SOME INSIGHTS ON BICATEGORIES OF FRACTIONS - III
-
EQUIVALENCES OF BICATEGORIES OF FRACTIONS

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Abstract. We fix any bicategory $\mathscr{A}$ together with a class of morphisms $\mathbf{W}_{\mathscr{A}}$, such that there is a bicategory of fractions $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$ (as described by D. Pronk). Given another such pair $(\mathscr{B}, \mathbf{W}_{\mathscr{B}})$ and any pseudofunctor $\mathcal{F}: \mathscr{A} \rightarrow \mathscr{B}$, we find necessary and sufficient conditions in order to have an induced equivalence of bicategories from $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$ to $\mathscr{B}\left[\mathbf{W}_{\mathscr{B}}^{-1}\right]$. In particular, this gives necessary and sufficient conditions in order to have an equivalence from any bicategory of fractions $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$ to any given bicategory $\mathscr{B}$.

Contents

Introduction

In 1996 Dorette Pronk introduced the notion of (*right*) *bicalculus of fractions* (see [Pr]), generalizing the concept of (right) calculus of fractions (described in 1967 by Pierre Gabriel and Michel Zisman, see [GZ]) from the framework of categories to that of bicategories. Pronk proved that given a bicategory $\mathscr{C}$ together with a class of morphisms $\mathbf{W}$ (satisfying a set of technical conditions called (BF)), there is a bicategory $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ (called (*right*) *bicategory of fractions*) and a pseudofunctor $\mathcal{U}_{\mathbf{W}}: \mathscr{C} \rightarrow \mathscr{C}\left[\mathbf{W}^{-1}\right]$. Such a pseudofunctor sends each element of $\mathbf{W}$ to an internal equivalence and is universal with respect to such property (see [Pr, Theorem 21]). In particular, the bicategory $\mathscr{C}\left[\mathbf{W}^{-1}\right]$ is unique up to equivalences of bicategories.

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In [T2, Definition 2.1] we introduced the notion of right saturated as follows: given any pair $(\mathscr{C},\mathbf{W})$ as above, we define the (right) saturation $\mathbf{W}_{\mathrm{sat}}$ of $\mathbf{W}$ as the class of all morphisms $f: B\rightarrow A$ in $\mathscr{C}$, such that there are a pair of objects C,D and a pair of morphisms $g: C\rightarrow B$, $h: D\rightarrow C$, such that both $f\circ g$ and $g\circ h$ belong to $\mathbf{W}$. Then we were able to prove that also $(\mathscr{C},\mathbf{W}_{\mathrm{sat}})$ admits a right bicategory of fractions $\mathscr{C}\left[\mathbf{W}_{\mathrm{sat}}^{-1}\right]$, that is equivalent to $\mathscr{C}\left[\mathbf{W}^{-1}\right]$. This allowed us to prove the following result (for the notations used below for bicategories of fractions, we refer directly to [Pr, § 2.2 and 2.3]).

Theorem 0.1. [T2, Theorems 0.2 and 0.3(A) and Corollary 0.4] *Let us fix any* 2 *pairs* $(\mathscr{A},\mathbf{W}_{\mathscr{A}})$ *and* $(\mathscr{B},\mathbf{W}_{\mathscr{B}})$, *both satisfying conditions* (BF), *and any pseudofunctor* $\mathcal{F}:\mathscr{A}\rightarrow\mathscr{B}$. *Then the following facts are equivalent:*

(*i*) $\mathcal{F}_1(\mathbf{W}_{\mathscr{A}})\subseteq\mathbf{W}_{\mathscr{B},\mathrm{sat}}$;

(*ii*) $\mathcal{F}_1(\mathbf{W}_{\mathscr{A},\mathrm{sat}})\subseteq\mathbf{W}_{\mathscr{B},\mathrm{sat}}$;

(*iii*) *there are a pseudofunctor* $\mathcal{G}$ *and a pseudonatural equivalence* κ *as below*

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "\mathcal{F}"] \arrow[d, "\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}"'] & & \mathscr{B} \arrow[d, "\mathcal{U}_{\mathbf{W}_{\mathscr{B}}}"] \\
\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right] \arrow[rr, dashed, "\mathcal{G}"'] \arrow[urr, phantom, "\swArrow\ \kappa"] & & \mathscr{B}\left[\mathbf{W}_{\mathscr{B}}^{-1}\right];
\end{tikzcd}$$

(*iv*) *there is a pair* $(\mathcal{G},\kappa)$ *as in* (*iii*), *such that the pseudofunctor* $\mu_{\kappa}:\mathscr{A}\rightarrow\mathrm{Cyl}\left(\mathscr{B}\left[\mathbf{W}_{\mathscr{B}}^{-1}\right]\right)$ *associated to* κ *sends each element of* $\mathbf{W}_{\mathscr{A}}$ *to an internal equivalence* (*here* $\mathrm{Cyl}(\mathscr{C})$ *is the bicategory of cylinders associated to any given bicategory* $\mathscr{C}$, *see* [B, pag. 60]).

Moreover, if any of the conditions above is satisfied, then there are a pseudofunctor

$$\widetilde{\mathcal{G}}:\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]\longrightarrow\mathscr{B}\left[\mathbf{W}_{\mathscr{B},\mathrm{sat}}^{-1}\right]$$

and a pseudonatural equivalence $\widetilde{\kappa}:\mathcal{U}_{\mathbf{W}_{\mathscr{B},\mathrm{sat}}}\circ\mathcal{F}\Rightarrow\widetilde{\mathcal{G}}\circ\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}$, *such that:*

- *the pseudofunctor* $\mu_{\widetilde{\kappa}}:\mathscr{A}\rightarrow\mathrm{Cyl}\left(\mathscr{B}\left[\mathbf{W}_{\mathscr{B},\mathrm{sat}}^{-1}\right]\right)$ *associated to* $\widetilde{\kappa}$ *sends each morphism of* $\mathbf{W}_{\mathscr{A}}$ *to an internal equivalence;*
- *for each object* $A_{\mathscr{A}}$, *we have* $\widetilde{\mathcal{G}}_0(A_{\mathscr{A}})=\mathcal{F}_0(A_{\mathscr{A}})$;
- *for each morphism* $(A'_{\mathscr{A}},\mathrm{w}_{\mathscr{A}},f_{\mathscr{A}}):A_{\mathscr{A}}\rightarrow B_{\mathscr{A}}$ *in* $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$, *we have*

$$\widetilde{\mathcal{G}}_1\Big(A'_{\mathscr{A}},\mathrm{w}_{\mathscr{A}},f_{\mathscr{A}}\Big)=\Big(\mathcal{F}_0(A'_{\mathscr{A}}),\mathcal{F}_1(\mathrm{w}_{\mathscr{A}}),\mathcal{F}_1(f_{\mathscr{A}})\Big);$$

- *for each* 2*-morphism*

$$\Big[A^3_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}},\alpha_{\mathscr{A}},\beta_{\mathscr{A}}\Big]:\Big(A^1_{\mathscr{A}},\mathrm{w}^1_{\mathscr{A}},f^1_{\mathscr{A}}\Big)\Longrightarrow\Big(A^2_{\mathscr{A}},\mathrm{w}^2_{\mathscr{A}},f^2_{\mathscr{A}}\Big)$$

in $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$, *we have*

$$\widetilde{\mathcal{G}}_2\Big(\Big[A^3_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}},\alpha_{\mathscr{A}},\beta_{\mathscr{A}}\Big]\Big)=\Big[\mathcal{F}_0(A^3_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}}), \tag{0.1}$$
$$\psi^{\mathcal{F}}_{\mathrm{w}^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}}}\odot\mathcal{F}_2(\alpha_{\mathscr{A}})\odot\Big(\psi^{\mathcal{F}}_{\mathrm{w}^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}}}\Big)^{-1},\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}}}\odot\mathcal{F}_2(\beta_{\mathscr{A}})\odot\Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}}}\Big)^{-1}\Big]$$

(*where the* 2*-morphisms* $\psi^{\mathcal{F}}_{\bullet}$ *are the associators of* $\mathcal{F}$).

In addition, given any pair $(\mathcal{G},\kappa)$ *as in* (*iv*), *the following facts are equivalent:*

(*a*) $\mathcal{G}:\mathscr{A}[\mathbf{W}_{\mathscr{A}}^{-1}]\rightarrow\mathscr{B}[\mathbf{W}_{\mathscr{B}}^{-1}]$ *is an equivalence of bicategories;*

(*b*) $\widetilde{\mathcal{G}}:\mathscr{A}[\mathbf{W}_{\mathscr{A}}^{-1}]\rightarrow\mathscr{B}[\mathbf{W}_{\mathscr{B},\mathrm{sat}}^{-1}]$ *is an equivalence of bicategories;*

Using the technical lemmas already proved in [T1] and [T2], in the present paper we will identify necessary and sufficient conditions such that (b) holds. Combining this with the previous theorem, we will then prove the following result.

Theorem 0.2. *Let us fix any 2 pairs $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ and $(\mathscr{B}, \mathbf{W}_{\mathscr{B}})$, both satisfying conditions* (BF)*, and any pseudofunctor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ such that $\mathcal{F}_1(\mathbf{W}_{\mathscr{A}}) \subseteq \mathbf{W}_{\mathscr{B},\mathrm{sat}}$. Moreover, let us consider any pair $(\mathcal{G}, \kappa)$ as in* Theorem 0.1(iv)*. Then $\mathcal{G}$ is an equivalence of bicategories if and only if $\mathcal{F}$ satisfies the following 5 conditions.*

(*A1*) *For any object $A_{\mathscr{B}}$, there are a pair of objects $A_{\mathscr{A}}$ and $A'_{\mathscr{B}}$ and a pair of morphisms $\mathrm{w}^1_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B}}$ and $\mathrm{w}^2_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$, as follows:*

$$\mathcal{F}_0(A_{\mathscr{A}}) \xleftarrow{\mathrm{w}^1_{\mathscr{B}}} A'_{\mathscr{B}} \xrightarrow{\mathrm{w}^2_{\mathscr{B}}} A_{\mathscr{B}}. \tag{0.2}$$

(*A2*) *Let us fix any triple of objects $A^1_{\mathscr{A}}, A^2_{\mathscr{A}}, A_{\mathscr{B}}$ and any pair of morphisms $\mathrm{w}^1_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B}}$ and $\mathrm{w}^2_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ as follows*

$$\mathcal{F}_0(A^1_{\mathscr{A}}) \xleftarrow{\mathrm{w}^1_{\mathscr{B}}} A_{\mathscr{B}} \xrightarrow{\mathrm{w}^2_{\mathscr{B}}} \mathcal{F}_0(A^2_{\mathscr{A}}). \tag{0.3}$$

Then there are an object $A^3_{\mathscr{A}}$, a pair of morphisms $\mathrm{w}^1_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\mathrm{w}^2_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A},\mathrm{sat}}$ as follows

$$A^1_{\mathscr{A}} \xleftarrow{\mathrm{w}^1_{\mathscr{A}}} A^3_{\mathscr{A}} \xrightarrow{\mathrm{w}^2_{\mathscr{A}}} A^2_{\mathscr{A}}$$

and a set of data $(A'_{\mathscr{B}}, \mathrm{z}^1_{\mathscr{B}}, \mathrm{z}^2_{\mathscr{B}}, \gamma^1_{\mathscr{B}}, \gamma^2_{\mathscr{B}})$ as follows

$$\begin{tikzcd}
 & & A_{\mathscr{B}} \arrow[dll, "\mathrm{w}^1_{\mathscr{B}}"'] \arrow[drr, "\mathrm{w}^2_{\mathscr{B}}"] & & \\
\mathcal{F}_0(A^1_{\mathscr{A}}) & \Downarrow \gamma^1_{\mathscr{B}} & A'_{\mathscr{B}} \arrow[u, "\mathrm{z}^1_{\mathscr{B}}"'] \arrow[d, "\mathrm{z}^2_{\mathscr{B}}"] & \Downarrow \gamma^2_{\mathscr{B}} & \mathcal{F}_0(A^2_{\mathscr{A}}), \\
 & & \mathcal{F}_0(A^3_{\mathscr{A}}) \arrow[ull, "\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})"] \arrow[urr, "\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})"'] & &
\end{tikzcd}$$

such that $\mathrm{z}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}}$ and both $\gamma^1_{\mathscr{B}}$ and $\gamma^2_{\mathscr{B}}$ are invertible.

