An Arrow-like theorem over median algebras

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Abstract. We present an Arrow-like theorem for aggregation functions over convervative median algebras. In doing so, we give a characterization of conservative median algebras by means of forbidden substructures and by providing their representation as chains.

1 Introduction and preliminaries

Informally, an aggregation function $f : \mathbf{A}^n \to \mathbf{B}$ is a mapping that preserves the structure of $\mathbf{A}$ into $\mathbf{B}$. Usually, $\mathbf{B}$ is taken equal to $\mathbf{A}$ and is equipped with a partial order so that aggregation functions are thought of as order-preserving maps [7]. In this paper, we are interested in aggregation functions $f : \mathbf{A}^n \to \mathbf{A}$ that satisfy the functional equation

$$f(\mathbf{m}(\mathbf{x}, \mathbf{y}, \mathbf{z})) \;=\; \mathbf{m}(f(\mathbf{x}), f(\mathbf{y}), f(\mathbf{z})), \tag{1.1}$$

where $\mathbf{A} = \langle A, \mathbf{m}\rangle$ is a *median algebra*, that is, an algebra with a single ternary operation $\mathbf{m}$, called a *median function*, that satisfies the equations

$$\begin{gathered} \mathbf{m}(x, x, y) = x, \\ \mathbf{m}(x, y, z) = \mathbf{m}(y, x, z) = \mathbf{m}(y, z, x), \\ \mathbf{m}(\mathbf{m}(x, y, z), t, u) = \mathbf{m}(x, \mathbf{m}(y, t, u), \mathbf{m}(z, t, u)), \end{gathered}$$

and that is extended to $\mathbf{A}^n$ componentwise. In particular, every median algebra satisfies the equation

$$\mathbf{m}(x, y, \mathbf{m}(x, y, z)) \;=\; \mathbf{m}(x, y, z). \tag{1.2}$$

An example of median function is the term function

$$\mathbf{m}(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (z \wedge y) \tag{1.3}$$

over a distributive lattice. The motivation for considering (1.1) is rooted in its natural interpretation in social choice: *the score of the median profile is the median of the scores of the profiles.*

Median algebras have been investigated by several authors (see [4, 9] for early references on median algebras and see [2, 10] for some surveys) who illustrated the deep interactions between median algebras, order theory and graph theory.

For instance, take an element a of a median algebra $\mathbf{A}$ and consider the relation $\leq_a$ defined on A by

$$x \leq_a y \quad\iff\quad \mathbf{m}(a, x, y) = x.$$

Endowed with this relation, $\mathbf{A}$ is a $\wedge$-semilattice order with bottom element a [13]: the associated operation $\wedge$ is defined by $x \wedge y = \mathbf{m}(a, x, y)$.

Semilattices constructed in this way are called *median semilattices*, and they coincide exactly with semilattices in which every principal ideal is a distributive lattice and in which any three elements have a join whenever each pair of them is bounded above. The operation $\mathbf{m}$ on $\mathbf{A}$ can be recovered from the median semilattice order $\leq_a$ using identity (1.3) where $\wedge$ and $\vee$ are defined with respect to $\leq_a$.

Note that if a median algebra $\mathbf{A}$ contains two elements 0 and 1 such that $\mathbf{m}(0, x, 1) = x$ for every $x \in A$, then $(A, \leq_0)$ is a distributive lattice order bounded by 0 and 1, and where $x \wedge y$ and $x \vee y$ are given by $\mathbf{m}(x, y, 0)$ and $\mathbf{m}(x, y, 1)$, respectively. Conversely, if $\mathbf{L} = \langle L, \vee, \wedge\rangle$ is a distributive lattice, then the term function defined by (1.3) is denoted by $\mathbf{m}_{\mathbf{L}}$ and gives rise to a median algebra on L, called the *median algebra associated with* $\mathbf{L}$. It is noteworthy that equations satisfied by median algebras of the form $\langle L, \mathbf{m}_{\mathbf{L}}\rangle$ are exactly those satisfied by median algebras. In particular, every median algebra satisfies the equation

$$\begin{aligned} \mathbf{m}(x, y, z) = \mathbf{m}\big(&\mathbf{m}\big(\mathbf{m}(x, y, z), x, t\big), \\ &\mathbf{m}\big(\mathbf{m}(x, y, z), z, t\big), \mathbf{m}\big(\mathbf{m}(x, y, z), y, t\big)\big). \end{aligned} \tag{1.4}$$

