ASYMPTOTIC INDEPENDENCE OF MULTIPLE WIENER-ITÔ INTEGRALS AND THE RESULTING LIMIT LAWS

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Abstract. We characterize the asymptotic independence between blocks consisting of multiple Wiener-Itô integrals. As a consequence of this characterization, we derive the celebrated fourth moment theorem of Nualart and Peccati, its multidimensional extension, and other related results on the multivariate convergence of multiple Wiener-Itô integrals, that involve Gaussian and non Gaussian limits. We give applications to the study of the asymptotic behavior of functions of short and long range dependent stationary Gaussian time series and establish the asymptotic independence for discrete non-Gaussian chaoses.

1. Introduction

Let $B = (B_t)_{t\in\mathbb{R}_+}$ be a standard one-dimensional Brownian motion, $q \geqslant 1$ be an integer, and let f be a symmetric element of $L^2(\mathbb{R}_+^q)$. Denote by $I_q(f)$ the q-tuple Wiener-Itô integral of f with respect to B. It is well known that multiple Wiener-Itô integrals of different orders are uncorrelated but not necessarily independent. In an important paper [17], Üstünel and Zakai gave the following characterization of the independence of multiple Wiener-Itô integrals.

Theorem 1.1 (Üstünel-Zakai)**.** *Let $p, q \geqslant 1$ be integers and let $f \in L^2(\mathbb{R}_+^p)$ and $g \in L^2(\mathbb{R}_+^q)$ be symmetric. Then, random variables $I_p(f)$ and $I_q(g)$ are independent if and only if*

$$\int_{\mathbb{R}_+^{p+q-2}} \left| \int_{\mathbb{R}_+} f(x_1, \ldots, x_{p-1}, u) g(x_{p+1}, \ldots, x_{p+q-2}, u)\, du \right|^2 dx_1 \ldots dx_{p+q-2} = 0. \tag{1.1}$$

Rosiński and Samorodnitsky [15] observed that multiple Wiener-Itô integrals are independent if and only if their squares are uncorrelated:

$$I_p(f) \perp\!\!\!\perp I_q(g) \iff \mathrm{Cov}(I_p(f)^2, I_q(g)^2) = 0. \tag{1.2}$$

This condition can be viewed as a generalization of the usual covariance criterion for the independence of jointly Gaussian random variables (the case of $p = q = 1$).

In the seminal paper [11], Nualart and Peccati discovered the following surprising central limit theorem.

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Theorem 1.2 (Nualart-Peccati)**.** *Let $F_n = I_q(f_n)$, where $q \geqslant 2$ is fixed and $f_n \in L^2(\mathbb{R}_+^q)$ are symmetric. Assume also that $E[F_n^2] = 1$ for all n. Then convergence in distribution of (F_n) to the standard normal law is equivalent to convergence of the fourth moment. That is, as $n \to \infty$,*

$$F_n \stackrel{\text{law}}{\to} N(0,1) \quad \Longleftrightarrow \quad E[F_n^4] \to 3. \tag{1.3}$$

Shortly afterwards, Peccati and Tudor [12] established a multidimensional extension of Theorem 1.2. Since the publication of these two important papers, many improvements and developments on this theme have been considered. In particular, Nourdin and Peccati [7] extended Theorem 1.2 to the case when the limit of F_n's is a centered gamma distributed random variable. We refer the reader to the book [8] for further information and details of the above results.

Heuristic argument linking Theorem 1.1 and Theorem 1.2 was given by Rosiński [14, pages 3–4], while addressing a question of Albert Shiryaev. Namely, let F and G be two i.i.d. centered random variables with fourth moment and unit variance. The link comes via a simple formula

$$\frac{1}{2}\mathrm{Cov}\big((F+G)^2, (F-G)^2\big) = E[F^4] - 3,$$

criterion (1.2), as well as the celebrated Bernstein's theorem that asserts that F and G are Gaussian if and only if $F+G$ and $F-G$ are independent. A rigorous argument to carry through this idea is based on a characterization of the asymptotic independence of multiple Wiener-Itô integrals, which is much more difficult to handle than the plain independence, and may also be of an independent interest. The covariance between the squares of multiple Wiener-Itô integrals plays the pivotal role in this characterization.

At this point we should also mention an extension of (1.2) to the multivariate setting. Let I be a finite set and $(q_i)_{i\in I}$ be a sequence of non-negative integers. Let $F_i = I_{q_i}(f_i)$ be a multiple Wiener-Itô integral of order q_i, $i \in I$. Consider a partition of I into disjoint blocks I_k, so that $I = \cup_{k=1}^d I_k$, and the resulting random vectors $(F_i)_{i\in I_k}$, $k = 1, \ldots, d$. Then

$$\{(F_i)_{i\in I_k} : k \leq d\} \text{ are independent} \Leftrightarrow \mathrm{Cov}(F_i^2, F_j^2) = 0 \ \forall i,j \text{ from different blocks.} \tag{1.4}$$

The proof of this criterion is similar to the proof of (1.2) in [15].

In this paper in Theorem 3.4 we establish an asymptotic version of (1.4) characterizing the asymptotic moment-independence between blocks of multiple Wiener-Itô integrals. As a consequence of this result, we deduce the fourth moment theorem of Nualart and Peccati [11] in Theorem 4.1, its multidimensional extension due to Peccati and Tudor [12] in Theorem 4.2, and some neat estimates on the speed of convergence in Theorem 4.3. Furthermore, we obtain new multidimensional extension of a theorem of Nourdin and Peccati [7] in Theorem 4.5, and give another new result on the bivariate convergence of vectors consisting of multiple Wiener-Itô integrals in Theorem 4.7. Proposition 5.3 applies Theorem 4.7 to establish the limit process for functions of short and long range dependent stationary Gaussian time series in the spirit of the celebrated Breuer-Major [2]

and Dobrushin-Major-Taqqu [4, 16] Theorems. In Theorem 5.4 we establish the asymptotic moment-independence for discrete non-Gaussian chaoses using some techniques of Mossel, O'Donnel and Oleszkiewicz [5].

The paper is organized as follows. In Section 2 we list some basic facts from Gaussian analysis and prove some lemmas needed in the present work. In particular, we establish Lemma 2.3, which a version of the Cauchy-Schwarz Inequality well suited to deal with contractions of functions, see (2.8). It is used in the proof of the main result, Theorem 3.4. Section 3 is devoted to the main results on the asymptotic independence. Section 4 gives some immediate consequences and related applications of the main result. Section 5 provides further applications to the study of short and long range dependent stochastic processes and multilinear random forms in non-Gaussian random variables.

2. Preliminaries

We will give here some basic elements of Gaussian analysis that are in the foundations of the present work. The reader is referred to the books [8, 10] for further details and ommited proofs.

Let $\mathfrak{H}$ be a real separable Hilbert space. For any $q \geqslant 1$ let $\mathfrak{H}^{\otimes q}$ be the qth tensor product of $\mathfrak{H}$ and denote by $\mathfrak{H}^{\odot q}$ the associated qth symmetric tensor product. We write $X = \{X(h), h \in \mathfrak{H}\}$ to indicate an isonormal Gaussian process over $\mathfrak{H}$, defined on some probability space $(\Omega, \mathcal{F}, P)$. This means that X is a centered Gaussian family, whose covariance is given in terms of the inner product of $\mathfrak{H}$ by $E\left[X(h)X(g)\right] = \langle h, g\rangle_{\mathfrak{H}}$. We also assume that $\mathcal{F}$ is generated by X.

For every $q \geqslant 1$, let $\mathcal{H}_q$ be the qth Wiener chaos of X, that is, the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables of the type $\{H_q(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q is the qth Hermite polynomial defined as

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}\big(e^{-\frac{x^2}{2}}\big). \tag{2.5}$$

We write by convention $\mathcal{H}_0 = \mathbb{R}$. For any $q \geqslant 1$, the mapping

$$I_q(h^{\otimes q}) = H_q(X(h)) \tag{2.6}$$

can be extended to a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$ equipped with the modified norm $\sqrt{q!}\,\|\cdot\|_{\mathfrak{H}^{\otimes q}}$ and the qth Wiener chaos $\mathcal{H}_q$. For $q = 0$ we write $I_0(c) = c$, $c \in \mathbb{R}$.

It is well known (Wiener chaos expansion) that $L^2(\Omega, \mathcal{F}, P)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_q$. Therefore, any square integrable random variable $F \in L^2(\Omega, \mathcal{F}, P)$ admits the following chaotic expansion

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{2.7}$$

where $f_0 = E[F]$, and the $f_q \in \mathfrak{H}^{\odot q}$, $q \geqslant 1$, are uniquely determined by F. For every $q \geqslant 0$ we denote by J_q the orthogonal projection operator on the qth Wiener chaos. In particular, if $F \in L^2(\Omega, \mathcal{F}, P)$ is as in (2.7), then $J_q F = I_q(f_q)$ for every $q \geqslant 0$.

Let $\{e_k,\, k \geqslant 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \ldots, p \wedge q$, the *contraction* of f and g of order r is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1,\ldots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}. \tag{2.8}$$

Notice that $f \otimes_r g$ is not necessarily symmetric: we denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g while, for $p = q$, $f \otimes_q g = \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}$. In the particular case where $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a measurable space and μ is a σ-finite and non-atomic measure, one has that $\mathfrak{H}^{\odot q} = L^2_s(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q})$ is the space of symmetric and square integrable functions on A^q. Moreover, for every $f \in \mathfrak{H}^{\odot q}$, $I_q(f)$ coincides with the q-tuple Wiener-Itô integral of f. In this case, (2.8) can be written as

$$\begin{aligned}(f \otimes_r g)(t_1, \ldots, t_{p+q-2r}) \;&=\; \int_{A^r} f(t_1, \ldots, t_{p-r}, s_1, \ldots, s_r) \\ &\quad \times g(t_{p-r+1}, \ldots, t_{p+q-2r}, s_1, \ldots, s_r) d\mu(s_1) \ldots d\mu(s_r).\end{aligned}$$

We have

$$\|f \otimes_r g\|^2 = \langle f \otimes_{p-r} f, g \otimes_{q-r} g \rangle \quad \text{for } r = 0, \ldots, p \wedge q, \tag{2.9}$$

where $\langle \cdot \rangle$ ($\| \cdot \|$, respectively) stands for inner product (the norm, respectively) in an appropriate tensor product space $\mathfrak{H}^{\otimes s}$. Also, the following *multiplication formula* holds: if $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then

$$I_p(f) I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g), \tag{2.10}$$

where $f \widetilde{\otimes}_r g$ denotes the symmetrization of $f \otimes_r g$.

We conclude these preliminaries by three useful lemmas, that will be needed throughout the sequel.

Lemma 2.1.

(i) *Multiple Wiener-Itô integral has all moments satisfying the following hypercontractivity-type inequality*

$$\big[E|I_p(f)|^r\big]^{1/r} \leq (r-1)^{p/2} \big[E|I_p(f)|^2\big]^{1/2}, \quad r \geqslant 2. \tag{2.11}$$

(ii) *If a sequence of distributions of $\{I_p(f_n)\}_{n \geq 1}$ is tight, then*

$$\sup_n E|I_p(f_n)|^r < \infty \quad \textit{for every } r > 0. \tag{2.12}$$

Proof. (i) Inequality (2.11) is well known and corresponds e.g. to [8, Corollary 2.8.14].
(ii) Combining (2.11) for $r = 4$ with Paley's inequality we get for every $\theta \in (0,1)$

$$P\big(|I_p(f)|^2 > \theta E|I_p(f)|^2\big) \geq (1-\theta)^2 \frac{\big(E|I_p(f)|^2\big)^2}{E|I_p(f)|^4} \geq (1-\theta)^2 9^{-p}. \tag{2.13}$$

By the assumption, there is an $M > 0$ such that $P(|I_p(f_n)|^2 > M) < 9^{-p-1}$, $n \geq 1$. By (2.13) with $\theta = 2/3$ and all n, we have

$$P\big(|I_p(f_n)|^2 > M\big) < 9^{-p-1} \leq P\big(|I_p(f_n)|^2 > (2/3)E|I_p(f_n)|^2\big).$$

As a consequence, $E|I_p(f_n)|^2 \leq (3/2)M$. Applying (2.11) we conclude (2.12). □

Lemma 2.2.