(*A3*) *Let us fix any pair of objects $B_{\mathscr{A}}, A_{\mathscr{B}}$ and any morphism $f_{\mathscr{B}} : A_{\mathscr{B}} \to \mathcal{F}_0(B_{\mathscr{A}})$. Then there are an object $A_{\mathscr{A}}$, a morphism $f_{\mathscr{A}} : A_{\mathscr{A}} \to B_{\mathscr{A}}$ and data $(A'_{\mathscr{B}}, \mathrm{v}^1_{\mathscr{B}}, \mathrm{v}^2_{\mathscr{B}}, \alpha_{\mathscr{B}})$ as follows*

$$\begin{tikzcd}
 & A_{\mathscr{B}} \arrow[dr, bend left, "f_{\mathscr{B}}"] & \\
A'_{\mathscr{B}} \arrow[ur, bend left, "\mathrm{v}^1_{\mathscr{B}}"] \arrow[dr, bend right, "\mathrm{v}^2_{\mathscr{B}}"'] & \Downarrow \alpha_{\mathscr{B}} & \mathcal{F}_0(B_{\mathscr{A}}), \\
 & \mathcal{F}_0(A_{\mathscr{A}}) \arrow[ur, bend right, "\mathcal{F}_1(f_{\mathscr{A}})"'] &
\end{tikzcd}$$

with $\mathrm{v}^1_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B}}$, $\mathrm{v}^2_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\alpha_{\mathscr{B}}$ invertible.

(*A4*) *Let us fix any pair of objects $A_{\mathscr{A}}, B_{\mathscr{A}}$, any pair of morphisms $f^1_{\mathscr{A}}, f^2_{\mathscr{A}} : A_{\mathscr{A}} \to B_{\mathscr{A}}$ and any pair of 2-morphisms $\gamma^1_{\mathscr{A}}, \gamma^2_{\mathscr{A}} : f^1_{\mathscr{A}} \Rightarrow f^2_{\mathscr{A}}$. Moreover, let us fix an object $A'_{\mathscr{B}}$ and a morphism $\mathrm{z}_{\mathscr{B}} : A'_{\mathscr{B}} \to \mathcal{F}_0(A_{\mathscr{A}})$ in $\mathbf{W}_{\mathscr{B}}$. If $\mathcal{F}_2(\gamma^1_{\mathscr{A}}) * i_{\mathrm{z}_{\mathscr{B}}} = \mathcal{F}_2(\gamma^2_{\mathscr{A}}) * i_{\mathrm{z}_{\mathscr{B}}}$, then there are an object $A'_{\mathscr{A}}$ and a morphism $\mathrm{z}_{\mathscr{A}} : A'_{\mathscr{A}} \to A_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, such that $\gamma^1_{\mathscr{A}} * i_{\mathrm{z}_{\mathscr{A}}} = \gamma^2_{\mathscr{A}} * i_{\mathrm{z}_{\mathscr{A}}}$.*

(*A5*) *Let us fix any triple of objects $A_{\mathscr{A}}, B_{\mathscr{A}}, A_{\mathscr{B}}$, any pair of morphisms $f^1_{\mathscr{A}}, f^2_{\mathscr{A}}: A_{\mathscr{A}} \to B_{\mathscr{A}}$, any morphism $\mathrm{v}_{\mathscr{B}}: A_{\mathscr{B}} \to \mathcal{F}_0(A_{\mathscr{A}})$ in $\mathbf{W}_{\mathscr{B}}$ and any 2-morphism*

$$
\begin{tikzcd}
 & \mathcal{F}_0(A_{\mathscr{A}}) \arrow[dr, bend left=10, "\mathcal{F}_1(f^1_{\mathscr{A}})"] \arrow[d, phantom, "\Downarrow\,\alpha_{\mathscr{B}}"] & \\
A_{\mathscr{B}} \arrow[ur, bend left=10, "\mathrm{v}_{\mathscr{B}}"] \arrow[dr, bend right=10, "\mathrm{v}_{\mathscr{B}}"'] & {} & \mathcal{F}_0(B_{\mathscr{A}}). \\
 & \mathcal{F}_0(A_{\mathscr{A}}) \arrow[ur, bend right=10, "\mathcal{F}_1(f^2_{\mathscr{A}})"'] &
\end{tikzcd}
$$

Then there are a pair of objects $A'_{\mathscr{A}}, A'_{\mathscr{B}}$, a triple of morphisms $\mathrm{v}_{\mathscr{A}}: A'_{\mathscr{A}} \to A_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, $\mathrm{z}_{\mathscr{B}}: A'_{\mathscr{B}} \to \mathcal{F}_0(A'_{\mathscr{A}})$ in $\mathbf{W}_{\mathscr{B}}$ and $\mathrm{z}'_{\mathscr{B}}: A'_{\mathscr{B}} \to A_{\mathscr{B}}$, a 2-morphism

$$
\begin{tikzcd}
 & A_{\mathscr{A}} \arrow[dr, bend left=10, "f^1_{\mathscr{A}}"] \arrow[d, phantom, "\Downarrow\,\alpha_{\mathscr{A}}"] & \\
A'_{\mathscr{A}} \arrow[ur, bend left=10, "\mathrm{v}_{\mathscr{A}}"] \arrow[dr, bend right=10, "\mathrm{v}_{\mathscr{A}}"'] & {} & B_{\mathscr{A}} \\
 & A_{\mathscr{A}} \arrow[ur, bend right=10, "f^2_{\mathscr{A}}"'] &
\end{tikzcd}
$$

and an invertible 2-morphism

$$
\begin{tikzcd}
 & \mathcal{F}_0(A'_{\mathscr{A}}) \arrow[dr, bend left=10, "\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})"] \arrow[d, phantom, "\Downarrow\,\sigma_{\mathscr{B}}"] & \\
A'_{\mathscr{B}} \arrow[ur, bend left=10, "\mathrm{z}_{\mathscr{B}}"] \arrow[dr, bend right=10, "\mathrm{z}'_{\mathscr{B}}"'] & {} & \mathcal{F}_0(A_{\mathscr{A}}), \\
 & A_{\mathscr{B}} \arrow[ur, bend right=10, "\mathrm{v}_{\mathscr{B}}"'] &
\end{tikzcd}
$$

such that $\alpha_{\mathscr{B}} * i_{\mathrm{z}'_{\mathscr{B}}}$ coincides with the following composition:

$$
\begin{tikzcd}[column sep=large]
 & & A_{\mathscr{B}} \arrow[ddrr, "\mathcal{F}_1(f^1_{\mathscr{A}})\circ \mathrm{v}_{\mathscr{B}}"] & \Downarrow\,\theta^{-1}_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathrm{v}_{\mathscr{B}},\mathrm{z}'_{\mathscr{B}}} & \\
 & \mathcal{F}_0(A_{\mathscr{A}}) \arrow[drrr, "\mathcal{F}_1(f^1_{\mathscr{A}})"] & \Downarrow\,\theta_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{z}_{\mathscr{B}}} & & \\
A'_{\mathscr{B}} \arrow[uurr, bend left, "\mathrm{z}'_{\mathscr{B}}"] \arrow[ur, bend left, "\mathrm{v}_{\mathscr{B}}\circ \mathrm{z}'_{\mathscr{B}}"] \arrow[ur, "\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})\circ \mathrm{z}_{\mathscr{B}}"'] \arrow[r, "\mathrm{z}_{\mathscr{B}}"] \arrow[dr, "\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})\circ \mathrm{z}_{\mathscr{B}}"] \arrow[dr, bend right, "\mathrm{v}_{\mathscr{B}}\circ \mathrm{z}'_{\mathscr{B}}"'] \arrow[ddrr, bend right, "\mathrm{z}'_{\mathscr{B}}"'] \arrow[u, phantom, "\Downarrow\,\sigma^{-1}_{\mathscr{B}}"] \arrow[d, phantom, "\Downarrow\,\sigma_{\mathscr{B}}"] & \mathcal{F}_0(A'_{\mathscr{A}}) \arrow[rrr, bend left, "\mathcal{F}_1(f^1_{\mathscr{A}})\circ\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})"] \arrow[rrr, bend right, "\mathcal{F}_1(f^2_{\mathscr{A}})\circ\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})"'] \arrow[rrr, phantom, "\Downarrow\,\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\odot\mathcal{F}_2(\alpha_{\mathscr{A}})\odot\left(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\right)^{-1}"] & & & \mathcal{F}_0(B_{\mathscr{A}}) \\
 & \mathcal{F}_0(A_{\mathscr{A}}) \arrow[urrr, "\mathcal{F}_1(f^2_{\mathscr{A}})"'] & \Downarrow\,\theta^{-1}_{\mathcal{F}_1(f^2_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{z}_{\mathscr{B}}} & & \\
 & & A_{\mathscr{B}} \arrow[uurr, "\mathcal{F}_1(f^2_{\mathscr{A}})\circ \mathrm{v}_{\mathscr{B}}"'] & \Downarrow\,\theta_{\mathcal{F}_1(f^2_{\mathscr{A}}),\mathrm{v}_{\mathscr{B}},\mathrm{z}'_{\mathscr{B}}} &
\end{tikzcd}
\tag{0.4}
$$

(*here the 2-morphisms of the form $\theta_\bullet$ are the associators of $\mathscr{B}$*).

In this way we have established an useful tool for checking if any 2 bicategories of fractions $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$ and $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B}}\right]$ are equivalent: it suffices to define a pseudofunctor $\mathcal{F}: \mathscr{A} \to \mathscr{B}$, such that $\mathcal{F}_1(\mathbf{W}_{\mathscr{A}}) \subseteq \mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and such that conditions (A1) – (A5) hold.

We are going to apply explicitly Theorem 0.2 in our next paper [T3], where $\mathscr{B}$ will be the 2-category of proper, effective, differentiable étale groupoids and $\mathbf{W}_{\mathscr{B}}$ will be the class of all Morita equivalences between such objects. The role of the bicategory

$\mathscr{A}$ will be played by a 2-category ($\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}$) whose objects will be reduced orbifold atlases; the class $\mathbf{W}_{\mathscr{A}}$ will be the class of all morphisms that are "refinements" of reduced orbifold atlases (see [T3, Definition 6.1]).

In the second part of this paper we are going to consider the following result about bicategories of fractions (a direct consequence of [Pr, Theorem 21]):

Theorem 0.3. [Pr] *Let us fix any pair $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ satisfying conditions* (BF)*, any bicategory $\mathscr{B}$ and any pseudofunctor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$. Then the following facts are equivalent:*

(1) $\mathcal{F}$ sends each morphism of $\mathbf{W}_{\mathscr{A}}$ to an internal equivalence of $\mathscr{B}$;
(2) there is a pseudofunctor $\overline{\mathcal{G}} : \mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right] \to \mathscr{B}$ and a pseudonatural equivalence of pseudofunctors $\overline{\kappa} : \mathcal{F} \Rightarrow \overline{\mathcal{G}} \circ \mathcal{U}_{\mathbf{W}_{\mathscr{A}}}$, whose associated pseudofunctor $\mathscr{A} \to$ $\mathrm{Cyl}(\mathscr{B})$ *sends each morphism of $\mathbf{W}_{\mathscr{A}}$ to an internal equivalence.*

As a consequence of Theorem 0.2, we are able to improve Theorem 0.3 as follows.

