Parameter estimation for α-fractional bridges

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Abstract: Let $\alpha, T > 0$. We study the asymptotic properties of a least squares estimator for the parameter α of a fractional bridge defined as $dX_t = -\alpha \frac{X_t}{T-t} dt + dB_t$, $0 \leqslant t < T$, where B is a fractional Brownian motion of Hurst parameter $H > \frac{1}{2}$. Depending on the value of α, we prove that we may have strong consistency or not as $t \to T$. When we have consistency, we obtain the rate of this convergence as well. Also, we compare our results to the (known) case where B is replaced by a standard Brownian motion W.

It is great pleasure for us to dedicate this paper to our friend David Nualart, in celebration of his 60th birthday and with all our admiration.

1 Introduction

Let W be a standard Brownian motion and let α be a non-negative real parameter. In recent years, the study of various problems related to the (so-called) α-Wiener bridge, that is, to the solution X to

$$X_0 = 0; \quad dX_t = -\alpha \frac{X_t}{T-t} dt + dW_t, \quad 0 \leqslant t < T, \tag{1}$$

has attracted interest. For a motivation and further references, we refer the reader to Barczy and Pap [2, 3], as well as Mansuy [6]. Because (1) is linear, it is immediate to solve it explicitely; one then gets the following formula:

$$X_t = (T-t)^\alpha \int_0^t (T-s)^{-\alpha} dW_s, \quad t \in [0, T),$$

the integral with respect to W being a Wiener integral.

An example of interesting problem related to X is the statistical estimation of α when one observes the whole trajectory of X. A natural candidate is the maximum likelihood estimator (MLE), which can be easily computed for this model, due to the specific form of (1): one gets

$$\hat{\alpha}_t = -\left(\int_0^t \frac{X_u}{T-u} dX_u\right) \Big/ \left(\int_0^t \frac{X_u^2}{(T-u)^2} du\right), \quad t < T. \tag{2}$$

In (2), the integral with respect to X must of course be understood in the Itô sense. On the other hand, at this stage it is worth noticing that $\hat{\alpha}_t$ coincides with a least squares estimator (LSE) as well; indeed, $\widehat{\alpha}_t$ (formally) minimizes

$$\alpha \mapsto \int_0^t \left| \dot{X}_u + \alpha \frac{X_u}{T-u} \right|^2 du.$$

Also, it is worth bearing in mind an alternative formula for $\widehat{\alpha}_t$, which is more easily amenable to analysis and which is immediately shown thanks to (1):

$$\alpha - \widehat{\alpha}_t \;=\; \left(\int_0^t \frac{X_u}{T-u} dW_u\right) \Big/ \left(\int_0^t \frac{X_u^2}{(T-u)^2} du\right). \tag{3}$$

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When dealing with (3) by means of a semimartingale approach, it is not very difficult to check that $\widehat{\alpha}_t$ is indeed a strongly consistent estimator of α. The next step generally consists in studying the second-order approximation. Let us describe what is known about this problem: as $t \to T$,

- if $0 < \alpha < \frac{1}{2}$ then
$$(T-t)^{\alpha-\frac{1}{2}}\big(\alpha - \widehat{\alpha}_t\big) \overset{\text{law}}{\longrightarrow} T^{\alpha-\frac{1}{2}}(1-2\alpha)\times \mathcal{C}(1), \tag{4}$$
with $\mathcal{C}(1)$ the standard Cauchy distribution, see [4, Theorem 2.8];
- if $\alpha = \frac{1}{2}$ then
$$|\log(T-t)|\big(\alpha - \widehat{\alpha}_t\big) \overset{\text{law}}{\longrightarrow} \frac{\int_0^T W_s dW_s}{\int_0^T W_s^2 ds}, \tag{5}$$
see [4, Theorem 2.5];
- if $\alpha > \frac{1}{2}$ then
$$\sqrt{|\log(T-t)|}\big(\alpha - \widehat{\alpha}_t\big) \overset{\text{law}}{\longrightarrow} \mathcal{N}(0, 2\alpha-1), \tag{6}$$
see [4, Theorem 2.11].

Thus, we have the full picture for the asymptotic behavior of the MLE/LSE associated to α-Wiener bridges.

In the present paper, our goal is to investigate what happens when, in (1), the standard Brownian motion W is replaced by a fractional Brownian motion B. More precisely, suppose from now on that $X = \{X_t\}_{t\in[0,T)}$ is the solution to
$$X_0 = 0; \quad dX_t = -\alpha\,\frac{X_t}{T-t}\,dt + dB_t, \quad 0 \leqslant t < T, \tag{7}$$
where B is a fractional Brownian motion with known parameter H, whereas $\alpha > 0$ is considered as an unknown parameter. Although X could have been defined for all H in $(0,1)$, for technical reasons and in order to keep the length of our paper within bounds we restrict ourself to the case $H \in (\frac{1}{2}, 1)$ in the sequel.

In order to estimate the unknown parameter α when the whole trajectory of X is observed, we continue to consider the estimator $\widehat{\alpha}_t$ given by (2). (It is no longer the MLE, but it is still a LSE.) Nevertheless, there is a major difference with respect to the standard Brownian motion case. Indeed, the process X being no longer a semimartingale, in (2) one cannot utilize the Itô integral to integrate with respect to it. However, because X has[§] γ-Hölder continuous paths on $[0,t]$ for all $\gamma \in (\frac{1}{2}, H)$ and all $t \in [0,T)$, one can choose, instead, the Young integral (see Section 2.3 for the main properties of this integral, notably its chain rule (17) and how (18) relies it Skorohod integral).

Let us now describe the results we prove in the present paper. First, in Theorem 1 we show that the (strong) consistency of $\widehat{\alpha}_t$ as $t \to T$ holds true if and only if $\alpha \leqslant \frac{1}{2}$. Then, depending on the precise value of $\alpha \in (0, \frac{1}{2}]$, we derive the asymptotic behavior of the error $\widehat{\alpha}_t - \alpha$. It turns out that, once adequately renormalized, this error converges either in law or almost surely, to a limit that we are able to compute explicitely. More specifically, we show in Theorem 2 the following convergences (below and throughout the paper, $\mathcal{C}(1)$ always stands for the standard Cauchy distribution and $\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ for the usual Beta function): as $t \to T$,

- if $0 < \alpha < 1-H$ then
$$(T-t)^{\alpha-H}\big(\alpha - \widehat{\alpha}_t\big) \overset{\text{law}}{\longrightarrow} T^{\alpha-H}(1-2\alpha)\sqrt{\frac{(H-\alpha)\beta(2-2H-\alpha, 2H-1)}{(1-H-\alpha)\beta(1-\alpha, 2H-1)}} \times \mathcal{C}(1); \tag{8}$$

[§] More precisely, we assume throughout the paper that we work with a suitable γ-Hölder continuous version of X, which is easily shown to exist by the Kolmogorov-Centsov theorem.

- if $\alpha = 1 - H$ then

$$\frac{(T-t)^{1-2H}}{\sqrt{|\log(T-t)|}}\big(\alpha - \widehat{\alpha}_t\big) \xrightarrow{\text{law}} T^{1-2H}(2H-1)^{\frac{3}{2}}\sqrt{\frac{2\,\beta(1-H,2H-1)}{\beta(H,2H-1)}} \times \mathcal{C}(1); \tag{9}$$

- if $1 - H < \alpha < \frac{1}{2}$ then

$$(T-t)^{2\alpha-1}\big(\alpha - \widehat{\alpha}_t\big) \xrightarrow{\text{a.s.}} (1-2\alpha)\int_0^T \frac{dB_u}{(T-u)^{1-\alpha}}\int_0^u \frac{dB_s}{(T-s)^\alpha} \Big/ \left(\int_0^T \frac{dB_s}{(T-s)^\alpha}\right)^2; \tag{10}$$

- if $\alpha = \frac{1}{2}$ then

$$|\log(T-t)|\big(\alpha - \widehat{\alpha}_t\big) \xrightarrow{\text{a.s.}} \frac{1}{2}. \tag{11}$$

When comparing the convergences (8) to (11) with those arising in the standard Brownian motion case (that is, (4) to (6)), we observe a new and interesting phenomenom when the parameter α ranges from $1 - H$ to $\frac{1}{2}$ (of course, this case is immaterial in the standard Brownian motion case).

We hope our proofs of (8) to (11) to be elementary. Indeed, except maybe the link (18) between Young and Skorohod integrals, they only involve soft arguments, often based on the mere derivation of suitable equivalent for some integrals. In particular, unlike the classical approach (as used, e.g., in [4]) we stress that, here, we use no tool coming from the semimartingale realm.

Before to conclude this introduction, we would like to mention the recent paper [5] by Hu and Nualart, which has been a valuable source of inspiration. More specifically, the authors of [5] study the estimation of the parameter $\alpha > 0$ arising in the fractional Ornstein-Uhlenbeck model, defined as $dX_t = -\alpha X_t dt + dB_t$, $t \geqslant 0$, where B is a fractional Brownian motion of (known) index $H \in (\frac{1}{2}, \frac{3}{4})$. They show the strong consistency of a least squares estimator $\widehat{\alpha}_t$ as $t \to \infty$ (with, however, a major difference with respect to us: they are forced to use Skorohod integral rather than Young integral to define $\widehat{\alpha}_t$, otherwise $\widehat{\alpha}_t \not\to \alpha$ as $t \to \infty$; unfortunately, this leads to an impossible-to-simulate estimator, and this is why they introduce an alternative estimator for α.) They then derive the associated rate of convergence as well, by exhibiting a central limit theorem. Their calculations are of completely different nature than ours because, to achieve their goal, the authors of [5] make use of the fourth moment theorem of Nualart and Peccati [8].

The rest of our paper is organized as follows. In Section 2 we introduce the needed material for our study, whereas Section 3 contains the precise statements and proofs of our results.

2 Basic notions for fractional Brownian motion

In this section, we briefly recall some basic facts concerning stochastic calculus with respect to fractional Brownian motion; we refer to [7] for further details. Let $B = \{B_t\}_{t\in[0,T]}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$, defined on some probability space $(\Omega, \mathcal{F}, P)$. (Here, and throughout the text, we do assume that $\mathcal{F}$ is the sigma-field generated by B.) This means that B is a centered Gaussian process with the covariance function $E[B_s B_t] = R_H(s,t)$, where

$$R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{12}$$

If $H = \frac{1}{2}$, then B is a Brownian motion. From (12), one can easily see that $E\big[|B_t - B_s|^2\big] = |t-s|^{2H}$, so B has $\gamma-$Hölder continuous paths for any $\gamma \in (0, H)$ thanks to the Kolmogorov-Centsov theorem.

