setting. But our aim in this paper is to show that this generalization is possible. Moreover, the Itô type formula which we obtain has a simple form: if F is a smooth function defined on H, we get that

$$F(X_t) = F(0) + \int_0^t \langle F'(X_s), \delta X_s \rangle + \frac{1}{2} \int_0^t \mathrm{Tr}(e^{2s\Delta} F''(X_s)) ds, \ t \in [0, T], \quad (1.5)$$

where the term $\int_0^t \langle F'(X_s), \delta X_s \rangle$ is a Skorokhod type integral that will be properly defined at Section 2. Notice also that the last term in (1.5) is the one that one could expect, since it corresponds to the Kolmogorov equation associated to (1.2) (see, for instance, [9] p. 257). Let us also mention that we wished to explain our approach by taking the simple example of the linear stochastic equation in dimension 1. But we believe that our method can be applied to some more general situations, and here is a list of possible extensions of our formulae:

1. The case of a general analytical operator A generating a C_0-semigroup $S(t)$ on a certain Hilbert space H. This would certainly require the use of the generalized Skorokhod integral introduced in [6].

2. The multiparametric setting (see [18] or [8] for a general presentation) of SPDEs, which can be related to the formulae obtained for the fractional Brownian sheet (see [17]).

3. The case of non-linear equations, that would amount to get some Itô's representations for processes defined informally by $Y = \int u(s, y) X(ds, dy)$, where u is a process satisfying some regularity conditions, and X is still the solution to equation (1.2).

We plan to report on these possible generalizations of our Itô's fomula in some subsequent papers.

Eventually, we would like to observe that a similar result to (1.5) has been obtained in [20], using another natural approach, namely the regularization of the kernel $e^{t\Delta}$ by an additional term $e^{\varepsilon\Delta}$, and then passing to the limit when $\varepsilon \to 0$. This method, that may be related to the one developed in [1] for the fractional Brownian case, leads however to some slightly different formulae, and we hope that our form of Itô's type formula (1.5) will give another point of view on this problem.

The paper will be organized as follows: in the next section, we will give some basic results about the Malliavin calculus with respect to the process X solution to (1.2). We will then prove the announced formula (1.5). At Section 3, we will state and prove the Tanaka type formula, for which we will use the space-time white noise setting for equation (1.2).

2 An Ito's type formula related to X

In this section, we will first recall some basic facts about Malliavin's calculus that we will use throughout the paper, and then establish our Itô's type formula.

2.1 Malliavin calculus notations and facts

Let us recall first that the process X solution to (1.2) is only $(1/4)^-$ Hölder continuous, which motivates the use of Malliavin calculus tools in order to get an Itô's type formula. This result is fairly standard, but we include it here for sake of completeness, since it is easily proven in our particular case.

Lemma 2.1 *We have, for some constants $0 < c_1 < c_2$, and for all $s, t \in [0, T]$:*

$$c_1 |t - s|^{1/2} \leq \mathrm{E}\left[|X_t - X_s|_H^2\right] \leq c_2 |t - s|^{1/2}.$$

Proof. A direct computation yields (recall that $\lambda_n = \pi^2 n^2$):

$$\mathrm{E}\left[|X_t - X_s|_H^2\right] = \sum_{n \geq 1} \int_0^s \left(e^{-\pi^2 n^2 (t-u)} - e^{-\pi^2 n^2 (s-u)}\right)^2 du + \sum_{n \geq 1} \int_s^t e^{-2\pi^2 n^2 (t-u)} du$$

$$= \sum_{n \geq 1} \frac{(1 - e^{-\pi^2 n^2 (t-s)})^2 (1 - e^{-2\pi^2 n^2 s})}{2\pi^2 n^2} + \sum_{n \geq 1} \frac{1 - e^{-2\pi^2 n^2 (t-s)}}{2\pi^2 n^2}$$

$$\leq \int_0^\infty \frac{(1 - e^{-\pi^2 x^2 (t-s)})^2}{2\pi^2 x^2} dx + \int_0^\infty \frac{1 - e^{-2\pi^2 x^2 (t-s)}}{2\pi^2 x^2} dx = \mathrm{cst} \, (t - s)^{1/2},$$

which gives the desired upper bound. The lower bound is obtained along the same lines.

$\square$

2.1.1 Malliavin calculus with respect to W

We will now recall some basic facts about the Malliavin calculus with respect to the cylindrical noise W. In fact, if we set $\mathcal{H}_{\mathrm{w}} := L^2([0, T]; H)$, with inner product $\langle \cdot \rangle_{\mathcal{H}_{\mathrm{w}}}$, then W can be seen as a Gaussian family $\{W(h); h \in \mathcal{H}_{\mathrm{w}}\}$, where

$$W(h) = \int_0^T \langle h(t), dW_t \rangle_H = \sum_{n \geq 1} \int_0^T \langle h(t), \mathbf{e}_n \rangle_H dW_t^n,$$

with covariance function

$$\mathrm{E}\left[W(h_1) W(h_2)\right] = \langle h_1, h_2 \rangle_{\mathcal{H}_{\mathrm{w}}},$$

which is nothing but a reformulation of (1.1). Then, as usual in the Malliavin calculus setting, the smooth functionals of W will be of the form

$$F = f\left(W(h_1), \ldots, W(h_d)\right), \quad d \geq 1, \, h_1, \ldots, h_d \in \mathcal{H}_{\mathrm{w}}, \, f \in C_b^\infty(\mathbb{R}^d),$$

and for this kind of functional, the Malliavin derivative is defined as an element of $\mathcal{H}_{\mathrm{w}}$ given by

$$D_t^W F = \sum_{i=1}^d \partial_i f\left(W(h_1), \ldots, W(h_d)\right) h_i(t).$$

It can be seen that D^W is a closable operator on $L^2(\Omega)$, and for $k \geq 1$, we will call $\mathbb{D}^{k,2}$ the closure of the set $\mathcal{S}$ of smooth functionals with respect to the norm

$$\|F\|_{k,2} = \|F\|_{L^2} + \sum_{j=1}^{k} \mathrm{E}\left[|D^{W,j}F|_{\mathcal{H}_W^{\otimes j}}\right].$$

If V is a separable Hilbert space, this construction can be generalized to a V-valued functional, leading to the definition of the spaces $\mathbb{D}^{k,2}(V)$ (see also [13] for a more detailed account on this topic). A chain rule for the derivative operator is also available: if $F = \{F^m; m \geq 1\} \in \mathbb{D}^{1,2}(V)$ and $\varphi \in C_b^1(V)$, then $\varphi(F) \in \mathbb{D}^{1,2}$, and

$$D_t^W\left(\varphi(F)\right) = \left\langle \nabla\varphi(F), D_t^W F \right\rangle_V = \sum_{m \geq 1} D_t^W F^m \partial_m \varphi(F). \tag{2.1}$$

The adjoint operator of D^W is called the divergence operator, usually denoted by δ^W, and defined by the duality relationship

$$\mathrm{E}\left[F\,\delta^W(u)\right] = \mathrm{E}\left[\left\langle D^W F, u \right\rangle_{\mathcal{H}_W}\right], \tag{2.2}$$

for a random variable $u \in \mathcal{H}_W$. The domain of δ^W is denoted by $\mathrm{Dom}(\delta^W)$, and we have that $\mathbb{D}^{1,2}(\mathcal{H}_W) \subset \mathrm{Dom}(\delta^W)$.

We will also need to consider the multiple integrals with respect to W, which can be defined in the following way: set $I_{0,T} = 1$, and if $h \in \mathcal{H}_W$, $I_{1,T}(h) = W(h)$. Next, if $m \geq 2$ and $h_1, \ldots, h_m \in \mathcal{H}_W$, we can define $I_{m,T}(\otimes_{j=1}^m h_j)$ recursively by

$$I_{m,T}(\otimes_{j=1}^m h_j) = I_{1,T}(u^{(m-1)}), \quad \text{where} \quad u^{(m-1)}(t) = \left[I_{m-1,t}(\otimes_{j=1}^{m-1} h_j)\right] h_m, \ t \leq T. \tag{2.3}$$

Let us observe at this point that the set of multiple integrals, that is

$$\mathcal{M} = \left\{ I_{m,T}(\otimes_{j=1}^m h_j); \ m \geq 0, h_1, \ldots, h_m \in \mathcal{H}_W \right\},$$

is dense in $L^2(\Omega)$ (see, for instance, Theorem 1.1.2 in [15]). We stress that we use a different normalization for the multiple integrals of order m, which is harmless for our purposes. Eventually, an easy application of the basic rules of Malliavin calculus yields that, for a given $m \geq 1$:

$$D_s^W I_{m,T}(h^{\otimes m}) = I_{m-1,T}(h^{\otimes m-1})h. \tag{2.4}$$

2.1.2 Malliavin calculus with respect to X

We will now give a brief account on the construction of the Malliavin calculus with respect to the process X: let $C(t,s)$ be the covariance operator associated to X, defined, for any $\mathbf{y}, \mathbf{z} \in H$ by

$$\mathrm{E}\left[\langle X_t, \mathbf{y} \rangle_H \langle X_s, \mathbf{z} \rangle_H\right] = \langle C(t,s)\mathbf{y}, \mathbf{z} \rangle_H, \ t, s > 0.$$

Notice that, in our case, $C(t,s)$ is a diagonal operator when expressed in the orthonormal basis $\{\mathbf{e}_n; n \geq 1\}$, whose n^{th} diagonal element is given by

$$[C(t,s)]_{n,n} = \frac{e^{-\lambda_n(t \vee s)} \sinh(\lambda_n(t \wedge s))}{2\lambda_n}, \ t, s > 0.$$

Now, the reproducing kernel Hilbert space $\mathcal{H}_X$ associated to X is defined as the closure of

$$\text{Span}\left\{\mathbf{1}_{[0,t]}\mathbf{y};\ t \in [0,T], \mathbf{y} \in H\right\},$$

with respect to the inner product

$$\left\langle\mathbf{1}_{[0,t]}\mathbf{y},\ \mathbf{1}_{[0,s]}\mathbf{z}\right\rangle_{\mathcal{H}_X} = \left\langle C(t,s)\mathbf{y},\mathbf{z}\right\rangle_H.$$

The Wiener integral of an element $h \in \mathcal{H}_X$ is now easily defined: $X(h)$ is a centered Gaussian random variable, and if $h_1, h_2 \in \mathcal{H}_X$,

$$\mathrm{E}\left[X(h_1)\,X(h_2)\right] = \langle h_1, h_2\rangle_{\mathcal{H}_X}.$$

In particular the previuous equality provide a natural isometry between $\mathcal{H}_X$ and the first chaos associated to X. Once these Wiener integrals are defined, one can proceed like in the case of the cylindrical Brownian motion, and construct a derivation operator D^X, some Sobolev spaces $\mathbb{D}^{k,2}_X(V)$, and a divergence operator δ^X.