Theorem 0.4. *Let us fix any pair $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ satisfying conditions* (BF)*, any bicategory $\mathscr{B}$ and any pseudofunctor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, such that $\mathcal{F}$ sends each morphism of $\mathbf{W}_{\mathscr{A}}$ to an internal equivalence of $\mathscr{B}$. Let us also fix any pair $(\overline{\mathcal{G}}, \overline{\kappa})$ as in* Theorem 0.3(2)*. Then $\overline{\mathcal{G}} : \mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right] \to \mathscr{B}$ is an equivalence of bicategories if and only if $\mathcal{F}$ satisfies the the following* 5 *conditions.*

(B1) Given any object $A_{\mathscr{B}}$, there are an object $A_{\mathscr{A}}$ and an internal equivalence $e_{\mathscr{B}} : \mathcal{F}_0(A_{\mathscr{A}}) \to A_{\mathscr{B}}$ (i.e. $\mathcal{F}_0$ is surjective up to internal equivalences).
(B2) Let us fix any pair of objects $A^1_{\mathscr{A}}, A^2_{\mathscr{A}}$ and any internal equivalence $e_{\mathscr{B}} : \mathcal{F}_0(A^1_{\mathscr{A}}) \to \mathcal{F}_0(A^2_{\mathscr{A}})$. Then there are an object $A^3_{\mathscr{A}}$, a pair of morphisms $\mathrm{w}^1_{\mathscr{A}} : A^3_{\mathscr{A}} \to A^1_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\mathrm{w}^2_{\mathscr{A}} : A^3_{\mathscr{A}} \to A^2_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A},\mathrm{sat}}$, an internal equivalence $e'_{\mathscr{B}} : \mathcal{F}_0(A^1_{\mathscr{A}}) \to \mathcal{F}_0(A^3_{\mathscr{A}})$ and a pair of invertible 2-morphisms as follows:

$$\begin{tikzcd}
 & & \mathcal{F}_0(A^2_{\mathscr{A}}) \\
\mathcal{F}_0(A^1_{\mathscr{A}}) \arrow[r, "e'_{\mathscr{B}}"] \arrow[urr, bend left, "e_{\mathscr{B}}"] \arrow[drr, bend right, "\mathrm{id}_{\mathcal{F}_0(A^1_{\mathscr{A}})}"'] & \mathcal{F}_0(A^3_{\mathscr{A}}) \arrow[ur, "\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})"] \arrow[dr, "\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})"'] \arrow[u, phantom, "\Downarrow \delta^2_{\mathscr{B}}"] \arrow[d, phantom, "\Downarrow \delta^1_{\mathscr{B}}"] & \\
 & & \mathcal{F}_0(A^1_{\mathscr{A}}).
\end{tikzcd}$$

(B3) Let us fix any pair of objects $B_{\mathscr{A}}$, $A_{\mathscr{B}}$ and any morphism $f_{\mathscr{B}} : A_{\mathscr{B}} \to \mathcal{F}_0(B_{\mathscr{A}})$. Then there are an object $A_{\mathscr{A}}$, a morphism $f_{\mathscr{A}} : A_{\mathscr{A}} \to B_{\mathscr{A}}$, an internal equivalence $e_{\mathscr{B}} : A_{\mathscr{B}} \to \mathcal{F}_0(A_{\mathscr{A}})$ and an invertible 2-morphism $\alpha_{\mathscr{B}} : f_{\mathscr{B}} \Rightarrow \mathcal{F}_1(f_{\mathscr{A}}) \circ e_{\mathscr{B}}$.
(B4) Let us fix any pair of objects $A_{\mathscr{A}}, B_{\mathscr{A}}$, any pair of morphisms $f^1_{\mathscr{A}}, f^2_{\mathscr{A}} : A_{\mathscr{A}} \to B_{\mathscr{A}}$ and any pair of 2-morphisms $\gamma^1_{\mathscr{A}}, \gamma^2_{\mathscr{A}} : f^1_{\mathscr{A}} \Rightarrow f^2_{\mathscr{A}}$, such that that $\mathcal{F}_2(\gamma^1_{\mathscr{A}}) = \mathcal{F}_2(\gamma^2_{\mathscr{A}})$. Then there are an object $A'_{\mathscr{A}}$ and a morphism $\mathrm{z}_{\mathscr{A}} : A'_{\mathscr{A}} \to A_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, such that $\gamma^1_{\mathscr{A}} * i_{\mathrm{z}_{\mathscr{A}}} = \gamma^2_{\mathscr{A}} * i_{\mathrm{z}_{\mathscr{A}}}$.
(B5) Let us fix any pair of objects $A_{\mathscr{A}}, B_{\mathscr{A}}$, any pair of morphisms $f^1_{\mathscr{A}}, f^2_{\mathscr{A}} : A_{\mathscr{A}} \to B_{\mathscr{A}}$ and any 2-morphism $\alpha_{\mathscr{B}} : \mathcal{F}_1(f^1_{\mathscr{A}}) \Rightarrow \mathcal{F}_1(f^2_{\mathscr{A}})$. Then there are an object $A'_{\mathscr{A}}$, a morphism $\mathrm{v}_{\mathscr{A}} : A'_{\mathscr{A}} \to A_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and a 2-morphism $\alpha_{\mathscr{A}} : f^1_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}} \Rightarrow f^2_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}}$, such that

$$\alpha_{\mathscr{B}} * i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})} = \psi^{\mathcal{F}}_{f^2_{\mathscr{A}}, \mathrm{v}_{\mathscr{A}}} \odot \mathcal{F}_2(\alpha_{\mathscr{A}}) \odot \left(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}}, \mathrm{v}_{\mathscr{A}}}\right)^{-1}.$$

Therefore, Theorem 0.4 fixes the incorrect statement of [Pr, Proposition 24]: such a Proposition was giving necessary and sufficient conditions such that $\overline{\mathcal{G}}$ as above is an equivalence of bicategories, but it turned out that such conditions were only sufficient but not necessary (we refer to the Appendix for more details about this).

In particular, Theorem 0.4 implies the following result.

Corollary 0.5. *Given any pair* $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ *satisfying conditions* (BF) *and any bicategory* $\mathscr{B}$*, the following facts are equivalent:*

(a) *there is an equivalence of bicategories* $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right] \rightarrow \mathscr{B}$*;*
(b) *there is a pseudofunctor* $\mathcal{F}: \mathscr{A} \rightarrow \mathscr{B}$ *such that*
- $\mathcal{F}$ *sends each morphism of* $\mathbf{W}_{\mathscr{A}}$ *to an internal equivalence of* $\mathscr{B}$*;*
- $\mathcal{F}$ *satisfies conditions* (B1) – (B5)*.*

Conditions (A) and (B) simplify considerably in the case when $\mathscr{A}$ and $\mathscr{B}$ are both 1-categories (considered as trivial bicategories). In that case, all the 2-morphisms in those conditions are trivial, thus all such conditions are much shorter to write; moreover, conditions (A4) and (B4) are automatically satisfied.

In all this paper we are going to use the axiom of choice, that we will assume from now on without further mention. The reason for this is twofold. First of all, the axiom of choice is used heavily in [Pr] in order to construct bicategories of fractions. In [T1, Corollary 0.6] we proved that under some restrictive hypothesis the axiom of choice is not necessary, but in the general case we need it in order to consider any of the bicategories of fractions mentioned above. Secondly, even in the cases when the axiom of choice is not necessary in order to construct the bicategories $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$ and $\mathscr{B}\left[\mathbf{W}_{\mathscr{B}}^{-1}\right]$, we will have to use often Theorem 0.1. Such a result was proved in [T2] using several times the universal property of any bicategory of fractions $\mathscr{C}\left[\mathbf{W}^{-1}\right]$, as stated in [Pr, Theorem 21]; the proof of that result in [Pr] requires most of the time the axiom of choice (since for each morphism w of $\mathbf{W}$ one has to choose a quasi-inverse for $\mathcal{U}_{\mathbf{W}}(\mathrm{w})$ in $\mathscr{C}\left[\mathbf{W}^{-1}\right]$), hence we have to assume the axiom of choice also in this paper.

1. Notations

Given any bicategory $\mathscr{C}$, we denote its objects by $A, B, \cdots$, its morphisms by $f, g, \cdots$ and its 2-morphisms by $\alpha, \beta, \cdots$ (we will use $A_{\mathscr{C}}, f_{\mathscr{C}}, \alpha_{\mathscr{C}}, \cdots$ if we have to recall that they belong to $\mathscr{C}$ when we are using more than one bicategory in the computations). Given any triple of morphisms $f: A \rightarrow B$, $g: B \rightarrow C$, $h: C \rightarrow D$ in $\mathscr{C}$, we denote by $\theta_{h,g,f}$ the associator $h \circ (g \circ f) \Rightarrow (h \circ g) \circ f$ that is part of the structure of $\mathscr{C}$; we denote by $\pi_f: f \circ \mathrm{id}_A \Rightarrow f$ and $\upsilon_f: \mathrm{id}_B \circ f \Rightarrow f$ the right and left unitors for $\mathscr{C}$ relative to any morphism f as above. We denote any pseudofunctor from $\mathscr{C}$ to another bicategory $\mathscr{D}$ by $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \psi^{\mathcal{F}}_{\bullet}, \sigma^{\mathcal{F}}_{\bullet}): \mathscr{C} \rightarrow \mathscr{D}$. Here for each pair of morphisms f, g as above, $\psi^{\mathcal{F}}_{g,f}$ is the associator from $\mathcal{F}_1(g \circ f)$ to $\mathcal{F}_1(g) \circ \mathcal{F}_1(f)$ and for each object A, $\sigma^{\mathcal{F}}_A$ is the unitor from $\mathcal{F}_1(\mathrm{id}_A)$ to $\mathrm{id}_{\mathcal{F}_0(A)}$.

For all the notations about axioms (BF1) – (BF5) and the construction of bicategories of fractions we refer either to the original reference [Pr] or to our previous paper [T1].

We recall from [St, (1.33)] that given any pair of bicategories $\mathscr{C}$ and $\mathscr{D}$, a pseudofunctor $\mathcal{M}: \mathscr{C} \rightarrow \mathscr{D}$ is a *weak equivalence of bicategories* (also known as *weak biequivalence*) if and only if the following conditions hold:

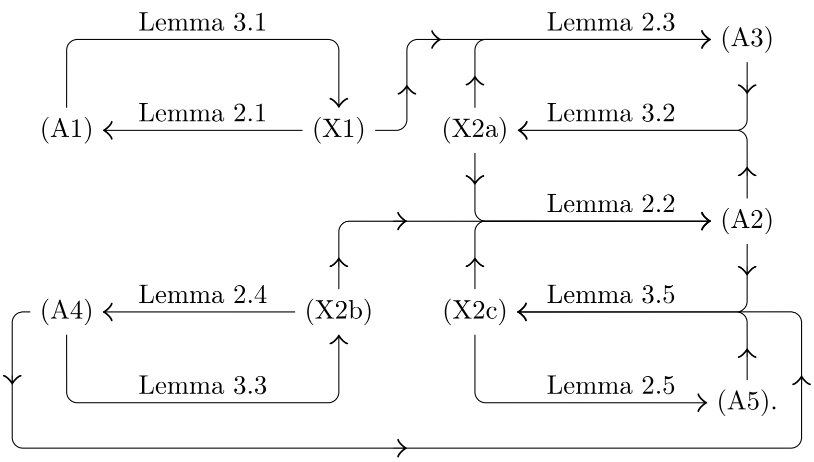
(X1) for each object $A_{\mathscr{D}}$, there are an object $A_{\mathscr{C}}$ and an internal equivalence from $\mathcal{M}_0(A_{\mathscr{C}})$ to $A_{\mathscr{D}}$ in $\mathscr{D}$;

(X2) for each pair of objects $A_{\mathscr{C}}, B_{\mathscr{C}}$, the functor $\mathcal{M}(A_{\mathscr{C}}, B_{\mathscr{C}})$ is an equivalence of categories, i.e.

