Asymptotic behaviour of the cross-variation of some integral long memory processes

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Abstract. We study the asymptotic behaviour of the cross-variation of two-dimensional processes having the form of a Young integral with respect to a fractional Brownian motion of index $H > \frac{1}{2}$. When H is smaller than or equal to $\frac{3}{4}$, we show asymptotic mixed normality. When H is stricly bigger than $\frac{3}{4}$, we obtain a limit that is expressed in terms of the difference of two independent Rosenblatt processes.

1 Introduction

1.1 Foreword and main results

In the near past, there have been many applications of stochastic differential equations (SDE) driven by fractional Brownian motion (fBm) in different areas of mathematical modelling. To name but a few, we mention the use of such equations as a model for meteorological phenomena [1, 11], protein dynamics [6], or noise in electrical networks [7].

Here, we consider more generally a two-dimensional stochastic process $\{X_t\}_{t\in[0,T]} = \{(X_t^{(1)}, X_t^{(2)})\}_{t\in[0,T]}$ of the form

$$X_t^{(i)} = x_i + \int_0^t \sigma_s^{i,1} dB_s^{(1)} + \int_0^t \sigma_s^{i,2} dB_s^{(2)}, \quad t \in [0,T],\, i = 1, 2. \tag{1.1}$$

In (1.1), $B = (B^{(1)}, B^{(2)})$ is a two-dimensional fractional Brownian motion of Hurst index $H > \frac{1}{2}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$,

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whereas $x = (x_1, x_2) \in \mathbb{R}^2$ and σ is a 2×2 matrix-valued process. The case where X solves a fractional SDE corresponds to $\sigma_t = \sigma(X_t)$, with $\sigma : \mathbb{R}^2 \to \mathcal{M}_2(\mathbb{R})$ deterministic. Since we are assuming that $H > \frac{1}{2}$, by imposing appropriate conditions on σ (see Section 2 for the details) we may and will assume throughout the text that $\int_0^t \sigma_s^{i,j} dB_s^{(j)}$ is understood in the Young [15] sense (see again Section 2 for the details).

In this paper, we are concerned with the asymptotic behaviour of the cross-variation associated to X on $[0, T]$, which is the sequence of stochastic processes defined as:

$$J_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} \Delta X_{k/n}^{(1)} \Delta X_{k/n}^{(2)}, \quad n \geqslant 1,\, t \in [0, T]. \tag{1.2}$$

Here, and the same anywhere else, we use the notation $\Delta X_{k/n}^{(i)}$ to indicate the increment $X_{k/n}^{(i)} - X_{(k-1)/n}^{(i)}$. We shall show the following two theorems. They might be of interest for solving problems arising from statistics, as for instance the problem of testing the hypothesis (H_0): "$\sigma^{1,2} = \sigma^{2,1} = 0$" in (1.1).

Theorem 1.1. *For any $t \in [0, T]$,*

$$n^{2H-1} J_n(t) \overset{\text{prob}}{\to} \int_0^t (\sigma_s^{1,1} \sigma_s^{1,2} + \sigma_s^{2,1} \sigma_s^{2,2}) ds \quad \text{as } n \to \infty. \tag{1.3}$$

Theorem 1.2. *Assume $\sigma^{1,2} = \sigma^{2,1} = 0$ and let*

$$a_n := \begin{cases} n^{2H-\frac{1}{2}} & \text{if } \frac{1}{2} < H < \frac{3}{4} \\ \frac{n}{\sqrt{\log n}} & \text{if } H = \frac{3}{4} \\ n & \text{if } \frac{3}{4} < H < 1 \end{cases}. \tag{1.4}$$

Then, as $n \to \infty$,

$$a_n\, J_n \overset{\mathcal{L}}{\to} \int_0^{\cdot} \sigma_s^{1,1} \sigma_s^{2,2} dZ_s \quad \text{in the Skorohod space } D[0, T]. \tag{1.5}$$

In (1.5), the definition of Z is according to the value of H. More precisely, Z equals $\frac{C_H}{2}$ times W when $H \in (\frac{1}{2}, \frac{3}{4}]$, with C_H given by (3.9)-(3.10) and W a Brownian motion independent of $\mathcal{F}$; and $Z = \frac{1}{2}\big(R^{(1)} - R^{(2)}\big)$ when $H \in (\frac{3}{4}, 1)$, with $R^{(k)}$ the Rosenblatt process constructed from the fractional Brownian motion

$$\beta^{(k)} = \frac{1}{\sqrt{2}} (B^{(1)} + (-1)^{k+1} B^{(2)}), \quad k = 1, 2,$$

see Definition 3.4 for the details.