(1) *Let $p, q \geqslant 1$, $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$. Then*

$$\|f \widetilde{\otimes} g\|^2 = \frac{p!q!}{(p+q)!} \sum_{r=0}^{p\wedge q} \binom{p}{r}\binom{q}{r} \|f \otimes_r g\|^2, \tag{2.14}$$

(2) *Let $q \geqslant 1$ and $f_1, f_2, f_3, f_4 \in \mathfrak{H}^{\odot q}$. Then*

$$(2q)!\langle f_1 \widetilde{\otimes} f_2, f_3 \widetilde{\otimes} f_4\rangle = \sum_{r=1}^{q-1} q!^2 \binom{q}{r}^2 \langle f_1 \otimes_r f_3, f_4 \otimes_r f_2\rangle + q!^2\big(\langle f_1, f_3\rangle\langle f_2, f_4\rangle + \langle f_1, f_4\rangle\langle f_2, f_3\rangle\big). \tag{2.15}$$

(3) *Let $q \geqslant 1$, $f \in \mathfrak{H}^{\odot(2q)}$ and $g \in \mathfrak{H}^{\odot q}$. We have*

$$\langle f \widetilde{\otimes}_q f, g \widetilde{\otimes} g\rangle = \frac{2q!^2}{(2q)!}\langle f \otimes_q f, g \otimes g\rangle + \frac{q!^2}{(2q)!} \sum_{r=1}^{q-1} \binom{q}{r}^2 \langle f \otimes_r g, g \otimes_r f\rangle. \tag{2.16}$$

Proof. Without loss of generality, we suppose throughout the proof that $\mathfrak{H}$ is equal to $L^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a measurable space and μ is a σ-finite measure without atoms.

(1) Let σ be a permutation of $\{1, \ldots, p+q\}$ (this fact is written in symbols as $\sigma \in \mathfrak{S}_{p+q}$). If $r \in \{0, \ldots, p \wedge q\}$ denotes the cardinality of $\{1, \ldots, p\} \cap \{\sigma(p+1), \ldots, \sigma(p+q)\}$, then it is readily checked that r is also the cardinality of $\{p+1, \ldots, p+q\} \cap \{\sigma(1), \ldots, \sigma(p)\}$ and that

$$\begin{aligned} &\int_{A^{p+q}} f(t_1, \ldots, t_p) g(t_{p+1}, \ldots, t_{p+q}) f(t_{\sigma(1)}, \ldots, t_{\sigma(p)}) g(t_{\sigma(p+1)}, \ldots, t_{\sigma(p+q)}) d\mu(t_1) \ldots d\mu(t_{p+q}) \\ = &\int_{A^{p+q-2r}} (f \otimes_r g)(x_1, \ldots, x_{p+q-2r})^2 d\mu(x_1) \ldots d\mu(x_{p+q-2r}) = \|f \otimes_r g\|^2. \end{aligned} \tag{2.17}$$

Moreover, for any fixed $r \in \{0, \ldots, p \wedge q\}$, there are $p!\binom{p}{r}q!\binom{q}{r}$ permutations $\sigma \in \mathfrak{S}_{p+q}$ such that $\{1, \ldots, p\} \cap \{\sigma(p+1), \ldots, \sigma(p+q)\} = r$. (Indeed, such a permutation is completely determined by the choice of: (a) r distinct elements $y_1, \ldots, y_r$ of $\{p+1, \ldots, p+q\}$; (b) $p-r$ distinct elements $y_{r+1}, \ldots, y_p$ of $\{1, \ldots, p\}$; (c) a bijection between $\{1, \ldots, p\}$ and

$\{y_1,\dots,y_p\}$; (d) a bijection between $\{p+1,\dots,p+q\}$ and $\{1,\dots,p+q\}\backslash\{y_1,\dots,y_p\}$.) Now, observe that the symmetrization of $f\otimes g$ is given by

$$f\widetilde{\otimes}g(t_1,\dots,t_{p+q})=\frac{1}{(p+q)!}\sum_{\sigma\in\mathfrak{S}_{p+q}}f(t_{\sigma(1)},\dots,t_{\sigma(p)})g(t_{\sigma(p+1)},\dots,t_{\sigma(p+q)}).$$

Therefore, using (2.17), we can write

$$\begin{aligned}\|f\widetilde{\otimes}g\|^2=\langle f\otimes g,f\widetilde{\otimes}g\rangle&=\frac{1}{(p+q)!}\sum_{\sigma\in\mathfrak{S}_{p+q}}\int_{A^{p+q}}f(t_1,\dots,t_p)g(t_{p+1},\dots,t_{p+q})\\&\quad\times f(t_{\sigma(1)},\dots,t_{\sigma(p)})g(t_{\sigma(p+1)},\dots,t_{\sigma(p+q)})d\mu(t_1)\dots d\mu(t_{p+q})\\&=\frac{1}{(p+q)!}\sum_{r=0}^{p\wedge q}\|f\otimes_r g\|^2\mathrm{Card}\{\sigma\in\mathfrak{S}_{p+q}:\,\{1,\dots,p\}\cap\{\sigma(p+1),\dots,\sigma(p+q)\}=r\}.\end{aligned}$$

and (2.14) follows.

(2) We proceed analogously. Indeed, we have

$$\begin{aligned}&\langle f_1\widetilde{\otimes}f_2,f_3\widetilde{\otimes}f_4\rangle=\langle f_1\otimes f_2,f_3\widetilde{\otimes}f_4\rangle\\&=\frac{1}{(2q)!}\sum_{\sigma\in\mathfrak{S}_{2q}}\int_{A^{2q}}f_1(t_1,\dots,t_q)f_2(t_{q+1},\dots,t_{2q})\\&\qquad\times f_3(t_{\sigma(1)},\dots,t_{\sigma(q)})f_4(t_{\sigma(q+1)},\dots,t_{\sigma(2q)})d\mu(t_1)\dots d\mu(t_{2q})\\&=\frac{1}{(2q)!}\sum_{r=0}^{q}\langle f_1\otimes_r f_3,f_4\otimes_r f_2\rangle\,\mathrm{Card}\{\sigma\in\mathfrak{S}_{2q}:\,\{\sigma(1),\dots,\sigma(q)\}\cap\{1,\dots,q\}=r\},\end{aligned}$$

from which we deduce (2.15).

(3) We have

$$\begin{aligned}(g\widetilde{\otimes}g)(t_1,\dots,t_{2q})&=\frac{1}{(2q)!}\sum_{\sigma\in\mathfrak{S}_{2q}}g(t_{\sigma(1)},\dots,t_{\sigma(q)})g(t_{\sigma(q+1)},\dots,t_{\sigma(2q)})\\&=\frac{1}{(2q)!}\sum_{r=0}^{q}\sum_{\substack{\sigma\in\mathfrak{S}_{2q}\\\{\sigma(1),\dots,\sigma(q)\}\cap\{1,\dots,q\}=r}}g(t_{\sigma(1)},\dots,t_{\sigma(q)})g(t_{\sigma(q+1)},\dots,t_{\sigma(2q)}),\end{aligned}$$

and

$$(f\otimes_q f)(t_1,\dots,t_{2q})=\int_{A^q}f(t_1,\dots,t_q,x_1,\dots,x_q)f(x_1,\dots,x_q,t_{q+1},\dots,t_{2q})d\mu(x_1)\dots d\mu(x_q),$$

so that

$$\begin{aligned}
&\langle f \,\widetilde{\otimes}_q\, f, g \,\widetilde{\otimes}\, g\rangle = \langle f \otimes_q f, g \,\widetilde{\otimes}\, g\rangle \\
= \;& \frac{1}{(2q)!}\sum_{r=0}^{q}\langle f\otimes_r g, g\otimes_r f\rangle \mathrm{Card}\{\sigma\in\mathfrak{S}_{2q}:\{\sigma(1),\dots,\sigma(q)\}\cap\{1,\dots,q\}=r\} \\
= \;& \frac{1}{(2q)!}\sum_{r=0}^{q}\binom{q}{r}^2 q!^2\langle f\otimes_r g, g\otimes_r f\rangle \\
= \;& \frac{q!^2}{(2q)!}\langle f\otimes_q g, g\otimes_q f\rangle + \frac{q!^2}{(2q)!}\langle f\otimes g, g\otimes f\rangle + \frac{1}{(2q)!}\sum_{r=1}^{q-1}\binom{q}{r}^2 q!^2\langle f\otimes_r g, g\otimes_r f\rangle.
\end{aligned}$$

Since $\langle f\otimes_q g, g\otimes_q f\rangle = \langle f\otimes g, g\otimes f\rangle = \langle f\otimes_q f, g\otimes g\rangle$, the desired conclusion (2.16) follows. □

Lemma 2.3 (Generalized Cauchy-Schwarz Inequality). *Assume that $\mathfrak{H} = L^2(A,\mathcal{A},\mu)$, where $(A,\mathcal{A})$ is a measurable space equipped with a σ-finite measure μ. For any integer $M\geqslant 1$, put $[M]=\{1,...,M\}$. Also, for every element $\mathbf{z}=(z_1,...,z_M)\in A^M$ and every nonempty set $c\subset[M]$, let $\mathbf{z}_c$ denote the element of $A^{|c|}$ (where $|c|$ is the cardinality of c) obtained by deleting from $\mathbf{z}$ the entries with index not contained in c. (For instance, if $M=5$ and $c=\{1,3,5\}$, then $\mathbf{z}_c=(z_1,z_3,z_5)$.) Let*

- *(α) $C,q\geqslant 2$ be integers, and let $c_1,...,c_q$ be nonempty subsets of $[C]$ such that each element of $[C]$ appears in exactly two of the c_i's (this implies that $\bigcup_i c_i=[C]$ and $\sum_i|c_i|=2C$);*
- *(β) let $h_1,...,h_q$ be functions such that $h_i\in L^2(\mu^{|c_i|}):=L^2(A^{|c_i|},\mathcal{A}^{|c_i|},\mu^{|c_i|})$ for every $i=1,...,q$ (in particular, each h_i is a function of $|c_i|$ variables).*

Then

$$\Big|\int_{A^C}\prod_{i=1}^{q}h_i(\mathbf{z}_{c_i})\,\mu^C(d\mathbf{z}_{[C]})\Big|\leqslant\prod_{i=1}^{q}\|h_i\|_{L^2(\mu^{|c_i|})}. \tag{2.18}$$

Moreover, if $c_0:=c_j\cap c_k\neq\emptyset$ for some $j\neq k$, then

$$\Big|\int_{A^C}\prod_{i=1}^{q}h_i(\mathbf{z}_{c_i})\,\mu^C(d\mathbf{z}_{[C]})\Big|\leqslant\|h_j\otimes_{c_0}h_k\|_{L^2(\mu^{|c_j\triangle c_k|})}\prod_{i\neq j,k}^{q}\|h_i\|_{L^2(\mu^{|c_i|})}, \tag{2.19}$$

where

$$h_j\otimes_{c_0}h_k(\mathbf{z}_{c_j\triangle c_k})=\int_{A^{|c_0|}}h_j(\mathbf{z}_{c_j})h_k(\mathbf{z}_{c_k})\,\mu^{|c_0|}(d\mathbf{z}_{c_0}).$$

(Notice that $h_j\otimes_{c_0}h_k=h_j\otimes_{|c_0|}h_k$ when h_j and h_k are symmetric.)

Proof. In the case $q=2$, (2.18) is just the Cauchy-Schwarz inequality and (2.19) is an equality. Assume that (2.18)–(2.19) hold for at most $q-1$ functions and proceed by induction. Among the sets $c_1,\dots,c_q$ at least two, say c_j and c_k, have nonempty intersection.

