On the absolute continuity of one-dimensional SDE's driven by a fractional Brownian motion

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Abstract

The problem of absolute continuity for a class of SDE's driven by a real fractional Brownian motion of any Hurst index is adressed. First, we give an elementary proof of the fact that the solution to the SDE has a positive density for all $t > 0$ when the diffusion coefficient does not vanish, echoing in the fractional Brownian framework the main result we had previously obtained for Marcus equations driven by Lévy processes [9]. Second, we extend in our setting the classical entrance-time criterion of Bouleau-Hirsch[2].

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1 Introduction

In this note we study the absolute continuity of the solutions at any time $t > 0$ to SDE's of the type:

$$X_t \;=\; x_0 \;+\; \int_0^t b(X_s)\, ds \;+\; \int_0^t \sigma(X_s) \diamond dB_s^H, \tag{1}$$

where b, σ are real functions and B^H is a linear fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$. In (1), $\diamond$ means a particular type of linear non-semimartingale integrators, the so-called Newton-Côtes integrator, which was recently introduced by one of us *et al.* [7] [8]. Roughly speaking, $\diamond$ is an operator defined through a limiting procedure involving the usual Newton-Côtes linear approximator (whose order depends on the roughness of the path B^H), and a forward-backward decomposition à la Russo-Vallois [12]. This gives a reasonable class of solutions to (1) as soon as σ is regular enough. We refer to [7] and [8] for more details on this topic.

The main interest of $\diamond$ is that it yields a first order Itô's formula: if $f : \mathbb{R}^2 \to \mathbb{R}$ is regular enough and $Y : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is a bounded variation process, then for every $t \geq 0$

$$f(B_t^H, Y_t) = f(0, Y_0) + \int_0^t f'_x(B_s^H, Y_s) \diamond dB_s^H + \int_0^t f'_y(B_s^H, Y_s) dY_s, \tag{2}$$

see [8] for details. This formula allows to solve (1) through Doss [5] and Sussmann [13]'s classical computations. More precisely, our solution X is given by $X_t = \varphi(B_t^H, Y_t)$ for every $t > 0$ where $(x,y) \mapsto \varphi(x,y)$ is the flow associated to σ:

$$\varphi'_x(x,y) = \sigma(\varphi(x,y)),\ \varphi(0,y) = y \text{ for every } (x,y) \in \mathbb{R}^2, \tag{3}$$

and Y is the solution to the random ODE

$$Y_t \;=\; x_0 \;+\; \int_0^t a(B_s^H, Y_s)\, ds,$$

with the notation

$$a(x,y) \;=\; \frac{b(\varphi(x,y))}{\varphi'_y(x,y)} \;=\; b(\varphi(x,y)) \exp\left\{ - \int_0^x \sigma'(\varphi(u,y))\, du \right\} \tag{4}$$

for every $(x,y) \in \mathbb{R}^2$. In the sequel, we will only refer to X as given by the above Doss-Sussmann transformation and we will study the absolute continuity with respect to the Lebesgue measure of X_t for any $t > 0$.

Our first result, which is given in Section 2, states that X_t has a positive density for every $t > 0$ as soon as σ does not vanish. Notice that in the much more difficult framework where the driving process of (1) is a non Gaussian Lévy process with infinitely many jumps, the same criterion was obtained in [9]. Here, the simple proof relies on a suitable Girsanov transformation [11] which reduces to the easy case when $b \equiv 0$, i.e. when $X_t = \varphi(B_t^H, x_0)$ for every $t > 0$. This positivity result is related to Proposition 6 in [1], where in a multidimensional setting but without drift, a sufficient condition (which becomes $\sigma(x_0) \neq 0$ in dimension one) under which X_t has a density for every $t > 0$ was given, as well as an equivalent of the density f_t at x_0 when $t \to 0$. We remark that in dimension one, a closed formula - see (5) below - can be readily obtained.