2.1 Space of deterministic integrands

We denote by $\mathcal{E}$ the set of step $\mathbb{R}-$valued functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = R_H(t,s).$$

We denote by $|\cdot|_{\mathcal{H}}$ the associated norm. The mapping $\mathbf{1}_{[0,t]} \mapsto B_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space associated with B. We denote this isometry by

$$\varphi \mapsto B(\varphi) = \int_0^T \varphi(s) dB_s. \tag{13}$$

When $H \in (\frac{1}{2}, 1)$, it follows from [9] that the elements of $\mathcal{H}$ may not be functions but distributions of negative order. It will be more convenient to work with a subspace of $\mathcal{H}$ which contains only functions. Such a space is the set $|\mathcal{H}|$ of all measurable functions φ on $[0,T]$ such that

$$|\varphi|^2_{|\mathcal{H}|} := H(2H-1) \int_0^T \int_0^T |\varphi(u)||\varphi(v)||u-v|^{2H-2} du dv < \infty.$$

If $\varphi, \psi \in |\mathcal{H}|$ then

$$E\big[B(\varphi)B(\psi)\big] = H(2H-1) \int_0^T \int_0^T \varphi(u)\psi(v)|u-v|^{2H-2} du dv. \tag{14}$$

We know that $(|\mathcal{H}|, \langle \cdot, \cdot \rangle_{|\mathcal{H}|})$ is a Banach space, but that $(|\mathcal{H}|, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is not complete (see, e.g., [9]). We have the dense inclusions $L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}$.

2.2 Malliavin derivative and Skorohod integral

Let $\mathcal{S}$ be the set of all smooth cylindrical random variables, which can be expressed as $F = f(B(\phi_1), \dots, B(\phi_n))$ where $n \geqslant 1$, $f : \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^\infty$-function such that f and all its derivatives have at most polynomial growth, and $\phi_i \in \mathcal{H}$, $i = 1, \dots, n$. The Malliavin derivative of F with respect to B is the element of $L^2(\Omega, \mathcal{H})$ defined by

$$D_s F \ = \ \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(\phi_1), \dots, B(\phi_n))\phi_i(s), \quad s \in [0,T].$$

In particular $D_s B_t = \mathbf{1}_{[0,t]}(s)$. As usual, $\mathbb{D}^{1,2}$ denotes the closure of the set of smooth random variables with respect to the norm

$$\|F\|^2_{1,2} \ = \ E[F^2] + E\big[|DF|^2_{\mathcal{H}}\big].$$

The Malliavin derivative D verifies the chain rule: if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^1_b$ and if $(F_i)_{i=1,\dots,n}$ is a sequence of elements of $\mathbb{D}^{1,2}$, then $\varphi(F_1, \dots, F_n) \in \mathbb{D}^{1,2}$ and we have, for any $s \in [0,T]$,

$$D_s \varphi(F_1, \dots, F_n) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F_1, \dots, F_n) D_s F_i.$$

The Skorohod integral δ is the adjoint of the derivative operator D. If a random variable $u \in L^2(\Omega, \mathcal{H})$ belongs to the domain of the Skorohod integral (denoted by domδ), that is, if it verifies

$$|E\langle DF, u\rangle_{\mathcal{H}}| \leqslant c_u \sqrt{E[F^2]} \quad \text{for any } F \in \mathcal{S},$$

then $\delta(u)$ is defined by the duality relationship

$$E[F\delta(u)] = E\big[\langle DF, u\rangle_{\mathcal{H}}\big],$$

for every $F \in \mathbb{D}^{1,2}$. In the sequel, when $t \in [0,T]$ and $u \in \text{dom}\delta$, we shall sometimes write $\int_0^t u_s \delta B_s$ instead of $\delta(u\mathbf{1}_{[0,t]})$. If $h \in \mathcal{H}$, notice moreover that $\int_0^T h_s \delta B_s = \delta(h) = B(h)$.

For every $q \geqslant 1$, let $\mathcal{H}_q$ be the qth Wiener chaos of B, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(B(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where H_q is the qth Hermite polynomial. The mapping $I_q(h^{\otimes q}) = H_q(B(h))$ provides a linear isometry between the symmetric tensor product $\mathcal{H}^{\odot q}$ (equipped with the modified norm $\|\cdot\|_{\mathcal{H}^{\odot q}} = \frac{1}{\sqrt{q!}}\|\cdot\|_{\mathcal{H}^{\otimes q}}$) and $\mathcal{H}_q$. Specifically, for all $f, g \in \mathcal{H}^{\odot q}$ and $q \geqslant 1$, one has

$$E\big[I_q(f)I_q(g)\big] = q!\langle f, g\rangle_{\mathcal{H}^{\otimes q}}. \tag{15}$$

On the other hand, it is well-known that any random variable Z belonging to $L^2(\Omega)$ admits the following chaotic expansion:

$$Z = E[Z] + \sum_{q=1}^{\infty} I_q(f_q), \tag{16}$$

where the series converges in $L^2(\Omega)$ and the kernels f_q, belonging to $\mathcal{H}^{\odot q}$, are uniquely determined by Z.

2.3 Young integral

For any $\gamma \in [0,1]$, we denote by $\mathscr{C}^\gamma([0,T])$ the set of γ-Hölder continuous functions, that is, the set of functions $f : [0,T] \to \mathbb{R}$ such that

$$|f|_\gamma := \sup_{0 \leqslant s < t \leqslant T} \frac{|f(t) - f(s)|}{(t-s)^\gamma} < \infty.$$

(Notice the calligraphic difference between a space $\mathscr{C}$ of Hölder continuous functions, and a space $\mathcal{C}$ of continuously differentiable functions!). We also set $|f|_\infty = \sup_{t\in[0,T]} |f(t)|$, and we equip $\mathscr{C}^\gamma([0,T])$ with the norm

$$\|f\|_\gamma := |f|_\gamma + |f|_\infty.$$

Let $f \in \mathscr{C}^\gamma([0,T])$, and consider the operator $T_f : \mathcal{C}^1([0,T]) \to \mathcal{C}^0([0,T])$ defined as

$$T_f(g)(t) = \int_0^t f(u)g'(u)du, \quad t \in [0,T].$$

It can be shown (see, e.g., [10, Section 2.2]) that, for any $\beta \in (1-\gamma, 1)$, there exists a constant $C_{\gamma,\beta,T} > 0$ depending only on γ, β and T such that, for any $g \in \mathscr{C}^\beta([0,T])$,

$$\left\| \int_0^\cdot f(u)g'(u)du \right\|_\beta \leqslant C_{\gamma,\beta,T}\|f\|_\gamma \|g\|_\beta.$$

We deduce that, for any $\gamma \in (0,1)$, any $f \in \mathscr{C}^\gamma([0,T])$ and any $\beta \in (1-\gamma,1)$, the linear operator $T_f : \mathcal{C}^1([0,T]) \subset \mathscr{C}^\beta([0,T]) \to \mathscr{C}^\beta([0,T])$, defined as $T_f(g) = \int_0^\cdot f(u)g'(u)du$, is continuous with respect to the norm $\|\cdot\|_\beta$. By density, it extends (in an unique way) to an operator defined on $\mathscr{C}^\beta$. As consequence, if $f \in \mathscr{C}^\gamma([0,T])$, if $g \in \mathscr{C}^\beta([0,T])$ and if $\gamma + \beta > 1$, then the (so-called) Young integral $\int_0^\cdot f(u)dg(u)$ is (well) defined as being $T_f(g)$.

The Young integral obeys the following chain rule. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^2$ function, and let $f, g \in \mathscr{C}^\gamma([0,T])$ with $\gamma > \frac{1}{2}$. Then $\int_0^\cdot \frac{\partial\phi}{\partial f}(f(u),g(u))df(u)$ and $\int_0^\cdot \frac{\partial\phi}{\partial g}(f(u),g(u))dg(u)$ are well-defined as Young integrals. Moreover, for all $t \in [0,T]$,

$$\phi(f(t),g(t)) = \phi(f(0),g(0)) + \int_0^t \frac{\partial\phi}{\partial f}(f(u),g(u))df(u) + \int_0^t \frac{\partial\phi}{\partial g}(f(u),g(u))dg(u). \tag{17}$$

2.4 Link between Young and Skorohod integrals

Assume $H > \frac{1}{2}$, and let $u = (u_t)_{t\in[0,T]}$ be a process with paths in $\mathscr{C}^\gamma([0,T])$ for some fixed $\gamma > 1-H$. Then, according to the previous section, the integral $\int_0^T u_s dB_s$ exists pathwise in the Young sense. Suppose moreover that u_t belongs to $\mathbb{D}^{1,2}$ for all $t \in [0,T]$, and that u satisfies

$$P\left(\int_0^T \int_0^T |D_s u_t||t-s|^{2H-2} ds dt < \infty\right) = 1.$$

Then $u \in \mathrm{dom}\delta$, and we have (see [1]), for all $t \in [0,T]$:

$$\int_0^t u_s dB_s = \int_0^t u_s \delta B_s + H(2H-1) \int_0^t \int_0^t D_s u_x |x-s|^{2H-2} ds dx. \tag{18}$$

In particular, notice that

$$\int_0^T \varphi_s dB_s = \int_0^T \varphi_s \delta B_s = B(\varphi) \tag{19}$$

when φ is non-random.

3 Statement and proofs of our main results

In all this section, we fix a fractional Brownian motion B of Hurst index $H \in (\frac{1}{2}, 1)$, as well as a parameter $\alpha > 0$. Let us consider the solution X to (7). It is readily checked that we have the following explicit expression for X_t:

$$X_t = (T-t)^\alpha \int_0^t (T-s)^{-\alpha} dB_s, \quad t \in [0,T), \tag{20}$$

where the integral can be understood either in the Young sense, or in the Skorohod sense, see indeed (19).