1.2 Link to the existing literature

Our results are close in spirit to those contained in [4] (which has been a strong source of inspiration to us), where central limit theorems for power variations of integral fractional processes are investigated.

As we will see our analysis of J_n, that requires similar but different efforts compared to [4] (as we are here dealing with a *two-dimensional* fractional Brownian motion on one hand and we also consider[1] the case where $H > \frac{3}{4}$ on the other hand), is actually greatly simplified by the use of a recent, nice result obtained in [3] about the asymptotic behaviour of weighted random sums.

1.3 Plan of the paper

The rest of the paper is as follows. Section 2 contains a thorough description of the framework in which our study takes place (in particular, we recall the definition of the Young integral and we provide its main properties). Section 3 gathers several preliminary results that will be essential for proving our main results. Finally, proofs of Theorems 1.1 and 1.2 are given in Section 4.

2 Our framework

In this section, we describe the framework used throughout the paper and we fix a parameter $\alpha \in (0,1)$.

We let C^α denote the set of Hölder continuous functions of index $\alpha \in (0,1)$, that is, the set of those functions $f : [0,T] \to \mathbb{R}$ satisfying

$$|f|_\alpha := \sup_{0 \leqslant s < t \leqslant T} \frac{|f(t) - f(s)|}{(t-s)^\alpha} < \infty. \tag{2.6}$$

Also, we set $\|f\|_\alpha := |f|_\alpha + |f|_\infty$, with $|f|_\infty = \sup_{0 \leqslant t \leqslant T} |f(t)|$.

For a fixed $f \in C^\alpha$, we consider the operator $T_f : \mathcal{C}^1 \to \mathcal{C}^1$ defined as

$$T_f(g)(t) = \int_0^t f(u) g'(u) du, \quad t \in [0,T].$$

Let $\beta \in (0,1)$ be such that $\alpha + \beta > 1$. Then T_f extends, in a unique way, to an operator $T_f : C^\beta \to C^\beta$, which further satisfies

$$\|T_f(g)\|_\beta \leqslant (1 + C_{\alpha,\beta})\,(1 + T^\beta) \|f\|_\alpha \|g\|_\beta,$$

with $C_{\alpha,\beta} = \frac{1}{2} \sum_{n=1}^\infty 2^{-n(\alpha+\beta-1)} < \infty$. See, e.g., [8, Theorem 3.1] for a proof.

[1]The authors of [4] did not consider the case where $H > \frac{3}{4}$ since, quoting them, “the problem is more involved because non-central limit theorems are required”.

Definition 2.1. *Let $\alpha, \beta \in (0,1)$ be such that $\alpha + \beta > 1$. Let $f \in C^\alpha$ and $g \in C^\beta$. The Young integral $\int_0^\cdot f(u)dg(u)$ is then defined as being $T_f(g)$.*

The Young integral satisfies (see, e.g., [8, inequality (3.3)]) that, for any $a, b \in [0,T]$ with $a < b$,

$$\left| \int_a^b (f(u) - f(a))dg(u) \right| \leqslant C_{\alpha,\beta} |f|_\alpha |g|_\beta \, (b-a)^{\alpha+\beta}. \tag{2.7}$$

As we said in the Introduction, we let $B = (B^{(1)}, B^{(2)})$ be a 2-dimensional fractional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. We assume further that $\mathcal{F}$ is the σ-field generated by B. We also suppose that the Hurst parameter H of B is the same for the two components and that it is strictly bigger than $\frac{1}{2}$.

