# A Short Note on the Bruinier-Kohnen Sign Equidistribution Conjecture and Halász' Theorem

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**Abstract**

In this note, we improve earlier results towards the Bruinier-Kohnen sign equidistribution conjecture for half-integral weight modular eigenforms in terms of natural density by using a consequence of Halász' Theorem. Moreover, applying a result of Serre we remove all unproved assumptions.

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By using the celebrated Sato-Tate theorem for integral weight modular eigenforms and the Shimura lift, in [4] and in [1] (together with Sara Arias-de-Reyna), we prove results related to the Bruinier-Kohnen sign equidistribution conjecture for modular eigenforms of half integral weight. In this note we improve one of our main results to a formulation in terms of natural density. Moreover, a theorem of Serre's allows us to remove all unproved assumptions.

The first improvement is due to the following application of Halász' Theorem that one of us learned from Kaisa Matomäki.

**Theorem 1.** *Let* $g : \mathbb{N} \to \{-1, 0, 1\}$ *be a multiplicative function. If* $\sum_{p, g(p)=0} \frac{1}{p}$ *converges and* $\sum_{p, g(p)=-1} \frac{1}{p}$ *diverges then*

$$\lim_{x\to\infty} \frac{|\{n \le x : g(n) \gtrless 0\}|}{|\{n \le x : g(n) \neq 0\}|} = \frac{1}{2}.$$

*Proof.* By Lemma 2.2 of [5], which is a consequence of Halász' Theorem (see [3] for details), there exists an absolute positive constant $C$ such that

$$\sum_{n \le x} g(n) \le C \cdot x \exp\left(-\frac{1}{4} \sum_{p \le x} \frac{1-g(p)}{p}\right)$$

for all $x \ge 2$. By assumption, we have $1 - g(p) \ge 0$ for all $p$ and $1 - g(p) = 2 > 1$ for any $p$ with $g(p) = -1$. We conclude that for $x \to \infty$, $\exp\left(-\frac{1}{4} \sum_{p \le x} \frac{1-g(p)}{p}\right)$ tends to 0. Hence for the average value of $g$, we have $\lim_{x\to\infty} \frac{\sum_{n \le x} g(n)}{x} = 0$ and therefore

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$$\sum_{n\le x} g(n) =\mid \{n\le x \mid g(n)>0\}\mid - \mid \{n \le x \mid g(n)<0\} \mid = o(x).$$

Since $\sum_{p,g(p)=0} \frac{1}{p}$ converges by assumption, we conclude again by Lemma 2.2 of [5] that for $x \to \infty$,

$$\frac{\sum_{n\le x}|g(n)|}{x} = \frac{|\{n\le x|g(n)>0\}|+|\{n\le x|g(n)<0\}|}{x}$$

tends to a positive limit, hence the assertion follows immediately. □

In order to state and prove the results towards the Bruinier-Kohnen conjecture, we introduce some notation to be used throughout the note. Let $k \geq 2$ and $4|N$ be integers and $\chi$ be a quadratic Dirichlet character modulo $N$. We denote the space of cusp forms of weight $k+1/2$ for the group $\Gamma_1(N)$ with character $\chi$ by $S_{k+1/2}(N,\chi)$ in the sense of Shimura, as in the main theorem in [8] on p. 458. For Hecke operators $T_{p^2}$ for primes $p \nmid N$, let $f = \sum_{n\geq 1} a(n)q^n \in S_{k+1/2}(N,\chi)$ be a non-zero cuspidal Hecke eigenform with real coefficients. For a fixed squarefree $t$ such that $a(t) \neq 0$, denote by $F_t$ the Shimura lift of $f$ with respect to $t$. It is a cuspidal Hecke eigenform of weight $2k$ for the group $\Gamma_0(N/2)$ with trivial character. By normalising $f$ we can and do assume $a(t) = 1$, in which case $F_t$ is normalised.

As in our previous treatments, the following theorem, of which we only state a weak version, is in the core of our approach. Its proof is based on the Sato-Tate theorem, see [2].

**Theorem 2.** *[4],[1] Assume the set-up above and define the set of primes*

$$\mathbb{P}_{>0} := \{p : a(tp^2) > 0\}$$

*and similarly $\mathbb{P}_{<0}$ and $\mathbb{P}_{=0}$ (depending on $f$ and $t$). Then the sets $\mathbb{P}_{>0}$ and $\mathbb{P}_{<0}$ have positive natural densities and the set $\mathbb{P}_{=0}$ has natural density 0.*

Due to its importance in the sequel, here we recall the following notion (Definition 2.2.1 of [1]).