For convenience, and because it will play an important role in the forthcoming computations, we introduce the following two processes related to X: for $t \in [0,T]$,

$$\xi_t = \int_0^t (T-s)^{-\alpha} dB_s; \tag{21}$$

$$\eta_t = \int_0^t dB_u (T-u)^{\alpha-1} \int_0^u dB_s (T-s)^{-\alpha} = \int_0^t (T-u)^{\alpha-1} \xi_u dB_u. \tag{22}$$

In particular, we observe that

$$X_t = (T-t)^\alpha \xi_t \quad \text{and} \quad \int_0^t \frac{X_u}{T-u} dB_u = \eta_t \quad \text{for } t \in [0,T). \tag{23}$$

When α is between 0 and H (resp. $1-H$ and H), in Lemma 4 (resp. Lemma 5) we shall actually show that the process ξ (resp. η) is well-defined on the whole interval $[0,T]$ (notice that we could have had a problem at $t = T$), and that it admits a continuous modification. This is why we may and will assume in the sequel, without loss of generality, that ξ (resp. η) is continuous when $0 < \alpha < H$ (resp. $1-H < \alpha < H$).

Recall the definition (2) of $\widehat{\alpha}_t$. By using (7) and then (23), as well as the definitions (21) and (22), we arrive to the following formula:

$$\alpha - \widehat{\alpha}_t = \frac{\int_0^t X_u (T-u)^{-1} dB_u}{\int_0^t X_u^2 (T-u)^{-2} ds} = \frac{\eta_t}{\int_0^t (T-u)^{2\alpha-2} \xi_u^2 du}.$$

Thus, in order to prove the convergences (8) to (11) of the introduction (that is, our main result!), we are left to study the (joint) asymptotic behaviors of η_t and $\int_0^t (T-u)^{2\alpha-2}\xi_u^2 du$ as $t \to T$. The asymptotic behavior of $\int_0^t (T-u)^{2\alpha-2}\xi_u^2 du$ is rather easy to derive (see Lemma 9), because it looks like a convergence *à la* Cesàro when $\alpha \leqslant \frac{1}{2}$. In contrast, the asymptotic behavior of η_t is more difficult to obtain, and will depend on the relative position of α with respect to $1-H$. It is actually the combination of Lemmas 3, 5, 6, 7, 8 that will allow to derive it for the full range of values of α.

We are now in position to prove our two main results, that we restate here as theorems for convenience.

Theorem 1 *We have $\widehat{\alpha}_t \overset{\text{prob.}}{\longrightarrow} \alpha \wedge \frac{1}{2}$ as $t \to T$. When $\alpha < H$ we have almost sure convergence as well.*

As a corollary, we find that $\widehat{\alpha}_t$ is a strong consistent estimator of α if and only if $\alpha \leqslant \frac{1}{2}$. The next result precises the associated rate of convergence in this case.

Theorem 2 *Let $G \sim \mathcal{N}(0,1)$ be independent of B, let $\mathcal{C}(1)$ stand for the standard Cauchy distribution, and let $\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ denote the usual Beta function.*

1. *Assume $\alpha \in (0, 1-H)$. Then, as $t \to T$,*

$$(T-t)^{\alpha-H}\big(\alpha - \widehat{\alpha}_t\big) \overset{\text{law}}{\longrightarrow} (1-2\alpha)\sqrt{H(2H-1)\frac{\beta(2-\alpha-2H, 2H-1)}{1-H-\alpha}} \times \frac{G}{\xi_T}$$

$$\overset{\text{law}}{=} T^{\alpha-H}(1-2\alpha)\sqrt{\frac{(H-\alpha)\beta(2-2H-\alpha, 2H-1)}{(1-H-\alpha)\beta(1-\alpha, 2H-1)}} \times \mathcal{C}(1).$$

2. *Assume $\alpha = 1-H$. Then, as $t \to T$,*

$$\frac{(T-t)^{1-2H}}{\sqrt{|\log(T-t)|}}\big(\alpha - \widehat{\alpha}_t\big) \overset{\text{law}}{\longrightarrow} (2H-1)^{\frac{3}{2}}\sqrt{2H\,\beta(1-H, 2H-1)} \times \frac{G}{\xi_T}$$

$$\overset{\text{law}}{=} T^{1-2H}(2H-1)^{\frac{3}{2}}\sqrt{\frac{2\,\beta(1-H, 2H-1)}{\beta(H, 2H-1)}} \times \mathcal{C}(1).$$

3. *Assume $\alpha \in \big(1-H, \frac{1}{2}\big)$. Then, as $t \to T$,*

$$(T-t)^{2\alpha-1}\big(\alpha - \widehat{\alpha}_t\big) \overset{\text{a.s.}}{\longrightarrow} \frac{(1-2\alpha)\,\eta_T}{(\xi_T)^2}.$$

4. *Assume $\alpha = \frac{1}{2}$. Then, as $t \to T$,*

$$|\log(T-t)|\big(\alpha - \widehat{\alpha}_t\big) \overset{\text{a.s.}}{\longrightarrow} \frac{1}{2}.$$

The rest of this section is devoted to the proofs of Theorems 1 and 2. Before to be in position to do so, we need to state and prove some auxiliary lemmas. In what follows we use the same symbol c for all constants whose precise value is not important for our consideration.

Lemma 3 *Let $\alpha, \beta \in (0,1)$ be such that $\alpha + \beta < 2H$. Then, for all $T > 0$,*

$$\int_0^T ds\,(T-s)^{-\beta}\int_0^T dr\,(T-r)^{-\alpha}|s-r|^{2H-2} = \int_0^T ds\,s^{-\beta}\int_0^T dr\,r^{-\alpha}|s-r|^{2H-2} < \infty.$$

Proof. By homogeneity, we first notice that

$$\int_0^T ds\, s^{-\beta} \int_0^T dr\, r^{-\alpha} |s-r|^{2H-2} = T^{2H-\alpha-\beta} \int_0^1 ds\, s^{-\beta} \int_0^1 dr\, r^{-\alpha} |s-r|^{2H-2},$$

so that it is not a loss of generality to assume in the proof that $T = 1$. If $\alpha + 1 < 2H$ then $\int_0^{1/s} r^{-\alpha}|1-r|^{2H-2}dr \leqslant cs^{-2H+1+\alpha}$, implying in turn

$$\int_0^1 ds\, s^{-\beta} \int_0^1 dr\, r^{-\alpha} |s-r|^{2H-2} = \int_0^1 ds\, s^{2H-\alpha-\beta-1} \int_0^{1/s} dr\, r^{-\alpha} |1-r|^{2H-2} \leqslant c \int_0^1 s^{-\beta} ds < \infty.$$

If $\alpha + 1 = 2H$, then $\int_0^{1/s} r^{1-2H}|1-r|^{2H-2}dr \leqslant c(1+|\log s|)$, implying in turn

$$\begin{aligned}
\int_0^1 ds\, s^{-\beta} \int_0^1 dr\, r^{-\alpha} |s-r|^{2H-2} &= \int_0^1 ds\, s^{-\beta} \int_0^1 dr\, r^{1-2H} |s-r|^{2H-2} \\
= \int_0^1 ds\, s^{-\beta} \int_0^{1/s} dr\, r^{1-2H} |1-r|^{2H-2} &\leqslant c \int_0^1 s^{-\beta}\big(1+|\log s|\big) ds < \infty.
\end{aligned}$$

Finally, if $\alpha + 1 > 2H$, then

$$\begin{aligned}
\int_0^1 ds\, s^{-\beta} \int_0^1 dr\, r^{-\alpha} |s-r|^{2H-2} &= \int_0^1 ds\, s^{2H-\alpha-\beta-1} \int_0^{1/s} dr\, r^{-\alpha} |1-r|^{2H-2} \\
&\leqslant \int_0^1 s^{2H-\alpha-\beta-1} ds \times \int_0^\infty r^{-\alpha} |1-r|^{2H-2} dr < \infty.
\end{aligned}$$

■

Lemma 4 *Assume $\alpha \in (0, H)$. Recall the definition (21) of ξ_t. Then $\xi_T := \lim_{t\to T} \xi_t$ exists in L^2. Moreover, for all $\varepsilon \in (0, H-\alpha)$, the process $\{\xi_t\}_{t\in[0,T]}$ admits a modification with $(H-\alpha-\varepsilon)$-Hölder continuous paths, still denoted ξ in the sequel. In particular, $\xi_t \to \xi_T$ almost surely as $t \to T$.*

Proof. Because $\alpha < H$, by Lemma 3 we have that $\int_0^T ds\, s^{-\alpha} \int_0^T du\, u^{-\alpha} |s-u|^{2H-2} < \infty$. For all $s \leqslant t < T$, we thus have, using (14) to get the first equality,

$$\begin{aligned}
E\left[(\xi_t - \xi_s)^2\right] &= H(2H-1) \int_s^t du (T-u)^{-\alpha} \int_s^t dv (T-v)^{-\alpha} |v-u|^{2H-2} \\
&= H(2H-1) \int_{T-t}^{T-s} du\, u^{-\alpha} \int_{T-t}^{T-s} dv\, v^{-\alpha} |v-u|^{2H-2} \\
&= H(2H-1) \int_0^{t-s} du\, (u+T-t)^{-\alpha} \int_0^{t-s} dv\, (v+T-t)^{-\alpha} |v-u|^{2H-2} \\
&\leqslant H(2H-1) \int_0^{t-s} du\, u^{-\alpha} \int_0^{t-s} dv\, v^{-\alpha} |v-u|^{2H-2} \\
&= H(2H-1)(t-s)^{2H-2\alpha} \int_0^1 du\, u^{-\alpha} \int_0^1 dv\, v^{-\alpha} |v-u|^{2H-2} = c(t-s)^{2H-2\alpha}.
\end{aligned}$$

By the Cauchy criterion, we deduce that $\xi_T := \lim_{t\to T} \xi_t$ exists in L^2. Moreover, because the process ξ is centered and Gaussian, the Kolmogorov-Centsov theorem applies as well, thus leading to the desired conclusion. ■

Lemma 5 *Assume $\alpha \in (1-H, H)$. Recall the definition (22) of η_t. Then $\eta_T := \lim_{t\to T} \eta_t$ exists in L^2. Moreover, there exists $\gamma > 0$ such that $\{\eta_t\}_{t\in[0,T]}$ admits a modification with γ-Hölder continuous paths, still denoted η in the sequel. In particular, $\eta_t \to \eta_T$ almost surely as $t \to T$.*