Let $\alpha \in (0,1)$ and let $\sigma^{i,j} : \Omega \times [0,T] \to \mathbb{R}$, $i,j = 1,2$, be four given stochastic processes that are measurable with respect to $\mathcal{F}$. We will assume throughout the text that the following two additional assumptions on α and $\sigma^{i,j}$ take place:

(A) $\alpha \in \left(\frac{1}{4} + \frac{H}{2}, H\right)$,

(B) For each pair $(i,j) \in \{1,2\}^2$, the random variable $\|\sigma^{i,j}\|_\alpha$ has moments of all orders.

Observe that $\alpha + H > 1$ due to both (**A**) and $H > \frac{1}{2}$, so that the integrals in (1.1) are well-defined in the Young sense. Also, recall the following variant of the Garcia-Rodemich-Rumsey Lemma [5]: for any $q > 1$, there exists a constant $c_{\alpha,q} > 0$ (depending only on α and q) such that

$$|B^{(i)}|_\alpha^q \leqslant c_{\alpha,q} \iint_{[0,T]^2} \frac{|B_u^{(i)} - B_v^{(i)}|^q}{|u-v|^{2+q\alpha}} dudv. \tag{2.8}$$

Using (2.8), one deduces that $|B^{(i)}|_\alpha$ has moments of all orders.

3 Preliminaries

3.1 Breuer-Major theorem

The next statement is a direct consequence of the celebrated Breuer-Major [2] theorem (see [8, Section 7.2] for a modern proof). We write 'fdd' to indicate the convergence of all the finite-dimensional distributions.

Theorem 3.1 (Breuer-Major)**.** *Let β be a (one-dimensional) fractional Brownian motion of index $H \in (0, \frac{3}{4}]$. Then, as $n \to \infty$ and with W a standard Brownian motion,*

(i) if $H < \frac{3}{4}$ then

$$\left\{ \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \left[(\beta_k - \beta_{k-1})^2 - 1 \right] \right\}_{t \in [0,T]} \xrightarrow{\text{fdd}} \frac{1}{2} \sum_{k \in \mathbb{Z}} \left(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H} \right)^2 \{W_t\}_{t \in [0,T]};$$

(ii) if $H = \frac{3}{4}$ then

$$\left\{ \frac{1}{\sqrt{n \log n}} \sum_{k=1}^{\lfloor nt \rfloor} \left[(\beta_k - \beta_{k-1})^2 - 1 \right] \right\}_{t \in [0,T]} \xrightarrow{\text{fdd}} \frac{3}{4} \log 2 \, \{W_t\}_{t \in [0,T]}.$$

By a scaling argument (to pass from k to k/n) and by using the seminal result of Peccati and Tudor [10] (to allow an extra F), one immediately deduces from Theorem 3.1 the following corollary.

Corollary 3.2. *Let $\beta = (\beta^{(1)}, \beta^{(2)})$ be a two-dimensional fractional Brownian motion of index $H \in (0, \frac{3}{4}]$. Then, as $n \to \infty$ and with W a (one-dimensional) standard Brownian motion independent of β, we have, for any random vector $F = (F_1, \ldots, F_d)$ measurable with respect to β,*

(i) if $H < \frac{3}{4}$ then

$$\left\{ F, n^{2H - \frac{1}{2}} \sum_{k=1}^{\lfloor nt \rfloor} \left[(\beta^{(1)}_{k/n} - \beta^{(1)}_{(k-1)/n})^2 - (\beta^{(2)}_{k/n} - \beta^{(2)}_{(k-1)/n})^2 \right] \right\}_{t \in [0,T]} \xrightarrow{\text{fdd}} \{F, C_H \, W_t\}_{t \in [0,T]},$$

where

$$C_H = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} \left(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H} \right)^2 \tag{3.9}$$