Of course, this non-vanishing condition on σ is not optimal. For instance, thinking of the equation $dX_t = X_t \diamond dB_t^H$ whose solution is $X_t = X_0 \exp B_t^H$, we see that the positivity assumption on σ is not necessary. Moreover, in the Brownian case $H = 1/2$, it is well-known that this criterion can be relaxed either into a condition of Hörmander type when σ is regular enough - see e.g. [10] p. 111, or into an optimal criterion involving the entrance time into $\{\sigma(x) \neq 0\}$ when σ has little regularity - see Theorem 6.3. in [2]. We did not try to go in the Hörmander direction, since the computations involving Newton-Côtes integrals become quite messy. Nevertheless, we were able to obtain a literal extension of Bouleau-Hirsch's criterion for *any* $H \in (0,1)$. This extension may seem a little surprising, since Bouleau-Hirsch's criterion bears a Markovian flavour, whereas the solution to our SDE is not Markovian in general. The proof, which is given in Section 3, consists in computing the Malliavin derivative of X_t via the Doss-Sussmann transformation, and then using a general non-degeneracy criterion of Nualart-Zakai.

Notice finally that the computation of this Malliavin derivative relies mainly on the existence of a Stratonovich change of variable formula. Hence, our Theorem B below could probably be extended to other type of "rough" equations driven by fBm, see e.g. [4] and [6]. In these two papers there are restrictions from below on the Hurst parameter of the driving fBm, but on the other hand this latter is allowed to be multidimensional. Since Bouleau-Hirsch's criterion also works in a multidimensional framework (with a more complicated formulation for the entrance-time), one may ask for a general fractional extension of this result. The present note can be viewed as a first attempt in this direction.

2 A non-vanishing criterion on the diffusion coefficient

The following theorem, whose proof is elementary, yields a first simple criterion on σ according to which X_t has a positive density on $\mathbb{R}$ for every $t > 0$.

Theorem A *If σ does not vanish, then X_t has a positive density on $\mathbb{R}$ for every $t > 0$.*

Proof. Considering $-B^H$ instead of B^H if necessary, we may suppose that $\sigma > 0$. Recalling that $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is the flow associated with σ, we first notice that for every fixed $y \in \mathbb{R}$, the function $x \mapsto \varphi(x, y)$ is a bijection onto $\mathbb{R}$. Indeed, $\varphi(\cdot, y)$ is clearly increasing and $\ell = \lim_{x\to+\infty} \varphi(x, y)$ exists in $\mathbb{R} \cup \{+\infty\}$. If $\ell \neq +\infty$, then $\lim_{x\to\infty} \varphi'_x(x, y) = \sigma(\ell) > 0$ and $\lim_{x\to+\infty} \varphi(x, y) = +\infty$. Similarly, we can show that $\lim_{x\to-\infty} \varphi(x, y) = -\infty$, which yields the desired property. We will denote by $\psi : \mathbb{R}^2 \to \mathbb{R}$ the inverse of φ, i.e. $\psi(x, y)$ is the unique solution to $\varphi(\psi(x, y), y) = x$.

(i) When $b \equiv 0$, we have $X_t = \varphi(B_t^H, x_0)$ for every $t > 0$ and we can write, for every $A \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned}\mathbb{P}(X_t \in A) \;=\; \mathbb{P}(B_t^H \in \psi(A, x_0)) \;&=\; \frac{1}{\sqrt{2\pi t^{2H}}} \int_{\psi(A,x_0)} \mathrm{e}^{-\frac{u^2}{2t^{2H}}}\, du \\ &=\; \frac{1}{\sqrt{2\pi t^{2H}}} \int_A \mathrm{e}^{-\frac{\varphi(v,x_0)^2}{2t^{2H}}} |\sigma(\varphi(v, x_0))| dv.\end{aligned}$$

Hence, X_t has an explicit positive density given by

$$f_{X_t}(v) \;=\; \frac{1}{\sqrt{2\pi t^{2H}}} \mathrm{e}^{-\frac{\varphi(v,x_0)^2}{2t^{2H}}} |\sigma(\varphi(v, x_0))|. \tag{5}$$

(ii) When $b \not\equiv 0$, we can first suppose that b has compact support, by an immediate approximation argument. Besides, for every $t > 0$, we have

$$X_t \;=\; \varphi(B_t^H, Y_t) = \varphi(B_t^H, \varphi(\psi(Y_t, x_0), x_0)) = \varphi(B_t^H + \psi(Y_t, x_0), x_0),$$

the last equality coming from the flow property of φ. Since b has compact support, it is easy to see from (4) and the bijection property of φ that Y_t is a bounded random variable for every $t > 0$. Hence $\psi(Y_t, x_0)$ is also bounded for every $t > 0$ and we can appeal to Girsanov's theorem for fBm (see Theorem 3.1 in [11]), which yields

$$X_t = \varphi(\tilde{B}_t^H, x_0),$$

where $\tilde{M}$ is a fBm under a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$. Hence we are reduced to the case $b \equiv 0$ and we can conclude from above that, under $\mathbb{Q}$, X_t has a positive density over $\mathbb{R}$. Since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, the same holds under $\mathbb{P}$.