Set $c_0 := c_j \cap c_k$, as above. Since c_0 does not have common elements with c_i for all $i \neq j, k$, by Fubini's theorem

$$\int_{A^C} \prod_{i=1}^{q} h_i(\mathbf{z}_{c_i})\, \mu^C(d\mathbf{z}_{[C]}) = \int_{A^{C-|c_0|}} h_j \otimes_{c_0} h_k(\mathbf{z}_{c_j \triangle c_k}) \prod_{i \neq j,k}^{q} h_i(\mathbf{z}_{c_i})\, \mu^{C-|c_0|}(d\mathbf{z}_{[C]\setminus c_0}). \quad (2.20)$$

Observe that every element of $[C] \setminus c_0$ belongs to exactly two of the $q-1$ sets: $c_j \triangle c_k$, c_i, $i \neq j, k$. Therefore, by the induction assumption, (2.18) implies (2.19), provided $c_j \triangle c_k \neq \emptyset$. When $c_j = c_k$, we have $h_j \otimes_{c_0} h_k = \langle h_j, h_k \rangle$ and (2.19) follows from (2.18) applied to the product of $q-2$ functions in (2.20). This proves (2.19), which in turn yields (2.18) by the Cauchy-Schwarz inequality. The proof is complete. □

3. The main results

The following theorem characterizes moment-independence of limits of multiple Wiener-Itô integrals.

Theorem 3.1. *Let $d \geqslant 2$, and let $q_1, \ldots, q_d$ be positive integers. Consider vectors*

$$(F_{1,n}, \ldots, F_{d,n}) = \big(I_{q_1}(f_{1,n}), \ldots, I_{q_d}(f_{d,n})\big), \quad n \geqslant 1,$$

with $f_{i,n} \in \mathfrak{H}^{\odot q_i}$. Assume that for some random vector $(U_1, \ldots, U_d)$,

$$(F_{1,n}, \ldots, F_{d,n}) \overset{\text{law}}{\to} (U_1, \ldots, U_d) \quad \text{as } n \to \infty. \quad (3.21)$$

Then U_i's admit moments of all orders and the following three conditions are equivalent:

- (α) *$U_1, \ldots, U_d$ are moment-independent, that is, $E[U_1^{k_1} \ldots U_d^{k_d}] = E[U_1^{k_1}] \ldots E[U_d^{k_d}]$ for all $k_1, \ldots, k_d \in \mathbb{N}$;*
- (β) *$\lim_{n\to\infty} \mathrm{Cov}(F_{i,n}^2, F_{j,n}^2) = 0$ for all $i \neq j$;*
- (γ) *$\lim_{n\to\infty} \|f_{i,n} \otimes_r f_{j,n}\| = 0$ for all $i \neq j$ and all $r = 1, \ldots, q_i \wedge q_j$;*

Moreover, if the distribution of each U_i is determined by its moments, then (a) is equivalent to that

- (δ) *$U_1, \ldots, U_d$ are independent.*

Remarks 3.2.

(1) Theorem 3.1 raises a question whether the moment-independence implies the usual independence under weaker conditions than the determinacy of the marginals. (Recall that a random variable having all moments is said to be determinate if any other random variable with the same moments has the same distribution.) The answer is negative in general, see [1, Theorem 5].

(2) Assume that $d = 2$ (for simplicity). In this case, (γ) becomes $\|f_{1,n} \otimes_r f_{2,n}\| \to 0$ for all $r = 1, \dots, q_1 \wedge q_2$. In view of Theorem 1.1 of Üstünel and Zakai, one may expect that (γ) could be replaced by a weaker condition (γ'): $\|f_{1,n} \otimes_1 f_{2,n}\| \to 0$.

However, the latter is false. To see it, consider a sequence $f_n \in \mathfrak{H}^{\odot 2}$ such that $\|f_n\|^2 = \frac{1}{2}$ and $\|f_n \otimes_1 f_n\| \to 0$. By Theorem 4.1 below, $F_n := I_2(f_n) \overset{\text{law}}{\to} U \sim N(0,1)$. Putting $f_{1,n} = f_{2,n} = f_n$, we observe that (γ') holds but (α) does not, as $(I_2(f_{1,n}), I_2(f_{2,n})) \overset{\text{law}}{\to} (U, U)$.

(3) Taking into account that assumptions (γ) and (δ) of Theorem 4.1 are equivalent, it is natural to wonder whether assumption (γ) of Theorem 3.1 is equivalent to its symmetrized version:

$$\lim_{n\to\infty} \|f_{i,n} \widetilde{\otimes}_r f_{j,n}\| = 0 \text{ for all } i \neq j \text{ and all } r = 1, \dots, q_i \wedge q_j.$$

The answer is negative in general, as is shown by the following counterexample. Let $f_1, f_2 : [0,1]^2 \to \mathbb{R}$ be symmetric functions given by

$$f_1(s,t) = \begin{cases} -1 & s,t \in [0,1/2] \\ 1 & \text{elsewhere} \end{cases} \quad \text{and} \quad f_2(s,t) = \begin{cases} -1 & s,t \in (1/2,1] \\ 1 & \text{elsewhere.} \end{cases}$$

Then $\langle f_1, f_2 \rangle = 0$ and

$$(f_1 \otimes_1 f_2)(s,t) = \begin{cases} -1 & \text{if } s \in [0,1/2] \text{ and } t \in (1/2,1] \\ 1 & \text{if } t \in [0,1/2] \text{ and } s \in (1/2,1] \\ 0 & \text{elsewhere,} \end{cases}$$

so that $f_1 \widetilde{\otimes}_1 f_2 \equiv 0$ and $\|f_1 \otimes_1 f_2\| = \sqrt{2}$.

(4) The condition of moment-independence, (α) of Theorem 3.1, can also be stated in terms of cumulants. Recall that the joint cumulant of random variables $X_1, \dots, X_m$ is defined by

$$\kappa(X_1, \dots, X_m) = (-i)^m \frac{\partial^m}{\partial t_1 \cdots \partial t_m} \log E[e^{i(t_1 X_1 + \cdots + t_m X_m)}]\Big|_{t_1 = 0, \dots, t_m = 0},$$

provided $E|X_1 \cdots X_m| < \infty$. When all X_i are equal to X, then $\kappa(X, \dots, X) = \kappa_m(X)$, the usual mth cumulant of X, see [6]. Then Theorem 3.1(α) is equivalent to

(α') *for all integers* $1 \le j_1 < \cdots < j_k \le d$, $k \ge 2$, *and* $m_1, \dots, m_k \ge 1$

$$\kappa(\underbrace{U_{j_1}, \dots, U_{j_1}}_{m_1}, \dots, \underbrace{U_{j_k}, \dots, U_{j_k}}_{m_k}) = 0. \tag{3.22}$$

Theorem 3.1 was proved in the first version of this paper [9]. Our proof of the crucial implication $(\gamma) \Rightarrow (\alpha)$ involved tedious combinatorial considerations. We are thankful to an anonymous referee who suggested a shorter and more transparent line of proof using Malliavin calculus. It significantly reduced the amount of combinatorial arguments of the original version but requires some basic facts from Malliavin calculus. We incorporated referee's suggestions and approach into the proof of a more general Theorem 3.4. Even

though Theorem 3.1 becomes a special case of Theorem 3.4 (see Corollary 3.6), we keep its original statement for a convenient reference.

Definition 3.3. *For each $n \geqslant 1$, let $F_n = (F_{i,n})_{i\in I}$ be a family of real-valued random variables indexed by a finite set I. Consider a partition of I into disjoint blocks I_k, so that $I = \cup_{k=1}^d I_k$. We say that vectors $(F_{i,n})_{i\in I_k}$, $k = 1, \ldots, d$ are asymptotically moment-independent if each $F_{i,n}$ admits moments of all orders and for any sequence $(\ell_i)_{i\in I}$ of non-negative integers,*

$$\lim_{n\to\infty} \Big\{ E\big[\prod_{i\in I} F_{i,n}^{\ell_i}\big] - \prod_{k=1}^{d} E\big[\prod_{i\in I_k} F_{i,n}^{\ell_i}\big] \Big\} = 0. \tag{3.23}$$

The next theorem characterizes the asymptotic moment-independence between blocks of multiple Wiener-Itô integrals.

Theorem 3.4. *Let I be a finite set and $(q_i)_{i\in I}$ be a sequence of non-negative integers. For each $n \geqslant 1$, let $F_n = (F_{i,n})_{i\in I}$ be a family of multiple Wiener-Itô integrals, where $F_{i,n} = I_{q_i}(f_{i,n})$ with $f_{i,n} \in \mathfrak{H}^{\odot q_i}$. Assume that for every $i \in I$*

$$\sup_n E\big[F_{i,n}^2\big] < \infty. \tag{3.24}$$

Given a partition of I into disjoint blocks I_k, the following conditions are equivalent:

- **(a)** *random vectors $(F_{i,n})_{i\in I_k}$, $k = 1, \ldots, d$ are asymptotically moment-independent;*
- **(b)** $\lim_{n\to\infty} \mathrm{Cov}(F_{i,n}^2, F_{j,n}^2) = 0$ *for every i, j from different blocks;*
- **(c)** $\lim_{n\to\infty} \|f_{i,n} \otimes_r f_{j,n}\| = 0$ *for every i, j from different blocks and $r = 1, \ldots, q_i \wedge q_j$.*

Proof: The implication **(a)** $\Rightarrow$ **(b)** is obvious.

To show **(b)** $\Rightarrow$ **(c)**, fix i, j belonging to different blocks. By (2.10) we have

$$F_{i,n}F_{j,n} = \sum_{r=0}^{q_i\wedge q_j} r! \binom{q_i}{r}\binom{q_j}{r} I_{q_i+q_j-2r}(f_{i,n}\widetilde{\otimes}_r f_{j,n}),$$

which yields

$$E[F_{i,n}^2F_{j,n}^2] = \sum_{r=0}^{q_i\wedge q_j} r!^2 \binom{q_i}{r}^2\binom{q_j}{r}^2 (q_i+q_j-2r)!\|f_{i,n}\widetilde{\otimes}_r f_{j,n}\|^2.$$

Moreover,

$$E[F_{i,n}^2]E[F_{j,n}^2] = q_i!q_j!\|f_{i,n}\|^2\|f_{j,n}\|^2.$$

Applying (2.14) to the second equality below, we evaluate $\mathrm{Cov}(F_{i,n}^2, F_{j,n}^2)$ as follows:

$$\mathrm{Cov}(F_{i,n}^2, F_{j,n}^2) = (q_i+q_j)!\|f_{i,n}\widetilde{\otimes}f_{j,n}\|^2 - q_i!q_j!\|f_{i,n}\|^2\|f_{j,n}\|^2 \tag{3.25}$$
$$+\sum_{r=1}^{q_i\wedge q_j} r!^2\binom{q_i}{r}^2\binom{q_j}{r}^2(q_i+q_j-2r)!\|f_{i,n}\widetilde{\otimes}_r f_{j,n}\|^2$$
$$= q_i!q_j!\sum_{r=1}^{q_i\wedge q_j}\binom{q_i}{r}\binom{q_j}{r}\|f_{i,n}\otimes_r f_{j,n}\|^2 + \sum_{r=1}^{q_i\wedge q_j} r!^2\binom{q_i}{r}^2\binom{q_j}{r}^2(q_i+q_j-2r)!\|f_{i,n}\widetilde{\otimes}_r f_{j,n}\|^2$$
$$\geqslant \max_{r=1,\ldots,q_i\wedge q_j}\|f_{i,n}\otimes_r f_{j,n}\|^2. \tag{3.26}$$

This bound yields the desired conclusion.