Proof. As a first step, fix $\beta_1, \beta_2 \in (1-H, H)$ and let us show that there exists $\varepsilon = \varepsilon(\beta_1, \beta_2, H) > 0$ and $c = c(\beta_1, \beta_2, H) > 0$ such that, for all $0 \leqslant s \leqslant t \leqslant T$,

$$\int_{[0,t]\times[s,t]} (T-u)^{-\beta_1}(T-v)^{-\beta_2}|u-v|^{2H-2}dudv \leqslant c(t-s)^{\varepsilon}. \tag{24}$$

Indeed, we have

$$\begin{aligned}
&\int_{[0,t]\times[s,t]} (T-u)^{-\beta_1}(T-v)^{-\beta_2}|u-v|^{2H-2}dudv = \int_{T-t}^{T} du\, u^{-\beta_1} \int_{T-t}^{T-s} dv\, v^{-\beta_2}|u-v|^{2H-2} \\
= &\int_0^t du(u+T-t)^{-\beta_1} \int_0^{t-s} dv(v+T-t)^{-\beta_2}|u-v|^{2H-2} \leqslant \int_0^t du\, u^{-\beta_1} \int_0^{t-s} dv\, v^{-\beta_2}|u-v|^{2H-2} \\
= &\int_0^{t-s} du\, u^{-\beta_1} \int_0^{t-s} dv\, v^{-\beta_2}|u-v|^{2H-2} + \int_{t-s}^{t} du\, u^{-\beta_1} \int_0^{t-s} dv\, v^{-\beta_2}(u-v)^{2H-2} \\
= &(t-s)^{2H-\beta_1-\beta_2} \int_0^1 du\, u^{-\beta_1} \int_0^1 dv\, v^{-\beta_2}|u-v|^{2H-2} + \int_{t-s}^{t} du\, u^{-\beta_1-\beta_2+2H-1} \int_0^{(t-s)/u} dv\, v^{-\beta_2}(1-v)^{2H-2} \\
\leqslant &c(t-s)^{2H-\beta_1-\beta_2} + c(t-s)^{1-\beta_2} \int_{t-s}^{t} du\, u^{-\beta_1}(u-t+s)^{2H-2} \quad \text{(see Lemma 3 for the first integral and} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{use } 1-v \geqslant 1-\tfrac{t-s}{u} \text{ for the second one)} \\
\leqslant &c(t-s)^{2H-\beta_1-\beta_2}\left(1 + \int_0^{s/(t-s)} (w+1)^{-\beta_1} w^{2H-2} dw\right) \\
= &c(t-s)^{2H-\beta_1-\beta_2} \times \begin{cases} 1 & \text{if } \beta_1 > 2H-1 \\ 1+|\log(t-s)| & \text{if } \beta_1 = 2H-1 \\ (t-s)^{-2H+1+\beta_1} & \text{if } \beta_1 < 2H-1 \end{cases} \\
\leqslant &c(t-s)^{\varepsilon} \quad \text{for some } \varepsilon \in (0, 1 \wedge (2H-\beta_1) - \beta_2),
\end{aligned}$$

hence (24) is shown.

Now, let $t < T$. Using (18), we can write

$$\eta_t = \int_0^t \xi_u (T-u)^{\alpha-1}\delta B_u + H(2H-1)\int_0^t du(T-u)^{\alpha-1}\int_0^u dv(T-v)^{-\alpha}(u-v)^{2H-2}. \tag{25}$$

(To have the right to write (25), according to Section 2.4 we must check that: (i) $u \to (T-u)^{\alpha-1}\xi_u$ belongs almost surely to $\mathscr{C}^{\gamma}([0,t])$ for some $\gamma > 1-H$; (ii) $\xi_u \in \mathbb{D}^{1,2}$ for all $u \in [0,t]$, and (iii) $\int_{[0,t]^2}(T-u)^{\alpha-1}|D_v\xi_u|\,|u-v|^{2H-2}dudv < \infty$ almost surely. To keep the length of this paper within bounds, we will do it completely here, and this will serve as a basis for the proof of the other instances where a similar verification should have been made as well. The main reason why (i) to (iii) are easy to check is because we are integrating on the compact interval $[0,t]$ with t *strictly less than* T.

Proof of (i). Firstly, $u \to (T-u)^{\alpha-1}$ is $\mathcal{C}^{\infty}$ and bounded on $[0,t]$. Secondly, for $u, v \in [0,t]$ with, say, $u < v$, we have

$$\begin{aligned}
E[(\xi_u - \xi_v)^2] &= H(2H-1)\int_u^v dx(T-x)^{-\alpha}\int_u^v dy(T-y)^{-\alpha}|y-x|^{2H-2} \\
&\leqslant (T-t)^{-2\alpha}H(2H-1)\int_u^v dx\int_u^v dy|y-x|^{2H-2} = (T-t)^{-2\alpha}|v-u|^{2H}.
\end{aligned}$$

Hence, by combining the Kolmogorov-Centsov theorem with the fact that ξ is Gaussian, we get that (almost) all the sample paths of ξ are θ-Hölderian on $[0,t]$ for any $\theta \in (0,H)$. Consequently, by choosing $\gamma \in (1-H, H)$ (which is possible since $H > 1/2$), the proof of (i) is concluded.

Proof of (ii). This is evident, using the representation (21) of ξ as well as the fact that $s \to (T-s)^{-\alpha}\mathbf{1}_{[0,t]}(s) \in |\mathcal{H}|$, see Section 2.1.

Proof of (iii). Here again, it is easy: indeed, we have $D_v\xi_u = (T-v)^{-\alpha}\mathbf{1}_{[0,u]}(v)$, so

$$\int_{[0,t]^2}(T-u)^{\alpha-1}|D_v\xi_u|\,|u-v|^{2H-2}dudv = \int_{[0,t]^2}(T-u)^{\alpha-1}(T-v)^{-\alpha}\,|u-v|^{2H-2}dudv < \infty.\Big)$$

Let us go back to the proof. We deduce from (25), after setting

$$\varphi_t(u,v) = \frac{1}{2}(T-u\vee v)^{\alpha-1}(T-u\wedge v)^{-\alpha}\,\mathbf{1}_{[0,t]^2}(u,v),$$

that

$$\eta_t = I_2(\varphi_t) + H(2H-1)\int_0^t du(T-u)^{\alpha-1}\int_0^u dv(T-v)^{-\alpha}(u-v)^{2H-2}.$$

Hence, because of (15),

$$E\left[(\eta_t-\eta_s)^2\right] = 2\|\varphi_t-\varphi_s\|^2_{\mathcal{H}^{\otimes 2}} + H^2(2H-1)^2\left(\int_s^t du(T-u)^{\alpha-1}\int_0^u dv(T-v)^{-\alpha}(u-v)^{2H-2}\right)^2. \tag{26}$$

We have, by observing that $\varphi_t-\varphi_s \in |\mathcal{H}|^{\odot 2}$,

$$\begin{aligned}
&\|\varphi_t-\varphi_s\|^2_{\mathcal{H}^{\otimes 2}}\\
&= H^2(2H-1)^2\int_{[0,T]^4}\big[\varphi_t(u,v)-\varphi_s(u,v)\big]\big[\varphi_t(x,y)-\varphi_s(x,y)\big]|u-x|^{2H-2}||v-y|^{2H-2}dudvdxdy\\
&= \frac{1}{4}H^2(2H-1)^2\int_{([0,t]^2\setminus[0,s]^2)^2}(T-u\vee v)^{\alpha-1}(T-x\vee y)^{\alpha-1}(T-u\wedge v)^{-\alpha}(T-x\wedge y)^{-\alpha}\\
&\qquad\qquad\times|u-x|^{2H-2}||v-y|^{2H-2}dudvdxdy.
\end{aligned}$$

Taking into account the form of the domain in the previous integral and using that $\varphi_t-\varphi_s$ is symmetric, we easily show that $\|\varphi_t-\varphi_s\|^2_{\mathcal{H}^{\otimes 2}}$ is upper bounded (up to constant, and without seeking for sharpness) by a sum of integrals of the type

$$\int_{[0,t]\times[s,t]\times[0,T]^2}(T-u)^{-\beta_1}(T-v)^{-\beta_2}(T-x)^{-\beta_3}(T-y)^{-\beta_4}|u-x|^{2H-2}|v-y|^{2H-2}dudvdxdy,$$

with $\beta_1,\beta_2,\beta_3,\beta_4 \in \{\alpha, 1-\alpha\}$. Hence, combining Lemma 3 with (24), we deduce that there exists $\varepsilon>0$ small enough and $c>0$ such that, for all $s,t\in[0,T]$,

$$\|\varphi_t-\varphi_s\|^2_{\mathcal{H}^{\otimes 2}} \leqslant c|t-s|^{\varepsilon}. \tag{27}$$

On the other hand, we can write, for all $s\leqslant t<T$,

$$\begin{aligned}
&\int_s^t du(T-u)^{\alpha-1}\int_0^u dv(T-v)^{-\alpha}(u-v)^{2H-2}\\
&= \int_{T-t}^{T-s}du\,u^{\alpha-1}\int_u^T dv\,v^{-\alpha}(v-u)^{2H-2}\\
&= \int_0^{t-s}du(u+T-t)^{\alpha-1}\int_u^t dv(v+T-t)^{-\alpha}(v-u)^{2H-2}\\
&\leqslant \int_0^{t-s}du\,u^{\alpha-1}\int_u^T dv\,v^{-\alpha}(v-u)^{2H-2}\\
&= (t-s)^{2H-1}\int_0^1 du\,u^{\alpha-1}\int_u^{\frac{T}{t-s}}dv\,v^{-\alpha}(v-u)^{2H-2}\\
&= (t-s)^{2H-1}\int_0^1 du\,u^{2H-2}\int_1^{\frac{T}{(t-s)u}}dv\,v^{-\alpha}(v-1)^{2H-2}.
\end{aligned} \tag{28}$$