(ii) if $H = \frac{3}{4}$ then

$$\left\{ F, \frac{n}{\sqrt{\log n}} \sum_{k=1}^{\lfloor nt \rfloor} \left[(\beta^{(1)}_{k/n} - \beta^{(1)}_{(k-1)/n})^2 - (\beta^{(2)}_{k/n} - \beta^{(2)}_{(k-1)/n})^2 \right] \right\}_{t\in[0,T]}$$
$$\xrightarrow{\text{fdd}} \left\{ F, C_{3/4} W_t \right\}_{t\in[0,T]},$$

where

$$C_{3/4} = \frac{3\sqrt{2}}{4} \log 2. \tag{3.10}$$

3.2 Taqqu's theorem and the Rosenblatt process

Taqqu's theorem [12] describes the fluctuations of the quadratic variation of the fractional Brownian motion when the Hurst index H is strictly bigger than $\frac{3}{4}$, that is, for the range of values which are not covered by the Breuer-Major Theorem 3.1. We state here a version that fits into our framework. With respect to the original statement, it is worthwhile noting that, in Theorem 3.3 (whose proof may be found in [9]), the convergence is in $L^2(\Omega)$ (and not only in law). This latter fact will reveal to be crucial in our proof of Theorem 1.2, as it will allow us to apply the main result of [3] recalled in Section 3.4.

Theorem 3.3 (Taqqu)**.** *Let β be a (one-dimensional) fractional Brownian motion of index $H \in (\frac{3}{4}, 1)$. Then, for any $t \in [0,T]$, the sequence*

$$n^{1-2H} \sum_{k=1}^{\lfloor nt \rfloor} \left[n^{2H} (\beta_{k/n} - \beta_{(k-1)/n})^2 - 1 \right] \tag{3.11}$$

converges in $L^2(\Omega)$ as $n \to \infty$.

Definition 3.4. *Let the assumption of Theorem 3.3 prevail and denote by R_t the limit of (3.11). The process $R = \{R_t\}_{t\in[0,T]}$ is called the Rosenblatt process constructed from β.*

For the main properties of the Rosenblatt process R, we refer the reader to Taqqu [13] or Tudor [14]. See also [8, Section 7.3]. An immediate corollary of Theorem 3.3 is as follows.

Corollary 3.5. *Let $\beta = (\beta^{(1)}, \beta^{(2)})$ be a two-dimensional fractional Brownian motion of index $H \in (\frac{3}{4}, 1)$. Then, for any $t \in [0, T]$,*

$$n \sum_{k=1}^{\lfloor nt \rfloor} \left[(\beta^{(1)}_{k/n} - \beta^{(1)}_{(k-1)/n})^2 - (\beta^{(2)}_{k/n} - \beta^{(2)}_{(k-1)/n})^2 \right] \xrightarrow{L^2(\Omega)} R^{(1)}_t - R^{(2)}_t$$

as $n \to \infty$, where $R^{(i)}$ is the Rosenblatt process constructed from the fractional Brownian motion $\beta^{(i)}$, $i = 1, 2$.

3.3 Two simple auxiliary lemmas

To complete the proofs of Theorems 1.1 and 1.2 we will, among other things, need the following two simple lemmas.

Lemma 3.6. *Let B and σ be as in Section 2. Then there exists a constant $C = C(\alpha, H, T, \sigma) > 0$ such that, for any $i, j = 1, 2$, any $n \geqslant 1$ and any $k \in \{1, ..., \lfloor nT \rfloor\}$,*

$$\left\| \int_{(k-1)/n}^{k/n} (\sigma^{i,j}_s - \sigma^{i,j}_{k/n}) dB^j_s \right\|_{L^2(\Omega)} \leqslant Cn^{-2\alpha}, \tag{3.12}$$

$$\left\| \int_{(k-1)/n}^{k/n} \sigma^{i,j}_s dB^j_s \right\|_{L^2(\Omega)} \leqslant Cn^{-H}. \tag{3.13}$$

Proof. Without loss of generality, we may and will assume that $i = j = 1$. Using (2.7) with $\beta = \alpha$, we have, almost surely,

$$\left| \int_{(k-1)/n}^{k/n} \left(\sigma^{1,1}_s - \sigma^{1,1}_{k/n} \right) dB^1_s \right| \leqslant C_{\alpha,\alpha} |\sigma^{1,1}|_\alpha |B^1|_\alpha n^{-2\alpha}.$$