Now we will prove $(\mathbf{c})\Rightarrow(\mathbf{a})$. We need to show (3.23) for fixed l_i. Writing $F_{i,n}^{l_i}$ as $\underbrace{F_{i,n}\times\ldots\times F_{i,n}}_{l_i}$ and enlarging I and I_k's accordingly, we may and do assume that all $l_i=1$. We will prove (3.23) by induction on $Q=\sum_{i\in I}q_i$. The formula holds when $Q=0$ or 1. Therefore, take $Q\geq 2$ and suppose that (3.23) holds whenever $\sum_{i\in I}q_i\leq Q-1$.

Fix $i_1\in I_1$ and set

$$X_n=\prod_{i\in I_1\setminus\{i_1\}} I_{q_i}(f_{i,n}),\quad Y_n=\prod_{j\in I\setminus I_1} I_{q_j}(f_{j,n}).$$

Assume that $q_1\geq 1$, otherwise the inductive step follows immediately. Let δ denote the divergence operator in the sense of Malliavin calculus and let D be the Malliavin derivative, see [10, Ch. 1.2-1.3]. Using the duality relation [10, Def. 1.3.1(ii)] and the product rule for the Malliavin derivative [3, Theorem 3.4] we get

$$\begin{aligned}E\big[\prod_{i\in I}F_{i,n}\big] &= E\big[I_{q_{i_1}}(f_{i_1,n})X_nY_n\big] = E\big[\delta(I_{q_{i_1}-1}(f_{i_1,n}))X_nY_n\big]\\ &= E\big[I_{q_{i_1}-1}(f_{i_1,n})\otimes_1 D(X_nY_n)\big]\\ &= E\big[Y_n\, I_{q_{i_1}-1}(f_{i_1,n})\otimes_1 DX_n\big] + E\big[X_n\, I_{q_{i_1}-1}(f_{i_1,n})\otimes_1 DY_n\big]\\ &= A_n+B_n.\end{aligned}$$

First we consider B_n. Using the product rule for DY_n we obtain

$$\begin{aligned}B_n &= \sum_{j\in I\setminus I_1} E\big[I_{q_{i_1}-1}(f_{i_1,n})\otimes_1 DF_{j,n}\prod_{i\in I\setminus\{i_1,j\}}F_{i,n}\big]\\ &= \sum_{j\in I\setminus I_1} q_jE\big[I_{q_{i_1}-1}(f_{i_1,n})\otimes_1 I_{q_j-1}(f_{j,n})\prod_{i\in I\setminus\{i_1,j\}}F_{i,n}\big].\end{aligned}$$

By the multiplication formula (2.10) we have

$$I_{q_{i_1}-1}(f_{i_1,n})\otimes_1 I_{q_j-1}(f_{j,n}) = \sum_{s=1}^{q_{i_1}\wedge q_j}(s-1)!\binom{q_{i_1}-1}{s-1}\binom{q_j-1}{s-1}I_{q_{i_1}+q_j-2s}(f_{i_1,n}\widetilde{\otimes}_s f_{j,n}).$$

Since i_1 and j belong to different blocks, condition (c) of the theorem applied to the above expansion yields that $I_{q_{i_1}-1}(f_{i_1,n}) \otimes_1 I_{q_j-1}(f_{j,n})$ converges to zero in L^2. Combining this with (3.24) and Lemma 2.1 we infer that $\lim_{n\to\infty} B_n = 0$.

Now we consider A_n. If $\mathrm{Card}(I_1) = 1$, then $X_n = 1$ by convention and so $A_n = 0$. Hence

$$\lim_{n\to\infty} \Big\{ E\big[\prod_{i\in I} F_{i,n}\big] - E\big[F_{i_1,n}\big] \prod_{k=2}^{d} E\big[\prod_{i\in I_k} F_{i,n}\big] \Big\} = \lim_{n\to\infty} B_n = 0.$$

Therefore, we now assume that $\mathrm{Card}(I_1) \geq 2$. Write $A_n = E\big[Z_n Y_n\big]$, where

$$\begin{aligned} Z_n &= I_{q_{i_1}-1}(f_{i_1,n}) \otimes_1 DX_n \\ &= \sum_{i\in I_1\setminus\{i_1\}} q_i I_{q_{i_1}-1}(f_{i_1,n}) \otimes_1 I_{q_i-1}(f_{i,n}) \prod_{j\in I_1\setminus\{i_1,i\}} F_{j,n} \\ &= \sum_{i\in I_1\setminus\{i_1\}} q_i \sum_{s=1}^{q_{i_1}\wedge q_i} (s-1)! \binom{q_{i_1}-1}{s-1}\binom{q_i-1}{s-1} I_{q_{i_1}+q_i-2s}(f_{i_1,n}\widetilde{\otimes}_s f_{i,n}) \prod_{j\in I\setminus\{i_1,i\}} F_{j,n}. \end{aligned}$$

Thus A_n is a linear combination of the terms

$$E\big[\big(I_{q_{i_1}+q_i-2s}(f_{i_1,n}\widetilde{\otimes}_s f_{i,n}) \prod_{j\in I_1\setminus\{i_1,i\}} F_{j,n}\big)\, Y_n\big],$$

where $i_1, i \in I_1$, $i_1 \neq i$, $1 \leq s \leq q_{i_1} \wedge q_i$. The term under expectation is a product of multiple integrals of orders summing to $\sum_{j\in I} q_j - 2s$. Therefore, the induction hypothesis applies provided

$$\lim_{n\to\infty} \big(f_{i_1,n}\widetilde{\otimes}_s f_{i,n}\big) \otimes_r f_{j,n} = 0 \tag{3.27}$$

for all $j \in I_k$ with $k \geqslant 2$ and all $r = 1, \ldots, (q_{i_1} + q_i - 2s) \wedge q_j$.

Suppose that (3.27) holds. Then by the induction hypothesis

$$\lim_{n\to\infty} \big\{ A_n - E[Z_n]E[Y_n] \big\} = 0.$$

Moreover,

$$E[Z_n] = E\big[I_{q_{i_1}-1}(f_{i_1,n}) \otimes_1 DX_n\big] = E[I_{q_{i_1}}(f_{i_1,n})X_n] = E\big[\prod_{i\in I_1} F_{i,n}\big].$$

Hence, by the induction hypothesis applied to Y_n and the uniform boundedness of all moments of $F_{i,n}$, we get

$$\lim_{n\to\infty} \Big\{ E\big[\prod_{i\in I} F_{i,n}\big] - \prod_{k=1}^{d} E\big[\prod_{i\in I_k} F_{i,n}\big] \Big\} = \lim_{n\to\infty} \Big\{ A_n - E[Z_n]E[Y_n] \Big\} = 0.$$

It remains to show (3.27). To this aim we will describe the structure of the terms under the limit (3.27). Without loss of generality we may assume that $\mathfrak{H} = L^2(\mu) := L^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a measurable space and μ is a σ-finite measure without atoms. Recall notation of Lemma 2.3. For every integer $M \geqslant 1$, put $[M] = \{1, ..., M\}$. Also, for every element $\mathbf{z} = (z_1, ..., z_M) \in A^M$ and every nonempty set $c \subset [M]$, we denote by $\mathbf{z}_c$ the

element of $A^{|c|}$ (where $|c|$ is the cardinality of c) obtained by deleting from $\mathbf{z}$ the entries with index not contained in c. (For instance, if $M = 5$ and $c = \{1, 3, 5\}$, then $\mathbf{z}_c = (z_1, z_3, z_5)$.)

Observe that $\left(f_{i_1,n} \widetilde{\otimes}_s f_{i,n}\right) \otimes_r f_{j,n}$ is a linear combination of functions $\psi(\mathbf{z}_{J_1})$, $\mathbf{z} \in A^M$ obtained as follows. Set $M = q_{i_1} + q_i + q_j - s - r$ and $M_0 = q_{i_1} + q_i - s$, so that $M > M_0 \geq 2$. Choose $b_1, b_2 \subset [M_0]$ such that $|b_1| = q_{i_1}$, $|b_2| = q_i$ and $|b_1 \cap b_2| = s$, and then choose $b_3 \subset [M]$ such that $|b_3| = q_j$ and $|b_3 \cap (b_1 \cup b_2)| = r$. It follows that $b_1 \cup b_2 \cup b_3 = [M]$ and $b_1 \cap b_2 \cap b_3 = \emptyset$. Therefore, each element of $[M]$ belongs exactly to one or two b_i's. Let

$$J = \{j \in [M] : j \text{ belongs to two sets } b_i\}$$

and put $J_1 = [M] \setminus J$. Then $\left(f_{i_1,n} \widetilde{\otimes}_s f_{i,n}\right) \otimes_r f_{j,n}$ is a linear combination of functions of the form

$$\psi(\mathbf{z}_{J_1}) = \int_{A^J} f_{i_1,n}(\mathbf{z}_{b_1}) f_{i,n}(\mathbf{z}_{b_2}) f_{j,n}(\mathbf{z}_{b_3})\, \mu^{|J|}(d\mathbf{z}_J),$$

where the summation goes over all choices b_1, b_2 under the constraint that the sets $b_1 \cap b_2$ and b_3 are fixed. This constraint makes J_1 unique, $|J_1| = q_{i_1} + q_i + q_j - 2s - 2r$.

Let $c_i = b_i \cap J$, $i = 1, 2, 3$ and notice that either $c_1 \cap c_3 \neq \emptyset$ or $c_2 \cap c_3 \neq \emptyset$ since $r \geq 1$. Suppose $c_0 = c_1 \cap c_3 \neq \emptyset$, the other case is identical. Applying Lemma 2.3 with $\mathbf{z}_{J_1}$ fixed we get

$$|\psi(\mathbf{z}_{J_1})|^2 \leq |f_{i_1,n} \otimes_{|c_0|} f_{j,n}(\mathbf{z}_{b_1 \triangle b_3})|^2 \int_{A^{|c_2|}} |f_{i,n}(\mathbf{z}_{b_2})|^2\, \mu^{|c_2|}(d\mathbf{z}_{c_2})$$

Since $b_1 \triangle b_3$ and $b_3 \setminus c_3$ make a disjoint partition of J_1, and additional integration with respect to $\mathbf{z}_{J_1}$ yields

$$\|\psi\|_{L^2(\mu^{|J_1|})} \leq \|f_{i_1,n} \otimes_{|c_0|} f_{j,n}\|_{L^2(\mu^{|b_1 \triangle b_3|})}\, \|f_{i,n}\|_{L^2(\mu^{|b_2|})} \to 0$$

as $n \to \infty$. This yields (3.27) and completes the proof of Theorem 3.4. □

Remark 3.5. Condition (**b**) of Theorem 3.4 is equivalent to

(**b'**) *for every* $1 \leq k \neq l \leq d$

$$\lim_{n\to\infty} \mathrm{Cov}(\|(F_{i,n})_{i\in I_k}\|^2, \|(F_{i,n})_{i\in I_l}\|^2) = 0,$$

where $\|\cdot\|$ denotes the Euclidean norms in $\mathbb{R}^{|I_k|}$ and $\mathbb{R}^{|I_l|}$ respectively.

Proof. Indeed, condition (**b**) of Theorem 3.4 implies (**b'**) and the converse follows from

$$\mathrm{Cov}(\|(F_{i,n})_{i\in I_k}\|^2, \|(F_{i,n})_{i\in I_l}\|^2) = \sum_{i\in I_k, j\in I_l} \mathrm{Cov}(F_{i,n}^2, F_{j,n}^2) \geq \mathrm{Cov}(F_{i,n}^2, F_{j,n}^2),$$

as the squares of multiple Wiener-Itô integrals are non-negatively correlated, cf. (3.26). □

The following corollary is useful to deduce the joint convergence in law from the convergence of marginals. It is stated for random vectors, as is Theorem 3.4, but it obviously applies in the setting of Theorem 3.1 when all vectors are one-dimensional.