Moreover, covering graphs (*i.e.*, undirected Hasse diagram) of median semilattices have been investigated and are, in a sense, equivalent to median graphs. Recall that a median graph is a (non necessarily finite) connected graph in which for any three vertices u, v, w there is exactly one vertex x that lies on a shortest path between u and v, on a shortest path between u and w and on a shortest path between v and w. In other words, x (the *median* of u, v and w) is the only vertex such that

$$\begin{gathered} d(u, v) = d(u, x) + d(x, v), \\ d(u, w) = d(u, x) + d(x, w), \\ d(v, w) = d(v, x) + d(x, w). \end{gathered}$$

Every median semilattice whose intervals are finite has a median covering graph [1] and conversely, every median graph is the covering graph of a median semilattice [1, 13]. This connection is deeper: median semilattices can be characterized among the ordered sets whose bounded chains are finite and in which any two elements are bounded below as the ones

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whose covering graph is median [3]. For further background see, e.g., [2].

Here we are particularly interested in solving equation (1.1) for median algebras $\mathbf{A}$ that are *conservative*, i.e., that satisfy

$$\mathbf{m}(x,y,z) \in \{x,y,z\}, \qquad x,y,z \in A. \tag{1.5}$$

This property essentially states that the aggregation procedure (in this case, a median) should pick one of its entries (e.g., the median candidate is one of the candidate).

Semilattices associated with conservative median algebras are called *conservative median semilattices*. It is not difficult to verify that a median algebra is conservative if and only if each of its subsets is a median subalgebra. Moreover, if $\mathbf{L}$ is a chain, then $\mathbf{m_L}$ satisfies (1.5); however the converse is not true. This fact was observed in §11 of [12], which presents the four element Boolean algebra as a counter-example.

The results of this paper are twofold. First, we present a description of conservative median algebras in terms of forbidden substructures (in complete analogy with BIRKHOFF's characterization of distributive lattices with M_5 and N_5 as forbidden substructures and KURATOWSKI's characterization of planar graphs in terms of forbidden minors), and that leads to a representation of conservative median algebras (with at least five elements) as chains. In fact, the only conservative median algebra that is not representable as a chain is the four element Boolean algebra.

Second, we characterize functions $f : \mathbf{B} \to \mathbf{C}$ that satisfy the equation

$$f(\mathbf{m}(x,y,z)) \;=\; \mathbf{m}(f(x), f(y), f(z)), \tag{1.6}$$

where $\mathbf{B}$ and $\mathbf{C}$ are finite products of (non necessarily finite) chains, as superposition of compositions of monotone maps with projection maps (Theorem 3.5). Particularized to aggregation functions $f : \mathbf{A}^n \to \mathbf{A}$, where $\mathbf{A}$ is a chain, we obtain an ARROW-like theorem: *f satisfies equation* (1.1) *if and only if it is dictatorial and monotone* (Corollary 3.6).

Throughout the paper we employ the following notation. For each positive integer n, we set $[n] = \{1,\ldots,n\}$. Algebras are denoted by bold roman capital letters $\mathbf{A}, \mathbf{B}, \mathbf{X}, \mathbf{Y}\ldots$ and their universes by italic roman capital letters $A, B, X, Y\ldots$. To simplify our presentation, we will keep the introduction of background to a minimum, and we will assume that the reader is familiar with the theory of lattices and ordered sets. We refer the reader to [6, 8] for further background. To improve the readability of the paper, we adopt the rather unusual convention that in any distributive lattice the empty set is a prime filter and a prime ideal. Proofs of the results presented in the third section are omitted because they rely on arguments involving a categorical duality that are beyond the scope of this paper.