Let us consider three cases. Assume first that $\alpha > 2H-1$: in this case,

$$\int_1^{\frac{T}{(t-s)u}} v^{-\alpha}(v-1)^{2H-2}dv \leqslant \int_1^{\infty} v^{-\alpha}(v-1)^{2H-2}dv < \infty;$$

leading, thanks to (28), to

$$\int_s^t du(T-u)^{\alpha-1}\int_0^u dv(T-v)^{-\alpha}(u-v)^{2H-2} \leqslant c(t-s)^{2H-1}.$$

The second case is when $\alpha = 2H-1$: we then have

$$\int_1^{\frac{T}{(t-s)u}} v^{-\alpha}(v-1)^{2H-2}dv \leqslant c\big(1+|\log(t-s)|+|\log u|\big)$$

so that, by (28),

$$\int_s^t du(T-u)^{\alpha-1}\int_0^u dv(T-v)^{-\alpha}(u-v)^{2H-2} \leqslant c(t-s)^{2H-1}\big(1+|\log(t-s)|\big).$$

Finally, the third case is when $\alpha < 2H-1$: in this case,

$$\int_1^{\frac{T}{(t-s)u}} v^{-\alpha}(v-1)^{2H-2}dv \leqslant c(t-s)^{\alpha-2H+1}u^{\alpha-2H+1};$$

so that, by (28),

$$\int_s^t du(T-u)^{\alpha-1}\int_0^u dv(T-v)^{-\alpha}(u-v)^{2H-2} \leqslant c(t-s)^{\alpha}.$$

To summarize, we have shown that there exists $c>0$ such that, for all $s,t\in[0,T]$,

$$\int_s^t du(T-u)^{\alpha-1}\int_0^u dv(T-v)^{-\alpha}(u-v)^{2H-2} \leqslant c\big(1+|\log(|t-s|)|\mathbf{1}_{\{\alpha=2H-1\}}\big)|t-s|^{(2H-1)\wedge\alpha}. \quad (29)$$

By inserting (27) and (29) into (26), we finally get that there exists $\varepsilon>0$ small enough and $c>0$ such that, for all $s,t\in[0,T]$,

$$E\left[(\eta_t-\eta_s)^2\right] \leqslant c|t-s|^{\varepsilon}.$$

By the Cauchy criterion, we deduce that $\eta_T := \lim_{t\to T}\eta_t$ exists in L^2. Moreover, because $\eta_t-\eta_s-E[\eta_t]+E[\eta_s]$ belongs to the second Wiener chaos of B (where all the L^p norms are equivalent), the Kolmogorov-Centsov theorem applies as well, thus leading to the desired conclusion. ■

Lemma 6 *Recall the definition (22) of η_t. For any $t\in[0,T)$, we have*

$$\begin{aligned}\eta_t &= \int_0^t (T-u)^{\alpha-1}dB_u \times \int_0^t (T-s)^{-\alpha}dB_s - \int_0^t \delta B_s\,(T-s)^{-\alpha}\int_0^s \delta B_u\,(T-u)^{\alpha-1}\\ &\quad -H(2H-1)\int_0^t ds\,(T-s)^{-\alpha}\int_0^s du\,(T-u)^{\alpha-1}(s-u)^{2H-2}.\end{aligned}$$

Proof. Fix $t\in[0,T)$. Applying the change of variable formula (17) to the right-hand side of the first equality in (22) leads to

$$\eta_t = \int_0^t (T-u)^{\alpha-1}dB_u \times \int_0^t (T-s)^{-\alpha}dB_s - \int_0^t dB_s\,(T-s)^{-\alpha}\int_0^s dB_u\,(T-u)^{\alpha-1}. \quad (30)$$

On the other hand, by (18) we have that

$$\int_0^t dB_s\,(T-s)^{-\alpha}\int_0^s dB_u\,(T-u)^{\alpha-1} \tag{31}$$
$$= \int_0^t \delta B_s\,(T-s)^{-\alpha}\int_0^s \delta B_u\,(T-u)^{\alpha-1} + H(2H-1)\int_0^t ds(T-s)^{-\alpha}\int_0^s du(T-u)^{\alpha-1}(s-u)^{2H-2}.$$

The desired conclusion follows. (We omit the justification of (30) and (31) because it suffices to proceed as in the proof (25).) ■

Lemma 7 *Let $\beta(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$ denote the usual Beta function, let Z be any $\sigma\{B\}$-measurable random variable satisfying $P(Z<\infty)=1$, and let $G\sim\mathcal{N}(0,1)$ be independent of B.*

1. *Assume $\alpha\in(0,1-H)$. Then, as $t\to T$,*

$$\left(Z,(T-t)^{1-H-\alpha}\int_0^t (T-u)^{\alpha-1}dB_u\right)\overset{\text{law}}{\longrightarrow}\left(Z,\sqrt{H(2H-1)\frac{\beta(2-\alpha-2H,2H-1)}{1-H-\alpha}}\,G\right). \tag{32}$$

2. *Assume $\alpha=1-H$. Then, as $t\to T$,*

$$\left(Z,\frac{1}{\sqrt{|\log(T-t)|}}\int_0^t (T-u)^{-H}dB_u\right)\overset{\text{law}}{\longrightarrow}\left(Z,\sqrt{2H(2H-1)\beta(1-H,2H-1)}\,G\right). \tag{33}$$

Proof. By a standard approximation procedure, we first notice that it is not a loss of generality to assume that Z belongs to $L^2(\Omega)$ (using e.g. that $Z\,\mathbf{1}_{\{|Z|\leqslant n\}}\overset{\text{a.s.}}{\longrightarrow} Z$ as $n\to\infty$).

1. Set $N=\sqrt{H(2H-1)\frac{\beta(2-\alpha-2H,2H-1)}{1-H-\alpha}}\,G$. For any $d\geqslant 1$ and any $s_1,\ldots,s_d\in[0,T)$, we shall prove that

$$\left(B_{s_1},\ldots,B_{s_d},(T-t)^{1-H-\alpha}\int_0^t (T-u)^{\alpha-1}dB_u\right)\overset{\text{law}}{\longrightarrow}\left(B_{s_1},\ldots,B_{s_d},N\right)\quad\text{as } t\to T. \tag{34}$$

Suppose for a moment that (34) has been shown, and let us proceed with the proof of (32). By the very construction of $\mathcal{H}$ and by reasoning by approximation, we deduce that, for any $l\geqslant 1$ and any $h_1,\ldots,h_l\in\mathcal{H}$ with unit norms,

$$\left(B(h_1),\ldots,B(h_l),(T-t)^{1-H-\alpha}\int_0^t (T-u)^{\alpha-1}dB_u\right)\overset{\text{law}}{\longrightarrow}\left(B(h_1),\ldots,B(h_l),N\right)\quad\text{as } t\to T.$$

This implies that, for any $l\geqslant 1$, any $h_1,\ldots,h_l\in\mathcal{H}$ with unit norms and any integers $q_1,\ldots,q_l\geqslant 0$,

$$\left(H_{q_1}(B(h_1)),\ldots,H_{q_l}(B(h_l)),(T-t)^{1-H-\alpha}\int_0^t (T-u)^{\alpha-1}dB_u\right)$$
$$\overset{\text{law}}{\longrightarrow}\quad\left(H_{q_1}(B(h_1)),\ldots,H_{q_l}(B(h_l)),N\right)\quad\text{as } t\to T,$$

with H_q the qth Hermite polynomial. Using now the very definition of the Wiener chaoses and by reasoning by approximation once again, we deduce that, for any $l\geqslant 1$, any integers $q_1,\ldots,q_l\geqslant 0$ and any $f_1\in\mathcal{H}^{\odot q_1},\ldots,f_l\in\mathcal{H}^{\odot q_l}$,

$$\left(I_{q_1}(f_1),\ldots,I_{q_l}(f_l),(T-t)^{1-H-\alpha}\int_0^t (T-u)^{\alpha-1}dB_u\right)\overset{\text{law}}{\longrightarrow}\left(I_{q_1}(f_1),\ldots,I_{q_l}(f_l),N\right)\quad\text{as } t\to T.$$

Thus, for any random variable $F\in L^2(\Omega)$ with a *finite* chaotic decomposition, we have

$$\left(F,(T-t)^{1-H-\alpha}\int_0^t (T-u)^{\alpha-1}dB_u\right)\overset{\text{law}}{\longrightarrow}(F,N)\quad\text{as } t\to T. \tag{35}$$

To conclude, let us consider the chaotic decomposition (16) of Z. By applying (35) to $F = E[Z] + \sum_{q=1}^n I_q(f_q)$ and then letting $n \to \infty$, we finally deduce that (32) holds true.

Now, let us proceed with the proof of (34). Because the left-hand side of (34) is a Gaussian vector, to get (34) it is sufficient to check the convergence of covariance matrices. Let us first compute the limiting variance of $(T-t)^{1-H-\alpha}\int_0^t (T-u)^{\alpha-1}dB_u$ as $t \to T$. By (14), for any $t \in [0,T)$ we have

$$\begin{aligned}
& E\left[\left((T-t)^{1-H-\alpha}\int_0^t (T-u)^{\alpha-1}dB_u\right)^2\right] \\
= \; & H(2H-1)(T-t)^{2-2H-2\alpha}\int_0^t ds(T-s)^{\alpha-1}\int_0^t du(T-u)^{\alpha-1}|s-u|^{2H-2} \\
= \; & H(2H-1)(T-t)^{2-2H-2\alpha}\int_{T-t}^T ds\, s^{\alpha-1}\int_{T-t}^T du\, u^{\alpha-1}|s-u|^{2H-2} \\
= \; & H(2H-1)\int_1^{\frac{T}{T-t}} ds\, s^{\alpha-1}\int_1^{\frac{T}{T-t}} du\, u^{\alpha-1}|s-u|^{2H-2} \\
\to \; & H(2H-1)\int_1^\infty ds\, s^{\alpha-1}\int_1^\infty du\, u^{\alpha-1}|s-u|^{2H-2} \quad \text{as } t \to T,
\end{aligned}$$

with

$$\begin{aligned}
& \int_1^\infty ds\, s^{\alpha-1}\int_1^\infty du\, u^{\alpha-1}|s-u|^{2H-2} = \int_1^\infty ds\, s^{2\alpha+2H-3}\int_{1/s}^\infty du\, u^{\alpha-1}|1-u|^{2H-2} \\
= \; & \int_1^\infty s^{2\alpha+2H-3}ds\int_1^\infty u^{\alpha-1}(u-1)^{2H-2}du + \int_1^\infty ds\, s^{2\alpha+2H-3}\int_{1/s}^1 du\, u^{\alpha-1}(1-u)^{2H-2} \\
= \; & \frac{\beta(2-\alpha-2H, 2H-1)}{2(1-H-\alpha)} + \int_0^1 du\, u^{\alpha-1}(1-u)^{2H-2}\int_{1/u}^\infty ds\, s^{2\alpha+2H-3} \\
= \; & \frac{\beta(2-\alpha-2H, 2H-1)}{1-H-\alpha}.
\end{aligned}$$