- (X2a) for each morphism $f_{\mathscr{D}} : \mathcal{M}_0(A_{\mathscr{C}}) \to \mathcal{M}_0(B_{\mathscr{C}})$, there are a morphism $f_{\mathscr{C}} : A_{\mathscr{C}} \to B_{\mathscr{C}}$ and an invertible 2-morphism $\alpha_{\mathscr{D}} : \mathcal{M}_1(f_{\mathscr{C}}) \Rightarrow f_{\mathscr{D}}$;
- (X2b) for each pair of morphisms $f^1_{\mathscr{C}}, f^2_{\mathscr{C}} : A_{\mathscr{C}} \to B_{\mathscr{C}}$ and for each pair of 2-morphisms $\alpha^1_{\mathscr{C}}, \alpha^2_{\mathscr{C}} : f^1_{\mathscr{C}} \Rightarrow f^2_{\mathscr{C}}$, if $\mathcal{M}_2(\alpha^1_{\mathscr{C}}) = \mathcal{M}_2(\alpha^2_{\mathscr{C}})$, then $\alpha^1_{\mathscr{C}} = \alpha^2_{\mathscr{C}}$;
- (X2c) for each pair $f^1_{\mathscr{C}}, f^2_{\mathscr{C}}$ as above and for each 2-morphism $\alpha_{\mathscr{D}} : \mathcal{M}_1(f^1_{\mathscr{C}}) \Rightarrow \mathcal{M}_1(f^2_{\mathscr{C}})$, there is a 2-morphism $\alpha_{\mathscr{C}} : f^1_{\mathscr{C}} \Rightarrow f^2_{\mathscr{C}}$ such that $\mathcal{M}_2(\alpha_{\mathscr{C}}) = \alpha_{\mathscr{D}}$.

Since in all this paper we assume the axiom of choice, then each weak equivalence of bicategories is a (*strong*) *equivalence of bicategories* (also known as biequivalence, see [PW, § 1]), i.e. it admits a quasi-inverse. Conversely, each strong equivalence of bicategories is a weak equivalence. So in the present setup we will simply write "equivalence of bicategories" meaning weak, equivalently strong, equivalence.

In the next 2 sections we will fix any set of data $(\mathscr{A}, \mathbf{W}_{\mathscr{A}}, \mathscr{B}, \mathbf{W}_{\mathscr{B}}, \mathcal{F})$ as in Theorem 0.1(i) and we will prove that conditions (X) for the pseudofunctor $\mathcal{M} := \widetilde{\mathcal{G}}$ are equivalent to conditions (A) for the pseudofunctor $\mathcal{F}$. To be more precise, the chain of implications that we are going to prove is the following:



In the special case when $\mathcal{F}$ sends each morphism of $\mathbf{W}_{\mathscr{A}}$ to an internal equivalence of $\mathscr{B}$ and $\mathbf{W}_{\mathscr{B}}$ is the class $\mathbf{W}_{\mathscr{B},\min}$ of quasi-units of $\mathscr{B}$ (see [PP, Definition 3.3]), we will show in Proposition 4.1 that each condition of type (A) above can be replaced by the analogous condition of type (B).

2. Necessity of conditions (A1) – (A5)

In all this section and in the next one, *we fix any 2 pairs* $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ *and* $(\mathscr{B}, \mathbf{W}_{\mathscr{B}})$ *that satisfy conditions* (BF) *and any pseudofunctor* $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, *such that* $\mathcal{F}_1(\mathbf{W}_{\mathscr{A}}) \subseteq \mathbf{W}_{\mathscr{B},\mathrm{sat}}$; we denote by $\widetilde{\mathcal{G}} : \mathscr{A}\Big[\mathbf{W}^{-1}_{\mathscr{A}}\Big] \to \mathscr{B}\Big[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\Big]$ the pseudofunctor mentioned in Theorem 0.1. In this section we will prove that conditions (X) for $\widetilde{\mathcal{G}}$ imply conditions (A) for $\mathcal{F}$.

Lemma 2.1. *If* $\widetilde{\mathcal{G}}$ *satisfies* (X1), *then* $\mathcal{F}$ *satisfies* (A1).

Proof. By (X1), given any object $A_{\mathscr{B}}$ there are an object $A_{\mathscr{A}}$ and an internal equivalence $\underline{e}_{\mathscr{B}}$ from $\widetilde{\mathcal{G}}_0(A_{\mathscr{A}}) = \mathcal{F}_0(A_{\mathscr{A}})$ to $A_{\mathscr{B}}$ in $\mathscr{B}[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}]$. By [T2, Proposition 0.1] applied to $(\mathscr{B},\mathbf{W}_{\mathscr{B}})$, there are a pseudofunctor $\mathcal{H}:\mathscr{B}[\mathbf{W}_{\mathscr{B},\mathrm{sat}}^{-1}]\to\mathscr{B}[\mathbf{W}_{\mathscr{B}}^{-1}]$ and a pseudonatural equivalence of pseudofunctors $\tau:\mathcal{U}_{\mathbf{W}_{\mathscr{B}}}\Rightarrow\mathcal{H}\circ\mathcal{U}_{\mathbf{W}_{\mathscr{B},\mathrm{sat}}}$. Since $\underline{e}_{\mathscr{B}}$ is an internal equivalence, then there is an internal equivalence in $\mathscr{B}\left[\mathbf{W}_{\mathscr{B}}^{-1}\right]$

$$\mathcal{H}_1(\underline{e}_{\mathscr{B}}):\ \mathcal{H}_0\circ\mathcal{F}_0(A_{\mathscr{A}})\longrightarrow\mathcal{H}_0(A_{\mathscr{B}}). \tag{2.1}$$

Moreover, since τ is a pseudonatural equivalence of pseudofunctors, then there are internal equivalences

$$\begin{aligned}\tau_{\mathcal{F}_0(A_{\mathscr{A}})}:\ \mathcal{F}_0(A_{\mathscr{A}})=\mathcal{U}_{\mathbf{W}_{\mathscr{B}},0}\circ\mathcal{F}_0(A_{\mathscr{A}})\longrightarrow\\ \longrightarrow\mathcal{H}_0\circ\mathcal{U}_{\mathbf{W}_{\mathscr{B},\mathrm{sat}},0}\circ\mathcal{F}_0(A_{\mathscr{A}})=\mathcal{H}_0\circ\mathcal{F}_0(A_{\mathscr{A}})\end{aligned} \tag{2.2}$$

and

$$\tau_{A_{\mathscr{B}}}:\ A_{\mathscr{B}}=\mathcal{U}_{\mathbf{W}_{\mathscr{B},\mathrm{sat}},0}(A_{\mathscr{B}})\longrightarrow\mathcal{H}_0\circ\mathcal{U}_{\mathbf{W}_{\mathscr{B},\mathrm{sat}},0}(A_{\mathscr{B}})=\mathcal{H}_0(A_{\mathscr{B}}). \tag{2.3}$$

Composing (2.2), (2.1) and any quasi-inverse for (2.3), we get an internal equivalence from $\mathcal{F}_0(A_{\mathscr{A}})$ to $A_{\mathscr{B}}$ in $\mathscr{B}\left[\mathbf{W}_{\mathscr{B}}^{-1}\right]$; then it suffices to apply [T2, Corollary 2.7] for $(\mathscr{B},\mathbf{W}_{\mathscr{B}})$. □

Lemma 2.2. *If* $\widetilde{\mathcal{G}}$ *satisfies* (X2a)*,* (X2b) *and* (X2c)*, then* $\mathcal{F}$ *satisfies* (A2)*.*

Proof. Let us fix any set of data as in (0.3) with $\mathrm{w}_{\mathscr{B}}^1$ in $\mathbf{W}_{\mathscr{B}}$ and $\mathrm{w}_{\mathscr{B}}^2\in\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. By (X2a) for $\mathcal{M}:=\widetilde{\mathcal{G}}$, there are a morphism $\underline{f}_{\mathscr{A}}:A_{\mathscr{A}}^1\to A_{\mathscr{A}}^2$ in $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$ and an invertible 2-morphism in $\mathscr{B}\left[\mathbf{W}_{\mathscr{B},\mathrm{sat}}^{-1}\right]$

$$\Gamma_{\mathscr{B}}:\ \widetilde{\mathcal{G}}_1\left(\underline{f}_{\mathscr{A}}\right)\Longrightarrow\left(A_{\mathscr{B}},\mathrm{w}_{\mathscr{B}}^1,\mathrm{w}_{\mathscr{B}}^2\right).$$

Since $\mathbf{W}_{\mathscr{B}}\subseteq\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\mathbf{W}_{\mathscr{B},\mathrm{sat},\mathrm{sat}}=\mathbf{W}_{\mathscr{B},\mathrm{sat},\mathrm{sat}}$ (see [T2, Remark 2.3 and Proposition 2.11(i)]), then by [T2, Corollary 2.7] for $(\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ we have that the target of $\Gamma_{\mathscr{B}}$ is an internal equivalence in $\mathscr{B}\left[\mathbf{W}_{\mathscr{B},\mathrm{sat}}^{-1}\right]$. Since $\Gamma_{\mathscr{B}}$ is an invertible 2-morphism, then we conclude that also $\widetilde{\mathcal{G}}_1(\underline{f}_{\mathscr{A}}):\mathcal{F}_0(A_{\mathscr{A}}^1)\to\mathcal{F}_0(A_{\mathscr{A}}^2)$ is an internal equivalence. So there are an internal equivalence $\underline{g}_{\mathscr{B}}:\mathcal{F}_0(A_{\mathscr{A}}^2)\to\mathcal{F}_0(A_{\mathscr{A}}^1)$ and invertible 2-morphisms in $\mathscr{B}\left[\mathbf{W}_{\mathscr{B},\mathrm{sat}}^{-1}\right]$ as follows

$$\Delta_{\mathscr{B}}:\ \mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}}^1)}\Longrightarrow\underline{g}_{\mathscr{B}}\circ\widetilde{\mathcal{G}}_1\left(\underline{f}_{\mathscr{A}}\right)\qquad\text{and}\qquad\Xi_{\mathscr{B}}:\ \widetilde{\mathcal{G}}_1\left(\underline{f}_{\mathscr{A}}\right)\circ\underline{g}_{\mathscr{B}}\Longrightarrow\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}}^2)}$$

(here the identity $\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}}^1)}$ belongs to $\mathscr{B}\left[\mathbf{W}_{\mathscr{B},\mathrm{sat}}^{-1}\right]$, so it is given by the triple $(\mathcal{F}_0(A_{\mathscr{A}}^1),\ \mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}}^1)},\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}}^1)})$, and analogously for $\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}}^2)}$). By (X2a) for $\underline{g}_{\mathscr{B}}$, there are a morphism $\underline{g}_{\mathscr{A}}:A_{\mathscr{A}}^2\to A_{\mathscr{A}}^1$ in $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$ and an invertible 2-morphism $\Omega_{\mathscr{B}}:\widetilde{\mathcal{G}}_1(\underline{g}_{\mathscr{A}})\Rightarrow\underline{g}_{\mathscr{B}}$ in $\mathscr{B}\left[\mathbf{W}_{\mathscr{B},\mathrm{sat}}^{-1}\right]$. Let us denote by

$$\Psi^{\widetilde{\mathcal{G}}}_{\underline{g}_{\mathscr{A}},\underline{f}_{\mathscr{A}}}:\ \widetilde{\mathcal{G}}_1\left(\underline{g}_{\mathscr{A}}\circ\underline{f}_{\mathscr{A}}\right)\Longrightarrow\widetilde{\mathcal{G}}_1\left(\underline{g}_{\mathscr{A}}\right)\circ\widetilde{\mathcal{G}}_1\left(\underline{f}_{\mathscr{A}}\right)$$

the associator of $\widetilde{\mathcal{G}}$ for the pair $(\underline{g}_{\mathscr{A}},\underline{f}_{\mathscr{A}})$ and by

$$\Sigma^{\widetilde{\mathcal{G}}}_{A_{\mathscr{A}}^1}:\ \widetilde{\mathcal{G}}_1\left(\mathrm{id}_{A_{\mathscr{A}}^1}\right)\Longrightarrow\mathrm{id}_{\widetilde{\mathcal{G}}_0(A_{\mathscr{A}}^1)}=\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}}^1)}$$

the unitor of $\widetilde{\mathcal{G}}$ for $A_{\mathscr{A}}^1$. Then it makes sense to consider the invertible 2-morphism

$$\widetilde{\Delta}_{\mathscr{B}}:=\left(\Psi^{\widetilde{\mathcal{G}}}_{\underline{g}_{\mathscr{A}},\underline{f}_{\mathscr{A}}}\right)^{-1}\odot\left(\Omega_{\mathscr{B}}^{-1}*i_{\widetilde{\mathcal{G}}_1(\underline{f}_{\mathscr{A}})}\right)\odot\Delta_{\mathscr{B}}\odot\Sigma^{\widetilde{\mathcal{G}}}_{A_{\mathscr{A}}^1}:\ \widetilde{\mathcal{G}}_1\left(\mathrm{id}_{A_{\mathscr{A}}^1}\right)\Longrightarrow\widetilde{\mathcal{G}}_1\left(\underline{g}_{\mathscr{A}}\circ\underline{f}_{\mathscr{A}}\right).$$