□

Remark Theorem A entails in particular that $\mathrm{Supp}\, \mathrm{X_t} = \mathbb{R}$ for every $t > 0$. Actually, this support property can be extended on the functional level: when σ does not vanish, it follows easily from Doss's arguments [5] that $\mathrm{Supp}\, X = \mathcal{C}_{x_0}$, where $X = \{X_t,\ t \geq 0\}$ is viewed as a random variable valued in $\mathcal{C}_{x_0}$, the set of continuous functions from $\mathbb{R}^+$ to $\mathbb{R}$ starting from x_0 endowed with the local supremum norm.

3 Extension of a result of Bouleau-Hirsch

In this section we extend Theorem A quite considerably, giving a necessary and sufficient condition on σ in the spirit of Bouleau-Hirsch's [2] criterion. However our arguments are somewhat more elaborate, and we first need to recall a few facts about the Gaussian analysis related to fractional Brownian motion. In order to simplify the presentation and without loss of generality, we will fix an horizon $T > 0$ to (1), hence we will define fBm on $[0, T]$ only.

3.1 Some recalls about fractional Brownian Motion

Let us give a few facts about the Gaussian structure of fBm and its Malliavin derivative process, following Sect. 3.1 in [11] and Chap. 1.2 in [10]. Set

$$R_H(t,s) := \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}),\ s,t \in [0,T].$$

Let $\mathcal{E}$ be the set of step-functions on $[0, T]$. Consider the Hilbert space $\mathcal{H}$ defined as the closure of $\mathcal{E}$ wih respect to the scalar product

$$\left(\mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\right)_{\mathcal{H}} \;=\; R_H(t,s).$$

More precisely, if we set

$$K_H(t,s) \;=\; \Gamma\left(H+1/2\right)^{-1}(t-s)^{H-1/2} F\left(H-1/2, 1/2-H; H+1/2, 1-t/s\right),$$

where F stands for the standard hypergeometric function, and define the linear operator K_H^* from $\mathcal{E}$ to $L^2([0,T])$ by

$$(K_H^*\varphi)(s) \;=\; K_H(T,s)\varphi(s) \;+\; \int_s^T (\varphi(r) - \varphi(s)) \frac{\partial K_H}{\partial r}(r,s)\, dr,$$

then $\mathcal{H}$ is isometric to $L^2([0,T])$ thanks to the equality

$$(\varphi, \rho)_{\mathcal{H}} \;=\; \int_0^T (K_H^*\varphi)(s)(K_H^*\rho)(s)\, ds. \tag{6}$$

B^H is a centred Gaussian process with covariance function $R_H(t,s)$, hence its associated Gaussian space is isometric to $\mathcal{H}$ through the mapping $\mathbf{1}_{[0,t]} \mapsto B_t^H$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with compact support and consider the random variable $F = f(B_{t_1}^H, \ldots, B_{t_n}^H)$ (we then say that F is a smooth random variable). The derivative process of F is the element of $L^2(\Omega, \mathcal{H})$ defined by

$$D_s F \;=\; \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B_{t_1}^H, \ldots, B_{t_n}^H)\mathbf{1}_{[0,t_i]}(s).$$

In particular $D_s B_t^H = \mathbf{1}_{[0,t]}(s)$. As usual, $\mathbb{D}^{1,1}$ is the closure of smooth random variables with respect to the norm

$$||F||_{1,1} \;=\; \mathbb{E}\left[|F|\right] + \mathbb{E}\left[||D.F||_{\mathcal{H}}\right]$$

and $\mathbb{D}^{1,1}_{loc}$ is it associated local domain, that is the set of random variables F such that there exists a sequence $\{(\Omega_n, F_n),\ n \geq 1\} \subset \mathcal{F} \times \mathbb{D}^{1,1}$ such that $\Omega_n \uparrow \Omega$ a.s. and $F = F_n$ a.s. on Ω_n (see [10] p. 45 for more details). We finally recall the following criterion which is due to Nualart-Zakai (see Theorem 2.1.3 in [10]) :