Thus,

$$\lim_{t\to T} E\left[\left((T-t)^{1-H-\alpha}\int_0^t (T-u)^{\alpha-1}dB_u\right)^2\right] = \frac{H(2H-1)}{1-H-\alpha}\,\beta(2-\alpha-2H, 2H-1).$$

On the other hand, by (14) we have, for any $v < t < T$,

$$\begin{aligned}
& E\left[B_v \times (T-t)^{1-H-\alpha}\int_0^t (T-u)^{\alpha-1}dB_u\right] \\
= \; & H(2H-1)(T-t)^{1-H-\alpha}\int_0^t du\,(T-u)^{\alpha-1}\int_0^v ds\,|u-s|^{2H-2} \\
= \; & H(T-t)^{1-H-\alpha}\int_0^t (T-u)^{\alpha-1}\big(u^{2H-1} + \operatorname{sign}(v-u)\times|v-u|^{2H-1}\big)du \\
\to \; & 0 \quad \text{as } t \to T,
\end{aligned}$$

because $\int_0^T (T-u)^{\alpha-1}\big(u^{2H-1} + \operatorname{sign}(v-u)\times|v-u|^{2H-1}\big)du < \infty$. Convergence (34) is then shown, and (32) follows.

2. By (14), for any $t \in [0 \vee (T-1), T)$ we have

$$
\begin{aligned}
& E\left[\left(\frac{1}{\sqrt{|\log(T-t)|}}\int_0^t (T-u)^{-H} dB_u\right)^2\right] \\
= \quad & \frac{H(2H-1)}{|\log(T-t)|}\int_0^t ds (T-s)^{-H}\int_0^t du (T-u)^{-H}|s-u|^{2H-2} \\
= \quad & \frac{H(2H-1)}{|\log(T-t)|}\int_{T-t}^T ds\, s^{-H}\int_{T-t}^T du\, u^{-H}|s-u|^{2H-2} \\
= \quad & \frac{2H(2H-1)}{|\log(T-t)|}\int_{T-t}^T ds\, s^{-H}\int_{T-t}^s du\, u^{-H}(s-u)^{2H-2} \\
= \quad & \frac{2H(2H-1)}{|\log(T-t)|}\int_{T-t}^T \frac{ds}{s}\int_{\frac{T-t}{s}}^1 du\, u^{-H}(1-u)^{2H-2} \\
= \quad & \frac{2H(2H-1)}{|\log(T-t)|}\int_{\frac{T-t}{T}}^1 du\, u^{-H}(1-u)^{2H-2}\int_{\frac{T-t}{u}}^T \frac{ds}{s} \\
= \quad & 2H(2H-1)\int_{\frac{T-t}{T}}^1 du\, u^{-H}(1-u)^{2H-2}\left(1+\frac{\log(Tu)}{|\log(T-t)|}\right).
\end{aligned}
$$

Because $\int_0^1 |\log(Tu)|u^{-H}(1-u)^{2H-2}du < \infty$, we get that

$$
E\left[\left(\frac{1}{\sqrt{|\log(T-t)|}}\int_0^t (T-s)^{-H} dB_s\right)^2\right] \to 2H(2H-1)\beta(1-H, 2H-1) \quad \text{as } t \to T.
$$

On the other hand, fix $v \in [0,T)$. For all $t \in [0 \vee (T-1), T)$, using (14) we can write

$$
\begin{aligned}
& E\left[B_v \times \frac{1}{\sqrt{|\log(T-t)|}}\int_0^t (T-u)^{-H} dB_u\right] \\
= \quad & \frac{H(2H-1)}{\sqrt{|\log(T-t)|}}\int_0^t du\, (T-u)^{-H}\int_0^v ds\, |u-s|^{2H-2} \\
= \quad & \frac{H}{\sqrt{|\log(T-t)|}}\int_0^T (T-u)^{\alpha-1}\big(u^{2H-1}+\mathrm{sign}(v-u)\times|v-u|^{2H-1}\big)du \\
\longrightarrow \quad & 0 \quad \text{as } t \to T,
\end{aligned}
$$

because $\int_0^T (T-u)^{\alpha-1}\big(u^{2H-1}+\mathrm{sign}(v-u)\times|v-u|^{2H-1}\big)du < \infty$. Thus, we have shown that, for any $d \geqslant 1$ and any $s_1, \ldots, s_d \in [0,T)$,

$$
\left(B_{s_1}, \ldots, B_{s_d}, (T-t)^{1-H-\alpha}\int_0^t (T-u)^{\alpha-1} dB_u\right) \xrightarrow{\text{law}} \left(B_{s_1}, \ldots, B_{s_d}, \sqrt{2H(2H-1)\beta(1-H, 2H-1)}\, G\right) \tag{36}
$$

as $t \to T$. Finally, the same reasoning as in point 1 above allows to go from (36) to (33). The proof of the lemma is concluded. ■

Lemma 8 *Assume $\alpha \in (0, 1-H]$. Then, as $t \to T$,*

$$
\limsup_{t\to T} E\left[\left(\int_0^t \delta B_u\, (T-u)^{-\alpha}\int_0^s \delta B_v\, (T-v)^{\alpha-1}\right)^2\right] < \infty.
$$

Proof. Set $\phi_t(u,v)=\frac{1}{2}(T-u\vee v)^{-\alpha}(T-u\wedge v)^{\alpha-1}\mathbf{1}_{[0,t]^2}(u,v)$. We have $\phi_t\in|\mathcal{H}|^{\odot 2}$ and $\int_0^t \delta B_u\,(T-u)^{-\alpha}\int_0^u \delta B_v\,(T-v)^{\alpha-1}=I_2(\phi_t)$, so that

$$
\begin{aligned}
&\limsup_{t\to T} E\left[\left(\int_0^t \delta B_u\,(T-u)^{-\alpha}\int_0^u \delta B_v\,(T-v)^{\alpha-1}\right)^2\right]=2\,\limsup_{t\to T}\|\phi_t\|^2_{\mathcal{H}^{\otimes 2}}\\
=\;& 2H^2(2H-1)^2\,\limsup_{t\to T}\int_{[0,T]^4}\phi_t(u,v)\phi_t(x,y)|u-x|^{2H-2}|v-y|^{2H-2}dudvdxdy\\
=\;& \frac{1}{2}H^2(2H-1)^2\int_{[0,T]^4}(T-u\vee v)^{-\alpha}(T-u\wedge v)^{\alpha-1}(T-x\vee y)^{-\alpha}(T-x\wedge y)^{\alpha-1}\\
&\qquad\qquad\times|u-x|^{2H-2}|v-y|^{2H-2}dudvdxdy\\
=\;& 2H^2(2H-1)^2\int_0^T du\,(T-u)^{-\alpha}\int_0^T dx\,(T-x)^{-\alpha}|u-x|^{2H-2}\int_0^u dv\,(T-v)^{\alpha-1}\\
&\qquad\qquad\times\int_0^x dv\,(T-y)^{\alpha-1}|v-y|^{2H-2}\\
=\;& 2H^2(2H-1)^2\int_0^T du\,u^{-\alpha}\int_0^T dx\,x^{-\alpha}|u-x|^{2H-2}\int_u^T dv\,v^{\alpha-1}\int_x^T dy\,y^{\alpha-1}|v-y|^{2H-2}\\
=\;& 2H^2(2H-1)^2\int_0^T du\,u^{-\alpha}\int_0^T dx\,x^{-\alpha}|u-x|^{2H-2}\int_u^T dv\,v^{2H+2\alpha-3}\int_{x/v}^{T/v} dy\,y^{\alpha-1}|1-y|^{2H-2}.
\end{aligned}
$$

Because $\alpha\leqslant 1-H$ and $H<1$, we have $\alpha<2-2H$, so that

$$
\int_{x/v}^{T/v} y^{\alpha-1}|1-y|^{2H-2}dy\leqslant\int_0^\infty y^{\alpha-1}|1-y|^{2H-2}dy<\infty.
$$

Moreover, because $2H+2\alpha-3\leqslant -1$ due to our assumption on α, we have

$$
\int_u^T dv\,v^{2H+2\alpha-3}\leqslant c\begin{cases} u^{2H+2\alpha-2} & \text{if } \alpha<1-H\\ 1+|\log u| & \text{if } \alpha=1-H\end{cases}.
$$

Consequently, if $\alpha=1-H$, then

$$
\begin{aligned}
&\int_0^T du\,u^{-\alpha}\int_0^T dx\,x^{-\alpha}|u-x|^{2H-2}\int_u^T dv\,v^{2H+2\alpha-3}\int_{x/v}^{T/v} dy\,y^{\alpha-1}|1-y|^{2H-2}\\
\leqslant\;& c\int_0^T du\,u^{H-1}\big(1+|\log u|\big)\int_0^T dx\,x^{H-1}|u-x|^{2H-2}\\
=\;& c\int_0^T du\,u^{4H-3}\big(1+|\log u|\big)\int_0^{T/u} dx\,x^{H-1}|1-x|^{2H-2}\\
\leqslant\;& c\int_0^T du\,u^{4H-3}\big(1+|\log u|\big)\times\begin{cases} 1 & \text{if } H<\frac{2}{3}\\ 1+|\log u| & \text{if } H=\frac{2}{3}\\ u^{2-3H} & \text{if } H>\frac{2}{3}\end{cases}\\
<\;& \infty,
\end{aligned}
$$

and the proof is concluded in this case. Assume now that $\alpha<1-H$. Then

$$
\begin{aligned}
&\int_0^T du\,u^{-\alpha}\int_0^T dx\,x^{-\alpha}|u-x|^{2H-2}\int_u^T dv\,v^{2H+2\alpha-3}\int_{x/v}^{T/v} dy\,y^{\alpha-1}|1-y|^{2H-2}\\
\leqslant\;& c\int_0^T du\,u^{2H+\alpha-2}\int_0^T dx\,x^{-\alpha}|u-x|^{2H-2}=c\int_0^T du\,u^{4H-3}\int_0^{T/u} dx\,x^{-\alpha}|1-x|^{2H-2}.
\end{aligned}
$$