Following the ideas contained in [2], we will now relate δ^X with a Skorokhod integral with respect to the Wiener process W. To this purpose, recall that $\mathcal{H}_W = L^2([0,T];H)$, and let us introduce the linear operators $G : \mathcal{H}_W \to \mathcal{H}_W$ defined by

$$Gh(t) = \int_0^t e^{(t-u)\Delta}h(u)du, \quad h \in \mathcal{H}_W, t \in [0,T] \tag{2.5}$$

and $G^* : \mathrm{Dom}(G^*) \to \mathcal{H}_W$ defined by

$$G^*h(t) = e^{(T-t)\Delta}h(t) + \int_t^T \Delta e^{(u-t)\Delta}[h(u) - h(t)]du,\ h \in \mathrm{Dom}(G^*), t \in [0,T]. \tag{2.6}$$

Observe that

$$\|\Delta e^{t\Delta}\|_{\mathrm{op}} \leq \sup_{\alpha \geq 0} \alpha e^{t\alpha} = \frac{1}{e\,t},\ \text{ for all } t \in (0,T]$$

and thus, it is easily seen from (2.6) that, for any $\varepsilon > 0$, $C^\varepsilon([0,T];H) \subset \mathrm{Dom}(G^*)$, where $C^\varepsilon([0,T];H)$ stands for the set of ε-Hölder continuous functions from $[0,T]$ to H. At a heuristic level, notice also that, formally, we have $X = G\dot{W}$, and thus, if $h : [0,T] \to H$ is regular enough,

$$\delta^X(h) = \int_0^T \langle h(t), \delta X_t\rangle = \int_0^T \langle h(t), GW(dt)\rangle_H. \tag{2.7}$$

Of course, the expression (2.7) is ill-defined, and in order to make it rigorous, we will need the following duality property:

Lemma 2.2 *For every $\varepsilon > 0$, $h, k \in C^\varepsilon([0,T];H)$ and $t \in [0,T]$, we have:*

$$\int_0^t \langle G^*h(s), k(s)\rangle_H\, ds = \int_0^t \langle h(s), Gk(ds)\rangle_H. \tag{2.8}$$

Proof. Without loss of generality, we can assume that h is given by $h(s) = \mathbf{1}_{[0,\tau]}(s)\mathbf{y}$ with $\tau \in [0,t]$ and $\mathbf{y} \in H$. Indeed, to obtain the general case, it suffices to use the linearity in (2.8) and the fact that the set of step functions is dense in $C^\varepsilon([0,T];H)$. Then we can write, on one hand:

$$
\int_0^t \langle h(s), Gk(ds) \rangle_H = \int_0^t \langle \mathbf{1}_{[0,\tau]}(s)\mathbf{y}, Gk(ds) \rangle_H
$$
$$
= \left\langle \mathbf{y}, \int_0^\tau Gk(ds) \right\rangle_H = \langle \mathbf{y}, Gk(\tau) \rangle_H = \int_0^\tau \langle \mathbf{y}, e^{(\tau-s)\Delta} k(s) \rangle_H ds.
$$

On the other hand, we have, by (2.6):

$$
\int_0^t \langle G^* h(s), k(s) \rangle_H ds
$$
$$
= \int_0^t \left\langle e^{(T-s)\Delta} h(s) + \int_s^T \Delta e^{(\sigma-s)\Delta}[h(\sigma) - h(s)]d\sigma, \ k(s) \right\rangle_H ds
$$
$$
= \int_0^\tau \left\langle e^{(T-s)\Delta}\mathbf{y} - \int_\tau^T \Delta e^{(\sigma-s)\Delta}\mathbf{y}\, d\sigma, \ k(s) \right\rangle_H ds
$$
$$
= \int_0^\tau \langle e^{(\tau-s)\Delta}\mathbf{y}, k(s) \rangle_H ds = \int_0^\tau \langle \mathbf{y}, e^{(\tau-s)\Delta} k(s) \rangle_H ds,
$$

where we have used the integration by parts and the fact that, if $h(t) = e^{t\Delta}\mathbf{y}$, then $h'(t) = \Delta e^{t\Delta}\mathbf{y}$ for any $t > 0$. The claim follows now easily. $\qquad\square$

Lemma 2.2 suggests, replacing k by $\dot{W}$ in (2.8), that the natural meaning for the quantities involved in (2.7) is, for $h \in C^\varepsilon([0,T];H)$,

$$
\delta^X(h) = \int_0^T \langle G^* h(t), dW_t \rangle_H.
$$

This transformation holds true for deterministic integrands like h, and we will now see how to extend it to a large class of random processes, thanks to Skorokhod integration.

Notice that G^* is an isometry between $\mathcal{H}_X$ and a closed subset of $\mathcal{H}_W$ (see also [2] p.772), which means that

$$
\mathcal{H}_X = (G^*)^{-1}(\mathcal{H}_W).
$$

We also have $\mathbb{D}_X^{1,2}(\mathcal{H}_X) = (G^*)^{-1}(\mathbb{D}^{1,2}(\mathcal{H}_W))$, which gives a nice characterization of this Sobolev space. However, it will be more convenient to check the smoothness conditions of a process u with respect to X in the following subset of $\mathbb{D}_X^{1,2}(\mathcal{H}_X)$: let $\hat{\mathbb{D}}_X^{1,2}(\mathcal{H}_X)$ be the set of H-valued stochastic processes $u = \{u_t,\ t \in [0,T]\}$ verifying

$$
\mathrm{E} \int_0^T |G^* u_t|_H^2\, dt < \infty \tag{2.9}
$$

and

$$
\mathrm{E} \int_0^T d\tau \int_0^T dt\, \|D_\tau^X G^* u_t\|_{\mathrm{op}}^2 = \mathrm{E} \int_0^T d\tau \int_0^T dt\, \|G^* D_\tau^X u_t\|_{\mathrm{op}}^2 < \infty, \tag{2.10}
$$

8

where $\|A\|_{\mathrm{op}} = \sup_{|\mathbf{y}|_H=1} |A\mathbf{y}|_H$. Then, for $u \in \hat{\mathbb{D}}_X^{1,2}(\mathcal{H}_X)$, we can define the Skorokhod integral of u with respect to X by:

$$\int_0^T \langle u_s, \delta X_s \rangle := \int_0^T \langle G^* u_s, \delta W_s \rangle_H, \qquad (2.11)$$

and it is easily checked that expression (2.11) makes sense. This will be the meaning we will give to a stochastic integral with respect to X. Let us insist again on the fact that this is a natural definition: if $g(s) = \sum_{j=1}^k \mathbf{1}_{[t_j,t_{j+1})}(s)\mathbf{y}_j$ is a step function with values in H, we have:

$$\int_0^T \langle g(s), \delta X_s \rangle = \sum_{j=1}^k \langle \mathbf{y}_j, X_{t_{j+1}} - X_{t_j} \rangle_H.$$

Indeed, if $\mathbf{y} \in H$ and $t \in [0,T]$, an obvious computation gives $G^*(\mathbf{1}_{[0,t]}\mathbf{y})(s) = \mathbf{1}_{[0,t]}(s)e^{(t-s)\Delta}\mathbf{y}$, and hence we can write:

$$\int_0^T \langle \mathbf{1}_{[0,t]}(s)\mathbf{y}, \delta X_s \rangle = \int_0^t \langle e^{(t-s)\Delta}\mathbf{y}, \delta W_s \rangle_H = \int_0^t \langle \mathbf{y}, e^{(t-s)\Delta}\delta W_s \rangle_H = \langle \mathbf{y}, X_t \rangle_H.$$

2.2 Itô's type formula

We are now in a position to state precisely and prove the main result of this section.

Theorem 2.3 *Let $F : H \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function with bounded first, second and third derivatives. Then $F'(X) \in Dom(\delta^X)$ and:*

$$F(X_t) = F(0) + \int_0^t \langle F'(X_s), \delta X_s \rangle + \frac{1}{2}\int_0^t \mathrm{Tr}(e^{2s\Delta}F''(X_s))ds, \ t \in [0,T]. \quad (2.12)$$

Remark 2.4 By a standard approximation argument, we could relax the assumptions on F, and consider a general $\mathcal{C}_b^2$ function $F : H \to \mathbb{R}$.