Theorem 1 (Nualart-Zakai) *If $F \in \mathbb{D}^{1,1}_{loc}$ and a.s. $||D.F||_{\mathcal{H}} > 0$, then F has a density with respect to Lebesgue measure on $\mathbb{R}$.*

3.2 Statement and proof of the main result

Let $J = \sigma^{-1}(\{0\})$ and $\operatorname{int} J$ be the interior of J. Consider the deterministic equation

$$x_t \;=\; x_0 \;+\; \int_0^t b\,(x_s)\,ds \tag{7}$$

and the deterministic time

$$t_x \;=\; \sup\{t \geq 0 : \;\; x_t \notin \operatorname{int} J\}.$$

When $H = 1/2$, it was proved by Bouleau-Hirsch (see e.g. Theorem 6.3. in [2]) that X_t has a density with respect to Lebesgue measure if and only if $t > t_x$. In particular X_t has a density for all t as soon as $\sigma(x_0) \neq 0$, which also follows from Hörmander's condition. Notice that Bouleau-Hirsch's result holds in a more general multidimensional context (but then t_x is the entrance time of X into the set where σ has maximal rank, and t_x is no more deterministic). In dimension 1, we aim to extend this result to fBm of *any* Hurst index :

Theorem B *Let $\{x_t,\ t \geq 0\}$, $\{X_t,\ t \geq 0\}$ and t_x be defined as above. Then X_t has a density with respect to Lebesgue measure if and only if $t > t_x$.*

We will need a lemma which extends Prop. 2.1.2 in [3], Chap. IV, to fBm.

Lemma 2 *With the above notations,*

$$t_x \;=\; \inf\{t > 0 : \;\; X_t \notin \operatorname{int} J\} \quad a.s.$$

Proof. According to (3), it is obvious that $\varphi(x,y) = y$ for all $x \in \mathbb{R}$ et $y \in J$ and then $\varphi'_y(x,y) = 1$ for all $x \in \mathbb{R}$ and $y \in \operatorname{int} J$. Set $\tau = \inf\{t > 0 : X_t \notin \operatorname{int} J\}$ and $T = \inf\{t > 0 : Y_t \notin \operatorname{int} J\}$. We have a.s.

- $t < T \Rightarrow \forall s \leq t$: $Y_s \in \operatorname{int} J \Rightarrow \forall s \leq t$: $X_s = \varphi(B_s^H, Y_s) = Y_s \Rightarrow \forall s \leq t$: $X_s \in \operatorname{int} J$ $\Rightarrow t \leq \tau$, which yields $T \leq \tau$.
- $t < \tau \Rightarrow \forall s \leq t$: $X_s \in \operatorname{int} J \Rightarrow \forall s \leq t$: $\varphi(B_s^H, X_s) = X_s = \varphi(B_s^H, Y_s) \Rightarrow \forall s \leq t$: $X_s = Y_s \Rightarrow \forall s \leq t$: $Y_s \in \operatorname{int} J \Rightarrow t \leq T$. Hence $\tau \leq T$.
- $t < t_x \Rightarrow \forall s \leq t$: $x_s \in \operatorname{int} J \Rightarrow \forall s \leq t$: $x'_s = b(x_s) = \frac{b \circ \varphi(B_s^H, x_s)}{\varphi'_y(B_s^H, x_s)} \Rightarrow \forall s \leq t$: $x_s = Y_s$ $\Rightarrow \forall s \leq t$: $Y_s \in \operatorname{int} J \Rightarrow t \leq T$, so that $t_x \leq T$.
- $t < T \Rightarrow \forall s \leq t$: $Y_s \in \operatorname{int} J \Rightarrow \forall s \leq t$: $Y'_s = \frac{b \circ \varphi(B_s^H, Y_s)}{\varphi'_y(B_s^H, Y_s)} = b(Y_s) \Rightarrow \forall s \leq t$: $Y_s = x_s$ $\Rightarrow \forall s \leq t$: $x_s \in \operatorname{int} J \Rightarrow t \leq t_x$, whence $T \leq t_x$.