Corollary 3.6. *Under notation of Theorem 3.4, let* $(U_i)_{i\in I}$ *be a random vector such that*

(i) $(F_{i,n})_{i\in I_k} \overset{\text{law}}{\to} (U_i)_{i\in I_k}$ *as* $n\to\infty$, *for each* k;
(ii) *vectors* $(U_i)_{i\in I_k}$, $k=1,\dots,d$ *are independent;*
(iii) *condition* (**b**) *or* (**c**) *of Theorem 3.4 holds [equivalently,* (β) *or* (γ) *of Theorem 3.1 when all* I_k *are singletons];*
(iv) $\mathcal{L}(U_i)$ *is determined by its moments for each* $i\in I$.

Then the joint convergence holds,

$$(F_{i,n})_{i\in I} \overset{\text{law}}{\to} (U_i)_{i\in I}, \quad n\to\infty.$$

Proof: By (i) the sequence $\{(F_{i,n})_{i\in I}\}_{n\geq 1}$ is tight. Let $(V_i)_{i\in I}$ be a random vector such that

$$(F_{i,n_j})_{i\in I} \overset{\text{law}}{\to} (V_i)_{i\in I}$$

as $n_j\to\infty$ along a subsequence. From Lemma 2.1(ii) we infer that condition (3.24) of Theorem 3.4 is satisfied. It follows that each V_i has all moments and $(V_i)_{i\in I_k} \overset{\text{law}}{=} (U_i)_{i\in I_k}$ for each k. By (iv), the laws of vectors $(U_i)_{i\in I}$ and $(V_i)_{i\in I}$ are determined by their joint moments, respectively, see [13, Theorem 3]. Under the assumption (iii), the vectors $(F_{i,n})_{i\in I_k}$, $k=1,\dots,d$ are asymptotically moment independent. Hence, for any sequence $(\ell_i)_{i\in I}$ of non-negative integers,

$$\begin{aligned} E\big[\prod_{i\in I} V_i^{\ell_i}\big] - E\big[\prod_{i\in I} U_i^{\ell_i}\big] &= E\big[\prod_{i\in I} V_i^{\ell_i}\big] - \prod_{k=1}^{d} E\big[\prod_{i\in I_k} U_i^{\ell_i}\big] \\ &= \lim_{n_j\to\infty} \Big\{ E\big[\prod_{i\in I} F_{i,n_j}^{\ell_i}\big] - \prod_{k=1}^{d} E\big[\prod_{i\in I_k} F_{i,n_j}^{\ell_i}\big] \Big\} = 0. \end{aligned}$$

Thus $(V_i)_{i\in I} \overset{\text{law}}{=} (U_i)_{i\in I}$. □

4. Applications

4.1. The fourth moment theorem of Nualart-Peccati.

We can give a short proof of the difficult and surprising part implication $(\beta)\Rightarrow(\alpha)$ of the fourth moment theorem of Nualart and Peccati [11], that we restate here for a convenience.

Theorem 4.1 (Nualart-Peccati)**.** *Let* (F_n) *be a sequence of the form* $F_n = I_q(f_n)$, *where* $q\geqslant 2$ *is fixed and* $f_n\in\mathfrak{H}^{\odot q}$. *Assume moreover that* $E[F_n^2] = q!\|f_n\|^2 = 1$ *for all* n. *Then, as* $n\to\infty$, *the following four conditions are equivalent:*

(α) $F_n \overset{\text{law}}{\to} N(0,1)$;
(β) $E[F_n^4]\to 3$;
(γ) $\|f_n\otimes_r f_n\|\to 0$ *for all* $r=1,\dots,q-1$;
(δ) $\|f_n \widetilde{\otimes}_r f_n\|\to 0$ *for all* $r=1,\dots,q-1$.

Proof of $(\beta) \Rightarrow (\alpha)$. Assume (β). Since the sequence (F_n) is bounded in $L^2(\Omega)$ by the assumption, it is relatively compact in law. Without loss of generality we may assume that $F_n \overset{\text{law}}{\to} Y$ and need to show that $Y \sim N(0,1)$. Let G_n be an independent copy of F_n of the form $G_n = I_q(g_n)$ with $f_n \otimes_1 g_n = 0$. This can easily be done by extending the underlying isonormal process to the direct sum $\mathfrak{H} \oplus \mathfrak{H}$. We then have

$$(I_q(f_n + g_n), I_q(f_n - g_n)) = (F_n + G_n, F_n - G_n) \overset{\text{law}}{\to} (Y + Z, Y - Z)$$

as $n \to \infty$, where Z stands for an independent copy of Y. Since

$$\frac{1}{2}\mathrm{Cov}[(F_n + G_n)^2, (F_n - G_n)^2] = E[F_n^4] - 3 \to 0,$$

$Y + Z$ and $Y - Z$ are moment-independent. (If they were independent, the classical Bernstein Theorem would conclude the proof.) However, in our case condition (α') in (3.22) says that

$$\kappa(\underbrace{Y + Z, \ldots, Y + Z}_{m_1}, \underbrace{Y - Z, \ldots, Y - Z}_{m_2}) = 0 \quad \text{for all } m_1, m_2 \geq 1.$$

Taking $n \geq 3$ we get

$$\begin{aligned} 0 &= \kappa(\underbrace{Y + Z, \ldots, Y + Z}_{n-2}, Y - Z, Y - Z) \\ &= \kappa(\underbrace{Y, \ldots, Y}_{n}) + \kappa(\underbrace{Z, \ldots, Z}_{n}) = 2\kappa_n(Y), \end{aligned}$$

where we used the multilinearity of κ and the fact that Y and Z are i.i.d. Since $\kappa_1(Y) = 0$, $\kappa_2(Y) = 1$, and $\kappa_n(Y) = 0$ for $n \geq 3$, we infer that $Y \sim N(0,1)$. □

4.2. **Generalizing a result of Peccati and Tudor.**

Applying our approach, one can add a further equivalent condition to a result of Peccati and Tudor [12]. As such, Theorem 4.2 turns out to be the exact multivariate equivalent of Theorem 4.1.

Theorem 4.2 (Peccati-Tudor)**.** *Let $d \geqslant 2$, and let $q_1, \ldots, q_d$ be positive integers. Consider vectors*

$$F_n = (F_{1,n}, \ldots, F_{d,n}) = \big(I_{q_1}(f_{1,n}), \ldots, I_{q_d}(f_{d,n})\big), \quad n \geqslant 1,$$

with $f_{i,n} \in \mathfrak{H}^{\odot q_i}$. Assume that, for $i, j = 1, \ldots, d$, as $n \to \infty$,

$$\mathrm{Cov}\big(F_{i,n}, F_{j,n}\big) \to \sigma_{ij}. \tag{4.28}$$

Let N be a centered Gaussian random vector with the covariance matrix $\Sigma = (\sigma_{ij})_{1 \leqslant i,j \leqslant d}$. Then the following two conditions are equivalent ($n \to \infty$):

(i) $F_n \overset{\text{law}}{\to} N$;

(ii) $E\big[\|F_n\|^4\big] \to E\big[\|N\|^4\big]$;

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

Proof. Only $(ii) \Rightarrow (i)$ has to be shown. Assume (ii). As in the proof of Theorem 4.1, we may assume that $F_n \overset{\text{law}}{\to} Y$ and must show that $Y \sim N_d(0,\Sigma)$. Let $G_n = (G_{1,n},\dots,G_{d,n})$ be an independent copy of F_n of the form $\big(I_{q_1}(g_{1,n}),\dots,I_{q_d}(g_{d,n})\big)$. Observe that

$$\frac{1}{2}\mathrm{Cov}\big(\,\|F_n+G_n\|^2,\|F_n-G_n\|^2\,\big) = E[\|F_n\|^4] - \big(E[\|F_n\|^2]\big)^2 - 2\sum_{i,j=1}^d \mathrm{Cov}(F_{i,n},F_{j,n})^2.$$

Using this identity for N and N' in place of F_n and G_n, where N' is an independent copy of N, we get

$$E[\|N\|^4] = \sum_{i,j=1}^d \big(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2\big). \tag{4.29}$$

Hence

$$\frac{1}{2}\mathrm{Cov}\big(\,\|F_n+G_n\|^2\,,\|F_n-G_n\|^2\,\big) = E[\|F_n\|^4] - E[\|N\|^4] + \sum_{i,j=1}^d \big[\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2 - \mathrm{Var}(F_{i,n})\mathrm{Var}(F_{j,n}) - 2\mathrm{Cov}(F_{i,n},F_{j,n})^2\big] \to 0.$$

By Remark 3.5, F_n+G_n and F_n-G_n are asymptotically moment-independent. Since one-dimensional projections of F_n+G_n and F_n-G_n are also asymptotically moment-independent, we can proceed by cumulants as above to determine the normality of Y. □

The following result associates neat estimates to Theorem 4.2.

Theorem 4.3. *Consider a vector*

$$F = (F_1,\dots,F_d) = (I_{q_1}(f_1),\dots,I_{q_d}(f_d))$$

with $f_i \in \mathfrak{H}^{\odot q_i}$, and let $\Sigma = (\sigma_{ij})_{1\leqslant i,j\leqslant d}$ be the covariance matrix of F, $\sigma_{ij} = E[F_iF_j]$. Let N be the associated Gaussian random vector, $N \sim N_d(0,\Sigma)$.

(1) *Assume that Σ is invertible. Then, for any Lipschitz function $h:\mathbb{R}^d\to\mathbb{R}$ we have*

$$\big|E[h(F)] - E[h(N)]\big| \leqslant \sqrt{d}\,\|\Sigma\|_{op}^{1/2}\|\Sigma^{-1}\|_{op}\|h\|_{Lip}\sqrt{E\|F\|^4 - E\|N\|^4},$$

where $\|\cdot\|_{op}$ denotes the operator norm of a matrix and $\|h\|_{Lip} = \sup_{x,y\in\mathbb{R}^d}\frac{|h(x)-h(y)|}{\|x-y\|}$.

(2) *For any C^2-function $h:\mathbb{R}^d\to\mathbb{R}$ we have*

$$\big|E[h(F)] - E[h(N)]\big| \leqslant \frac{1}{2}\|h''\|_\infty\sqrt{E\|F\|^4 - E\|N\|^4},$$

where $\|h''\|_\infty = \max_{1\leqslant i,j\leqslant d}\ \sup_{x\in\mathbb{R}^d}\left|\frac{\partial^2 h}{\partial x_i\partial x_j}(x)\right|$.

Proof: The proof is divided into three steps.