Remark 2.5 As it was already said in the introduction, if $\mathrm{Tr}(F''(x))$ is uniformly bounded in $x \in H$, one can take limits in equation (1.4) as $N \to \infty$ to obtain:

$$F(X_t) = F(0) + \int_0^t \langle F'(X_s), dX_s \rangle_H + \frac{1}{2}\int_0^t \mathrm{Tr}(F''(X_s))ds, \ t \in [0,T]. \quad (2.13)$$

Here, the stochastic integral is naturally defined by

$$\int_0^t \langle F'(X_s), dX_s \rangle_H := \mathrm{L}^2 - \lim_{N\to\infty} \sum_{n=1}^N \int_0^t \partial_n F(X_s)dX_s^n.$$

In this case, the stochastic integrals in formulae (2.12) and (2.13) are obviously related by a simple algebraic equality. However, our formula (2.12) remains valid for any $\mathcal{C}_b^2$ function F, without any hypothesis on the trace of F''.

Proof of Theorem 2.3. For simplicity, assume that $F(0) = 0$. We will split the proof into several steps.

Step 1: strategy of the proof. Recall (see Section 2.1.1) that the set $\mathcal{M}$ is a total subset of $L^2(\Omega)$ and $\mathcal{M}$ itself is generated by the random variables of the form $\delta^{\mathrm{w}}(h^{\otimes m})$, $m \in \mathbb{N}$, with $h \in \mathcal{H}_{\mathrm{w}}$. Then, in order to obtain (2.12), it is sufficient to show:

$$\mathrm{E}[Y_m F(X_t)] = \mathrm{E}\left[Y_m \int_0^t \langle F'(X_s), \delta X_s \rangle\right] + \frac{1}{2}\mathrm{E}\left[Y_m \int_0^t \mathrm{Tr}(e^{2s\Delta}F''(X_s))ds\right], \quad (2.14)$$

where $Y_0 \equiv 1$ and, for $m \geq 1$, $Y_m = \delta^{\mathrm{w}}(h^{\otimes m})$ with $h \in \mathcal{H}_{\mathrm{w}}$. This will be done in Steps 2 and 3. The proof of the fact that $F'(X) \in \hat{\mathbb{D}}_{\mathrm{x}}^{1,2}(\mathcal{H}_{\mathrm{x}})$ is postponed at Step 4.

Step 2: the case $m = 0$. Set $\varphi(t, \mathbf{y}) = \mathrm{E}[F(e^{t\Delta}\mathbf{y} + X_t)]$, with $\mathbf{y} \in H$. Then, the Kolmogorov equation given *e.g.* in [9] p. 257, states that

$$\partial_t \varphi = \frac{1}{2}\mathrm{Tr}(\partial_{\mathbf{yy}}^2 \varphi) + \langle \Delta\mathbf{y}, \partial_{\mathbf{y}}\varphi \rangle_H. \quad (2.15)$$

Furthermore, in our case, we have:

$$\partial_{\mathbf{yy}}^2 \varphi(t, \mathbf{y}) = e^{2t\Delta}\mathrm{E}[F''(e^{t\Delta}\mathbf{y} + X_t)],$$

and since F'' is bounded:

$$\left|\mathrm{Tr}\left(\partial_{\mathbf{yy}}^2 \varphi(t, \mathbf{y})\right)\right| \leq \mathrm{cst}\sum_{n \geq 1} e^{-2\lambda_n t} \leq \frac{\mathrm{cst}}{t^{1/2}} \text{ for all } t > 0,$$

which means in particular that $\int_0^t \mathrm{Tr}\left(\partial_{\mathbf{yy}}^2 \varphi(s, \mathbf{y})\right) ds$ is a convergent integral. Then, applying (2.15) with $\mathbf{y} = 0$, we obtain:

$$\mathrm{E}[F(X_t)] = \varphi(t, 0) = \int_0^t \partial_s \varphi(s, 0)ds$$

$$= \frac{1}{2}\int_0^t \mathrm{Tr}(\partial_{\mathbf{yy}}^2 \varphi(s, 0))ds = \frac{1}{2}\int_0^t \mathrm{E}[\mathrm{Tr}(e^{2s\Delta}F''(X_s))]ds, \quad (2.16)$$

and thus, (2.14) is verified for $m = 0$.

Step 3: the general case. For the sake of readability, we will prove (2.14) only for $m = 2$, the general case $m \geq 1$ being similar, except for some cumbersome notations. Let us recall first that, according to (2.3), we can write, for $t \geq 0$:

$$Y_2 = \delta^{\mathrm{w}}(h^{\otimes 2}) = \int_0^T \langle u_t, \delta W_t \rangle_H = \delta^{\mathrm{w}}(u) \quad \text{with} \quad u_t = \left(\int_0^t \langle h(s), \delta W_s \rangle_H\right)h(t). \quad (2.17)$$

On the other hand, thanks to (1.3) and (2.1), it is readily seen that:

$$D_{s_1}^{\mathrm{w}}F(X_t) = \sum_{n \geq 1} e^{-\lambda_n(t-s_1)}\partial_n F(X_t)\mathbf{1}_{[0,t]}(s_1)\,\mathbf{e}_n \quad (2.18)$$

10

and

$$D_{s_2}^{\mathrm{W}}(D_{s_1}^{\mathrm{W}}F(X_t)) = \sum_{n,r\geq 1} e^{-\lambda_n(t-s_1)}e^{-\lambda_r(t-s_2)}\partial_{nr}^2 F(X_t)\mathbf{1}_{[0,t]}(s_1)\mathbf{1}_{[0,t]}(s_2)\,\mathbf{e}_n\otimes\mathbf{e}_r, \quad (2.19)$$

where $\partial^2 F(\mathbf{y})$ is interpreted as a quadratic form, for any $\mathbf{y}\in H$. Now, set

$$(G_{nr}^{\otimes 2}h)(t) := \frac{1}{2}\left(\int_0^t h^n(s_1)e^{-\lambda_n(t-s_1)}ds_1\right)\left(\int_0^t h^r(s_2)e^{-\lambda_r(t-s_2)}ds_2\right). \quad (2.20)$$

Putting together (2.17) and (2.18), we get:

$$\mathrm{E}[Y_2 F(X_t)] = \mathrm{E}[\delta^{\mathrm{W}}(u)F(X_t)] = \int_0^t ds_1\,\mathrm{E}\left[\langle u_{s_1}, D_{s_1}^{\mathrm{W}}F(X_t)\rangle_H\right]$$

$$= \int_0^t ds_1\,\mathrm{E}\left[\langle \delta^{\mathrm{W}}(\mathbf{1}_{[0,s_1]}h)h(s_1), D_{s_1}^{\mathrm{W}}F(X_t)\rangle_H\right]$$

$$= \sum_{n\geq 1}\int_0^t ds_1\,\mathrm{E}[\delta^{\mathrm{W}}(\mathbf{1}_{[0,s_1]}h)h^n(s_1)D_{s_1}^{n,W}F(X_t)]$$

$$= \sum_{n\geq 1}\int_0^t ds_1\,\mathrm{E}\left[\int_0^t ds_2\,\langle \mathbf{1}_{[0,s_1]}(s_2)h(s_2), h^n(s_1)D_{s_2}^{\mathrm{W}}\left(D_{s_1}^{n,\mathrm{W}}F(X_t)\right)\rangle_H\right],$$

where we have written $D_{s_1}^{n,\mathrm{W}}F(X_t)$ for the nth component in H of $D_{s_1}^{\mathrm{W}}F(X_t)$. Thus, invoking (2.19) and (2.20), we obtain

$$\mathrm{E}[Y_2 F(X_t)] = \sum_{n,r\geq 1}\int_0^t ds_1\int_0^{s_1} ds_2\,h^r(s_2)h^n(s_1)e^{-\lambda_n(t-s_1)}e^{-\lambda_r(t-s_2)}\mathrm{E}\left[\partial_{nr}^2 F(X_t)\right]$$

$$(2.21)$$

$$= \sum_{n,r\geq 1}(G_{nr}^{\otimes 2}h)(t)\mathrm{E}[\partial_{nr}^2 F(X_t)].$$

Let us differentiate now this expression with respect to t: setting $\psi_{nr}(s,\mathbf{y}) := \mathrm{E}[\partial_{nr}^2 F(e^{s\Delta}\mathbf{y}+X_s)]$, we have

$$\mathrm{E}[Y_2 F(X_t)] = A_1 + A_2,$$

where

$$A_1 := \sum_{n,r\geq 1}\int_0^t \mathrm{E}[\partial_{nr}^2 F(X_s)](G_{nr}^{\otimes 2}h)(ds) \quad\text{and}\quad A_2 := \sum_{n,r\geq 1}\int_0^t (G_{nr}^{\otimes 2}h)(s)\partial_s\psi_{nr}(s,0)ds.$$

Let us show now that

$$A_1 = E\left[Y_2\int_0^T \langle F'(X_s)\mathbf{1}_{[0,t]}(s), \delta X_s\rangle\right] \equiv \hat{A}_1.$$

Indeed, assume for the moment that $F'(X)\in\mathrm{Dom}(\delta)$. Then, the integration by parts (2.2) yields, starting from $\hat{A}_1$:

$$\hat{A}_1 = \mathrm{E}\left[Y_2\int_0^T \langle G^*F'(X_s)\mathbf{1}_{[0,t]}(s), \delta W_s\rangle_H\right]$$

$$= \mathrm{E}\left[\int_0^T \langle D_s^W Y_2, G^*F'(X_s)\mathbf{1}_{[0,t]}(s)\rangle_H ds\right],$$

11

and according to (2.4), we get

$$
\begin{aligned}
\hat{A}_1 &= \mathrm{E}\left[\delta^W(h)\int_0^T \langle h(s), G^*F'(X_s)\mathbf{1}_{[0,t]}(s)\rangle_H ds\right] \\
&= \int_0^t \langle Gh(ds), \mathrm{E}[\delta^W(h)F'(X_s)]\rangle_H \\
&= \sum_{n\geq 1}\int_0^t Gh^n(ds_1)\mathrm{E}\left[\int_0^T \langle h(s_2), D^W_{s_2}(\partial_n F(X_{s_1}))\rangle_H ds_2\right] \\
&= \sum_{n,r\geq 1}\int_0^t \mathrm{E}[\partial^2_{nr}F(X_{s_1})]Gh^n(ds_1)\int_0^{s_1} h^r(s_2)e^{-\lambda_r(s_1-s_2)}ds_2.
\end{aligned}
$$