By (X2b) and (X2c) for $\mathcal{M} := \widetilde{\mathcal{G}}$, there is a unique invertible 2-morphism $\Delta_{\mathscr{A}}$: $\mathrm{id}_{A^1_{\mathscr{A}}} \Rightarrow \underline{g}_{\mathscr{A}} \circ \underline{f}_{\mathscr{A}}$ in $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$, such that $\widetilde{\mathcal{G}}_2(\Delta_{\mathscr{A}}) = \widetilde{\Delta}_{\mathscr{B}}$. Analogously, there is an invertible 2-morphism $\Xi_{\mathscr{A}} : \underline{f}_{\mathscr{A}} \circ \underline{g}_{\mathscr{A}} \Rightarrow \mathrm{id}_{A^2_{\mathscr{A}}}$. This proves that $\underline{f}_{\mathscr{A}}$ is an internal equivalence in $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$. By [T2, Corollary 2.7] for $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$, we get that necessarily $\underline{f}_{\mathscr{A}}$ has the following form

$$\underline{f}_{\mathscr{A}} = \Big(A^1_{\mathscr{A}} \xleftarrow{\mathrm{w}^1_{\mathscr{A}}} A^3_{\mathscr{A}} \xrightarrow{\mathrm{w}^2_{\mathscr{A}}} A^2_{\mathscr{A}}\Big),$$

with $\mathrm{w}^1_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\mathrm{w}^2_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A},\mathrm{sat}}$. Now we use the description of $\widetilde{\mathcal{G}}_1$ in Theorem 0.1 and [T1, Proposition 0.8(ii)] for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$; so we have that $\Gamma_{\mathscr{B}}^{-1}$ is represented by a set of data as in the internal part of the following diagram:

$$\begin{tikzcd}
 & A_{\mathscr{B}} \arrow[dl, "\mathrm{w}^1_{\mathscr{B}}"'] \arrow[dr, "\mathrm{w}^2_{\mathscr{B}}"] & \\
\mathcal{F}_0(A^1_{\mathscr{A}}) \arrow[r, phantom, "\Downarrow \eta^1_{\mathscr{B}}"] & \overline{A}_{\mathscr{B}} \arrow[u, "\mathrm{u}^1_{\mathscr{B}}"'] \arrow[d, "\mathrm{u}^2_{\mathscr{B}}"] \arrow[r, phantom, "\Downarrow \eta^2_{\mathscr{B}}"] & \mathcal{F}_0(A^2_{\mathscr{A}}), \\
 & \mathcal{F}_0(A^3_{\mathscr{A}}) \arrow[ul, "\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})"] \arrow[ur, "\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})"'] &
\end{tikzcd}$$

such that $\mathrm{w}^1_{\mathscr{B}} \circ \mathrm{u}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and both $\eta^1_{\mathscr{B}}$ and $\eta^2_{\mathscr{B}}$ are invertible. Since $\mathrm{w}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}} \subseteq \mathbf{W}_{\mathscr{B},\mathrm{sat}}$, then by [T2, Proposition 2.11(ii)] we get that $\mathrm{u}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. By definition of $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$, there are an object $A'_{\mathscr{B}}$ and a morphism $\mathrm{t}_{\mathscr{B}} : A'_{\mathscr{B}} \to \overline{A}_{\mathscr{B}}$, such that $\mathrm{u}^1_{\mathscr{B}} \circ \mathrm{t}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}}$. Then in order to conclude that (A2) holds, it suffices to define for each $m = 1, 2$ $\mathrm{z}^m_{\mathscr{B}} := \mathrm{u}^m_{\mathscr{B}} \circ \mathrm{t}_{\mathscr{B}}$ and

$$\gamma^m_{\mathscr{B}} := \theta^{-1}_{\mathcal{F}_1(\mathrm{w}^m_{\mathscr{A}}),\mathrm{u}^2_{\mathscr{B}},\mathrm{t}_{\mathscr{B}}} \odot \Big(\eta^m_{\mathscr{B}} * i_{\mathrm{t}_{\mathscr{B}}}\Big) \odot \theta_{\mathrm{w}^m_{\mathscr{B}},\mathrm{u}^1_{\mathscr{B}},\mathrm{t}_{\mathscr{B}}} : \ \mathrm{w}^m_{\mathscr{B}} \circ \mathrm{z}^1_{\mathscr{B}} \Longrightarrow \mathcal{F}_1(\mathrm{w}^m_{\mathscr{A}}) \circ \mathrm{z}^2_{\mathscr{B}}.$$

□

Lemma 2.3. *If* $\widetilde{\mathcal{G}}$ *satisfies* (X1) *and* (X2a), *then* $\mathcal{F}$ *satisfies* (A3).

Proof. Let us fix any pair of objects $B_{\mathscr{A}}, A_{\mathscr{B}}$ and any morphism $f_{\mathscr{B}} : A_{\mathscr{B}} \to \mathcal{F}_0(B_{\mathscr{A}})$. By (X1) and Lemma 2.1, (A1) holds for $\mathcal{F}$, so there are a pair of objects $\overline{A}_{\mathscr{A}}, \overline{A}_{\mathscr{B}}$ and a pair of morphism $\mathrm{w}^1_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B}}$ and $\mathrm{w}^2_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ as follows

$$\mathcal{F}_0(\overline{A}_{\mathscr{A}}) \xleftarrow{\mathrm{w}^1_{\mathscr{B}}} \overline{A}_{\mathscr{B}} \xrightarrow{\mathrm{w}^2_{\mathscr{B}}} A_{\mathscr{B}}.$$

Let us consider the morphism in $\mathscr{B}\left[\mathbf{W}_{\mathscr{B},\mathrm{sat}}^{-1}\right]$ defined as follows

$$\underline{f}_{\mathscr{B}} := \Big(\mathcal{F}_0(\overline{A}_{\mathscr{A}}) \xleftarrow{\mathrm{w}^1_{\mathscr{B}}} \overline{A}_{\mathscr{B}} \xrightarrow{f_{\mathscr{B}} \circ \mathrm{w}^2_{\mathscr{B}}} \mathcal{F}_0(B_{\mathscr{A}})\Big).$$

By condition (X2a), there are a morphism $\underline{f}_{\mathscr{A}} : \overline{A}_{\mathscr{A}} \to B_{\mathscr{A}}$ in $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$ and an invertible 2-morphism

$$\Gamma_{\mathscr{B}} : \ \widetilde{\mathcal{G}}_1\left(\underline{f}_{\mathscr{A}}\right) \Longrightarrow \underline{f}_{\mathscr{B}}$$

in $\mathscr{B}\left[\mathbf{W}_{\mathscr{B},\mathrm{sat}}^{-1}\right]$. By the description of bicategories of fractions in [Pr, § 2.2], $\underline{f}_{\mathscr{A}}$ is given by a triple $(A_{\mathscr{A}}, \mathrm{w}_{\mathscr{A}}, f_{\mathscr{A}})$ as follows

$$\underline{f}_{\mathscr{A}} := \Big(\overline{A}_{\mathscr{A}} \xleftarrow{\quad \mathrm{w}_{\mathscr{A}} \quad} A_{\mathscr{A}} \xrightarrow{\quad f_{\mathscr{A}} \quad} B_{\mathscr{A}}\Big)$$

with $\mathrm{w}_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, so by the description of $\widetilde{\mathcal{G}}_1$ in Theorem 0.1, the source of $\Gamma_{\mathscr{B}}$ is the triple $(\mathcal{F}_0(A_{\mathscr{A}}), \mathcal{F}_1(\mathrm{w}_{\mathscr{A}}), \mathcal{F}_1(f_{\mathscr{A}}))$. By [T1, Proposition 0.8(ii)] for the pair $(\mathscr{C}, \mathbf{W}) := (\mathscr{B}, \mathbf{W}_{\mathscr{B}})$, $\Gamma^{-1}_{\mathscr{B}}$ is represented by a set of data in $\mathscr{B}$ as follows

$$\begin{tikzcd}
 & & \overline{A}_{\mathscr{B}} \arrow[dll, "\mathrm{w}^1_{\mathscr{B}}"'] \arrow[drr, "f_{\mathscr{B}} \circ \mathrm{w}^2_{\mathscr{B}}"] & & \\
\mathcal{F}_0(\overline{A}_{\mathscr{A}}) & \Downarrow \gamma^1_{\mathscr{B}} & \widetilde{A}_{\mathscr{B}} \arrow[u, "\mathrm{z}^1_{\mathscr{B}}"'] \arrow[d, "\mathrm{z}^2_{\mathscr{B}}"] & \Downarrow \gamma^2_{\mathscr{B}} & \mathcal{F}_0(B_{\mathscr{A}}), \\
 & & \mathcal{F}_0(A_{\mathscr{A}}) \arrow[ull, "\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})"] \arrow[urr, "\mathcal{F}_1(f_{\mathscr{A}})"'] & &
\end{tikzcd}$$

such that $\mathrm{w}^1_{\mathscr{B}} \circ \mathrm{z}^1_{\mathscr{B}}$ belong to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and both $\gamma^1_{\mathscr{B}}$ and $\gamma^2_{\mathscr{B}}$ are invertible. Since $\mathrm{w}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}} \subseteq \mathbf{W}_{\mathscr{B},\mathrm{sat}}$, then by [T2, Proposition 2.11(ii)] we get that $\mathrm{z}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. Therefore, by axiom (BF2) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ (see [T2, Lemma 2.8]), $\mathrm{w}^2_{\mathscr{B}} \circ \mathrm{z}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. So by definition of $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ there are an object $A'_{\mathscr{B}}$ and a morphism $\mathrm{t}_{\mathscr{B}} : A'_{\mathscr{B}} \to \widetilde{A}_{\mathscr{B}}$, such that $\mathrm{v}^1_{\mathscr{B}} := (\mathrm{w}^2_{\mathscr{B}} \circ \mathrm{z}^1_{\mathscr{B}}) \circ \mathrm{t}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}}$. Since $\mathrm{w}^2_{\mathscr{B}} \circ \mathrm{z}^1_{\mathscr{B}}$ belong to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$, then by [T2, Proposition 2.11(ii)] we get that $\mathrm{t}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$.

Using (BF5) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$ (see again [T2, Lemma 2.8]) and $(\gamma^1_{\mathscr{B}})^{-1}$, we get that $\mathcal{F}_1(\mathrm{w}_{\mathscr{A}}) \circ \mathrm{z}^2_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. In all this section we are assuming that $\mathcal{F}_1(\mathbf{W}_{\mathscr{A}}) \subseteq \mathbf{W}_{\mathscr{B},\mathrm{sat}}$, so $\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. Hence by [T2, Proposition 2.11(ii)] we get that $\mathrm{z}^2_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. So by (BF2) also the morphism $\mathrm{v}^2_{\mathscr{B}} := \mathrm{z}^2_{\mathscr{B}} \circ \mathrm{t}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. Then we set

$$\alpha_{\mathscr{B}} := \theta^{-1}_{\mathcal{F}_1(f_{\mathscr{A}}),\mathrm{z}^2_{\mathscr{B}},\mathrm{t}_{\mathscr{B}}} \odot \Big(\gamma^2_{\mathscr{B}} * i_{\mathrm{t}_{\mathscr{B}}}\Big) \odot \Big(\theta_{f_{\mathscr{B}},\mathrm{w}^2_{\mathscr{B}},\mathrm{z}^1_{\mathscr{B}}} * i_{\mathrm{t}_{\mathscr{B}}}\Big) \odot \theta_{f_{\mathscr{B}},\mathrm{w}^2_{\mathscr{B}} \circ \mathrm{z}^1_{\mathscr{B}},\mathrm{t}_{\mathscr{B}}} :$$
$$f_{\mathscr{B}} \circ \mathrm{v}^1_{\mathscr{B}} \Longrightarrow \mathcal{F}_1(f_{\mathscr{A}}) \circ \mathrm{v}^2_{\mathscr{B}}.$$

This suffices to conclude that (A3) holds for $\mathcal{F}$. □

Lemma 2.4. *If $\widetilde{\mathcal{G}}$ satisfies* (X2b), *then $\mathcal{F}$ satisfies* (A4).