Let us distinguish three different cases. First, if $\alpha < 2H-1$ then

$$\int_0^T du\, u^{4H-3} \int_0^{T/u} dx\, x^{-\alpha}|1-x|^{2H-2} \leqslant c \int_0^T u^{2H-2+\alpha} du < \infty.$$

Second, if $\alpha = 2H-1$ then

$$\begin{aligned}\int_0^T du\, u^{4H-3} \int_0^{T/u} dx\, x^{-\alpha}|1-x|^{2H-2} &= \int_0^T du\, u^{4H-3} \int_0^{T/u} dx\, x^{1-2H}|1-x|^{2H-2} \\ &\leqslant c \int_0^T u^{4H-3}\big(1+|\log u|\big) du < \infty.\end{aligned}$$

Third, if $\alpha > 2H-1$ then

$$\int_0^T du\, u^{4H-3} \int_0^{T/u} dx\, x^{-\alpha}|1-x|^{2H-2} \leqslant \int_0^T u^{4H-3} du \int_0^\infty x^{-\alpha}|1-x|^{2H-2} dx < \infty.$$

Thus, in all the possible cases we see that $\limsup_{t\to T} E\left[\left(\int_0^t \delta B_u\,(T-u)^{-\alpha} \int_0^u \delta B_v\,(T-v)^{\alpha-1}\right)^2\right]$ is finite, and the proof of the lemma is done. ■

Lemma 9 *Assume $\alpha \in (0,H)$, and recall the definition (21) of ξ_t. Then, as $t \to T$:*

1. *if $0 < \alpha < \frac{1}{2}$, then*
$$(T-t)^{1-2\alpha} \int_0^t \xi_s^2 (T-s)^{2\alpha-2}\, ds \overset{\text{a.s.}}{\to} \frac{\xi_T^2}{1-2\alpha};$$

2. *if $\alpha = \frac{1}{2}$, then*
$$\frac{1}{|\log(T-t)|} \int_0^t \frac{\xi_s^2}{T-s}\, ds \overset{\text{a.s.}}{\to} \xi_T^2;$$

3. *if $\frac{1}{2} < \alpha < H$, then*
$$\int_0^t \xi_s^2 (T-s)^{2\alpha-2}\, ds \overset{\text{a.s.}}{\to} \int_0^T \xi_s^2 (T-s)^{2\alpha-2}\, ds < \infty.$$

Proof. 1. Using the $\left(\frac{H}{2}-\frac{\alpha}{2}\right)$-Hölderianity of ξ (Lemma 4), we can write

$$\begin{aligned}&\left|(T-t)^{1-2\alpha} \int_0^t \xi_s^2 (T-s)^{2\alpha-2}\, ds - \frac{\xi_T^2}{1-2\alpha}\right| \\ \leqslant\ & (T-t)^{1-2\alpha} \int_0^t \left|\xi_s^2 - \xi_T^2\right| (T-s)^{2\alpha-2}\, ds + (T-t)^{1-2\alpha} \frac{T^{2\alpha-1}}{1-2\alpha}\xi_T^2 \\ \leqslant\ & c|\xi|_\infty (T-t)^{1-2\alpha} \int_0^t (T-s)^{\frac{H}{2}+\frac{3\alpha}{2}-2}\, ds + (T-t)^{1-2\alpha} \frac{T^{2\alpha-1}}{1-2\alpha}\xi_T^2 \\ \leqslant\ & c|\xi|_\infty \big((T-t)^{\frac{H}{2}-\frac{\alpha}{2}} + (T-t)^{1-2\alpha} T^{\frac{H}{2}+\frac{3\alpha}{2}-1}\big) + (T-t)^{1-2\alpha} \frac{T^{2\alpha-1}}{1-2\alpha}\xi_T^2 \\ \to\ & 0 \quad \text{almost surely as } t \to T.\end{aligned}$$

2. Using the $(\frac{H}{2}-\frac{1}{4})$-Hölderianity of ξ (Lemma 4), we can write

$$
\begin{aligned}
&\left|\frac{1}{|\log(T-t)|}\int_0^t \frac{\xi_s^2}{T-s}ds - \xi_T^2\right| \\
\leqslant\ & \frac{1}{|\log(T-t)|}\int_0^t \frac{\left|\xi_s^2-\xi_T^2\right|}{T-s}ds + \frac{\log(T)}{|\log(T-t)|}\xi_T^2 \\
\leqslant\ & \frac{c|\xi|_\infty}{|\log(T-t)|}\int_0^t (T-s)^{\frac{H}{2}-\frac{5}{4}}ds + \frac{\log(T)}{|\log(T-t)|}\xi_T^2 \\
\leqslant\ & \frac{c|\xi|_\infty}{|\log(T-t)|}\big(T^{\frac{H}{2}-\frac{1}{4}} + (T-t)^{\frac{H}{2}-\frac{1}{4}}\big) + \frac{\log(T)}{|\log(T-t)|}\xi_T^2 \\
\to\ & 0 \quad \text{almost surely as } t\to T.
\end{aligned}
$$

3. By Lemma 4, the process ξ is continuous on $[0,T]$, hence integrable. Moreover, $s\mapsto (T-s)^{2\alpha-2}$ is integrable at $s=T$ because $\alpha>\frac{1}{2}$. The convergence in point 3 is then clear, with a finite limit. ■

We are now ready to prove Theorems 1 and 2.

Proof of Theorem 1. Fix $\alpha>0$. Thanks to the change of variable formula (17) (which can be well applied here, as is easily shown by proceeding as in the proof (25)), we can write, for any $t\in[0,T)$:

$$
\begin{aligned}
\frac{1}{2}(T-t)^{2\alpha-1}\xi_t^2 &= \frac{1-2\alpha}{2}\int_0^t (T-u)^{2\alpha-2}\xi_u^2du + \int_0^t (T-u)^{2\alpha-1}\xi_u d\xi_u \\
&= \frac{1-2\alpha}{2}\int_0^t (T-u)^{2\alpha-2}\xi_u^2du + \eta_t,
\end{aligned}
$$

so that

$$
\alpha-\widehat{\alpha}_t = \frac{\xi_t^2}{2(T-t)^{1-2\alpha}\int_0^t \xi_u^2(T-u)^{2\alpha-2}du} + \alpha - \frac{1}{2}. \tag{37}
$$

When $\alpha\in(0,\frac{1}{2})$, we have $(T-t)^{1-2\alpha}\int_0^t \xi_u^2(T-u)^{2\alpha-2}du \overset{\text{a.s.}}{\to} \frac{\xi_T^2}{1-2\alpha}$ (resp. $\xi_t^2 \overset{\text{a.s.}}{\to} \xi_T^2$) as $t\to T$ by Lemma 9 (resp. Lemma 4); hence, as desired one gets that $\alpha-\widehat{\alpha}_t \overset{\text{a.s.}}{\to} 0$ as $t\to T$.

When $\alpha=\frac{1}{2}$, the identity (37) becomes

$$
\alpha-\widehat{\alpha}_t = \frac{\xi_t^2}{2\int_0^t \xi_u^2(T-u)^{-1}du}; \tag{38}
$$

as $t\to T$, we have $\int_0^t \xi_u^2(T-u)^{-1}du \overset{\text{a.s.}}{\sim} |\log(T-t)|\xi_T^2$ (resp. $\xi_t^2 \overset{\text{a.s.}}{\to} \xi_T^2$) by Lemma 9 (resp. Lemma 4). Hence, here again we have $\alpha-\widehat{\alpha}_t \overset{\text{a.s.}}{\to} 0$ as $t\to T$.

Suppose now that $\alpha\in(\frac{1}{2},H)$. As $t\to T$, we have $\int_0^t \xi_u^2(T-u)^{2\alpha-2}du \overset{\text{a.s.}}{\to} \int_0^T \xi_u^2(T-u)^{2\alpha-2}du$ (resp. $\xi_t^2 \overset{\text{a.s.}}{\to} \xi_T^2$) by Lemma 9 (resp. Lemma 4). Hence (37) yields this time that $\alpha-\widehat{\alpha}_t \overset{\text{a.s.}}{\to} \alpha-\frac{1}{2}$ as $t\to T$, that is $\widehat{\alpha}_t \overset{\text{a.s.}}{\to} \frac{1}{2}$.

Assume finally that $\alpha \geqslant H$. By (15), we have

$$
\begin{aligned}
E\big[(T-t)^{2\alpha-1}\xi_t^2\big] &= (T-t)^{2\alpha-1}\int_0^t du(T-u)^{-\alpha}\int_0^t dv(T-v)^{-\alpha}|v-u|^{2H-2} \\
&= (T-t)^{2\alpha-1}\int_{T-t}^{T} du\, u^{-\alpha}\int_{T-t}^{T} dv\, v^{-\alpha}|v-u|^{2H-2} \\
&= (T-t)^{2H-1}\int_1^{\frac{T}{T-t}} du\, u^{-\alpha}\int_1^{\frac{T}{T-t}} dv\, v^{-\alpha}|v-u|^{2H-2} \\
&= (T-t)^{2H-1}\int_{\frac{T-t}{T}}^{1} du\, u^{\alpha-2H}\int_{\frac{T-t}{T}}^{1} dv\, v^{\alpha-2H}|v-u|^{2H-2} \\
&\leqslant (T-t)^{2H-1}\int_{\frac{T-t}{T}}^{1} du\, u^{2\alpha-2H-1}\int_0^{\frac{1}{u}} dv\, v^{\alpha-2H}|v-1|^{2H-2} \\
&\leqslant c(T-t)^{2H-1}\int_{\frac{T-t}{T}}^{1} du\, u^{2\alpha-2H-1}\times\begin{cases} 1 & \text{if } \alpha<1 \\ 1+|\log u| & \text{if } \alpha=1 \\ u^{1-\alpha} & \text{if } \alpha>1 \end{cases} \\
&\leqslant c(T-t)^{2H-1}\times\begin{cases} |\log(T-t)| & \text{if } \alpha=H \\ 1 & \text{if } \alpha>H \end{cases} \\
&\longrightarrow 0 \text{ as } t\to T.
\end{aligned}
$$

Hence, having a look at (37) and because $\int_0^t \xi_u^2(T-u)^{2\alpha-2}du \overset{\text{a.s.}}{\to} \int_0^T \xi_u^2(T-u)^{2\alpha-2}du \in (0,\infty]$ as $t\to T$, we deduce that $\alpha-\widehat{\alpha}_t \overset{\text{prob.}}{\to} \alpha-\frac{1}{2}$ as $t\to T$, that is $\widehat{\alpha}_t \overset{\text{prob.}}{\to} \frac{1}{2}$.