Now, symmetrizing this expression in n,r we get

$$
\hat{A}_1 = \frac{1}{2}\sum_{n,r\geq 1}\int_0^t \mathrm{E}[\partial^2_{nr}F(X_{s_1})]\left[Gh^n(ds_1)\int_0^{s_1} h^r(s_2)e^{-\lambda_r(s_1-s_2)}ds_2\right.
$$
$$
\left. + Gh^r(ds_1)\int_0^{s_1} h^n(s_2)e^{-\lambda_n(s_1-s_2)}ds_2\right],
$$

and a simple use of (2.20) yields

$$
\hat{A}_1 = \sum_{n,r\geq 1}\int_0^t \mathrm{E}[\partial^2_{nr}F(X_{s_1})](G^{\otimes 2}_{nr}h)(ds_1) = A_1. \tag{2.22}
$$

Set now

$$
\hat{A}_2 = \mathrm{E}\left[Y_2\int_0^t \mathrm{Tr}(e^{2s\Delta}F''(X_s))ds\right],
$$

and let us show that $2A_2 = \hat{A}_2$. Indeed, using the same reasoning which was used to obtain (2.21), we can write:

$$
\begin{aligned}
\hat{A}_2 &= \mathrm{Tr}\left(\int_0^t e^{2s\Delta}\mathrm{E}[Y_2\,F''(X_s)]ds\right) \\
&= \mathrm{Tr}\left(\int_0^t e^{2s\Delta}\sum_{n,r\geq 1}(G^{\otimes 2}_{nr}h)(s)\mathrm{E}[\partial^2_{nr}F''(X_s)]\right) = 2A_2, \tag{2.23}
\end{aligned}
$$

by applying relation (2.16) to $\partial^2_{nr}F$. Thus, putting together (2.23) and (2.22), our Itô type formula is proved, except for one point whose proof has been omitted up to now, namely the fact that $F'(X)\in \mathrm{Dom}(\delta^x)$.

Step 4: To end the proof, it suffices to show that $F'(X)\in \hat{\mathbb{D}}^{1,2}_x(\mathcal{H}_x)$. To this purpose, we first verify (2.9), and we start by observing that

$$
\mathrm{E}\int_0^T |G^*F'(X_s)|^2_H ds \leq \mathrm{cst}\left(\int_0^T \mathrm{E}\left[|e^{(T-s)\Delta}F'(X_s)|^2_H\right]ds\right.
$$
$$
\left. + \int_0^T \mathrm{E}\left[\left(\int_s^T |\Delta e^{(t-s)\Delta}(F'(X_t)-F'(X_s))|_H dt\right)^2\right]ds.\right)
$$

12

Clearly, the hypothesis "F' is bounded" means, in our context, that:

$$\sup_{\mathbf{y} \in H} |F'(\mathbf{y})|_H^2 = \sup_{\mathbf{y} \in H} \sum_{n \geq 1} (\partial_n F(\mathbf{y}))^2 < \infty.$$

Then, we easily get

$$\mathrm{E} \int_0^T \left[|e^{(T-s)\Delta} F'(X_s)|_H^2 \right] ds = \int_0^T \sum_{n \geq 1} e^{-2\lambda_n (T-s)} \mathrm{E} \left[(\partial_n F(X_s))^2 \right] ds < \infty.$$

On the other hand, we also have that

$$
\begin{aligned}
|\Delta e^{(t-s)\Delta}(F'(X_t) - F'(X_s))|_H^2 &= \sum_{n \geq 1} \lambda_n^2 e^{-2\lambda_n (t-s)} (\partial_n F(X_t) - \partial_n F(X_s))^2 \\
&\leq \sup_{\alpha \geq 0} \{\alpha^2 e^{-2\alpha(t-s)}\} |F'(X_t) - F'(X_s)|_H^2 \\
&\leq \mathrm{cst}\, (t-s)^{-2} |X_t - X_s|_H^2 \sup_{\mathbf{y} \in H} \|F''(\mathbf{y})\|_{\mathrm{op}}^2.
\end{aligned}
$$

Thus, we can write:

$$\mathrm{E} \int_0^T \left[\left(\int_s^T |\Delta e^{(t-s)\Delta}(F'(X_t) - F'(X_s))|_H dt \right)^2 \right] ds \leq \mathrm{cst} \int_0^T f_T(s) ds,$$

with f_T given by

$$f_T(s) := \mathrm{E} \left\{ \left(\int_s^T (t-s)^{-1} |X_t - X_s|_H dt \right)^2 \right\}. \tag{2.24}$$

Fix now $\varepsilon > 0$ and consider the positive measure $\nu_s(dt) = (t-s)^{-1/2-2\varepsilon} dt$. Invoking Lemma 2.1, we get that

$$
\begin{aligned}
f_T(s) &= \mathrm{E} \left\{ \left(\int_s^T (t-s)^{-1/2+2\varepsilon} |X_t - X_s|_H \, \nu_s(dt) \right)^2 \right\} \\
&\leq \mathrm{cst}\, \nu_s([s,T]) \int_s^T (t-s)^{-1+4\varepsilon} \mathrm{E}(|X_t - X_s|_H^2) \nu_s(dt) \\
&\leq \mathrm{cst}\, (T-s)^{1/2-2\varepsilon} \int_s^T (t-s)^{-1+2\varepsilon} dt = \mathrm{cst}\, (T-s)^{1/2}.
\end{aligned}
$$

Hence, f_T is bounded on $[0, T]$ and (2.9) is verified.

We verify now (2.10). Notice first that $F'(X_t) \in H$, and thus $D^{\mathrm{W}} F'(X_t)$ can be interpreted as an operator valued random variable. Furthermore, thanks to (1.3), we can compute, for $\tau \in [0, T]$:

$$D_\tau^W F'(X_t) = \sum_{n \geq 1} D_\tau^W [\partial_n F(X_t)] \mathbf{e}_n = \sum_{n,r \geq 1} e^{-\lambda_r (t-\tau)} \partial_{nr}^2 F(X_t) \mathbf{1}_{[0,t]}(\tau) \, \mathbf{e}_n \otimes \mathbf{e}_r.$$

13

Hence $\|D_\tau^W F'(X_s)\|_{\mathrm{op}}^2 \le \|F''(X_s)\|_{\mathrm{op}}^2$ and

$$\mathrm{E} \int_0^T d\tau \left(\int_0^T ds \|e^{(T-s)\Delta} D_\tau^W F'(X_s)\|_{\mathrm{op}} \right)^2$$
$$\le \mathrm{E} \int_0^T d\tau \left(\int_0^T ds \|e^{(T-s)\Delta}\|_{\mathrm{op}} \|D_\tau^W F'(X_s)\|_{\mathrm{op}} \right)^2 < \infty, \quad (2.25)$$

according to the fact that $\|e^{(T-s)\Delta}\|_{\mathrm{op}} \le 1$. On the other hand, since X_t is $\mathcal{F}_t$-adapted, we get

$$\mathrm{E} \int_0^T d\tau \int_0^T ds \left(\int_s^T dt \|\Delta e^{(t-s)\Delta} (D_\tau^W F'(X_t) - D_\tau^W F'(X_s))\|_{\mathrm{op}} \right)^2 = B_1 + B_2,$$
$$(2.26)$$

with

$$B_1 \; := \; \mathrm{E} \int_0^T d\tau \int_0^\tau ds \left(\int_\tau^T dt \|\Delta e^{(t-s)\Delta} D_\tau^W F'(X_t)\|_{\mathrm{op}} \right)^2$$

$$B_2 \; := \; \mathrm{E} \int_0^T d\tau \int_\tau^T ds \left(\int_s^T dt \|\Delta e^{(t-s)\Delta} (D_\tau^W F'(X_t) - D_\tau^W F'(X_s))\|_{\mathrm{op}} \right)^2.$$

Moreover, for $\mathbf{y} \in H$ such that $|\mathbf{y}|_H = 1$ and $t > \tau$, we have:

$$|\Delta e^{(t-s)\Delta} D_\tau^W F'(X_t)\mathbf{y}|_H^2 = \sum_{n\ge1} \lambda_n^2 e^{-2\lambda_n(t-s)} \left(\sum_{r\ge1} e^{-\lambda_r(t-\tau)} \partial_{nr}^2 F(X_t)\mathbf{y}_r \right)^2$$
$$\le \sup_{\alpha\ge0}\{\alpha^2 e^{-2\alpha(t-s)}\} \sum_{n,r\ge1} e^{-2\lambda_r(t-\tau)}(\partial_{nr}^2 F(X_t))^2 \sum_{r\ge1} \mathbf{y}_r^2 \le \frac{\mathrm{cst}}{(t-s)^2},$$

and thus
$$\|\Delta e^{(t-s)\Delta} D_\tau^W F'(X_t)\|_{\mathrm{op}} \le \mathrm{cst}(t-s)^{-1},$$

from which we deduce easily

$$B_1 = \mathrm{E} \int_0^T d\tau \int_0^\tau ds \left(\int_\tau^T dt \|\Delta e^{(t-s)\Delta} D_\tau^W F'(X_t)\|_{\mathrm{op}} \right)^2 < \infty. \quad (2.27)$$