**Definition 3.** *Let $S$ be a set of primes. It is called* weakly regular *if there is $a \in \mathbb{R}$ (called the* Dirichlet density *of $S$) and a function $g(z)$ which is holomorphic on $\{\mathrm{Re}(z) > 1\}$ and continuous (in particular, finite) on $\{\mathrm{Re}(z) \geq 1\}$ such that*

$$\sum_{p\in S} \frac{1}{p^z} = a \log\left(\frac{1}{z-1}\right) + g(z).$$

The second improvement of this paper is the observation that a result of Serre's allows us to prove directly that the set $\mathbb{P}_{=0}$ is always weakly regular. This approach avoids the use of Sato-Tate equidistribution and consequently does not depend on any unproved error terms for it. It only applies to $\mathbb{P}_{=0}$ and hence does not seem to give us the weak regularity of the other sets $\mathbb{P}_{>0}$ and $\mathbb{P}_{<0}$.

**Proposition 4.** *Assume the setup above. Then the set $\mathbb{P}_{=0}$ is weakly regular of density zero.*

*Proof.* Let $F = F_t = \sum_{n=1}^{\infty} A(n)q^n$ be the Shimura lift of $f$ with respect to $t$. If $F$ has CM, then the result has been proved in Theorem 4.1.1(c) of [1]. So let us assume that $F$ has no CM. Due to the assumption $a(t) = 1$ we have the formula

$$A(p) = a(tp^2) + \epsilon(p)p^{k-1}$$

for all primes $p$, where $\epsilon$ is an at most quadratic (due to the assumption that all coefficients are real) Dirichlet character of modulus $2tN^2$ (see e.g. equation (4.1) of [1]). Consequently, we have the inclusion

$$\begin{aligned}\{p < x : p \nmid 2tN, p \in \mathbb{P}_{=0}\} &= \{p < x : p \nmid 2tN, a(tp^2) = 0\}\\ &\subseteq \{p < x : p \nmid 2tN, A(p) = p^{k-1}\} \cup \{p < x : p \nmid 2tN, A(p) = -p^{k-1}\}.\end{aligned}$$

By Corollaire 1 of Théorème 15 in [7] (with $h(T) = T^{k-1}$ and $h(T) = -T^{k-1}$), it follows that

$$\#\{p < x : p \in \mathbb{P}_{=0}\} = o(\frac{x}{\log(x)^{9/8}}).$$

Consequently, Corollary 2.2.4 of [1] implies that $\mathbb{P}_{=0}$ is weakly regular of density zero. $\square$

We now use the application of Halász' theorem and the weak regularity of $\mathbb{P}_{=0}$ to prove the equidistribution result we are after in terms of natural density. In [1] we needed regularity to achieve this goal.

**Theorem 5.** *Assume the setup above. Then the sets* $\{n \in \mathbb{N} | a(tn^2) > 0\}$ *and* $\{n \in \mathbb{N} | a(tn^2) < 0\}$ *have equal positive natural density, that is, both are precisely half of the natural density of the set* $\{n \in \mathbb{N} | a(tn^2) \neq 0\}$.

*Proof.* Let $g(n) = \begin{cases} 1 & \text{if } a(tn^2) > 0, \\ 0 & \text{if } a(tn^2) = 0, \\ -1 & \text{if } a(tn^2) < 0. \end{cases}$ Due to the relations $a(tn^2m^2)a(t) = a(tn^2)a(tm^2)$ for $gcd(n, m) = 1$ (see p. 453 of [8]), it is clear that $g(n)$ is multiplicative. Since $\mathbb{P}_{=0}$ is weakly regular of density zero by Proposition 4, it follows that $\sum_{p \in \mathbb{P}_{=0}} \frac{1}{p}$ is finite. Moreover, the fact that $\mathbb{P}_{<0}$ is of positive density implies that $\sum_{p \in \mathbb{P}_{<0}} \frac{1}{p}$ diverges. Thus the result follows from Theorem 1. $\square$

In [1] we obtained the same conclusion under the additional assumption of the Generalised Riemann Hypothesis (GRH) because we needed to achieve the regularity of $\mathbb{P}_{=0}$, which we could derive from the very strong error term in Sato-Tate proved in [6], under the assumption of GRH.

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