2 Characterizations of conservative median algebras

Let $\mathbf{C}_0 = \langle C_0, \leq_0, c_0\rangle$ and $\mathbf{C}_1 = \langle C_1, \leq_1, c_1\rangle$ be chains with bottom elements c_0 and c_1. The *$\perp$-coalesced sum* $\mathbf{C}_0 \perp \mathbf{C}_1$ of $\mathbf{C}_0$ and $\mathbf{C}_1$ is the poset obtained by amalgamating c_0 and c_1 in the disjoint union of C_0 and C_1. Formally,

$$\mathbf{C}_0 \perp \mathbf{C}_1 = \langle C_0 \sqcup C_1 \;/\!\equiv,\; \leq\;\rangle,$$

where $\sqcup$ is the disjoint union, where $\equiv$ is the equivalence generated by $\{(c_0, c_1)\}$ and where $\leq$ is defined by

$$x/\!\equiv \;\leq\; y/\!\equiv \iff (x \in \{c_0, c_1\} \text{ or } x \leq_0 y \text{ or } x \leq_1 y).$$

Proposition 2.1. *The partially ordered sets $\mathbf{A}_1,\ldots,\mathbf{A}_4$ depicted in Fig. 1 are not conservative median semilattices.*

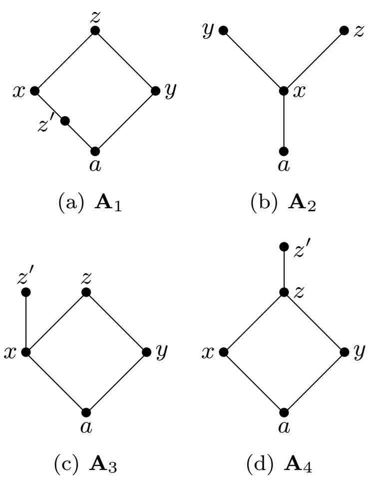


Figure 1. Examples of $\wedge$-semilattices that are not conservative.

Proof. The poset $\mathbf{A}_1$ is a bounded lattice (also denoted by N_5 in the literature on lattice theory, e.g., in [6, 8]) that is not distributive. In $\mathbf{A}_2$ the center is equal to the median of the other three elements. The poset $\mathbf{A}_3$ contains a copy of $\mathbf{A}_2$, and $\mathbf{A}_4$ is a distributive lattice that contains a copy of the dual of $\mathbf{A}_2$ and thus it is not conservative as a median algebra. ☐

The following Theorem provides descriptions of conservative semilattices with at least five elements, both in terms of forbidden substructures and in the form of representations by chains. Note that any semillatice with at most four elements is conservative, but the poset depicted in Fig. 1(b).

Theorem 2.2. *Let $\mathbf{A}$ be a median algebra with $|A| \geq 5$. The following conditions are equivalent.*

(1) $\mathbf{A}$ is conservative.
(2) For every $a \in A$ the ordered set $\langle A, \leq_a\rangle$ does not contain a copy of the poset depicted in Fig. 1(b).
(3) There is an $a \in A$ and lower bounded chains $\mathbf{C}_0$ and $\mathbf{C}_1$ such that $\langle A, \leq_a\rangle$ is isomorphic to $\mathbf{C}_0 \perp \mathbf{C}_1$.
(4) For every $a \in A$, there are lower bounded chains $\mathbf{C}_0$ and $\mathbf{C}_1$ such that $\langle A, \leq_a\rangle$ is isomorphic to $\mathbf{C}_0 \perp \mathbf{C}_1$.

Proof. (1) $\implies$ (2): Follows from Proposition 2.1.

(2) $\implies$ (1): Suppose that $\mathbf{A}$ is not conservative, that is, there are $a, b, c, d \in A$ such that $d := \mathbf{m}(a,b,c) \notin \{a,b,c\}$. Clearly, a, b and c must be pairwise distinct. By (1.2), a and b are $\leq_c$-incomparable, and $d <_c a$ and $d <_c b$. Moreover, $c <_c d$ and thus $\langle \{a,b,c,d\}, \leq_c\rangle$ is a copy of A_2 in $\langle A, \leq_c\rangle$.

(1) $\implies$ (4): Let $a \in A$. First, suppose that for every $x, y \in A \setminus \{a\}$ we have $\mathbf{m}(x,y,a) \neq a$. Since $\mathbf{A}$ is conservative,

for every $x, y \in A$, either $x \leq_a y$ or $y \leq_a x$. Thus $\leq_a$ is a chain with bottom element a, and we can choose $\mathbf{C}_1 = \langle A, \leq_a, a\rangle$ and $\mathbf{C}_2 = \langle \{a\}, \leq_a, a\rangle$.