We also have, for $\mathbf{y} \in H$ such that $|\mathbf{y}|_H = 1$ and $t > s > \tau$:

$$|\Delta e^{(t-s)\Delta}(D_\tau^W F'(X_t) - D_\tau^W F'(X_s))\mathbf{y}|_H^2$$
$$= \sum_{n\ge1} \lambda_n^2 e^{-2\lambda_n(t-s)} \left[\sum_{r\ge1} \left(e^{-\lambda_r(t-\tau)} \partial_{nr}^2 F(X_t) - e^{-\lambda_r(s-\tau)} \partial_{nr}^2 F(X_s) \right) \mathbf{y}_r \right]^2$$
$$\le \sup_{\alpha\ge0}\{\alpha^2 e^{-2\alpha(t-s)}\} \sum_{n,r\ge1} \left(e^{-\lambda_r(t-\tau)} \partial_{nr}^2 F(X_t) - e^{-\lambda_r(s-\tau)} \partial_{nr}^2 F(X_s) \right)^2.$$

14

But, F'' and F''' being bounded, we can write:

$$\sum_{n,r\geq 1}\left(e^{-\lambda_r(t-\tau)}\partial_{nr}^2 F(X_t) - e^{-\lambda_r(s-\tau)}\partial_{nr}^2 F(X_s)\right)^2$$

$$\leq \text{cst}\sum_{n,r\geq 1}\left(e^{-\lambda_r(t-\tau)} - e^{-\lambda_r(s-\tau)}\right)^2\left(\partial_{nr}^2 F(X_t)\right)^2$$

$$+ \text{cst}\sum_{n,r\geq 1}\left(\partial_{nr}^2 F(X_t) - \partial_{nr}^2 F(X_s)\right)^2 e^{-2\lambda_r(s-\tau)}$$

$$\leq \text{cst}\sup_{\alpha\geq 0}\left(e^{-\alpha(t-\tau)} - e^{-\alpha(s-\tau)}\right)^2 \|F''(X_t)\|_{\text{op}}^2 + \text{cst}\|F''(X_t) - F''(X_s)\|_{\text{op}}^2$$

$$\leq \text{cst}\left\{(t-s)^2 + |X_t - X_s|_H^2\right\},$$

and consequently,

$$\|\Delta e^{(t-s)\Delta}(D_\tau^W F'(X_t) - D_\tau^W F'(X_s))\|_{\text{op}} \leq \text{cst}(t-s)^{-1}|X_t - X_s|_H$$

and

$$B_2 = \text{E}\int_0^T d\tau \int_\tau^T ds\left(\int_s^T dt\|\Delta e^{(t-s)\Delta}(D_\tau^W F'(X_t) - D_\tau^W F'(X_s))\|_{\text{op}}\right)^2$$

$$\leq \text{cst}\int_0^T d\tau \int_\tau^T ds f_T(s) \quad (2.28)$$

with f_T given by (2.24). By boundedness of f_T, and putting together (2.25), (2.26), (2.27) and (2.28), we obtain that (2.10) holds true, which ends the proof of our theorem.

$\square$

3   A Tanaka's type formula related to X

In this section, we will make a step towards a definition of the local time associated to the stochastic heat equation: we will establish a Tanaka's type formula related to X, for which we will need a little more notation. Let us denote $C_c(]0,1[)$ the set of real functions defined on $]0,1[$, with compact support. Let $\{G_t(x,y); t\geq 0, x,y\in[0,1]\}$ be the Dirichlet heat kernel on $[0,1]$, that is the fundamental solution to the equation

$$\partial_t h(t,x) = \partial_{xx}^2 h(t,x), \quad t\in[0,T], x\in[0,1], \qquad h(t,0) = h(t,1) = 0, \quad t\in[0,T].$$

Notice that, following the notations of Section 1, $G_t(x,y)$ can be decomposed as

$$G_t(x,y) = \sum_{n\geq 1} e^{-\lambda_n t}\mathbf{e}_n(x)\mathbf{e}_n(y). \quad (3.1)$$

Now, we can state:

Theorem 3.1 *Let $\varphi \in C_c(]0,1[)$ and $F_\varphi : H \to \mathbb{R}$ given by $F_\varphi(\ell) = \int_0^1 |\ell(x)|\varphi(x)dx$. Then:*

$$F_\varphi(X_t) = F_\varphi(0) + \int_0^t \left\langle F_\varphi'(X_s), \delta X_s \right\rangle + L_t^\varphi, \tag{3.2}$$

where $[F_\varphi'(\ell)](\tilde{\ell}) = \int_0^1 \text{sign}(\ell(x))\varphi(x)\tilde{\ell}(x)dx$ and L_t^φ is given by

$$L_t^\varphi = \int_0^t \int_0^1 \delta_0(X_s(x)) \, G_{2s}(x,x) \, \varphi(x)dx \, ds, \tag{3.3}$$

where δ_0 stands for the Dirac measure at 0, and $\delta_0(X_s(x))$ has to be understood as a distribution on the Wiener space associated to W.

3.1 An approximation result

In order to perform the computations leading to Tanaka's formula (3.2), it will be convenient to change a little our point of view on equation (1.2), which will be done in the next subsection.

3.1.1 The Walsh setting

We have already mentioned that the Brownian sheet W could be interpreted as the space-time white noise on $[0,T] \times [0,1]$, which means that W can be seen as a Gaussian family $\{W(h); h \in \mathcal{H}_w\}$, with

$$W(h) = \int_0^T \int_0^1 h(t,x)W(dt,dx), \qquad h \in \mathcal{H}_w$$

and

$$\mathrm{E}\left[W(h_1)W(h_2)\right] = \int_0^T \int_0^1 h_1(t,x)h_2(t,x) \, dtdx, \qquad h_1, h_2 \in \mathcal{H}_w,$$

and where we recall that $\mathcal{H}_w = L^2([0,T] \times [0,1])$. Associated to this Gaussian family, we can construct again a derivative operator, a divergence operator, some Sobolev spaces, that we will simply denote respectively by $D, \delta, \mathbb{D}^{k,2}$. These objects coincide in fact with the ones introduced at Section 2.1.1. Notice for instance that, for a given $m \geq 1$, and for a functional $F \in \mathbb{D}^{m,2}$, $D^m F$ will be considered as a random function on $([0,T] \times [0,1])^m$, denoted by $D^m_{(s_1,y_1),\ldots,(s_m,y_m)}F$. We will also deal with the multiple integrals with respect to W, that can be defined as follows: for $m \geq 1$ and $f_m : ([0,T] \times [0,1])^m \to \mathbb{R}$ such that $f_m(t_1, x_1, \ldots, t_m, x_m)$ is symmetric with respect to $(t_1, \ldots, t_m)$, we set

$$I_m(f_m) = m! \int_{0<t_1<\ldots<t_m<T} \int_{[0,1]^m} f(t_1, x_1, \ldots, t_m, x_m)W(dt_1,dx_1)\ldots W(dt_m,dx_m).$$

Eventually, we will use the negative Sobolev space $\mathbb{D}^{-1,2}$ in the sense of Watanabe, which can be defined as the dual space of $\mathbb{D}^{1,2}$ in $L^2(\Omega)$. We refer to [15] or [14] for a detailed account on the Malliavin calculus with respect to W. Notice in particular that the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ considered here is generated by the random variables $\{W(\mathbf{1}_{[0,s]} \times \mathbf{1}_A); s \leq t, A \text{ Borel set in } [0,1]\}$, which is useful for a correct definition

of $I_m(f_m)$. Then, the isometry relationship between multiple integrals can be read as:

$$\mathrm{E}\left[I_m(f_m)I_p(g_p)\right] = \begin{cases} 0 & \text{if } m \neq p \\ m! \left\langle f_m, g_m \right\rangle_{\mathcal{H}_{\mathrm{W}}^{\otimes m}} & \text{if } m = p, \end{cases}, \quad m, p \in \mathbb{N}$$

where $\mathcal{H}_{\mathrm{W}}^{\otimes m}$ has to be interpreted as $L^2(([0,T] \times [0,1])^m)$.

In this context, the stochastic convolution X can also be written according to Walsh's point of view (see [18]): set

$$G_{t,x}(s,y) := G_{t-s}(x,y)\mathbf{1}_{[0,t]}(s), \tag{3.4}$$

then, for $t \in [0,T]$ and $x \in [0,1]$, $X_t(x)$ is given by

$$X_t(x) = \int_0^T \int_0^1 G_{t,x}(s,y)W(ds,dy) = I_1\left(G_{t,x}\right). \tag{3.5}$$

3.1.2 A regularization procedure

For simplicity, we will only prove (3.2) for $t = T$. Now, we will get formula (3.2) by a natural method: we will first regularize the absolute value function $|\cdot|$ in order to apply the Itô formula (2.12), and then we pass to the limit as the regularization step tends to 0. To complete this program, we will use the following classical bounds (see for instance [3], p. 268) on the Dirichlet heat kernel: for all $\eta > 0$, their exist two constants $0 < c_1 < c_2$ such that, for all $x, y \in [\eta, 1 - \eta]$, we have:

$$c_1 t^{-1/2} \leq G_t(x,y) \leq c_2 t^{-1/2}. \tag{3.6}$$

from which we deduce that uniformly in $(t,x) \in [0,T] \times [\eta, 1 - \eta]$,

$$c_1 t^{1/2} \leq \int_0^t \int_\eta^{1-\eta} G_s(x,y)^2 ds dy \leq c_2 t^{1/2}. \tag{3.7}$$

Fix $\varphi \in C_c(]0,1[)$ and assume that φ has support in $[\eta, 1 - \eta]$. For $\varepsilon > 0$, let $F_\varepsilon : H \to \mathbb{R}$ be defined by

$$F_\varepsilon(\ell) = \int_0^1 \sigma_\varepsilon(\ell(x))\varphi(x)dx, \quad \text{with } \sigma_\varepsilon : \mathbb{R} \to \mathbb{R} \text{ given by } \sigma_\varepsilon = |\cdot| * p_\varepsilon,$$

where $p_\varepsilon(x) = (2\pi\varepsilon)^{-1/2}e^{-x^2/(2\varepsilon)}$ is the Gaussian kernel on $\mathbb{R}$ with variance $\varepsilon > 0$. For $t \in [0,T]$, let us also define the random variable

$$Z_t^\varepsilon = \mathrm{Tr}\left(e^{2t\Delta}F_\varepsilon''(X_t)\right) = \int_0^1 G_{2t}(x,x)\varphi(x)\sigma_\varepsilon''(X_t(x))dx. \tag{3.8}$$

We prove here the following convergence result:

Lemma 3.2 *If Z_t^ε is defined by (3.8), $\int_0^T Z_t^\varepsilon dt$ converges in L^2, as $\varepsilon \to 0$, towards the random variable L_T^φ defined by (3.3).*

17

Proof. Following the idea of [7], we will show this convergence result by means of the Wiener chaos decomposition of $\int_0^T Z_t^\varepsilon dt$, which will be computed firstly.