Using Cauchy-Schwarz inequality, one deduces

$$\begin{aligned} & E\left[\left(\int_{(k-1)/n}^{k/n} \left(\sigma^{1,1}_s - \sigma^{1,1}_{k/n} \right) dB^1_s \right)^2 \right] \\ \leqslant\; & C^2_{\alpha,\alpha} \sqrt{E\left[\|\sigma^{1,1}\|^4_\alpha \right]} \sqrt{[E|B^1|^4_\alpha]}\, n^{-4\alpha} = Cn^{-4\alpha}, \end{aligned}$$

thus yielding (3.12). On the other hand, one has

$$\begin{aligned}
&\left\| \int_{(k-1)/n}^{k/n} \sigma_s^{i,j} dB_s^j \right\|_{L^2(\Omega)} \\
\leqslant\ & \left\| \int_{(k-1)/n}^{k/n} \left(\sigma_s^{i,j} - \sigma_{k/n}^{i,j} \right) dB_s^j \right\|_{L^2(\Omega)} + \left\| \sigma_{k/n}^{i,j} \Delta B_{k/n}^j \right\|_{L^2(\Omega)} \\
\leqslant\ & Cn^{-2\alpha} + Cn^{-2H}, \quad \text{by (3.12) and because of } (\mathbf{B}) \\
\leqslant\ & Cn^{-H}, \quad \text{using } (\mathbf{A}),
\end{aligned}$$

which is the desired claim (3.13). □

Lemma 3.7. *Let $g, h : [0,T] \to \mathbb{R}$ be two continuous functions, let $\gamma \in \mathbb{R}$, and let us write $\Delta h_{k/n}$ to denote the increment $h(k/n) - h((k-1)/n)$. If*

$$\forall t \in [0,T] \cap \mathbb{Q} : \quad \lim_{n\to\infty} n^\gamma \sum_{k=1}^{\lfloor nT \rfloor} \mathbf{1}_{[0,t]}(k/n) \left(\Delta h_{k/n} \right)^2 = t, \tag{3.14}$$

then, for all $t \in [0,T]$,

$$\lim_{n\to\infty} n^\gamma \sum_{k=1}^{\lfloor nT \rfloor} g(k/n) \mathbf{1}_{[0,t]}(k/n) \left(\Delta h_{k/n} \right)^2 = \int_0^t g(s) ds.$$

Proof. Since $t \mapsto n^\gamma \sum_{k=1}^n \mathbf{1}_{[0,t]}(k/n) \left(\Delta h_{k/n} \right)^2$ is non-decreasing, it is straightforward to deduce from (3.14) that, for *all* $t \in [0,T]$,

$$\lim_{n\to\infty} n^\gamma \sum_{k=1}^{\lfloor nT \rfloor} \mathbf{1}_{[0,t]}(k/n) \left(\Delta h_{k/n} \right)^2 = t.$$

Otherwise stated, the cumulative distribution function (cdf) of the compactly supported measure

$$\nu_n(dx) = n^\gamma \sum_{k=1}^{\lfloor nT \rfloor} \left(\Delta h_{k/n} \right)^2 \delta_{k/n}(dx),$$

where δ_a stands for the Dirac mass at a, converges pointwise to the cdf of the Lebesgue measure on $[0,T]$. Since g is continuous, it is then a routine exercise to deduce that our desired claim holds true. □

3.4 Asymptotic behaviour of weighted random sums, following Corcuera, Nualart and Podolskij [3]

The following result represents a central ingredient in the proof of both Theorems 1.1 and 1.2.

Proposition 3.8. *Let $u = \{u_t\}_{t\in[0,T]}$ be a Hölder continuous process with index $\alpha > \frac{1}{2}$, set*

$$K_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} u_{k/n} \Delta B^{(1)}_{k/n} \Delta B^{(2)}_{k/n}, \quad t \in [0,T],$$

and let a_n be given by (1.4). Then, as $n \to \infty$,

$$a_n\, K_n \xrightarrow{\mathcal{L}} \int_0^{\cdot} u_s dZ_s \quad \text{in the Skorohod space } D[0,T]. \tag{3.15}$$

Here, Z is as in the statement of Theorem 1.2.