Proof. Let us fix any set of data $(A_{\mathscr{A}}, B_{\mathscr{A}}, f^1_{\mathscr{A}}, f^2_{\mathscr{A}}, \gamma^1_{\mathscr{A}}, \gamma^2_{\mathscr{A}}, A'_{\mathscr{B}}, \mathrm{z}_{\mathscr{B}})$ as in (A4) and let us suppose that $\mathcal{F}_2(\gamma^1_{\mathscr{A}}) * i_{\mathrm{z}_{\mathscr{B}}} = \mathcal{F}_2(\gamma^2_{\mathscr{A}}) * i_{\mathrm{z}_{\mathscr{B}}}$. For each $m = 1, 2$, we consider the 2-morphism in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$

$$\Gamma^m_{\mathscr{A}} := \Big[A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, \mathrm{id}_{A_{\mathscr{A}}}, i_{\mathrm{id}_{A_{\mathscr{A}}} \circ \mathrm{id}_{A_{\mathscr{A}}}}, \gamma^m_{\mathscr{A}} * i_{\mathrm{id}_{A_{\mathscr{A}}}}\Big] :$$
$$\Big(A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, f^1_{\mathscr{A}}\Big) \Longrightarrow \Big(A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, f^2_{\mathscr{A}}\Big).$$

By the description of $\widetilde{\mathcal{G}}_2$ in Theorem 0.1 and the description of 2-morphisms in a bicategory of fractions, see [Pr, § 2.3], for each $m = 1, 2$ we have

$$\widetilde{\mathcal{G}}_2(\Gamma^m_{\mathscr{A}}) = \Big[\mathcal{F}_0(A_{\mathscr{A}}), \mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}}), \mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}}),$$
$$\psi^{\mathcal{F}}_{\mathrm{id}_{A_{\mathscr{A}}},\mathrm{id}_{A_{\mathscr{A}}}} \odot \mathcal{F}_2\Big(i_{\mathrm{id}_{A_{\mathscr{A}}} \circ \mathrm{id}_{A_{\mathscr{A}}}}\Big) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{id}_{A_{\mathscr{A}}},\mathrm{id}_{A_{\mathscr{A}}}}\Big)^{-1},$$

$$\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{id}_{A_{\mathscr{A}}}} \odot \mathcal{F}_2\Big(\gamma^m_{\mathscr{A}} * i_{\mathrm{id}_{A_{\mathscr{A}}}}\Big) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{id}_{A_{\mathscr{A}}}}\Big)^{-1}\Big] =$$
$$= \Big[\mathcal{F}_0(A_{\mathscr{A}}), \mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}}), \mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}}), i_{\mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}})\circ\mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}})}, \mathcal{F}_2(\gamma^m_{\mathscr{A}}) * i_{\mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}})}\Big] =$$
$$= \Big[A'_{\mathscr{B}}, \mathrm{z}_{\mathscr{B}}, \mathrm{z}_{\mathscr{B}}, i_{\mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}})\circ\mathrm{z}_{\mathscr{B}}}, \mathcal{F}_2(\gamma^m_{\mathscr{A}}) * i_{\mathrm{z}_{\mathscr{B}}}\Big].$$

Since $\mathcal{F}_2(\gamma^1_{\mathscr{A}}) * i_{\mathrm{z}_{\mathscr{B}}} = \mathcal{F}_2(\gamma^2_{\mathscr{A}}) * i_{\mathrm{z}_{\mathscr{B}}}$, then we conclude that $\widetilde{\mathcal{G}}_2(\Gamma^1_{\mathscr{A}}) = \widetilde{\mathcal{G}}_2(\Gamma^2_{\mathscr{A}})$. By (X2b) for $\widetilde{\mathcal{G}}$, we get that $\Gamma^1_{\mathscr{A}} = \Gamma^2_{\mathscr{A}}$. Then by [T1, Proposition 0.7] for the pair $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$, there are an object $A'_{\mathscr{A}}$ and a morphism $\mathrm{z}_{\mathscr{A}} : A'_{\mathscr{A}} \to A_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, such that

$$\Big(\gamma^1 * i_{\mathrm{id}_{A_{\mathscr{A}}}}\Big) * i_{\mathrm{z}_{\mathscr{A}}} = \Big(\gamma^2 * i_{\mathrm{id}_{A_{\mathscr{A}}}}\Big) * i_{\mathrm{z}_{\mathscr{A}}}.$$

Then the claim follows immediately. □

Lemma 2.5. *If $\widetilde{\mathcal{G}}$ satisfies* (X2c)*, then $\mathcal{F}$ satisfies* (A5)*.*

Proof. Let us fix any set of data $(A_{\mathscr{A}}, B_{\mathscr{A}}, A_{\mathscr{B}}, f^1_{\mathscr{A}}, f^2_{\mathscr{A}}, \mathrm{v}_{\mathscr{B}}, \alpha_{\mathscr{B}})$ as in (A5). Then let us consider the 2-morphism in $\mathscr{B}\Big[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\Big]$

$$\Gamma_{\mathscr{B}} := \Big[A_{\mathscr{B}}, \mathrm{v}_{\mathscr{B}}, \mathrm{v}_{\mathscr{B}}, i_{\mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}})\circ\mathrm{v}_{\mathscr{B}}}, \alpha_{\mathscr{B}}\Big] : \Big(\mathcal{F}_0(A_{\mathscr{A}}), \mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}}), \mathcal{F}_1(f^1_{\mathscr{A}})\Big) \Longrightarrow$$
$$\Longrightarrow \Big(\mathcal{F}_0(A_{\mathscr{A}}), \mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}}), \mathcal{F}_1(f^2_{\mathscr{A}})\Big). \tag{2.4}$$

We can interpret $\Gamma_{\mathscr{B}}$ as a 2-morphism from $\widetilde{\mathcal{G}}_1(A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, f^1_{\mathscr{A}})$ to $\widetilde{\mathcal{G}}_1(A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, f^2_{\mathscr{A}})$, so by (X2c) there is a 2-morphism

$$\Gamma_{\mathscr{A}} : \Big(A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, f^1_{\mathscr{A}}\Big) \Longrightarrow \Big(A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, f^2_{\mathscr{A}}\Big)$$

in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$, such that $\widetilde{\mathcal{G}}_2(\Gamma_{\mathscr{A}}) = \Gamma_{\mathscr{B}}$. We apply [T1, Lemma 6.1] for $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$, $\alpha := i_{\mathrm{id}_{A_{\mathscr{A}}}\circ\mathrm{id}_{A_{\mathscr{A}}}}$ and $\Gamma := \Gamma_{\mathscr{A}}$; then there are an object $A'_{\mathscr{A}}$, a morphism $\mathrm{v}_{\mathscr{A}} : A'_{\mathscr{A}} \to A_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and a 2-morphism $\gamma_{\mathscr{A}} : f^1_{\mathscr{A}} \circ (\mathrm{id}_{A_{\mathscr{A}}} \circ \mathrm{v}_{\mathscr{A}}) \Rightarrow f^2_{\mathscr{A}} \circ (\mathrm{id}_{A_{\mathscr{A}}} \circ \mathrm{v}_{\mathscr{A}})$, such that $\Gamma_{\mathscr{A}}$ is represented by the data in the following diagram

$$\begin{tikzcd}
& A_{\mathscr{A}} \arrow[dl, "\mathrm{id}_{A_{\mathscr{A}}}"'] \arrow[dr, "f^1_{\mathscr{A}}"] & \\
A_{\mathscr{A}} & A'_{\mathscr{A}} \arrow[u, "\mathrm{id}_{A_{\mathscr{A}}}\circ\mathrm{v}_{\mathscr{A}}"'] \arrow[d, "\mathrm{id}_{A_{\mathscr{A}}}\circ\mathrm{v}_{\mathscr{A}}"] \arrow[l, phantom, "\Downarrow\,\varepsilon_{\mathscr{A}}"] \arrow[r, phantom, "\Downarrow\,\gamma_{\mathscr{A}}"] & B_{\mathscr{A}}, \\
& A_{\mathscr{A}} \arrow[ul, "\mathrm{id}_{A_{\mathscr{A}}}"] \arrow[ur, "f^2_{\mathscr{A}}"'] &
\end{tikzcd} \tag{2.5}$$

where

$$\varepsilon_{\mathscr{A}} := \theta^{-1}_{\mathrm{id}_{A_{\mathscr{A}}},\mathrm{id}_{A_{\mathscr{A}}},\mathrm{v}_{\mathscr{A}}} \odot \Big(i_{\mathrm{id}_{A_{\mathscr{A}}}\circ\mathrm{id}_{A_{\mathscr{A}}}} * i_{\mathrm{v}_{\mathscr{A}}}\Big) \odot \theta_{\mathrm{id}_{A_{\mathscr{A}}},\mathrm{id}_{A_{\mathscr{A}}},\mathrm{v}_{\mathscr{A}}} = i_{(\mathrm{id}_{A_{\mathscr{A}}}\circ\mathrm{id}_{A_{\mathscr{A}}})\circ\mathrm{v}_{\mathscr{A}}}.$$

If we denote by $\upsilon_{\mathrm{v}_{\mathscr{A}}}$ the unitor $\mathrm{id}_{A_{\mathscr{A}}} \circ \mathrm{v}_{\mathscr{A}} \Rightarrow \mathrm{v}_{\mathscr{A}}$, then we can define

$$\alpha_{\mathscr{A}} := \Big(i_{f^2_{\mathscr{A}}} * \upsilon_{\mathrm{v}_{\mathscr{A}}}\Big) \odot \gamma_{\mathscr{A}} \odot \Big(i_{f^1_{\mathscr{A}}} * \upsilon^{-1}_{\mathrm{v}_{\mathscr{A}}}\Big) : \; f^1_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}} \Longrightarrow f^2_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}}$$

and we get easily that

$$\Gamma_{\mathscr{A}} = \Big[A'_{\mathscr{A}}, \mathrm{v}_{\mathscr{A}}, \mathrm{v}_{\mathscr{A}}, i_{\mathrm{id}_{A_{\mathscr{A}}}\circ\mathrm{v}_{\mathscr{A}}}, \alpha_{\mathscr{A}}\Big].$$

So by (0.1) we have

$$\begin{aligned}\widetilde{\mathcal{G}}_2(\Gamma_{\mathscr{A}})=\Big[&\mathcal{F}_0(A'_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\\ &\psi^{\mathcal{F}}_{\mathrm{id}_{A_{\mathscr{A}}},\mathrm{v}_{\mathscr{A}}}\odot\mathcal{F}_2(i_{\mathrm{id}_{A_{\mathscr{A}}}\circ\mathrm{v}_{\mathscr{A}}})\odot\Big(\psi^{\mathcal{F}}_{\mathrm{id}_{A_{\mathscr{A}}},\mathrm{v}_{\mathscr{A}}}\Big)^{-1},\\ &\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\odot\mathcal{F}_2(\alpha_{\mathscr{A}})\odot\Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\Big)^{-1}\Big]=\\ =\Big[&\mathcal{F}_0(A'_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),i_{\mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}})\circ\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})},\\ &\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\odot\mathcal{F}_2(\alpha_{\mathscr{A}})\odot\Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\Big)^{-1}\Big].\end{aligned}\tag{2.6}$$