Stroock's formula states that any random variable $F \in \cap_{k \geq 1} \mathbb{D}^{k,2}$ can be expanded as

$$F = \sum_{m=0}^\infty \frac{1}{m!} I_m \left(\mathrm{E}\left[D^m F \right] \right).$$

In our case, a straightforward computation yields, for any $t \in [0,T]$ and $m \geq 1$,

$$D^m_{(s_1,y_1),\ldots,(s_m,y_m)} Z_t^\varepsilon$$
$$= \int_0^1 G_{2t}(x,x)\varphi(x) G_{t,x}^{\otimes m}((s_1,y_1),\ldots,(s_m,y_m)) \sigma_\varepsilon^{(m+2)}(X_t(x)) dx.$$

Moreover, since $\sigma_\varepsilon'' = \delta_0 * p_\varepsilon$, we have

$$\mathrm{E}\left[\sigma_\varepsilon^{(m+2)}(X_t(x)) \right] = m! \left(\varepsilon + v(t,x) \right)^{-m/2} p_{\varepsilon+v(t,x)}(0) H_m(0),$$

where $v(t,x)$ denotes the variance of the centered Gaussian random variable $X_t(x)$ and H_m is the mth Hermite polynomial:

$$H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} \left(e^{-\frac{x^2}{2}} \right),$$

verifying $H_m(0) = 0$ if m is odd and $H_m(0) = \frac{(-1)^{m/2}}{2^{m/2}(m/2)!}$ if m is even. Thus, the Wiener chaos decomposition of $\int_0^T Z_t^\varepsilon dt$ is given by

$$\int_0^T Z_t^\varepsilon dt$$
$$= \sum_{m \geq 1} \int_0^T dt \int_0^1 dx\, G_{2t}(x,x)\varphi(x) \left(\varepsilon + v(t,x) \right)^{-m/2} p_{\varepsilon+v(t,x)}(0) H_m(0) I_m(G_{t,x}^{\otimes m})$$
$$= \sum_{m \geq 1} \int_0^T dt \int_0^1 dx\, \beta_{m,\varepsilon}(t,x) I_m(G_{t,x}^{\otimes m}), \tag{3.9}$$

with

$$\beta_{m,\varepsilon}(t,x) := G_{2t}(x,x)\varphi(x) \left(\varepsilon + v(t,x) \right)^{-m/2} p_{\varepsilon+v(t,x)}(0) H_m(0), \quad m \geq 1.$$

We will now establish the L^2-convergence of $\int_0^T Z_t^\varepsilon dt$, using (3.9). For this purpose let us notice that each term

$$\int_0^T dt \int_0^1 dx\, \beta_{m,\varepsilon}(t,x) I_m(G_{t,x}^{\otimes m})$$

converges in $L^2(\Omega)$, as $\varepsilon \to 0$, towards

$$\int_0^T dt \int_0^1 dx\, G_{2t}(x,x)\varphi(x) v(t,x)^{-m/2} p_{v(t,x)}(0) H_m(0) I_m(G_{t,x}^{\otimes m}).$$

Thus, setting

$$\alpha_{m,\varepsilon} := \mathrm{E}\left\{\left(\int_0^T dt \int_0^1 dx\, \beta_{m,\varepsilon}(t,x) I_m(G_{t,x}^{\otimes m})\right)^2\right\},$$

the L^2-convergence of $\int_0^T Z_t^\varepsilon dt$ will be proven once we show that

$$\lim_{M\to\infty} \sup_{\varepsilon>0} \sum_{m\geq M} \alpha_{m,\varepsilon} = 0, \tag{3.10}$$

and hence once we control the quantity $\alpha_{m,\varepsilon}$ uniformly in ε. We can write

$$\alpha_{m,\varepsilon} = \int_{[0,T]^2} dt_1 dt_2 \int_{[0,1]^2} dx_1 dx_2\, \beta_{m,\varepsilon}(t_1,x_1)\beta_{m,\varepsilon}(t_2,x_2) \mathrm{E}\{I_m(G_{t_1,x_1}^{\otimes m})I_m(G_{t_2,x_2}^{\otimes m})\}.$$

Moreover

$$\mathrm{E}\{I_m(G_{t_1,x_1}^{\otimes m})I_m(G_{t_2,x_2}^{\otimes m})\} = m! \left\langle G_{t_1,x_1}^{\otimes m}, G_{t_2,x_2}^{\otimes m}\right\rangle_{L^2([0,T]\times[0,1])^m}$$

$$= m! \left(\int_{[0,T]\times[0,1]} G_{t_1-s}(x_1,y)\mathbf{1}_{[0,t_1]}(s) G_{t_2-s}(x_2,y)\mathbf{1}_{[0,t_2]}(s) ds dy\right)^m$$

$$=: m! \left(R(t_1,x_1,t_2,x_2)\right)^m.$$

Using (3.6), we can give a rough upper bound on $\beta_{m,\varepsilon}(t,x)$:

$$|\beta_{m,\varepsilon}(t,x)| \leq |G_{2t}(x,x)|\,|\varphi(x)| \frac{1}{v(t,x)^{\frac{m+1}{2}}} \frac{\mathrm{cst}}{2^{\frac{m}{2}}(\frac{m}{2})!} \leq \frac{\mathrm{cst}\,|\varphi(x)|}{2^{\frac{m}{2}}(\frac{m}{2})!\, t^{\frac{1}{2}} v(t,x)^{\frac{m+1}{2}}}.$$

Then, thanks to the fact that $\varphi = 0$ outside $[\eta, 1-\eta]$, we get

$$\alpha_{m,\varepsilon} \leq c_m \int_{([0,T]\times[\eta,1-\eta])^2} dt_1 dt_2 dx_1 dx_2 \frac{|R(t_1,x_1,t_2,x_2)|^m\,|\varphi(x_1)|\,|\varphi(x_2)|}{t_1^{1/2} t_2^{1/2} v(t_1,x_1)^{(m+1)/2} v(t_2,x_2)^{(m+1)/2}},$$

with

$$c_m = \frac{\mathrm{cst}\,m!}{2^m\,[(m/2)!]^2} \leq \frac{\mathrm{cst}}{\sqrt{m}},$$

by Stirling formula. Assume, for instance, $t_1 \leq t_2$. Invoking the decomposition (3.1) of $G_t(x,y)$ and the fact that $\{\mathbf{e}_n; n \geq 1\}$ is an orthogonal family, we obtain

$$R(t_1,x_1,t_2,x_2) = \int_0^{t_1} ds \int_0^1 dy\, G_{t_1-s}(x_1,y)G_{t_2-s}(x_2,y)$$

$$= \int_0^{t_1} ds \int_0^1 dy \left(\sum_{n\geq 1} e^{-\lambda_n(t_1-s)}\mathbf{e}_n(x_1)\mathbf{e}_n(y)\right)\left(\sum_{r\geq 1} e^{-\lambda_r(t_2-s)}\mathbf{e}_r(x_2)\mathbf{e}_r(y)\right)$$

$$= \sum_{n\geq 1} \mathbf{e}_n(x_1)\mathbf{e}_n(x_2) \int_0^{t_1} ds\, e^{-\lambda_n[(t_1-s)+(t_2-s)]} = \sum_{n\geq 1} \frac{2}{\lambda_n}\mathbf{e}_n(x_1)\mathbf{e}_n(x_2)e^{-\lambda_n t_2}\sinh(\lambda_n t_1),$$

and using the same kind of arguments, we can write, for $k = 1,2$:

$$v(t_k,x_k) = \sum_{n\geq 1} \frac{2}{\lambda_n}\mathbf{e}_n(x_k)^2 e^{-\lambda_n t_k}\sinh(\lambda_n t_k).$$

19

Now Cauchy-Schwarz's inequality gives

$$R(t_1, x_1, t_2, x_2)$$

$$\leq \left\{ \sum_{n\geq 1} \frac{2}{\lambda_n} \mathbf{e}_n(x_1)^2 e^{-\lambda_n t_2} \sinh(\lambda_n t_1) \right\}^{1/2} \left\{ \sum_{n\geq 1} \frac{2}{\lambda_n} \mathbf{e}_n(x_2)^2 e^{-\lambda_n t_2} \sinh(\lambda_n t_1) \right\}^{1/2}$$

$$\leq \left\{ \sum_{n\geq 1} \frac{2}{\lambda_n} \mathbf{e}_n(x_1)^2 e^{-\lambda_n t_2} \sinh(\lambda_n t_1) \right\}^{1/2} v(t_2, x_2)^{1/2}.$$