The proof of our Proposition 3.8 heavily relies on a nice result taken from Corcuera, Nualart and Podolskij [3]. Actually, we will need a slight extension of the result of [3], that we state here for convenience (and also because we do not share the same notation). The only difference between Theorem 3.9 as stated below and its original version appearing in [3] is that Z is allowed not to be a Brownian motion. A careful inspection of the proof given in [3] indeed reveals that the Brownian feature of Z plays actually no role; the only property of Z which is used is that the sum of its Hölder exponent and that of u is strictly bigger than 1, see (**H1**).

Theorem 3.9 (Corcuera, Nualart, Podolskij)**.** *The underlying probability space is $(\Omega, \mathcal{F}, P)$. Let $u = \{u_t\}_{t\in[0,T]}$ be a Hölder continuous process with index $\alpha \in (0,1)$, and let $\xi = \{\xi_{k,n}\}_{n\in\mathbb{N},\, 1\leqslant k\leqslant \lfloor nT \rfloor}$ be a family of random variables. Set*

$$g_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} \xi_{k,n}, \quad t \in [0,T].$$

Assume the following two hypotheses on the double sequence ξ:

(H1) $\{g_n(t)\}_{t\in[0,T]} \overset{\text{f.d.d.}}{\to} \{Z(t)\}_{t\in[0,T]}$ *$\mathcal{F}$-stably, where Z is Hölder continuous with index β such that $\alpha + \beta > 1$.*

(H2) *There is a constant $C > 0$ such that, for any $1 \leqslant i < j \leqslant [nT]$,*

$$E\left[\left(\sum_{k=i+1}^{j} \xi_{k,n}\right)^4\right] \leqslant C\left(\frac{j-i}{n}\right)^2.$$

Then

$$\sum_{k=1}^{\lfloor n\cdot\rfloor} u_{\frac{k}{n}}\, \xi_{k,n} \xrightarrow{\mathcal{L}} \int_0^{\cdot} u_s dZ_s \quad \textit{in the Skorohod space } D[0,T],$$

where $\int_0^{\cdot} u_s dZ_s$ is understood as a Young integral.

Armed with Theorem 3.9, we are now ready to prove Proposition 3.8.

Proof of Proposition 3.8. Set $\xi_{k,n} = a_n \Delta B^{(1)}_{k/n} \Delta B^{(2)}_{k/n}$ and $g_n(t) = \sum_{k=1}^{\lfloor nt\rfloor} \xi_{k,n}$, $t \in [0,T]$. We shall check the two assumptions (**H1**) and (**H2**) of Theorem 3.9.

Step 1: Checking (**H1**). We make use of the rotation trick. More precisely, let $\beta^{(1)} = \frac{1}{\sqrt{2}}(B^{(1)} + B^{(2)})$ and $\beta^{(2)} = \frac{1}{\sqrt{2}}(B^{(1)} - B^{(2)})$, so that $\xi_{k,n} = \frac{a_n}{2}\left(\left(\Delta\beta^{(1)}_{k/n}\right)^2 - \left(\Delta\beta^{(2)}_{k/n}\right)^2\right)$. It is easy to check that $\beta^{(1)}$ and $\beta^{(2)}$ are two independent fractional Brownian motions of index H. As a result, assumption (**H1**) is satisfied thanks to Corollary 3.2 (resp. Corollary 3.5) when $H \leqslant \frac{3}{4}$ (resp. $H > \frac{3}{4}$).

Step 2: Checking (**H2**). Since all the $L^p(\Omega)$-norms are equivalent inside a given Wiener chaos (here: the second Wiener chaos), it suffices to check the existence of a constant $C > 0$ such that, for any $1 \leqslant i < j \leqslant [nT]$,

$$E\left[\left(\sum_{k=i+1}^{j} \xi_{k,n}\right)^2\right] \leqslant C\,\frac{j-i}{n}. \tag{3.16}$$