Step 1: Recall that for a Lipschitz function $h : \mathbb{R}^d \to \mathbb{R}$ [8, Theorem 6.1.1] yields

$$\big|E[h(F)] - E[h(N)]\big| \leqslant \sqrt{d}\,\|\Sigma\|_{op}^{1/2}\|\Sigma^{-1}\|_{op}\|h\|_{Lip}\sqrt{\sum_{i,j=1}^{d} E\left\{\big(\sigma_{ij} - \frac{1}{q_j}\langle DF_i, DF_j\rangle\big)^2\right\}},$$

while for a C^2-function with bounded Hessian [8, Theorem 6.1.2] gives

$$\big|E[h(F)] - E[h(N)]\big| \leqslant \frac{1}{2}\|h''\|_{\infty}\sqrt{\sum_{i,j=1}^{d} E\left\{\big(\sigma_{ij} - \frac{1}{q_j}\langle DF_i, DF_j\rangle\big)^2\right\}}.$$

Step 2: We claim that for any $i, j = 1, \ldots, d$,

$$E\left\{\big(\sigma_{ij} - \frac{1}{q_j}\langle DF_i, DF_j\rangle\big)^2\right\} \leqslant \mathrm{Cov}(F_i^2, F_j^2) - 2\sigma_{ij}^2.$$

Indeed, by [8, identity (6.2.4)] and the fact that $\sigma_{ij} = 0$ if $q_i \neq q_j$, we have

$$\begin{aligned}
&E\left\{\big(\sigma_{ij} - \frac{1}{q_j}\langle DF_i, DF_j\rangle\big)^2\right\} \\
&= \begin{cases} q_i^2 \sum_{r=1}^{q_i\wedge q_j} (r-1)!^2 \binom{q_i-1}{r-1}^2 \binom{q_j-1}{r-1}^2 (q_i+q_j-2r)! \|f_i \widetilde{\otimes}_r f_j\|^2 & \text{if } q_i \neq q_j \\ q_i^2 \sum_{r=1}^{q_i-1} (r-1)!^2 \binom{q_i-1}{r-1}^4 (2q_i-2r)! \|f_i \widetilde{\otimes}_r f_j\|^2 & \text{if } q_i = q_j \end{cases} \\
&\leqslant \begin{cases} \sum_{r=1}^{q_i\wedge q_j} r!^2 \binom{q_i}{r}^2 \binom{q_j}{r}^2 (q_i+q_j-2r)! \|f_i \widetilde{\otimes}_r f_j\|^2 & \text{if } q_i \neq q_j \\ \sum_{r=1}^{q_i-1} r!^2 \binom{q_i}{r}^4 (2q_i-2r)! \|f_i \widetilde{\otimes}_r f_j\|^2 & \text{if } q_i = q_j \end{cases}.
\end{aligned}$$

On the other hand, from (3.25) we have

$$\begin{aligned}
&\mathrm{Cov}(F_i^2, F_j^2) - 2\sigma_{ij}^2 \\
&= \begin{cases} q_i! q_j! \sum_{r=1}^{q_i\wedge q_j} \binom{q_i}{r}\binom{q_j}{r} \|f_i \otimes_r f_j\|^2 \\ \qquad + \sum_{r=1}^{q_i\wedge q_j} r!^2 \binom{q_i}{r}^2 \binom{q_j}{r}^2 (q_i+q_j-2r)! \|f_i \widetilde{\otimes}_r f_j\|^2 & \text{if } q_i \neq q_j \\ q_i!^2 \sum_{r=1}^{q_i-1} \binom{q_i}{r}^2 \|f_i \otimes_r f_j\|^2 \\ \qquad + \sum_{r=1}^{q_i-1} r!^2 \binom{q_i}{r}^4 (2q_i-2r)! \|f_i \widetilde{\otimes}_r f_j\|^2 & \text{if } q_i = q_j \end{cases}.
\end{aligned}$$

The claim follows immediately.

Step 3: Applying (4.29) we get

$$E\|F\|^4 - E\|N\|^4 = \sum_{i,j=1}^{d} \big(E[F_i^2 F_j^2] - \sigma_{ii}\sigma_{jj} - 2\sigma_{ij}^2\big)$$
$$= \sum_{i,j=1}^{d} \big\{\mathrm{Cov}(F_i^2, F_j^2) - 2\sigma_{ij}^2\big\}.$$

Combining Steps 1-3 gives the desired conclusion. □

4.3. **A multivariate version of the convergence towards χ^2.**

Here we will prove a multivariate extension of a result of Nourdin and Peccati [7]. Such an extension was an open problem as far as we know.

In what follows, $G(\nu)$ will denote a random variable with the centered χ^2 distribution having $\nu > 0$ degrees of freedom. When ν is an integer, then $G(\nu) \overset{\text{law}}{=} \sum_{i=1}^{\nu}(N_i^2 - 1)$, where $N_1, \ldots, N_\nu$ are i.i.d. standard normal random variables. In general, $G(\nu)$ is a centered gamma random variable with a shape parameter $\nu/2$ and scale parameter 2. Nourdin and Peccati [7] established the following theorem.

Theorem 4.4 (Nourdin-Peccati)**.** *Fix $\nu > 0$ and let $G(\nu)$ be as above. Let $q \geqslant 2$ be an even integer, and let $F_n = I_q(f_n)$ be such that $\lim_{n\to\infty} E[F_n^2] = E[G(\nu)^2] = 2\nu$. Set $c_q = 4\left[(q/2)!\right]^3 [q!]^{-2}$. Then, the following four assertions are equivalent, as $n \to \infty$:*

- (α) $F_n \overset{\text{law}}{\to} G(\nu)$;
- (β) $E[F_n^4] - 12E[F_n^3] \to E[G(\nu)^4] - 12E[G(\nu)^3] = 12\nu^2 - 48\nu$;
- (γ) $\|f_n \widetilde{\otimes}_{q/2} f_n - c_q \times f_n\| \to 0$, *and* $\|f_n \otimes_r f_n\| \to 0$ *for every* $r = 1, ..., q-1$ *such that* $r \neq q/2$;
- (δ) $\|f_n \widetilde{\otimes}_{q/2} f_n - c_q \times f_n\| \to 0$, *and* $\|f_n \widetilde{\otimes}_r f_n\| \to 0$ *for every* $r = 1, ..., q-1$ *such that* $r \neq q/2$.

The following is our multivariate extension of this theorem.

Theorem 4.5. *Let $d \geqslant 2$, let $\nu_1, \ldots, \nu_d$ be positive reals, and let $q_1, \ldots, q_d \geqslant 2$ be even integers. Consider vectors*

$$F_n = (F_{1,n}, \ldots, F_{d,n}) = \big(I_{q_1}(f_{1,n}), \ldots, I_{q_d}(f_{d,n})\big), \quad n \geqslant 1,$$

with $f_{i,n} \in \mathfrak{H}^{\odot q_i}$, such that $\lim_{n\to\infty} E[F_{i,n}^2] = 2\nu_i$ for every $i = 1, \ldots, d$. Assume that:

- (i) $E[F_{i,n}^4] - 12E[F_{i,n}^3] \to 12\nu_i^2 - 48\nu_i$ *for every i;*
- (ii) $\lim_{n\to\infty} \mathrm{Cov}(F_{i,n}^2, F_{j,n}^2) = 0$ *whenever $q_i = q_j$ for some $i \neq j$;*
- (ii) $\lim_{n\to\infty} E[F_{i,n}^2 F_{j,n}] = 0$ *whenever $q_j = 2q_i$.*

Then

$$(F_{1,n},\dots,F_{d,n}) \overset{\text{law}}{\to} (G(\nu_1),\dots,G(\nu_d))$$

where $G(\nu_1),\dots,G(\nu_d)$ are independent random variables having centered χ^2 distributions with $\nu_1,\dots,\nu_d$ degrees of freedom, respectively.

Proof. Using the well-known Carleman's condition, it is easy to check that the law of $G(\nu)$ is determined by its moments. By Corollary 3.6 it is enough to show that condition (γ) of Theorem 3.1 holds.

Fix $1 \leqslant i \neq j \leqslant d$ as well as $1 \leqslant r \leqslant q_i \wedge q_j$. Switching i and j if necessary, assume that $q_i \leqslant q_j$. From Theorem 4.4(γ) we get that $f_{k,n} \otimes_r f_{k,n} \to 0$ for each $1 \leqslant k \leqslant d$ and every $1 \leqslant r \leqslant q_k - 1$, except when $r = q_k/2$. Using the identity

$$\|f_{i,n} \otimes_r f_{j,n}\|^2 = \langle f_{i,n} \otimes_{q_i - r} f_{i,n}, f_{j,n} \otimes_{q_j - r} f_{j,n}\rangle \tag{4.30}$$

(see (2.9)) together with the Cauchy-Schwarz inequality we infer that condition (γ) of Theorem 3.1 holds for all values of r, i and j, except of the cases: $r = q_i = q_j$, $r = q_i/2 = q_j/2$, and $r = q_i = q_j/2$. Assumption (i) together with (3.26) show that $f_{i,n} \otimes_r f_{j,n} \to 0$ for all $1 \leqslant r \leqslant q_i = q_j$. Thus, it remains to verify condition (γ) of Theorem 3.1 when $r = q_i = q_j/2$. Lemma 2.2 (identity (2.16) therein) yields

$$\begin{aligned} &\langle f_{j,n} \widetilde{\otimes}_{q_i} f_{j,n}, f_{i,n} \widetilde{\otimes} f_{i,n}\rangle \\ = \;& \frac{2q_i!^2}{q_j!}\langle f_{j,n} \otimes_{q_i} f_{j,n}, f_{i,n} \otimes f_{i,n}\rangle + \frac{q_i!^2}{q_j!}\sum_{s=1}^{q_i-1}\binom{q_i}{s}^2 \langle f_{j,n} \otimes_s f_{i,n}, f_{i,n} \otimes_s f_{j,n}\rangle. \end{aligned}$$

Using (4.30) and Theorem 4.4 and a reasoning as above, it is straightforward to show that the sum $\sum_{s=1}^{q_i-1}\binom{q_i}{s}^2\langle f_{j,n} \otimes_s f_{i,n}, f_{i,n} \otimes_s f_{j,n}\rangle$ tends to zero as $n \to \infty$. On the other hand, the condition on the q_i-th contraction in Theorem 4.4(δ) yields that $f_{j,n} \widetilde{\otimes}_{q_i} f_{j,n} - c_{q_j} f_{j,n} \to 0$ as $n \to \infty$. Moreover, we have

$$\langle f_{j,n}, f_{i,n} \widetilde{\otimes} f_{i,n}\rangle = \frac{1}{q_j!}E[F_{j,n}F_{i,n}^2],$$

which tends to zero by assumption (ii). All these facts together imply that $\langle f_{j,n} \otimes_{q_i} f_{j,n}, f_{i,n} \otimes f_{i,n}\rangle \to 0$ as $n \to \infty$. Using (4.30) for $r = q_i$ we get $f_{i,n} \otimes_{q_i} f_{j,n} \to 0$, showing that condition (γ) of Theorem 3.1 holds true in the last remaining case. The proof of the theorem is complete. □

Example 4.6. Consider $F_n = (F_{1,n}, F_{2,n}) = (I_{q_1}(f_{1,n}), I_{q_2}(f_{2,n}))$, where $2 \leq q_1 \leq q_2$ are even integers. Suppose that

$$\begin{aligned} &E[F_{1,n}^2] \to 1, \quad E[F_{1,n}^4] - 6E[F_{1,n}^3] \to -3, \quad \text{and} \\ &E[F_{2,n}^2] \to 2, \quad E[F_{2,n}^4] - 6E[F_{2,n}^3] \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

When $q_1 = q_2$ or $q_2 = 2q_1$ we require additionally:

$$\mathrm{Cov}(F_{1,n}^2, F_{2,n}^2) \to 0 \ (q_1 = q_2), \quad E[F_{1,n}^2 F_{2,n}] \to 0 \ (q_2 = 2q_1).$$

Then Theorem 4.5 (the case $\nu_1 = 2$, $\nu_2 = 4$) gives

$$F_n \overset{\text{law}}{\to} (V_1 - 1, V_2 + V_3 - 2)$$

where V_1, V_2, V_3 are i.i.d. standard exponential random variables.

4.4. **Bivariate convergence.**

Theorem 4.7. *Let $p_1, \dots, p_r, q_1, \dots, q_s$ be positive integers. Assume further that $\min p_i \geqslant \max q_j$. Consider*

$$(F_{1,n}, \dots, F_{r,n}, G_{1,n}, \dots, G_{s,n}) = \big(I_{p_1}(f_{1,n}), \dots, I_{p_r}(f_{r,n}), I_{q_1}(g_{1,n}), \dots, I_{q_s}(g_{s,n})\big), \quad n \geqslant 1,$$

with $f_{i,n} \in \mathfrak{H}^{\odot p_i}$ and $g_{j,n} \in \mathfrak{H}^{\odot q_j}$. Suppose that as $n \to \infty$

$$F_n = (F_{1,n}, \dots, F_{r,n}) \overset{\text{law}}{\to} N \quad \text{and} \quad G_n = (G_{1,n}, \dots, G_{s,n}) \overset{\text{law}}{\to} V, \tag{4.31}$$

where $N \sim N_r(0, \Sigma)$, the marginals of V are determined by their moments, and N, V are independent. If $E[F_{i,n}G_{j,n}] \to 0$ (which trivially holds when $p_i \neq q_j$) for all i, j, then

$$(F_n, G_n) \overset{\text{law}}{\to} (N, V) \tag{4.32}$$

jointly, as $n \to \infty$.