Introduce the expression

$$A(t_1, t_2, x_1) := \sum_{n\geq 1} \frac{2}{\lambda_n} \mathbf{e}_n(x_1)^2 e^{-\lambda_n t_2} \sinh(\lambda_n t_1) = \int_0^{t_1} G_{t_1+t_2-2s}(x_1, x_1) ds.$$

We have obtained that $R(t_1, x_1, t_2, x_2) \leq A(t_1, t_2, x_1)^{1/2} v(t_2, x_2)^{1/2}$. Notice that (3.7) yields $c_1 t^{1/2} \leq v(t, x) \leq c_2 t^{1/2}$ uniformly in $x \in [\eta, 1-\eta]$. Thus, we obtain

$$\alpha_{m,\varepsilon} \leq \frac{\text{cst}}{\sqrt{m}} \int_{([0,T]\times[\eta,1-\eta])^2} dt_1 dt_2 dx_1 dx_2 \frac{v(t_2, x_2)^{m/2} A(t_1, t_2, x_1)^{m/2} |\varphi(x_1)| |\varphi(x_2)|}{t_1^{1/2} t_2^{1/2} v(t_1, x_1)^{(m+1)/2} v(t_2, x_2)^{(m+1)/2}},$$

and hence

$$\alpha_{m,\varepsilon} \leq \frac{\text{cst}}{\sqrt{m}} \int_{([0,T]\times[\eta,1-\eta])^2} dt_1 dt_2 dx_1 dx_2 \frac{t_2^{m/4}}{t_1^{1/2} t_2^{1/2} t_1^{(m+1)/2} t_2^{(m+1)/2}}$$

$$\left(\int_0^{t_1} G_{t_1+t_2-2s}(x_1, x_1) ds \right)^{m/2}.$$

Hence, according to (3.6), we get

$$\alpha_{m,\varepsilon} \leq \frac{\text{cst}}{\sqrt{m}} \int_0^T t_1^{-(m+3)/4} dt_1 \int_{t_1}^T t_2^{-3/4} \left[(t_2+t_1)^{1/2} - (t_2-t_1)^{1/2} \right]^{m/2} dt_2$$

$$\leq \frac{\text{cst}}{\sqrt{m}} \int_0^T t_1^{-(m+3)/4} dt_1 \int_{t_1}^T t_2^{-3/4} \frac{t_1^{m/2}}{t_2^{m/4}} dt_2 \leq \frac{\text{cst}}{\sqrt{m}} \int_0^T t_1^{(m-3)/4} dt_1 \int_{t_1}^T \frac{dt_2}{t_2^{(m+3)/4}} \leq \frac{\text{cst}}{m^{3/2}}.$$

Consequently, the series $\sum_m \alpha_{m,\varepsilon}$ converges uniformly in $\varepsilon > 0$, which gives immediately (3.10).

Thus, we obtain that $\int_0^T Z_t^\varepsilon dt \to \mathcal{Z}$ in $L^2(\Omega)$, as $\varepsilon \to 0$, where

$$\mathcal{Z} := \sum_{m\geq 1} \int_0^T dt \int_0^1 dx \, G_{2t}(x,x) \varphi(x) v(t,x)^{-m/2} p_{v(t,x)}(0) H_m(0) I_m(G_{t,x}^{\otimes m}).$$

To finish the proof we need to identify $\mathcal{Z}$ with (3.3). First, let us give the precise meaning of (3.3). Using (3.5), we can write

$$L_T^\varphi = \int_0^T \int_0^1 \delta_0(W(G_{t,x})) G_{2t}(x,x) \varphi(x) dx dt,$$

20

where we recall that δ_0 stands for the Dirac measure at 0, and we will show that $L_T^\varphi \in \mathbb{D}^{-1,2}$ (this latter space has been defined at Section 3.1.1). Indeed, (see also [16], p. 259), for any random variable $U \in \mathbb{D}^{1,2}$, with obvious notation for the Sobolev norm of U, we have

$$|\mathrm{E}\left(U\delta_0(W(G_{t,x}))\right)| \leq \frac{\|U\|_{1,2}}{|G_{t,x}|_{\mathcal{H}_{\mathrm{W}}}} \leq \mathrm{cst}\frac{\|U\|_{1,2}}{s^{1/4}},$$

using (3.4) and (3.7). This yields

$$|\mathrm{E}\left(UL_T^\varphi\right)| \leq \mathrm{cst} \int_0^T \int_\eta^{1-\eta} \frac{\|U\|_{1,2}}{t^{1/4}}|G_{2t}(x,x)|\,|\varphi(x)|dxdt < \infty,$$

according to (3.6). Similarly, $\int_0^T Z_t^\varepsilon dt \in \mathbb{D}^{-1,2}$, since

$$\int_0^T Z_t^\varepsilon dt = \int_0^T \int_0^1 \sigma_\varepsilon''(W(G_{t,x}))G_{2t}(x,x)\varphi(x)dxdt$$

and the same reasoning applies. Moreover $\int_0^T Z_t^\varepsilon dt \to L_T^\varphi$ in $\mathbb{D}^{-1,2}$ as $\varepsilon \to 0$. Indeed, for any random variable $U \in \mathbb{D}^{1,2}$,

$$\mathrm{E}\left\{U\left(\int_0^T Z_t^\varepsilon dt - L_T^\varphi\right)\right\} = \int_0^T \int_0^1 dxdtG_{2t}(x,x)\varphi(x)$$
$$\times \mathrm{E}\left\{U\left[\sigma_\varepsilon''(W(G_{t,x})) - \delta_0(W(G_{t,x}))\right]\right\}$$

and, as in [16],

$$\mathrm{E}\left\{U\left[\sigma_\varepsilon''(W(G_{t,x})) - \delta_0(W(G_{t,x}))\right]\right\}$$
$$= \mathrm{E}\left\{\frac{1}{|G_{t,x}|_{\mathcal{H}_{\mathrm{W}}}^2}U\langle D^W[\sigma_\varepsilon'(W(G_{t,x})) - \mathrm{sgn}(W(G_{t,x}))], G_{t,x}\rangle_{\mathcal{H}_{\mathrm{W}}}\right\}$$
$$= \frac{1}{|G_{t,x}|_{\mathcal{H}_{\mathrm{W}}}^2}\mathrm{E}\left\{(\sigma_\varepsilon' - \mathrm{sgn})(W(G_{t,x}))\delta^W(UG_{t,x})\right\}.$$

By Cauchy-Schwarz inequality, the right hand side is bounded by

$$\frac{1}{|G_{t,x}|_{\mathcal{H}_{\mathrm{W}}}^2}\left\{\mathrm{E}\,|(\sigma_\varepsilon' - \mathrm{sgn})(W(G_{t,x}))|^2\right\}^{\frac{1}{2}}\left\{\mathrm{E}\left|UW(G_{t,x}) - \langle G_{t,x}, D^WU\rangle_{\mathcal{H}_{\mathrm{W}}}\right|^2\right\}^{\frac{1}{2}}$$

and the conclusion follows using again (3.6) and (3.7), and also the fact that $\sigma_\varepsilon' \to$ sgn, as $\varepsilon \to 0$.

Finally, it is clear that $L_T^\varphi = \mathcal{Z}$ in $\mathbb{D}^{-1,2}$, and also in the L^2-sense. The proof of Lemma 3.2 is now complete.

$\square$

3.2 Proof of Theorem 3.1

In order to prove relation (3.2) (only for $t = T$ for simplicity), let us take up our regularization procedure: for any $\varepsilon > 0$, we have, according to (2.12), that

$$F_\varepsilon(X_T) = F_\varepsilon(0) + \int_0^T \langle F_\varepsilon'(X_t), \delta X_t \rangle + \int_0^T Z_t^\varepsilon dt. \tag{3.11}$$

We have seen that $\int_0^T Z_t^\varepsilon dt \to L_T^\varphi$ as $\varepsilon \to 0$, in $L^2(\Omega)$. Since it is obvious that $F_\varepsilon(X_T) - F_\varepsilon(0)$ converges in $L^2(\Omega)$ to $F_\varphi(X_T) - F_\varphi(0)$, a simple use of formula (3.11) shows that $\int_0^T \langle F_\varepsilon'(X_t), \delta X_t \rangle$ converges. In order to obtain (3.2), it remains to prove that

$$\lim_{\varepsilon \to 0} \int_0^T \langle F_\varepsilon'(X_t), \delta X_t \rangle = \int_0^T \langle F_\varphi'(X_t), \delta X_t \rangle. \tag{3.12}$$

But, from standard Malliavin calculus results (see, for instance, Lemma 1, p. 304 in [7]), in order to prove (3.12), it is sufficient to show that

$$G^* V^\varepsilon \to G^* V \text{ as } \varepsilon \to 0, \text{ in } L^2([0,T] \times \Omega; H), \tag{3.13}$$

with

$$V^\varepsilon(t) = F_\varepsilon'(X_t) = \sigma_\varepsilon'(X_t)\varphi \in H \text{ and } V(t) = \text{sgn}(X_t)\varphi \in H.$$

We will now prove (3.13) through several steps, adapting in our context the approach used in [7].

Step 1. To begin with, let us first establish the following result:

Lemma 3.3 *For $s, t \in (0, T)$, $x \in [\eta, 1 - \eta]$ and $a \in \mathbb{R}$,*

$$P(X_t(x) > a, X_s(x) < a) \leq \text{cst} \, (t - s)^{1/4} s^{-1/2}, \tag{3.14}$$

where the constant depends only on T, a and η.