Using the independence of $B^{(1)}$ and $B^{(2)}$, one computes that

$$E\left[\left(\sum_{k=i+1}^{j} \xi_{k,n}\right)^2\right] = a_n^2\, n^{-4H} \sum_{k,k'=i+1}^{j} \rho(k-k')^2,$$

with $\rho(r) = \frac{1}{2}\big(|r+1|^{2H} + |r-1|^{2H} - 2|r|^{2H}\big)$. As a result, for any $1 \leqslant i < j \leqslant [nT]$,

$$E\left[\left(\sum_{k=i+1}^{j} \xi_{k,n}\right)^2\right] \leqslant a_n^2\, n^{-4H}(j-i) \sum_{r=-[nT]}^{[nT]} \rho(r)^2.$$

It is straightforward to show that $a_n^2\, n^{1-4H} \sum_{r=-[nT]}^{[nT]} \rho(r)^2 = O(1)$ as $n \to \infty$. Thus, (3.16) is satisfied, and so is (**H2**).

To conclude the proof of Proposition 3.8, it remains to apply Theorem 3.9 with $\xi_{k,n} = a_n \Delta B_{k/n}^{(1)} \Delta B_{k/n}^{(2)}$. $\square$

4 Proof of our main results

4.1 Proof of Theorem 1.1

We divide it into several steps.

Step 1. Recall J_n from (1.2). One can write

$$\begin{aligned} J_n(t) &= \sum_{k=1}^{\lfloor nt \rfloor} \left(\int_{(k-1)/n}^{k/n} \sigma_s^{1,1} dB_s^1 + \int_{(k-1)/n}^{k/n} \sigma_s^{1,2} dB_s^2 \right) \\ &\qquad \times \left(\int_{(k-1)/n}^{k/n} \sigma_s^{2,1} dB_s^1 + \int_{(k-1)/n}^{k/n} \sigma_s^{2,2} dB_s^2 \right) \\ &=: A_n(t) + R_{1,n}(t) + R_{2,n}(t), \end{aligned}$$

with

$$A_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} \left(\sigma_{k/n}^{1,1} \Delta B_{k/n}^1 + \sigma_{k/n}^{1,2} \Delta B_{k/n}^2 \right) \left(\sigma_{k/n}^{2,1} \Delta B_{k/n}^1 + \sigma_{k/n}^{2,2} \Delta B_{k/n}^2 \right), \tag{4.17}$$

$$\begin{aligned} R_{1,n}(t) &= \sum_{k=1}^{\lfloor nt \rfloor} \left(\int_{(k-1)/n}^{k/n} \sigma_s^{1,1} dB_s^1 + \int_{(k-1)/n}^{k/n} \sigma_s^{1,2} dB_s^2 \right) \\ &\quad \times \left(\int_{(k-1)/n}^{k/n} \left(\sigma_s^{2,1} - \sigma_{k/n}^{2,1} \right) dB_s^1 + \int_{(k-1)/n}^{k/n} \left(\sigma_s^{2,2} - \sigma_{k/n}^{2,2} \right) dB_s^2 \right), \end{aligned} \tag{4.18}$$

$$\begin{aligned} R_{2,n}(t) &= \sum_{k=1}^{\lfloor nt \rfloor} \left(\sigma_{k/n}^{2,1} \Delta B_{k/n}^1 + \sigma_{k/n}^{2,2} \Delta B_{k/n}^2 \right) \\ &\quad \times \left(\int_{(k-1)/n}^{k/n} \left(\sigma_s^{1,1} - \sigma_{k/n}^{1,1} \right) dB_s^1 + \int_{(k-1)/n}^{k/n} \left(\sigma_s^{1,2} - \sigma_{k/n}^{1,2} \right) dB_s^2 \right). \end{aligned} \tag{4.19}$$