Proof. We will show that condition (**c**) of Theorem 3.4 holds. By (2.12) we may and do assume that $E[F_{i,n}^2] = 1$ for all i and n. By Theorem 4.1(γ), $\|f_{i,n} \otimes_r f_{i,n}\| \to 0$ for all $r = 1, \dots, p_i - 1$. Observe that

$$\|f_{i,n} \otimes_r g_{j,n}\|^2 = \langle f_{i,n} \otimes_{p_i - r} f_{i,n}, g_{j,n} \otimes_{q_j - r} g_{j,n} \rangle$$

so that $\|f_{i,n} \otimes_r g_{j,n}\| \to 0$ for $1 \leq r \leq p_i \wedge q_j = q_j$, except possibly when $r = p_i = q_j$. But in this latter case,

$$p_i! \|f_{i,n} \otimes_r g_{j,n}\| = p_i! \big|\langle f_{i,n}, g_{j,n} \rangle\big| = \big|E[F_{i,n}G_{j,n}]\big| \to 0$$

by the assumption. Corollary 3.6 concludes the proof. □

Theorem 4.7 admits the following immediate corollary.

Corollary 4.8. *Let $p \geqslant q$ be positive integers. Consider two stochastic processes $F_n = (I_p(f_{t,n}))_{t \in T}$ and $G_n = (I_q(g_{t,n}))_{t \in T}$, where $f_{t,n} \in \mathfrak{H}^{\odot p}$ and $g_{t,n} \in \mathfrak{H}^{\odot q}$. Suppose that as $n \to \infty$*

$$F_n \overset{\text{f.d.d.}}{\to} X \quad \text{and} \quad G_n \overset{\text{f.d.d.}}{\to} Y,$$

where X is centered and Gaussian, the marginals of Y are determined by their moments, and X, Y are independent. If $E[I_p(f_{t,n})I_q(g_{s,n})] \to 0$ (which trivially holds when $p \neq q$) for all $s, t \in T$, then

$$(F_n, G_n) \overset{\text{f.d.d.}}{\to} (X, Y)$$

jointly, as $n \to \infty$.

5. Further Applications

5.1. **Partial sums associated with Hermite polynomials.** Consider a centered stationary Gaussian sequence $\{G_k\}_{k\geqslant 1}$ with unit variance. For any $k \geqslant 0$, denote by

$$r(k) = E[G_1 G_{1+k}]$$

the covariance between G_1 and G_{1+k}. We extend r to $\mathbb{Z}_-$ by symmetry, that is, $r(k) = r(-k)$. For any integer $q \geqslant 1$, we write

$$S_{q,n}(t) = \sum_{k=1}^{\lfloor nt \rfloor} H_q(G_k), \quad t \geqslant 0,$$

to indicate the partial sums associated with the subordinated sequence $\{H_q(G_k)\}_{k\geqslant 1}$. Here, H_q denotes the qth Hermite polynomial given by (2.5).

The following result is a summary of the main finding in Breuer and Major [2].

Theorem 5.1. *If $\sum_{k\in\mathbb{Z}} |r(k)|^q < \infty$ then, as $n \to \infty$,*

$$\frac{S_{q,n}}{\sqrt{n}} \overset{\text{f.d.d.}}{\to} a_q\, B,$$

where B is a standard Brownian motion and $a_q = \left[q! \sum_{k\in\mathbb{Z}} r(k)^q\right]^{1/2}$.

Assume further that the covariance function r has the form

$$r(k) = k^{-D} L(k), \quad k \geqslant 1,$$

with $D > 0$ and $L : (0,\infty) \to (0,\infty)$ a function which is slowly varying at infinity and bounded away from 0 and infinity on every compact subset of $[0,\infty)$. The following result is due to Taqqu [16].

Theorem 5.2. *If $0 < D < \frac{1}{2}$ then, as $n \to \infty$,*

$$\frac{S_{2,n}}{n^{1-D} L(n)} \overset{\text{f.d.d.}}{\to} b_D\, R_{1-D},$$

where $b_D = [(1-D)(1-2D)]^{-1/2}$ and R_H is a Rosenblatt process of parameter $H = 1-D$, defined as

$$R_H(t) = c_H\, I_2\left(f_H(t,\cdot)\right), \quad t \geqslant 0,$$

with

$$f_H(t,x,y) = \int_0^t (s-x)_+^{\frac{H}{2}-1} (s-y)_+^{\frac{H}{2}-1} ds, \quad t \geqslant 0,\, x, y \in \mathbb{R},$$

$c_H > 0$ an explicit constant such that $E[R_H(1)^2] = 1$, and the double Wiener-Itô integral I_2 is with respect to a two-sided Brownian motion B.

Let $q \geqslant 3$ be an integer. The following result is a consequence of Corollary 4.8 and Theorems 5.1 and 5.2. It gives the asymptotic behavior (after proper renormalization of each coordinate) of the pair $(S_{q,n}, S_{2,n})$ when $D \in \left(\frac{1}{q}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$. Since what follows is just mean to be an illustration, we will not consider the remaining case, that is, when

$D \in \big(0, \frac{1}{q}\big)$; it is an interesting problem, but to answer it would be out of the scope of the present paper.

Proposition 5.3. *Let $q \geqslant 3$ be an integer, and let the constants a_p and b_D be given by Theorems 5.1 and 5.2, respectively.*

(1) *If $D \in (\frac{1}{2}, \infty)$ then*

$$\left(\frac{S_{q,n}}{\sqrt{n}}, \frac{S_{2,n}}{\sqrt{n}}\right) \overset{\text{f.d.d.}}{\to} (a_q\, B_1, a_2\, B_2)\,,$$

where (B_1, B_2) is a standard Brownian motion in $\mathbb{R}^2$.

(2) *If $D \in \big(\frac{1}{q}, \frac{1}{2}\big)$ then*

$$\left(\frac{S_{q,n}}{\sqrt{n}}, \frac{S_{2,n}}{n^{1-D}L(n)}\right) \overset{\text{f.d.d.}}{\to} (a_q\, B, b_D\, R_{1-D})\,,$$

where B is a Brownian motion independent of the Rosenblatt process R_{1-D} of parameter $1-D$.

Proof: Let us first introduce a specific realization of the sequence $\{G_k\}_{k\geqslant 1}$ that will allow one to use the results of this paper. The space

$$\mathcal{H} := \overline{\text{span}\{G_1, G_2, \ldots\}}^{L^2(\Omega)}$$

being a real separable Hilbert space, it is isometrically isomorphic to either $\mathbb{R}^N$ (for some finite $N \geqslant 1$) or $L^2(\mathbb{R}_+)$. Let us assume that $\mathcal{H} \simeq L^2(\mathbb{R}_+)$, the case where $\mathcal{H} \simeq \mathbb{R}^N$ being easier to handle. Let $\Phi : \mathcal{H} \to L^2(\mathbb{R}_+)$ be an isometry. Set $e_k = \Phi(G_k)$ for each $k \geqslant 1$. We have

$$r(k-l) = E[G_k G_l] = \int_0^\infty e_k(x) e_l(x) dx, \quad k, l \geqslant 1. \tag{5.33}$$

If $B = (B_t)_{t\in\mathbb{R}_+}$ denotes a standard Brownian motion, we deduce that

$$\{G_k\}_{k\geqslant 1} \overset{\text{law}}{=} \left\{\int_0^\infty e_k(t) dB_t\right\}_{k\geqslant 1},$$

these two sequences being indeed centered, Gaussian and having the same covariance structure. Using (2.6) we deduce that $S_{q,n}$ has the same distribution than $I_q\big(\sum_{k=1}^n e_k^{\otimes q}\big)$ (with I_q the q-tuple Wiener-Itô integral associated to B).

Hence, to reach the conclusion of point 1 it suffices to combine Corollary 4.8 with Theorem 5.1. For point 2, just use Corollary 4.8 and Theorem 5.2, together with the fact that the distribution of $R_H(t)$ is determined by its moments (as is the case for any double Wiener-Itô integral). □

5.2. **Moment-independence for discrete homogeneous chaos.** To develop the next application we will need the following basic ingredients:

(i) A sequence $\mathbf{X} = (X_1, X_2, \ldots)$ of i.i.d. random variables, with mean 0, variance 1 and all moments finite.

(ii) Two positive integers q_1, q_2 as well as two sequences $a_{k,n} : \mathbb{N}^{q_k} \to \mathbb{R}$, $n \geqslant 1$ of real-valued functions satisfying for all $i_1, \ldots, i_{q_k} \geq 1$ and $k = 1, 2$,
   (a) [symmetry] $a_{k,n}(i_1, \ldots, i_{q_k}) = a_{k,n}(i_{\sigma(1)}, \ldots, i_{\sigma(q_k)})$ for every permutation σ;
   (b) [vanishing on diagonals] $a_{k,n}(i_1, \ldots, i_{q_k}) = 0$ whenever $i_r = i_s$ for some $r \neq s$;
   (c) [unit-variance] $q_k! \sum_{i_1,\ldots,i_{q_k}=1}^{\infty} a_{k,n}(i_1, \ldots, i_{q_k})^2 = 1$.

Consider

$$Q_{k,n}(\mathbf{X}) = \sum_{i_1,\ldots,i_{q_k}=1}^{\infty} a_{k,n}(i_1, \ldots, i_{q_k}) X_{i_1} \ldots X_{i_{q_k}}, \quad n \geqslant 1, \quad k = 1, 2. \tag{5.34}$$

This series converges in $L^2(\Omega)$, $E[Q_{k,n}(\mathbf{X})] = 0$ and $E[Q_{k,n}(\mathbf{X})^2] = 1$. We have the following result.

Theorem 5.4. *As $n \to \infty$, assume that the contribution of each X_i to $Q_{k,n}(\mathbf{X})$ is uniformly negligible, that is,*

$$\sup_{i \geq 1} \sum_{i_2,\ldots,i_{q_k}=1}^{\infty} a_{k,n}(i, i_2, \ldots, i_{q_k})^2 \to 0, \quad k = 1, 2, \tag{5.35}$$

and that, for any $r = 1, \ldots, q_1 \wedge q_2$,

$$\sum_{i_1,\ldots,i_{q_1+q_2-2r}=1}^{\infty} \left(\sum_{l_1,\ldots,l_r=1}^{\infty} a_{1,n}(l_1, \ldots, l_r, i_1, \ldots, i_{q_1-r}) a_{2,n}(l_1, \ldots, l_r, i_{q_1-r+1}, \ldots, i_{q_1+q_2-2r}) \right)^2 \to 0. \tag{5.36}$$

Then $Q_{1,n}(\mathbf{X})$ and $Q_{2,n}(\mathbf{X})$ are asymptotically moment-independent.

Proof: Fix $M, N \geqslant 1$. We want to prove that, as $n \to \infty$,

$$E[Q_{1,n}(\mathbf{X})^M Q_{2,n}(\mathbf{X})^N] - E[Q_{1,n}(\mathbf{X})^M] E[Q_{2,n}(\mathbf{X})^N] \to 0. \tag{5.37}$$

The proof is divided into three steps.