Proof. The proof is similar to the one given for Lemma 4, p. 309 in [7]. Indeed, the first part of that proof can be invoked in our case since $(X_t(x), X_s(x))$ is a centered Gaussian vector (with covariance $\omega(s, t, x)$). Hence we can write

$$P(X_t(x) > a, X_s(x) < a) \leq \frac{1 + |a|\rho\sqrt{2\pi}}{2\pi} \sqrt{\frac{v(t,x)v(s,x)}{\omega(s,t,x)^2} - 1}, \tag{3.15}$$

where

$$\rho^2 = \frac{E[(X_t(x) - X_s(x))^2]}{v(t,x)v(s,x) - \omega(s,t,x)^2}. \tag{3.16}$$

Furthermore, it is a simple computation to show that

$$\omega(s, t, x) = E[X_t(x)X_s(x)] \geq \text{cst} \, s^{1/2}. \tag{3.17}$$

Indeed, using again (3.6) we deduce that

$$E[X_t(x)X_s(x)] = \int_0^s du \int_0^1 dy \, G_{t-u}(x, y)G_{s-u}(x, y)$$

22

$$\geq \int_0^s du \int_\eta^{1-\eta} dy\, G_{t-u}(x,y)G_{s-u}(x,y) \geq \operatorname{cst} \int_0^s \frac{du}{\sqrt{(t-u)(s-u)}}$$

$$= \operatorname{cst} \int_0^{\frac{s}{t-s}} \frac{du}{\sqrt{(1+w)w}} \geq \operatorname{cst} \sqrt{\frac{t-s}{t}} \int_0^{\frac{s}{t-s}} \frac{du}{\sqrt{u}} = \operatorname{cst} \sqrt{\frac{s}{t}} \geq \operatorname{cst} \sqrt{s}.$$

Moreover, one can observe, as in [7], that

$$v(t,x)v(s,x) - \omega(s,t,x)^2 \leq \operatorname{E}\left[(X_t(x) - X_s(x))^2\right] \operatorname{E}\left[X_s(x)^2\right].$$

Consequently,

$$\sqrt{\frac{v(t,x)v(s,x)}{\omega(s,t,x)^2} - 1} \leq \operatorname{cst} (t-s)^{1/4} s^{-1/4},$$

since it is well-known that

$$\operatorname{E}\left[(X_t(x) - X_s(x))^2\right] \leq \operatorname{cst} (t-s)^{1/2}.$$

Eventually, following again [7], we get that

$$\rho \sqrt{\frac{v(t,x)v(s,x)}{\omega(s,t,x)^2} - 1} = \frac{\sqrt{\operatorname{E}\left[(X_t(x) - X_s(x))^2\right]}}{\omega(s,t,x)}.$$

Inequality (3.14) follows now easily.

$\square$

Step 2. We shall prove that $G^*V \in \operatorname{L}^2([0,T] \times \Omega; H)$. First, using the fact that $\left\| e^{(T-t)\Delta} \right\|_{\operatorname{op}} \leq 1$, we remark that

$$\operatorname{E}\left[\int_0^T \left| e^{(T-t)\Delta} \operatorname{sgn}(X_t)\varphi \right|_H^2 dt\right] \leq \operatorname{E}\left[\int_0^T \left\| e^{(T-t)\Delta} \right\|_{\operatorname{op}}^2 |\operatorname{sgn}(X_t)\varphi|_H^2 dt\right] < \infty.$$

Now, let us denote by A the quantity

$$A := \operatorname{E}\left[\int_0^T \left| \int_t^T \Delta e^{(r-t)\Delta} \left(\operatorname{sgn}(X_r)\varphi - \operatorname{sgn}(X_t)\varphi\right) dr \right|_H^2 dt\right].$$

We have

$$A \leq \operatorname{E}\left[\int_0^T \left(\int_t^T \left\| \Delta e^{(r-t)\Delta} \right\|_{\operatorname{op}} |\operatorname{sgn}(X_r)\varphi - \operatorname{sgn}(X_t)\varphi|_H \, dr\right)^2 dt\right],$$

with

$$\operatorname{sgn}(X_r(x)) - \operatorname{sgn}(X_t(x)) = 2\left(U_{r,t}^+(x) - U_{r,t}^-(x)\right)$$

where $U_{r,t}^+(x) = \mathbf{1}_{\{X_r(x)>0,\, X_t(x)<0\}}$ and $U_{r,t}^-(x) = \mathbf{1}_{\{X_r(x)<0,\, X_t(x)>0\}}$. Thus

$$A \leq \operatorname{cst} \int_0^T dt\, \operatorname{E}\left[\left(\int_t^T \frac{dr}{r-t}\left(\int_0^1 dx \left[\left(U_{r,t}^+(x) - U_{r,t}^-(x)\right)\varphi(x)\right]^2\right)^{1/2}\right)^2\right]$$

$$\leq \operatorname{cst} \int_0^T dt\, \operatorname{E}\left[\left(\int_t^T \frac{dr}{r-t}\left(\int_0^1 dx\, U_{r,t}^+(x)\, \varphi(x)^2\right)^{1/2}\right)^2\right].$$

Then $A \leq \text{cst} \int_0^T A_t dt$ with

$$A_t := \int_t^T \frac{dr_2}{r_2 - t} \int_t^T \frac{dr_1}{r_1 - t} \, \mathrm{E} \left[\left(\int_0^1 U_{r_1,t}^+(x) \varphi(x)^2 dx \right)^{1/2} \right.$$
$$\left. \left(\int_0^1 U_{r_2,t}^+(x) \varphi(x)^2 dx \right)^{1/2} \right],$$

which gives

$$
\begin{aligned}
A_t &\leq \int_t^T \frac{dr_2}{r_2 - t} \int_t^T \frac{dr_1}{r_1 - t} \left(\int_{[0,1]^2} dx_1 dx_2 \varphi(x_1)^2 \varphi(x_2)^2 \mathrm{E} \left[U_{r_1,t}^+(x_1) U_{r_2,t}^+(x_2) \right] \right)^{1/2} \\
&\leq \int_t^T \frac{dr_2}{r_2 - t} \int_t^T \frac{dr_1}{r_1 - t} \left(\int_0^1 dx_1 \varphi(x_1)^2 \mathrm{E} \left[U_{r_1,t}^+(x_1) \right]^{1/2} \right)^{1/2} \\
&\qquad\qquad \left(\int_0^1 dx_2 \varphi(x_2)^2 \mathrm{E} \left[U_{r_2,t}^+(x_2) \right]^{1/2} \right)^{1/2} \\
&= \left[\int_t^T \frac{dr}{r - t} \left(\int_0^1 dx \, \varphi(x)^2 \, \mathrm{P} \left[X_r(x) > 0, \, X_t(x) < 0 \right]^{1/2} \right)^{1/2} \right]^2.
\end{aligned}
$$

Plugging (3.14) into this last inequality, we easily get that $G^*V \in \mathrm{L}^2([0,T] \times \Omega, H)$. The remainder of the proof follows now closely the steps developed in [7] and the details are left to the reader.

$\square$

References

[1] E. Alòs, O. Mazet, D. Nualart. Stochastic calculus with respect to fractional Brownian motion with Hurst parameter lesser than $1/2$. *Stoch. Proc. Appl.* **86**, 121-139, 2000.

[2] E. Alòs, O. Mazet, D. Nualart. Stochastic calculus with respect to Gaussian processes. *Ann. Probab.* **29**, 766-801, 2001.

[3] M. van den Berg. Gaussian bounds for the Dirichlet heat kernel. *J. Funct. Anal.* **88**, 267-278, 1990.

[4] R. Carmona, B. Rozovskii (Eds). Stochastic partial differential equations: six perspectives. Providence, Rhode Island : AMS xi, 334 pages, 1999.

[5] S. Cerrai. Second order PDE's in finite and infinite dimension. A probabilistic approach. *Lect. Notes in Math.* **1762**, 330 pages, 2001.

[6] P. Cheridito, D. Nualart. Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter H in (0,1/2). Preprint.

[7] L. Coutin, D. Nualart, C. Tudor. Tanaka formula for the fractional Brownian motion. *Stoch. Proc. Appl.* **94**, 301-315, 2001.

[8] R. Dalang. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.* **4**, no. 6, 29 pp, 1999.

[9] G. Da Prato, J. Zabczyk. Stochastic equations in infinite dimensions. *Cambridge University Press*, xviii, 454 pages, 1992.

[10] G. Da Prato, J. Zabczyk. Ergodicity for infinite dimensional systems, xii, 339 pages, 1996. *Cambridge University Press*.

[11] J. Guerra, D. Nualart. The $1/H$-variation of the divergence integral with respect to the fractional Brownian motion for $H > 1/2$ and fractional Bessel processes. *Preprint Barcelona*, 2004.

[12] Y. Hu, D. Nualart. Some processes associated with fractional Bessel processes. *Preprint Barcelona*, 2004.

[13] J. León, D. Nualart. Stochastic evolution equations with random generators. *Ann. Probab* **26**, no 1, 149-186, 1998.

[14] P. Malliavin. Stochastic analysis. *Springer-Verlag*, 343 pages, 1997.

[15] D. Nualart. The Malliavin calculus and related topics. *Springer-Verlag*, 266 pages, 1995.

[16] D. Nualart, J. Vives. Smoothness of Brownian local times and related functionals *Potential Anal.* **1**, no. 3, 257-263, 1992

[17] C. Tudor, F. Viens. Itô formula and local time for the fractional Brownian sheet. *Electron. J. Probab.* **8**, no. 14, 31 pp, 2003.

[18] J. Walsh. An introduction to stochastic partial differential equations. *Lecture Notes in Math.* **1180**, 1986.

[19] L. Zambotti. A reflected stochastic heat equation as symmetric dynamics with respect to the 3-d Bessel bridge. *J. Funct. Anal.* **180**, no. 1, 195-209, 2001.

[20] L. Zambotti. Itô-Tanaka formula for SPDEs. Preprint, 2004.