Step 2. Let us prove the convergence of $n^{2H-1}R_{i,n}(t)$, $i=1,2$, $t\in[0,T]$, in $L^1(\Omega)$ towards zero. Using Cauchy-Schwarz and Lemma 3.6, we see that

$$\begin{aligned}\|R_{1,n}(t)\|_{L^1(\Omega)} &\leqslant \sum_{k=1}^{\lfloor nt\rfloor}\left\|\int_{(k-1)/n}^{k/n}\sigma_s^{1,1}dB_s^1+\int_{(k-1)/n}^{k/n}\sigma_s^{1,2}dB_s^2\right\|_{L^2(\Omega)}\\ &\times \left\|\int_{(k-1)/n}^{k/n}\left(\sigma_s^{2,1}-\sigma_{k/n}^{2,1}\right)dB_s^1+\int_{(k-1)/n}^{k/n}\left(\sigma_s^{2,2}-\sigma_{k/n}^{2,2}\right)dB_s^2\right\|_{L^2(\Omega)}\\ &\leqslant Cn^{-(H+2\alpha-1)}.\end{aligned}$$

Thanks to our assumption (**A**), one deduces that $n^{2H-1}\|R_{1,n}(t)\|_{L^1(\Omega)}\to 0$ as $n\to\infty$. Similarly, one proves that $n^{2H-1}\|R_{2,n}(t)\|_{L^1(\Omega)}\to 0$.

Step 3. Let us now consider A_n. One has

$$\begin{aligned}A_n(t) &= \sum_{k=1}^{\lfloor nt\rfloor}\left(\sigma_{k/n}^{1,1}\Delta B_{k/n}^1+\sigma_{k/n}^{1,2}\Delta B_{k/n}^2\right)\left(\sigma_{k/n}^{2,1}\Delta B_{k/n}^1+\sigma_{k/n}^{2,2}\Delta B_{k/n}^2\right)\\ &=: A_{1,n}(t)+A_{2,n}(t)+S_n(t),\end{aligned}$$

with

$$A_{i,n}(t) = \sum_{k=1}^{\lfloor nt\rfloor}\sigma_{k/n}^{1,i}\sigma_{k/n}^{2,i}\left(\Delta B_{k/n}^i\right)^2,\quad i=1,2, \tag{4.20}$$

$$S_n(t) = \sum_{k=1}^{\lfloor nt\rfloor}\left(\sigma_{k/n}^{1,1}\sigma_{k/n}^{2,2}+\sigma_{k/n}^{1,2}\sigma_{k/n}^{2,1}\right)\Delta B_{k/n}^1\Delta B_{k/n}^2. \tag{4.21}$$

Using Proposition 3.8 and whatever the value of H compared to $\frac{3}{4}$, one immediately checks that $n^{2H-1}S_n(t)$ converges in law to zero, thus in probability. On the other hand, fix $i\in\{1,2\}$ and recall the well-known fact that, for any $t\in[0,T]$,

$$\lim_{n\to\infty} n^{2H-1}\sum_{k=1}^{\lfloor nT\rfloor}\mathbf{1}_{[0,t]}(k/n)\left(\Delta B_{k/n}^i\right)^2 = t\quad\text{almost surely.}$$

We then deduce that, with probability 1, assumption (3.14) holds true with $h=B^i$ and $\gamma=2H-1$. Lemma 3.7 applies and yields that

$$n^{2H-1}A_{i,n}(t)\to\int_0^t\sigma_s^{1,i}\sigma_s^{2,i}ds\quad\text{almost surely.}$$

Step 4. Plugging together the conclusions of Steps 1 to 3 completes the proof of Theorem 1.1. □

4.2 Proof of Theorem 1.2

Recall from the previous section that $J_n = A_{1,n} + A_{2,n} + S_n + R_{1,n} + R_{2,n}$, with $A_{i,n}$, S_n, $R_{1,n}$ and $R_{2,n}$ given by (4.20), (4.21), (4.18) and (4.19) respectively. Using the estimates of Step 2 in the previous section, we easily obtain that, under (**A**), $a_n\, R_{i,n}(t)$ tends to zero in $L^1(\Omega)$ as $n \to \infty$, $i = 1, 2$, $t \in [0, T]$. Moreover, the quantities $A_{1,n}$ and $A_{2,n}$ given by (4.20) equal zero when $\sigma^{1,2} = \sigma^{2,1} = 0$. As a result, the asymptotic behavior of $a_n\, J_n$ is the same as that of $a_n\, S_n$, and the desired conclusion follows directly from Proposition 3.8. $\square$

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