Suppose now that there are $x, y \in A \setminus \{a\}$ such that $\mathbf{m}(x, y, a) = a$, that is, $x \wedge y = a$. We show that

$$z \neq a \implies \big(\mathbf{m}(x, z, a) \neq a \text{ or } \mathbf{m}(y, z, a) \neq a\big), \quad z \in A. \quad (2.1)$$

For the sake of a contradiction, suppose that $\mathbf{m}(x, z, a) = a$ and $\mathbf{m}(y, z, a) = a$ for some $z \neq a$. By equation (1.4), we have

$$\begin{aligned}\mathbf{m}(x, y, z) = \mathbf{m}\big(&\mathbf{m}(\mathbf{m}(x, y, z), x, a),\\ &\mathbf{m}\big(\mathbf{m}(x, y, z), z, a\big), \mathbf{m}\big(\mathbf{m}(x, y, z), y, a\big)\big). \quad (2.2)\end{aligned}$$

Assume that $\mathbf{m}(x, y, z) = x$. Then (2.2) is equivalent to

$$x = \mathbf{m}(x, \mathbf{m}(x, z, a), \mathbf{m}(x, y, a)) = a,$$

which yields the desired contradiction. By symmetry, we derive the same contradiction in the case $\mathbf{m}(x, y, z) \in \{y, z\}$.

We now prove that

$$z \neq a \implies \big(\mathbf{m}(x, z, a) = a \text{ or } \mathbf{m}(y, z, a) = a\big), \quad z \in A. \quad (2.3)$$

For the sake of a contradiction, suppose that $\mathbf{m}(x, z, a) \neq a$ and $\mathbf{m}(y, z, a) \neq a$ for some $z \neq a$. Since $\mathbf{m}(x, y, a) = a$ we have that $z \notin \{x, y\}$.

If $\mathbf{m}(x, z, a) = z$ and $\mathbf{m}(y, z, a) = y$, then $y \leq_a z \leq_a x$ which contradicts $x \wedge y = a$. Similarly, if $\mathbf{m}(x, z, a) = z$ and $\mathbf{m}(y, z, a) = z$, then $z \leq_a x$ and $z \leq_a y$ which also contradicts $x \wedge y = a$. The case $\mathbf{m}(x, z, a) = x$ and $\mathbf{m}(y, z, a) = z$ leads to similar contradictions.

Hence $\mathbf{m}(x, z, a) = x$ and $\mathbf{m}(y, z, a) = y$, and the $\leq_a$-median semilattice arising from the subalgebra $\mathbf{B} = \{a, x, y, z\}$ of $\mathbf{A}$ is the median semilattice associated with the four element Boolean algebra. Let $z' \in A \setminus \{a, x, y, z\}$. By (2.1) and symmetry we may assume that $\mathbf{m}(x, z', a) \in \{x, z'\}$. First, suppose that $\mathbf{m}(x, z', a) = z'$. Then $\langle \{a, x, y, z, z'\}, \leq_a\rangle$ is N_5 (Fig. 1(a)) which is not a median semilattice. Suppose then that $\mathbf{m}(x, z', a) = x$. In this case, the restriction of $\leq_a$ to $\{a, x, y, z, z'\}$ is depicted in Fig. 1(c) or 1(d), which contradicts Proposition 2.1, and the proof of (2.3) is thus complete.

Now, let $C_0 = \{z \in A \mid (x, z, a) \neq a\}$, $C_1 = \{z \in A \mid (y, z, a) \neq a\}$ and let $\mathbf{C}_0 = \langle C_0, \leq_a, a\rangle$ and $\mathbf{C}_1 = \langle C_1, \leq_a, a\rangle$. It follows from (2.1) and (2.3) that $\langle \mathbf{A}, \leq_a\rangle$ is isomorphic to $\mathbf{C}_0 \bot \mathbf{C}_1$.

(4) $\implies$ (3): Trivial.