Finally, this proves that a.s. $t_x = T = \tau$, and completes the proof of the Lemma.

□

Proof of Theorem B. Suppose first that $t > t_x$. Recall that $X_t = \varphi(B_t^H, Y_t)$, where φ is given by (3) and Y is the unique solution to

$$Y_s \;=\; x_0 \;+\; \int_0^s L_u^{-1} b(X_u)\,du,$$

where we set

$$L_u \;=\; \varphi'_y(B_u^H, Y_u) \;=\; \exp\left[\int_0^{B_u^H} \sigma'(\varphi(z, Y_u))\,dz\right]$$

for every $u \geq 0$ - the second equality being an obvious consequence of (3). Notice that $L_u > 0$ a.s. for every $u \geq 0$. We will also use the notation

$$M_u \;=\; \varphi''_{yy}(B^H_u, Y_u) \;=\; L_u \int_0^{B^H_u} \sigma''(\varphi(z, Y_u))\varphi'_y(z, Y_u)\, dz,$$

the second equality coming readily from (3) as well.

We now differentiate the random variables X_u, $u \leq t$. Fixing $s \in [0, t]$ once and for all, the Chain Rule (see Prop. 1.2.2 in [10]) yields

$$D_s X_u \;=\; (\sigma(X_u) + L_u D_s Y_u)\, \mathbf{1}_{[0,u]}(s).$$

In particular, setting $N_u = L_u^{-1} D_s X_u$ for every $u \leq t$, we get

$$N_t \;=\; L_t^{-1}\sigma(X_t) + D_s Y_t.$$

Itô's formula (2) entails

$$L_t \;=\; 1 \;+\; \int_0^t L_u \sigma'(X_u) \diamond dB^H_u \;+\; \int_0^t M_u\, dY_u$$

and

$$L_t^{-1}\sigma(X_t) \;\;=\;\; L_s^{-1}\sigma(X_s) \;+\; \int_s^t \left(\sigma'(X_u) - L_u^{-2} M_u \sigma(X_u)\right)\, dY_u.$$

On the other hand, differentiating Y_t yields

$$D_s Y_t = \int_s^t N_u b'(X_u)\, du \;-\; \int_s^t \left(\sigma'(X_u) - L_u^{-2} M_u \sigma(X_u) + L_u^{-1} M_u N_u\right) dY_u.$$

Putting everything together, we get

$$N_t = L_s^{-1}\sigma(X_s) \exp\left[\int_s^t \left(b'(X_u) - L_u^{-2} M_u b(X_u)\right)\, du\right].$$

Hence,

$$D_s X_t \;\;=\;\; \sigma(X_s) \exp\left[\int_s^t b'(X_u)\, du\right] \left(\frac{L_t}{L_s} \exp - \left[\int_s^t L_u^{-1} M_u\, dY_u\right]\right).$$

Notice that by Itô's formula

$$L_u = \exp\left[\int_0^u \sigma'(X_v) \diamond dB^H_v \;+\; \int_0^u L_v^{-1} M_v\, dY_v\right],$$

so that

$$D_s X_t \;\;=\;\; \sigma(X_s) \exp\left[\int_s^t b'(X_u)\, du \;+\; \int_0^t \sigma'(X_u) \diamond dB^H_u\right].$$

Now since $t > t_x$, it follows from Lemma 2 and the a.s. continuity of $s \mapsto \sigma(X_s)$ that the function $s \mapsto D_s X_t$ does not vanish on a subset of $[0, t]$ with positive Lebesgue measure. It is then not difficult to see that the same holds for the function $s \mapsto (K_H^* D.X_t)(s)$. Using (6), we obtain

$$||D.X_t||_{\mathcal{H}}^2 \;=\; (D.X_t, D.X_t)_{\mathcal{H}} \;=\; \int_0^T (K_H^* D.X_t)^2(s)\, ds \;>\; 0 \quad \text{a.s.}$$

Thanks to Theorem 1, we can conclude that X_t has a density with respect to Lebesgue measure.

Suppose finally that $t \leq t_x$. Then it follows by uniqueness that $X_t = x_t$ a.s. where x_t is deterministic, so that X_t cannot have a density. This completes the proof of Theorem B.

□

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