Step 1. In this step we show that

$$E[Q_{1,n}(\mathbf{X})^M Q_{2,n}(\mathbf{X})^N] - E[Q_{1,n}(\mathbf{G})^M Q_{2,n}(\mathbf{G})^N] \to 0 \quad \text{as } n \to \infty. \tag{5.38}$$

Following the approach of Mossel, O'Donnel and Oleszkiewicz [5], we will use the Lindeberg replacement trick. Let $\mathbf{G} = (G_1, G_2, \ldots)$ be a sequence of i.i.d. $N(0, 1)$ random variables independent of $\mathbf{X}$. For a positive integer s, set $\mathbf{W}^{(s)} = (G_1, \ldots, G_s, X_{s+1}, X_{s+2}, \ldots)$, and

put $\mathbf{W}^{(0)} = \mathbf{X}$. Fix $s \geq 1$ and write for $k = 1, 2$ and $n \geq 1$,

$$\begin{aligned} U_{k,n,s} &= \sum_{\substack{i_1,\ldots,i_{q_k} \\ i_1 \neq s,\ldots,i_{q_k} \neq s}} a_{k,n}(i_1,\ldots,i_{q_k}) W_{i_1}^{(s)} \ldots W_{i_{q_k}}^{(s)}, \\ V_{k,n,s} &= \sum_{\substack{i_1,\ldots,i_{q_k} \\ \exists j:\, i_j = s}} a_{k,n}(i_1,\ldots,i_{q_k}) W_{i_1}^{(s)} \ldots \widehat{W_s^{(s)}} \ldots W_{i_{q_k}}^{(s)} \\ &= q_k \sum_{i_2,\ldots,i_{q_k}=1}^{\infty} a_{k,n}(s, i_2,\ldots,i_{q_k}) W_{i_2}^{(s)} \ldots W_{i_{q_k}}^{(s)}, \end{aligned}$$

where $\widehat{W_s^{(s)}}$ means that the term $W_s^{(s)}$ is dropped (observe that this notation bears no ambiguity: indeed, since $a_{k,n}$ vanishes on diagonals, each string $i_1,\ldots,i_{q_k}$ contributing to the definition of $V_{k,n,s}$ contains the symbol s exactly once). For each s and k, note that $U_{k,n,s}$ and $V_{k,n,s}$ are independent of the variables X_s and G_s, and that

$$Q_{k,n}(\mathbf{W}^{(s-1)}) = U_{k,n,s} + X_s V_{k,n,s} \quad \text{and} \quad Q_{k,n}(\mathbf{W}^{(s)}) = U_{k,n,s} + G_s V_{k,n,s}.$$

By the binomial formula, using the independence of X_s from $U_{k,n,s}$ and $V_{k,n,s}$, we have

$$\begin{aligned} &E[Q_{1,n}(\mathbf{W}^{(s-1)})^M Q_{2,n}(\mathbf{W}^{(s-1)})^N] \\ &\qquad = \sum_{i=0}^{M} \sum_{j=0}^{N} \binom{M}{i} \binom{N}{j} E[U_{1,n,s}^{M-i} U_{2,n,s}^{N-j} V_{1,n,s}^{i} V_{2,n,s}^{j}] E[X_s^{i+j}]. \end{aligned}$$

Similarly,

$$\begin{aligned} &E[Q_{1,n}(\mathbf{W}^{(s)})^M Q_{2,n}(\mathbf{W}^{(s)})^N] \\ &\qquad = \sum_{i=0}^{M} \sum_{j=0}^{N} \binom{M}{i} \binom{N}{j} E[U_{1,n,s}^{M-i} U_{2,n,s}^{N-j} V_{1,n,s}^{i} V_{2,n,s}^{j}] E[G_s^{i+j}]. \end{aligned}$$

Therefore

$$\begin{aligned} &E[Q_{1,n}(\mathbf{W}^{(s-1)})^M Q_{2,n}(\mathbf{W}^{(s-1)})^N] - E[Q_{1,n}(\mathbf{W}^{(s)})^M Q_{2,n}(\mathbf{W}^{(s)})^N] \\ &\qquad = \sum_{i+j \geq 3} \binom{M}{i} \binom{N}{j} E[U_{1,n,s}^{M-i} U_{2,n,s}^{N-j} V_{1,n,s}^{i} V_{2,n,s}^{j}] \big(E[X_s^{i+j}] - E[G_s^{i+j}] \big). \end{aligned}$$

Now, observe that Propositions 3.11, 3.12 and 3.16 of [5] imply that both $(U_{1,n,s})_{n,s\geq 1}$ and $(U_{2,n,s})_{n,s\geq 1}$ are uniformly bounded in all $L^p(\Omega)$ spaces. It also implies that, for any $p \geqslant 3$, $k = 1, 2$ and $n, s \geq 1$,

$$E[|V_{k,n,s}|^p]^{1/p} \leqslant C_p \, E[V_{k,n,s}^2]^{1/2},$$

where C_p depends only on p. Hence, for $0 \leq i \leq M$, $0 \leq j \leq N$, $i + j \geq 3$, we have

$$\big| E[U_{1,n,s}^{M-i} U_{2,n,s}^{N-j} V_{1,n,s}^{i} V_{2,n,s}^{j}] \big| \leq C \, E[V_{1,n,s}^2]^{i/2} \, E[V_{2,n,s}^2]^{j/2}, \tag{5.39}$$

where C does not depend on $n, s \geq 1$. Since $E[X_i] = E[G_i] = 0$ and $E[X_i^2] = E[G_i^2] = 1$, we get

$$E[V_{k,n,s}^2] = q_k q_k! \sum_{i_2,\ldots,i_{q_k}=1}^{\infty} a_{k,n}(s, i_2, \ldots, i_{q_k})^2.$$

When $i \geq 3$, then (5.39) is bounded from above by

$$C \left(\sup_{i \geq 1} \sum_{i_2,\ldots,i_{q_1}=1}^{\infty} a_{1,n}(i, i_2, \ldots, i_{q_1})^2 \right)^{(i-2)/2} \sum_{i_2,\ldots,i_{q_1}=1}^{\infty} a_{1,n}(s, i_2, \ldots, i_{q_1})^2,$$

where C does not depend on $n, s \geq 1$, and we get a similar bound when $j \geq 3$. If $i = 2$, then $j \geq 1$ ($i + j \geq 3$), so (5.39) isz bounded from above by

$$C \left(\sup_{i \geq 1} \sum_{i_2,\ldots,i_{q_2}=1}^{\infty} a_{2,n}(i, i_2, \ldots, i_{q_2})^2 \right)^{j/2} \sum_{i_2,\ldots,i_{q_1}=1}^{\infty} a_{1,n}(s, i_2, \ldots, i_{q_1})^2,$$

and we have a similar bound when $j = 2$. Taking into account assumption (5.35) we infer that the upper-bound for (5.39) is of the form

$$C\epsilon_n \sum_{k=1}^{2} \sum_{i_2,\ldots,i_{q_k}=1}^{\infty} a_{k,n}(s, i_2, \ldots, i_{q_k})^2,$$

where $\lim_{n\to\infty} \epsilon_n = 0$ and C is independent of n, s. We conclude that

$$\begin{aligned} \big|E[Q_{1,n}(\mathbf{W}^{(s-1)})^M Q_{2,n}(\mathbf{W}^{(s-1)})^N] - E[Q_{1,n}(\mathbf{W}^{(s)})^M Q_{2,n}(\mathbf{W}^{(s)})^N]\big| \\ \leq C\epsilon_n \sum_{k=1}^{2} \sum_{i_2,\ldots,i_{q_k}=1}^{\infty} a_{k,n}(s, i_2, \ldots, i_{q_k})^2, \end{aligned}$$

where C does not depend on n, s. Since, for fixed k, n, $Q_{k,n}(\mathbf{W}^{(s)}) \to Q_{k,n}(\mathbf{G})$ in $L^2(\Omega)$ as $s \to \infty$, by Propositions 3.11, 3.12 and 3.16 of [5], the convergence holds in all $L^p(\Omega)$. Hence

$$\begin{aligned} &\big|E[Q_{1,n}(\mathbf{X})^M Q_{2,n}(\mathbf{X})^N] - E[Q_{1,n}(\mathbf{G})^M Q_{2,n}(\mathbf{G})^N]\big| \\ &\qquad \leq \sum_{s=1}^{\infty} \big|E[Q_{1,n}(\mathbf{W}^{(s-1)})^M Q_{2,n}(\mathbf{W}^{(s-1)})^N] - E[Q_{1,n}(\mathbf{W}^{(s)})^M Q_{2,n}(\mathbf{W}^{(s)})^N]\big| \\ &\qquad \leq C\epsilon_n \sum_{k=1}^{2} \sum_{i_1,\ldots,i_{q_k}=1}^{\infty} a_{k,n}(i_1, i_2, \ldots, i_{q_k})^2 = C\big((q_1!)^{-1} + (q_2!)^{-1}\big)\epsilon_n. \end{aligned}$$

This proves (5.38).

Step 2. We show that $n \to \infty$,

$$E[Q_{1,n}(\mathbf{X})^M] - E[Q_{1,n}(\mathbf{G})^M] \to 0 \quad \text{and} \quad E[Q_{2,n}(\mathbf{X})^N] - E[Q_{2,n}(\mathbf{G})^N] \to 0. \tag{5.40}$$

The proof is similar to Step 1 (and easier). Thus, we omit it.

Step 3. Without loss of generality we may and do assume that $G_k = B_k - B_{k-1}$, where B is a standard Brownian motion. For $k = 1, 2$ and $n \geqslant 1$, due to the multiplication formula (2.10), $Q_{k,n}(\mathbf{G})$ is a multiple Wiener-Itô integral of order q_k with respect to B:

$$Q_{k,n}(\mathbf{G}) = I_{q_k}\left(\sum_{i_1,\ldots,i_{q_k}=1}^{\infty} a_{k,n}(i_1,\ldots,i_{q_k})\mathbf{1}_{[i_1-1,i_1]\times\ldots\times[i_{q_k}-1,i_{q_k}]}\right).$$

In this setting, condition (5.36) coincides with condition (γ) of Theorem 3.1 (or $(\mathbf{c})$ of Theorem 3.4). Therefore,

$$E[Q_{1,n}(\mathbf{G})^M Q_{2,n}(\mathbf{G})^N] - E[Q_{1,n}(\mathbf{G})^M]E[Q_{2,n}(\mathbf{G})^N] \to 0. \tag{5.41}$$

Combining (5.38), (5.40) and (5.41) we get the desired conclusion (5.37). □

Remark 5.5. The conclusion of Theorem 5.4 may fail if either (5.35) or (5.36) are not satisfied. It follows from Step 3 above that the theorem fails when (5.36) does not hold and $\mathbf{X}$ is Gaussian. Theorem 5.4 also fails when (5.35) is not satisfied, (5.36) holds, and $\mathbf{X}$ is a Rademacher sequence, as we can see from the following counterexample. Consider $q_1 = q_2 = 2$, and set

$$\begin{aligned} a_{1,n}(i,j) &= \frac{1}{4}\big(\mathbf{1}_{\{1\}}(i)\mathbf{1}_{\{2\}}(j) + \mathbf{1}_{\{2\}}(i)\mathbf{1}_{\{1\}}(j) + \mathbf{1}_{\{1\}}(i)\mathbf{1}_{\{3\}}(j) + \mathbf{1}_{\{3\}}(i)\mathbf{1}_{\{1\}}(j)\big) \\ a_{2,n}(i,j) &= \frac{1}{4}\big(\mathbf{1}_{\{2\}}(i)\mathbf{1}_{\{4\}}(j) + \mathbf{1}_{\{4\}}(i)\mathbf{1}_{\{2\}}(j) - \mathbf{1}_{\{3\}}(i)\mathbf{1}_{\{4\}}(j) - \mathbf{1}_{\{4\}}(i)\mathbf{1}_{\{3\}}(j)\big). \end{aligned}$$

Then $Q_{1,n}(\mathbf{X}) = \frac{1}{2}X_1(X_2 + X_3)$ and $Q_{2,n}(\mathbf{X}) = \frac{1}{2}X_4(X_2 - X_3)$, where X_i are i.i.d. with $P(X_i = 1) = P(X_i = -1) = 1/2$. It is straightforward to check that (5.36) holds and obviously (5.35) is not satisfied. Since $Q_{1,n}(\mathbf{X})Q_{2,n}(\mathbf{X}) = 0$, we get

$$0 = E[Q_{1,n}(\mathbf{X})^2 Q_{2,n}(\mathbf{X})^2] \neq E[Q_{1,n}(\mathbf{X})^2]E[Q_{2,n}(\mathbf{X})^2],$$

implying in particular that $Q_{1,n}(\mathbf{X})$ and $Q_{2,n}(\mathbf{X})$ are (asymptotically) moment-dependent.

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