(3) $\implies$ (1): Let $x, y, z \in \mathbf{C}_0 \bot \mathbf{C}_1$. If $x, y, z \in C_i$ for some $i \in \{0, 1\}$ then $\mathbf{m}(x, y, z) \in \{x, y, z\}$. Otherwise, if $x, y \in C_i$ and $z \notin C_i$, then $\mathbf{m}(x, y, z) \in \{x, y\}$. □

The equivalence between (3) and (1) gives rise to the following representation of conservative median algebras.

Theorem 2.3. *Let $\mathbf{A}$ be a median algebra with $|A| \geq 5$. Then $\mathbf{A}$ is conservative if and only if there is a totally ordered set $\mathbf{C}$ such that $\mathbf{A}$ is isomorphic to $\langle \mathbf{C}, \mathbf{m}_{\mathbf{C}}\rangle$.*

Proof. Sufficiency is trivial. For necessity, consider the universe of $\mathbf{C}_0 \bot \mathbf{C}_1$ in condition (3) endowed with $\leq$ defined by $x \leq y$ if $x \in C_1$ and $y \in C_0$ or $x, y \in C_0$ and $x \leq_0 y$ or $x, y \in C_1$ and $y \leq_1 x$. □

As stated in the next result, the totally ordered set $\mathbf{C}$ given in Theorem 2.3 is unique, up to (dual) isomorphism.

Theorem 2.4. *Let $\mathbf{A}$ be a median algebra. If $\mathbf{C}$ and $\mathbf{C}'$ are two chains such that $\mathbf{A} \cong \langle \mathbf{C}, \mathbf{m}_{\mathbf{C}}\rangle$ and $\mathbf{A} \cong \langle \mathbf{C}', \mathbf{m}_{\mathbf{C}'}\rangle$, then $\mathbf{C}$ is order isomorphic or dual order isomorphic to $\mathbf{C}'$.*

3 Homomorphisms between conservative median algebras

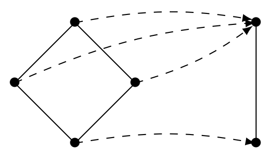
In view of Theorem 2.3 and Theorem 2.4, we introduce the following notation. Given a conservative median algebra $\mathbf{A}$ ($|A| \geq 5$), we denote a chain representation of $\mathbf{A}$ by $\mathbf{C}(\mathbf{A})$, that is, $\mathbf{C}(\mathbf{A})$ is a chain such that $\mathbf{A} \cong \langle \mathbf{C}(\mathbf{A}), \mathbf{m}_{\mathbf{C}(\mathbf{A})}\rangle$, and we denote the corresponding isomorphism by $f_{\mathbf{A}} : \mathbf{A} \to \langle \mathbf{C}(\mathbf{A}), \mathbf{m}_{\mathbf{C}(\mathbf{A})}\rangle$. If $f : \mathbf{A} \to \mathbf{B}$ is a map between two conservative median algebras with at least five elements, the map $f' : \mathbf{C}(\mathbf{A}) \to \mathbf{C}(\mathbf{B})$ defined as $f' = f_{\mathbf{B}} \circ f \circ f_{\mathbf{A}}^{-1}$ is said to be *induced by* f.

A function $f : \mathbf{A} \to \mathbf{B}$ between median algebras $\mathbf{A}$ and $\mathbf{B}$ is called a *median homomorphism* if it satisfies equation (1.6). We use the terminology introduced above to characterize median homomorphisms between conservative median algebras. Recall that a map between two posets is *monotone* if it is isotone or antitone.

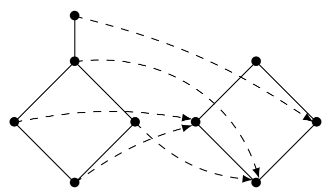
Theorem 3.1. *Let $\mathbf{A}$ and $\mathbf{B}$ be two conservative median algebras with at least five elements. A map $f : \mathbf{A} \to \mathbf{B}$ is a median homomorphism if and only if the induced map $f' : \mathbf{C}(\mathbf{A}) \to \mathbf{C}(\mathbf{B})$ is monotone.*

Corollary 3.2. *Let $\mathbf{C}$ and $\mathbf{C}'$ be two chains. A map $f : \mathbf{C} \to \mathbf{C}'$ is a median homomorphism if and only if it is monotone.*

Remark 3.3. Note that Corollary 3.2 only holds for chains. Indeed, Fig. 2(a) gives an example of a monotone map that is not a median homomorphism, and Fig. 2(b) gives an example of median homomorphism that is not monotone.