The proof of Theorem 1 is done. ■

Proof of Theorem 2. 1. Assume that α belongs to $(0,1-H)$. We have, by using Lemma 6 to go from the first to the second line,

$$
\begin{aligned}
&(T-t)^{\alpha-H}\big(\alpha-\widehat{\alpha}_t\big) = \frac{(T-t)^{1-H-\alpha}\eta_t}{(T-t)^{1-2\alpha}\int_0^t \xi_s^2(T-s)^{2\alpha-2}ds} \\
=\ & \frac{(T-t)^{1-H-\alpha}\int_0^t (T-u)^{\alpha-1}dB_u\int_0^t (T-s)^{-\alpha}dB_s}{(T-t)^{1-2\alpha}\int_0^t \xi_s^2(T-s)^{2\alpha-2}ds} \\
&-\frac{(T-t)^{1-H-\alpha}\int_0^t \delta B_s(T-s)^{-\alpha}\int_0^s \delta B_u(T-u)^{\alpha-1}}{(T-t)^{1-2\alpha}\int_0^t \xi_s^2(T-s)^{2\alpha-2}ds} \\
&-H(2H-1)\frac{(T-t)^{1-H-\alpha}\int_0^t ds\,(T-s)^{-\alpha}\int_0^s du\,(T-u)^{\alpha-1}(s-u)^{2H-2}}{(T-t)^{1-2\alpha}\int_0^t \xi_s^2(T-s)^{2\alpha-2}ds}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1-2\alpha}{\xi_T}(T-t)^{1-H-\alpha}\int_0^t (T-u)^{\alpha-1}dB_u \\
&\qquad\qquad \times\frac{\int_0^t (T-s)^{-\alpha}dB_s}{\xi_T}\times\frac{\xi_T^2}{(1-2\alpha)(T-t)^{1-2\alpha}\int_0^t\xi_s^2(T-s)^{2\alpha-2}ds} \\
&\quad -\frac{(T-t)^{1-H-\alpha}\int_0^t\delta B_s(T-s)^{-\alpha}\int_0^s\delta B_u(T-u)^{\alpha-1}}{(T-t)^{1-2\alpha}\int_0^t\xi_s^2(T-s)^{2\alpha-2}ds} \\
&\quad -H(2H-1)\frac{(T-t)^{1-H-\alpha}\int_0^t ds\,(T-s)^{-\alpha}\int_0^s du\,(T-u)^{\alpha-1}(s-u)^{2H-2}}{(T-t)^{1-2\alpha}\int_0^t\xi_s^2(T-s)^{2\alpha-2}ds} \\
&= a_t\times b_t\times c_t-d_t-e_t, \qquad (39)
\end{aligned}
$$

with clear definitions for a_t to e_t. Lemma 7 yields

$$
a_t\overset{\text{law}}{\to}(1-2\alpha)\sqrt{H(2H-1)\frac{\beta(2-\alpha-2H,2H-1)}{1-H-\alpha}}\times\frac{G}{\xi_T}\quad\text{as } t\to T,
$$

where $G\sim\mathcal{N}(0,1)$ is independent of B, whereas Lemma 4 (resp. Lemma 9) implies that $b_t\overset{\text{a.s.}}{\to}1$ (resp. $c_t\overset{\text{a.s.}}{\to}1$) as $t\to T$. On the other hand, by combining Lemma 9 with Lemma 8 (resp. Lemma 3), we deduce that $d_t\overset{\text{prob.}}{\to}0$ (resp. $e_t\overset{\text{prob.}}{\to}0$) as $t\to T$. By plugging all these convergences together we get that, as $t\to T$,

$$
(T-t)^{\alpha-H}\big(\widehat{\alpha}_t-\alpha\big)\overset{\text{law}}{\to}(1-2\alpha)\sqrt{H(2H-1)\frac{\beta(2-\alpha-2H,2H-1)}{1-H-\alpha}}\times\frac{G}{\xi_T}.
$$

Because it is well-known that the ratio of two independent $\mathcal{N}(0,1)$-random variables is $\mathcal{C}(1)$-distributed, to conclude it remains to compute the variance σ^2 of $\xi_T\sim\mathcal{N}(0,\sigma^2)$. By (15), we have:

$$
\begin{aligned}
E[\xi_T^2] &= H(2H-1)\int_0^T du(T-u)^{-\alpha}\int_0^T dv(T-v)^{-\alpha}|v-u|^{2H-2} \\
&= H(2H-1)\int_0^T du\,u^{-\alpha}\int_0^T dv\,v^{-\alpha}|v-u|^{2H-2} \\
&= 2H(2H-1)\int_0^T du\,u^{-\alpha}\int_0^u dv\,v^{-\alpha}(u-v)^{2H-2} \\
&= 2H(2H-1)\int_0^T u^{2H-2\alpha-1}du\int_0^1 v^{-\alpha}(1-v)^{2H-2}dv \\
&= \frac{H(2H-1)}{H-\alpha}T^{2H-2\alpha}\beta(1-\alpha,2H-1), \qquad (40)
\end{aligned}
$$

and the proof of the first part of Theorem 2 is done.

2. Assume that $\alpha=1-H$. The proof follows the same lines as in point 1 above. The counterpart of decomposition (39) is here:

$$
\begin{aligned}
\frac{(T-t)^{1-2H}}{\sqrt{|\log(T-t)|}}\big(\alpha-\widehat{\alpha}_t\big) &= \frac{2H-1}{\xi_T\sqrt{|\log(T-t)|}}\int_0^t(T-s)^{-H}dB_s \\
&\qquad \times\frac{\int_0^t(T-u)^{H-1}dB_u}{\xi_T}\times\frac{\xi_T^2}{(2H-1)(T-t)^{2H-1}\int_0^t\xi_s^2(T-s)^{-2H}ds}
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{\int_0^t \delta B_s (T-s)^{H-1} \int_0^s \delta B_u (T-u)^{-H}}{\sqrt{|\log(T-t)|}(T-t)^{2H-1}\int_0^t \xi_s^2 (T-s)^{-2H} ds} \\
& -H(2H-1)\frac{\int_0^t ds\,(T-s)^{H-1}\int_0^s du\,(T-u)^{-H}(s-u)^{2H-2}}{\sqrt{|\log(T-t)|}(T-t)^{2H-1}\int_0^t \xi_s^2 (T-s)^{-2H} ds} \\
= \; & \widetilde{a}_t \times \widetilde{b}_t \times \widetilde{c}_t - \widetilde{d}_t - \widetilde{e}_t.
\end{aligned}
$$

Lemma 7 yields

$$
\widetilde{a}_t \overset{\text{law}}{\to} (2H-1)^{\frac{3}{2}}\sqrt{2H\,\beta(1-H, 2H-1)} \times \frac{G}{\xi_T} \quad \text{as } t \to T,
$$

where $G \sim \mathcal{N}(0,1)$ is independent of B, whereas Lemma 4 (resp. Lemma 9) implies that $\widetilde{b}_t \overset{\text{a.s.}}{\to} 1$ (resp. $\widetilde{c}_t \overset{\text{a.s.}}{\to} 1$) as $t \to T$. On the other hand, by combining Lemma 9 with Lemma 8 (resp. Lemma 3), we deduce that $\widetilde{d}_t \overset{\text{prob.}}{\to} 0$ (resp. $\widetilde{e}_t \overset{\text{prob.}}{\to} 0$) as $t \to T$. By plugging all these convergences together we get that, as $t \to T$,

$$
\frac{(T-t)^{1-2H}}{\sqrt{|\log(T-t)|}}\big(\widehat{\alpha}_t - \alpha\big) \overset{\text{law}}{\to} (2H-1)^{\frac{3}{2}}\sqrt{2H\,\beta(1-H, 2H-1)} \times \frac{G}{\xi_T}.
$$

Moreover, by (40) we have that $\xi_T \sim \mathcal{N}\big(0, HT^{4H-2}\beta(H, 2H-1)\big)$. Thus,

$$
(2H-1)^{\frac{3}{2}}\sqrt{2H\,\beta(1-H, 2H-1)} \times \frac{G}{\xi_T} \overset{\text{law}}{=} T^{1-2H}(2H-1)^{\frac{3}{2}}\sqrt{\frac{2\,\beta(1-H, 2H-1)}{\beta(H, 2H-1)}} \times \mathcal{C}(1),
$$

and the convergence in point 2 is shown.

3. Assume that α belongs to $(1-H, \frac{1}{2})$. Using the decomposition

$$
(T-t)^{2\alpha-1}\big(\alpha - \widehat{\alpha}_t\big) = \frac{\eta_t}{(T-t)^{1-2\alpha}\int_0^t \xi_u^2 (T-u)^{2\alpha-2} du},
$$

we immediately see that the second part of Theorem 2 is an obvious consequence of Lemmas 5 and 9.

4. Assume that $\alpha = \frac{1}{2}$. Recall the identity (38) for this particular value of α:

$$
\alpha - \widehat{\alpha}_t = \frac{\xi_t^2}{2\int_0^t \xi_u^2 (T-u)^{-1} du}.
$$

As $t \to T$, we have $\xi_t^2 \overset{\text{a.s.}}{\to} \xi_T^2$ by Lemma 4, whereas $\int_0^t \xi_u^2 (T-u)^{-1} du \overset{\text{a.s.}}{\sim} |\log(T-t)|\xi_T^2$ by Lemma 9. Therefore, we deduce as announced that $|\log(T-t)|\big(\alpha - \widehat{\alpha}_t\big) \overset{\text{a.s.}}{\to} \frac{1}{2}$ as $t \to T$. ∎

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