Now by (BF3) for $(\mathscr{B},\mathbf{W}_{\mathscr{B}})$ there is a set of data as in the upper part of the following diagram

$$\begin{tikzcd}
 & & \widetilde{A}_{\mathscr{B}} \arrow[dll, "\mathrm{r}_{\mathscr{B}}"'] \arrow[drr, "\mathrm{s}_{\mathscr{B}}"] \arrow[d, phantom, "\overset{\rho_{\mathscr{B}}}{\Rightarrow}"] & & \\
\mathcal{F}_0(A'_{\mathscr{A}}) \arrow[rr, "\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})"'] & & \mathcal{F}_0(A_{\mathscr{A}}) & & A_{\mathscr{B}}, \arrow[ll, "\mathrm{v}_{\mathscr{B}}"]
\end{tikzcd}$$

such that $\mathrm{r}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}}$ and $\rho_{\mathscr{B}}$ is invertible. Then using (2.6) and the description of 2-morphisms in [Pr, § 2.3], we get

$$\begin{aligned}\widetilde{\mathcal{G}}_2(\Gamma_{\mathscr{A}})=\Big[&\widetilde{A}_{\mathscr{B}},\mathrm{v}_{\mathscr{B}}\circ\mathrm{s}_{\mathscr{B}},\mathrm{v}_{\mathscr{B}}\circ\mathrm{s}_{\mathscr{B}},i_{\mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}})\circ(\mathrm{v}_{\mathscr{B}}\circ\mathrm{s}_{\mathscr{B}})},\\ &\Big(i_{\mathcal{F}_1(f^2_{\mathscr{A}})}*\rho_{\mathscr{B}}\Big)\odot\theta^{-1}_{\mathcal{F}_1(f^2_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{r}_{\mathscr{B}}}\odot\Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}*i_{\mathrm{r}_{\mathscr{B}}}\Big)\odot\Big(\mathcal{F}_2(\alpha_{\mathscr{A}})*i_{\mathrm{r}_{\mathscr{B}}}\Big)\odot\\ &\odot\Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}*i_{\mathrm{r}_{\mathscr{B}}}\Big)^{-1}\odot\theta_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{r}_{\mathscr{B}}}\odot\Big(i_{\mathcal{F}_1(f^1_{\mathscr{A}})}*\rho^{-1}_{\mathscr{B}}\Big)\Big].\end{aligned}\tag{2.7}$$

Moreover, from (2.4) we get

$$\begin{aligned}\Gamma_{\mathscr{B}}=\Big[&\widetilde{A}_{\mathscr{B}},\mathrm{v}_{\mathscr{B}}\circ\mathrm{s}_{\mathscr{B}},\mathrm{v}_{\mathscr{B}}\circ\mathrm{s}_{\mathscr{B}},i_{\mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}})\circ(\mathrm{v}_{\mathscr{B}}\circ\mathrm{s}_{\mathscr{B}})},\\ &\theta^{-1}_{\mathcal{F}_1(f^2_{\mathscr{A}}),\mathrm{v}_{\mathscr{B}},\mathrm{s}_{\mathscr{B}}}\odot\Big(\alpha_{\mathscr{B}}*i_{\mathrm{s}_{\mathscr{B}}}\Big)\odot\theta_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathrm{v}_{\mathscr{B}},\mathrm{s}_{\mathscr{B}}}\Big].\end{aligned}\tag{2.8}$$

Since $\widetilde{\mathcal{G}}_2(\Gamma_{\mathscr{A}})=\Gamma_{\mathscr{B}}$, then using [T1, Proposition 0.7] for $(\mathscr{B},\mathbf{W}_{\mathscr{B},\mathrm{sat}})$ together with (2.7) and (2.8), we get that there are an object $A'_{\mathscr{B}}$ and a morphism $\mathrm{t}_{\mathscr{B}}:A'_{\mathscr{B}}\rightarrow\widetilde{A}_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$, such that

$$\begin{aligned}&\Big(\theta^{-1}_{\mathcal{F}_1(f^2_{\mathscr{A}}),\mathrm{v}_{\mathscr{B}},\mathrm{s}_{\mathscr{B}}}\odot\Big(\alpha_{\mathscr{B}}*i_{\mathrm{s}_{\mathscr{B}}}\Big)\odot\theta_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathrm{v}_{\mathscr{B}},\mathrm{s}_{\mathscr{B}}}\Big)*i_{\mathrm{t}_{\mathscr{B}}}=\\ &=\Big(\Big(i_{\mathcal{F}_1(f^2_{\mathscr{A}})}*\rho_{\mathscr{B}}\Big)\odot\theta^{-1}_{\mathcal{F}_1(f^2_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{r}_{\mathscr{B}}}\odot\Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}*i_{\mathrm{r}_{\mathscr{B}}}\Big)\odot\Big(\mathcal{F}_2(\alpha_{\mathscr{A}})*i_{\mathrm{r}_{\mathscr{B}}}\Big)\odot\\ &\odot\Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}*i_{\mathrm{r}_{\mathscr{B}}}\Big)^{-1}\odot\theta_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{r}_{\mathscr{B}}}\odot\Big(i_{\mathcal{F}_1(f^1_{\mathscr{A}})}*\rho^{-1}_{\mathscr{B}}\Big)\Big)*i_{\mathrm{t}_{\mathscr{B}}}.\end{aligned}\tag{2.9}$$

By definition of $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$, without loss of generality we can assume that $\mathrm{t}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}}$ (and not only to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$) and that it still verifies (2.9). Then we define a morphism $\mathrm{z}_{\mathscr{B}}:=\mathrm{r}_{\mathscr{B}}\circ\mathrm{t}_{\mathscr{B}}:A'_{\mathscr{B}}\rightarrow\mathcal{F}_0(A'_{\mathscr{A}})$ (this morphism belongs to $\mathbf{W}_{\mathscr{B}}$ by axiom (BF2)) and a morphism $\mathrm{z}'_{\mathscr{B}}:=\mathrm{s}_{\mathscr{B}}\circ\mathrm{t}_{\mathscr{B}}:A'_{\mathscr{B}}\rightarrow A_{\mathscr{B}}$. Moreover, we define an invertible 2-morphism

$$\sigma_{\mathscr{B}} := \theta^{-1}_{\mathrm{v}_{\mathscr{B}},\mathrm{s}_{\mathscr{B}},\mathrm{t}_{\mathscr{B}}} \odot \Big(\rho_{\mathscr{B}} * i_{\mathrm{t}_{\mathscr{B}}}\Big) \odot \theta_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{r}_{\mathscr{B}},\mathrm{t}_{\mathscr{B}}} : \ \mathcal{F}_1(\mathrm{v}_{\mathscr{A}}) \circ \mathrm{z}_{\mathscr{B}} \Longrightarrow \mathrm{v}_{\mathscr{B}} \circ \mathrm{z}'_{\mathscr{B}} .$$

Then identity (2.9) implies that $\alpha_{\mathscr{B}} * i_{\mathrm{z}'_{\mathscr{B}}}$ coincides with the composition (0.4), so (A5) holds. □

3. Sufficiency of conditions (A1) – (A5)

In this section we assume all the hypothesis and notations on $\mathscr{A}, \mathbf{W}_{\mathscr{A}}, \mathscr{B}, \mathbf{W}_{\mathscr{B}}$ and $\mathcal{F}$ of the previous section and we prove that conditions (A) for $\mathcal{F}$ imply conditions (X) for $\widetilde{\mathcal{G}}$.

Lemma 3.1. *If $\mathcal{F}$ satisfies* (A1)*, then $\widetilde{\mathcal{G}}$ satisfies* (X1)*.*

Proof. Using [T2, Corollary 2.7] for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$, [T2, Proposition 2.11(i)] and the fact that $\mathbf{W}_{\mathscr{B}} \subseteq \mathbf{W}_{\mathscr{B},\mathrm{sat}}$, we get that the data of (0.2) give an internal equivalence in $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$ from $\mathcal{F}_0(A_{\mathscr{A}}) = \widetilde{\mathcal{G}}_0(A_{\mathscr{A}})$ to $A_{\mathscr{B}}$, so condition (X1) holds for $\widetilde{\mathcal{G}}$. □

Lemma 3.2. *If $\mathcal{F}$ satisfies* (A2) *and* (A3)*, then $\widetilde{\mathcal{G}}$ satisfies* (X2a)*.*

Proof. Let us fix any pair of objects $A_{\mathscr{A}}, B_{\mathscr{A}}$ and any morphism

$$\underline{f}_{\mathscr{B}} := \Big(\mathcal{F}_0(A_{\mathscr{A}}) \xleftarrow{\mathrm{w}_{\mathscr{B}}} A_{\mathscr{B}} \xrightarrow{f_{\mathscr{B}}} \mathcal{F}_0(B_{\mathscr{A}})\Big)$$

in $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$. We need to prove that there are a morphism $\underline{f}_{\mathscr{A}} : A_{\mathscr{A}} \to B_{\mathscr{A}}$ in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$ and an invertible 2-morphism $\Gamma_{\mathscr{B}} : \widetilde{\mathcal{G}}_1(\underline{f}_{\mathscr{A}}) \Rightarrow \underline{f}_{\mathscr{B}}$ in $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$.

By definition of morphisms in $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$, the morphism $\mathrm{w}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$, so by definition of right saturated there are an object $\widetilde{A}_{\mathscr{B}}$ and a morphism $\mathrm{w}'_{\mathscr{B}} : \widetilde{A}_{\mathscr{B}} \to A_{\mathscr{B}}$, such that $\mathrm{w}_{\mathscr{B}} \circ \mathrm{w}'_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}}$. By (A3) applied to $f_{\mathscr{B}} \circ \mathrm{w}'_{\mathscr{B}}$, there are an object $A'_{\mathscr{A}}$, a morphism $f_{\mathscr{A}} : A'_{\mathscr{A}} \to B_{\mathscr{A}}$ and data $(A'_{\mathscr{B}}, \mathrm{v}^1_{\mathscr{B}}, \mathrm{v}^2_{\mathscr{B}}, \alpha_{\mathscr{B}})$ as follows

$$\begin{tikzcd} & \widetilde{A}_{\mathscr{B}} \arrow[dr, "f_{\mathscr{B}}\circ \mathrm{w}'_{\mathscr{B}}"] \arrow[d, phantom, "\Downarrow \alpha_{\mathscr{B}}"] & \\ A'_{\mathscr{B}} \arrow[ur, "\mathrm{v}^1_{\mathscr{B}}"] \arrow[dr, "\mathrm{v}^2_{\mathscr{B}}"'] & {} & \mathcal{F}_0(B_{\mathscr{A}}), \\ & \mathcal{F}_0(A'_{\mathscr{A}}) \arrow[ur, "\mathcal{F}_1(f_{\mathscr{A}})"'] & \end{tikzcd}$$

with $\mathrm{v}^1_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B}}$, $\mathrm{v}^2_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\alpha_{\mathscr{B}}$ invertible. By (BF2) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B}})$, we have that $(\mathrm{w}_{\mathscr{B}} \circ \mathrm{w}'_{\mathscr{B}}) \circ \mathrm{v}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}}$. So we can apply (A2) to the set of data

$$\mathcal{F}_0(A_{\mathscr{A}}) \xleftarrow{(\mathrm{w}_{\mathscr{B}} \circ \mathrm{w}'_{\mathscr{B}})\circ \mathrm{v}^1_{\mathscr{B}}} A'_{\mathscr{B}} \xrightarrow{\mathrm{v}^2_{\mathscr{B}}} \mathcal{F}_0(A'_{\mathscr{A}}).$$