(a) A monotone map which is not a median homomorphism.



(b) A median homomorphism which is not monotone.

Figure 2. Examples for Remark 3.3.

Since the class of conservative median algebras is clearly closed under homomorphic images, we obtain the following corollary.

Corollary 3.4. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two median algebras and* $f : \mathbf{A} \to \mathbf{B}$*. If* $\mathbf{A}$ *is conservative, and if* $|A|, |f(A)| \geq 5$*, then* f *is a median homomorphism if and only if* $f(\mathbf{A})$ *is a conservative median subalgebra of* $\mathbf{B}$ *and the induced map* $f' : \mathbf{C}(\mathbf{A}) \to \mathbf{C}(f(\mathbf{A}))$ *is monotone.*

We are actually able to lift the previous result to finite products of chains. If $f_i : A_i \to A'_i$ ($i \in [n]$) is a family of maps, let $(f_1, \ldots, f_n) : A_1 \times \cdots \times A_n \to A'_1 \times \cdots \times A'_n$ be defined by

$$(f_1, \ldots, f_n)(x_1, \ldots, x_n) := (f_1(x_1), \ldots, f_n(x_n)).$$

If $A = A_1 \times \cdots \times A_n$ and $i \in [n]$, then we denote the projection map from A onto A_i by π_i^A, or simply by π_i if there is no danger of ambiguity.

The following theorem characterizes median homomorphisms between finite products of chains.

Theorem 3.5. *Let* $\mathbf{A} = \mathbf{C}_1 \times \cdots \times \mathbf{C}_k$ *and* $\mathbf{B} = \mathbf{D}_1 \times \cdots \times \mathbf{D}_n$ *be two finite products of chains. Then* $f : \mathbf{A} \to \mathbf{B}$ *is a median homomorphism if and only if there exist* $\sigma : [n] \to [k]$ *and monotone maps* $f_i : \mathbf{C}_{\sigma(i)} \to \mathbf{D}_i$ *for* $i \in [k]$ *such that* $f = (f_{\sigma(i)}, \ldots, f_{\sigma(k)})$.

As an immediate consequence, it follows that aggregation functions compatible with median functions on ordinal scales are dictorial.

Corollary 3.6. *Let* $\mathbf{C}_1, \ldots, \mathbf{C}_n$ *and* $\mathbf{D}$ *be chains. A map* $f : \mathbf{C}_1 \times \cdots \times \mathbf{C}_n \to \mathbf{D}$ *is a median homomorphism if and only if there is a* $j \in [n]$ *and a monotone map* $g : \mathbf{C}_j \to \mathbf{D}$ *such that* $f = g \circ \pi_j$.

In the particular case of Boolean algebras (*i.e.*, powers of a two element chain), Theorem 3.5 can be restated as follows.

Corollary 3.7. *Assume that* $f : \mathbf{A} \to \mathbf{B}$ *is a map between two finite Boolean algebras* $\mathbf{A} \cong \mathbf{2}^n$ *and* $\mathbf{B} \cong \mathbf{2}^m$.

1. *The map* f *is a median homomorphism if and only if there are* $\sigma : [m] \to ([n] \cup \{\perp\})$ *and* $\epsilon : [m] \to \{\mathrm{id}, \neg\}$ *such that*

$$f : (x_1, \ldots, x_n) \mapsto (\epsilon_1 x_{\sigma_1}, \ldots, \epsilon_m x_{\sigma_m}),$$

where $x_\perp$ *is defined as the constant map* 0.

In particular,

2. *A map* $f\colon \mathbf{A} \to \mathbf{2}$ *is a median homomorphism if and only if it is a constant function, a projection map or the negation of a projection map.*
3. *A map* $f : \mathbf{A} \to \mathbf{A}$ *is a median isomorphism if and only if there is a permutation* σ *of* $[n]$ *and an element* ϵ *of* $\{\mathrm{id}, \neg\}^n$ *such that* $f(x_1, \ldots, x_n) = (\epsilon_1 x_{\sigma(1)}, \ldots, \epsilon_n x_{\sigma(n)})$ *for any* $(x_1, \ldots, x_n)$ *in* A.

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