Therefore there are an object $A^3_{\mathscr{A}}$, a pair of morphisms $\mathrm{w}_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\mathrm{w}'_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A},\mathrm{sat}}$ as follows

$$A_{\mathscr{A}} \xleftarrow{\mathrm{w}_{\mathscr{A}}} A^3_{\mathscr{A}} \xrightarrow{\mathrm{w}'_{\mathscr{A}}} A'_{\mathscr{A}}$$

and a set of data in $\mathscr{B}$ as in the internal part of the following diagram

$$\begin{tikzcd}
 & & A'_{\mathscr{B}} \arrow[dll, "(\mathrm{w}_{\mathscr{B}}\circ \mathrm{w}'_{\mathscr{B}})\circ \mathrm{v}^1_{\mathscr{B}}"'] \arrow[drr, "\mathrm{v}^2_{\mathscr{B}}"] & & \\
\mathcal{F}_0(A_{\mathscr{A}}) \arrow[rr, phantom, "\Downarrow\,\gamma^1_{\mathscr{B}}"] & & \overline{A}_{\mathscr{B}} \arrow[u, "\mathrm{z}^1_{\mathscr{B}}"] \arrow[d, "\mathrm{z}^2_{\mathscr{B}}"] \arrow[rr, phantom, "\Downarrow\,\gamma^2_{\mathscr{B}}"] & & \mathcal{F}_0(A'_{\mathscr{A}}), \\
 & & \mathcal{F}_0(A^3_{\mathscr{A}}) \arrow[ull, "\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})"] \arrow[urr, "\mathcal{F}_1(\mathrm{w}'_{\mathscr{A}})"'] & &
\end{tikzcd}$$

with $\mathrm{z}^1_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B}}$ and both $\gamma^1_{\mathscr{B}}$ and $\gamma^2_{\mathscr{B}}$ invertible. Then we define a pair of invertible 2-morphisms in $\mathscr{B}$

$$\rho^1_{\mathscr{B}} := \theta^{-1}_{\mathrm{w}_{\mathscr{B}},\mathrm{w}'_{\mathscr{B}},\mathrm{v}^1_{\mathscr{B}}\circ \mathrm{z}^1_{\mathscr{B}}} \odot \theta^{-1}_{\mathrm{w}_{\mathscr{B}}\circ \mathrm{w}'_{\mathscr{B}},\mathrm{v}^1_{\mathscr{B}},\mathrm{z}^1_{\mathscr{B}}} \odot \left(\gamma^1_{\mathscr{B}}\right)^{-1} :$$
$$\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})\circ \mathrm{z}^2_{\mathscr{B}} \Longrightarrow \mathrm{w}_{\mathscr{B}}\circ(\mathrm{w}'_{\mathscr{B}}\circ(\mathrm{v}^1_{\mathscr{B}}\circ \mathrm{z}^1_{\mathscr{B}}))$$

and

$$\rho^2_{\mathscr{B}} := \theta^{-1}_{f_{\mathscr{B}},\mathrm{w}'_{\mathscr{B}},\mathrm{v}^1_{\mathscr{B}}\circ \mathrm{z}^1_{\mathscr{B}}} \odot \theta^{-1}_{f_{\mathscr{B}}\circ \mathrm{w}'_{\mathscr{B}},\mathrm{v}^1_{\mathscr{B}},\mathrm{z}^1_{\mathscr{B}}} \odot \Big(\alpha^{-1}_{\mathscr{B}} * i_{\mathrm{z}^1_{\mathscr{B}}}\Big)\odot$$
$$\odot\, \theta_{\mathcal{F}_1(f_{\mathscr{A}}),\mathrm{v}^2_{\mathscr{B}},\mathrm{z}^1_{\mathscr{B}}} \odot \Big(i_{\mathcal{F}_1(f_{\mathscr{A}})} * \left(\gamma^2_{\mathscr{B}}\right)^{-1}\Big) \odot \theta^{-1}_{\mathcal{F}_1(f_{\mathscr{A}}),\mathcal{F}_1(\mathrm{w}'_{\mathscr{A}}),\mathrm{z}^2_{\mathscr{B}}} \odot \Big(\psi^{\mathcal{F}}_{f_{\mathscr{A}},\mathrm{w}'_{\mathscr{A}}} * i_{\mathrm{z}^2_{\mathscr{B}}}\Big) :$$
$$\mathcal{F}_1(f_{\mathscr{A}}\circ \mathrm{w}'_{\mathscr{A}})\circ \mathrm{z}^2_{\mathscr{B}} \Longrightarrow f_{\mathscr{B}}\circ(\mathrm{w}'_{\mathscr{B}}\circ(\mathrm{v}^1_{\mathscr{B}}\circ \mathrm{z}^1_{\mathscr{B}})).$$

Then the following 2-morphism is invertible in $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$

$$\Gamma_{\mathscr{B}} := \Big[\overline{A}_{\mathscr{B}}, \mathrm{z}^2_{\mathscr{B}}, \mathrm{w}'_{\mathscr{B}}\circ(\mathrm{v}^1_{\mathscr{B}}\circ \mathrm{z}^1_{\mathscr{B}}), \rho^1_{\mathscr{B}}, \rho^2_{\mathscr{B}}\Big] :$$
$$\widetilde{\mathcal{G}}_1\Big(A^3_{\mathscr{A}}, \mathrm{w}_{\mathscr{A}}, f_{\mathscr{A}}\circ \mathrm{w}'_{\mathscr{A}}\Big) = \Big(\mathcal{F}_0(A^3_{\mathscr{A}}), \mathcal{F}_1(\mathrm{w}_{\mathscr{A}}), \mathcal{F}_1(f_{\mathscr{A}}\circ \mathrm{w}'_{\mathscr{A}})\Big) \Longrightarrow \underline{f}_{\mathscr{B}}.$$

This suffices to conclude that (X2a) holds for $\widetilde{\mathcal{G}}$. □

Lemma 3.3. *Let us suppose that $\mathcal{F}$ satisfies* (A4). *Let us fix any pair of objects $A_{\mathscr{A}}, B_{\mathscr{A}}$ and any pair of morphisms $(A^m_{\mathscr{A}}, \mathrm{w}^m_{\mathscr{A}}, f^m_{\mathscr{A}}) : A_{\mathscr{A}} \to B_{\mathscr{A}}$ for $m = 1, 2$ in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$. Moreover, let us fix any pair of 2-morphisms*

$$\Gamma^m_{\mathscr{A}} : \Big(A^1_{\mathscr{A}}, \mathrm{w}^1_{\mathscr{A}}, f^1_{\mathscr{A}}\Big) \Longrightarrow \Big(A^2_{\mathscr{A}}, \mathrm{w}^2_{\mathscr{A}}, f^2_{\mathscr{A}}\Big) \quad \text{for} \quad m = 1, 2$$

in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$ and let us suppose that $\widetilde{\mathcal{G}}_2(\Gamma^1_{\mathscr{A}}) = \widetilde{\mathcal{G}}_2(\Gamma^2_{\mathscr{A}})$. Then $\Gamma^1_{\mathscr{A}} = \Gamma^2_{\mathscr{A}}$, i.e. $\widetilde{\mathcal{G}}$ satisfies condition (X2b).

Proof. By [T1, Proposition 0.7] for the pair $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$, there are an object $A^3_{\mathscr{A}}$, a pair of morphisms $\mathrm{v}^1_{\mathscr{A}} : A^3_{\mathscr{A}} \to A^1_{\mathscr{A}}$ and $\mathrm{v}^2_{\mathscr{A}} : A^3_{\mathscr{A}} \to A^2_{\mathscr{A}}$, an invertible 2-morphism $\alpha_{\mathscr{A}} : \mathrm{w}^1_{\mathscr{A}}\circ \mathrm{v}^1_{\mathscr{A}} \Rightarrow \mathrm{w}^2_{\mathscr{A}}\circ \mathrm{v}^2_{\mathscr{A}}$ and a pair of 2-morphisms $\gamma^m_{\mathscr{A}} : f^1_{\mathscr{A}}\circ \mathrm{v}^1_{\mathscr{A}} \Rightarrow f^2_{\mathscr{A}}\circ \mathrm{v}^2_{\mathscr{A}}$ for $m = 1, 2$, such that

$$\Gamma^m_{\mathscr{A}} = \Big[A^3_{\mathscr{A}}, \mathrm{v}^1_{\mathscr{A}}, \mathrm{v}^2_{\mathscr{A}}, \alpha_{\mathscr{A}}, \gamma^m_{\mathscr{A}}\Big] \quad \text{for} \quad m = 1, 2. \tag{3.1}$$

Then we have the following identity in $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$:

$$\Big[\mathcal{F}_0(A^3_{\mathscr{A}}), \mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}}), \mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}}), \psi^{\mathcal{F}}_{\mathrm{w}^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}}} \odot \mathcal{F}_2(\alpha_{\mathscr{A}}) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{w}^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}}}\Big)^{-1},$$

$$\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}}} \odot \mathcal{F}_2(\gamma^1_{\mathscr{A}}) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}}}\Big)^{-1}\Big] \overset{(0.1)}{=} \widetilde{\mathcal{G}}_2(\Gamma^1_{\mathscr{A}}) = \widetilde{\mathcal{G}}_2(\Gamma^2_{\mathscr{A}}) \overset{(0.1)}{=}$$
$$= \Big[\mathcal{F}_0(A^3_{\mathscr{A}}), \mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}}), \mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}}), \psi^{\mathcal{F}}_{\mathrm{w}^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}}} \odot \mathcal{F}_2(\alpha_{\mathscr{A}}) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{w}^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}}}\Big)^{-1},$$
$$\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}}} \odot \mathcal{F}_2(\gamma^2_{\mathscr{A}}) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}}}\Big)^{-1}\Big].$$

Then by [T1, Proposition 0.7] for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{sat}})$, there are an object $A^1_{\mathscr{B}}$ and a morphism $\mathrm{z}^1_{\mathscr{B}} : A^1_{\mathscr{B}} \to \mathcal{F}_0(A^3_{\mathscr{A}})$ in $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$, such that

$$\Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}}} \odot \mathcal{F}_2(\gamma^1_{\mathscr{A}}) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}}}\Big)^{-1}\Big) * i_{\mathrm{z}^1_{\mathscr{B}}} =$$
$$= \Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}}} \odot \mathcal{F}_2(\gamma^2_{\mathscr{A}}) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}}}\Big)^{-1}\Big) * i_{\mathrm{z}^1_{\mathscr{B}}}. \tag{3.2}$$

By definition of $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$, there are an object $A^2_{\mathscr{B}}$ and a morphism $\mathrm{z}^2_{\mathscr{B}} : A^2_{\mathscr{B}} \to A^1_{\mathscr{B}}$, such that $\mathrm{z}^1_{\mathscr{B}} \circ \mathrm{z}^2_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}}$. Then from (3.2) we get easily

$$\mathcal{F}_2(\gamma^1_{\mathscr{A}}) * i_{\mathrm{z}^1_{\mathscr{B}} \circ \mathrm{z}^2_{\mathscr{B}}} = \mathcal{F}_2(\gamma^2_{\mathscr{A}}) * i_{\mathrm{z}^1_{\mathscr{B}} \circ \mathrm{z}^2_{\mathscr{B}}}.$$

By (A4) there are an object $A^4_{\mathscr{A}}$ and a morphism $\mathrm{z}_{\mathscr{A}} : A^4_{\mathscr{A}} \to A^3_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, such that $\gamma^1_{\mathscr{A}} * i_{\mathrm{z}_{\mathscr{A}}} = \gamma^2_{\mathscr{A}} * i_{\mathrm{z}_{\mathscr{A}}}$; using (3.1), this implies easily that $\Gamma^1_{\mathscr{A}} = \Gamma^2_{\mathscr{A}}$. □

In order to prove that conditions (A) imply also condition (X2c), we need firstly to prove a preliminary lemma as follows:

Lemma 3.4. *Let us suppose that* (A4) *and* (A5) *hold. Let us fix any set of data* $(A_{\mathscr{A}}, B_{\mathscr{A}}, A_{\mathscr{B}}, f^1_{\mathscr{A}}, f^2_{\mathscr{A}}, \mathrm{v}_{\mathscr{B}}, \alpha_{\mathscr{B}})$ *as in* (A5) *and let us suppose that* $\alpha_{\mathscr{B}}$ *is* invertible. *Then there is a set of data* $(A'_{\mathscr{A}}, A'_{\mathscr{B}}, \mathrm{v}_{\mathscr{A}}, \mathrm{z}_{\mathscr{B}}, \mathrm{z}'_{\mathscr{B}}, \alpha_{\mathscr{A}}, \sigma_{\mathscr{B}})$ *satisfying all the conditions of* (A5) *and such that* $\alpha_{\mathscr{A}}$ *is* invertible.

Proof. Using (A5), there are a pair of objects $\widetilde{A}'_{\mathscr{A}}$, $\widetilde{A}'_{\mathscr{B}}$, a triple of morphisms $\widetilde{\mathrm{v}}_{\mathscr{A}} : \widetilde{A}'_{\mathscr{A}} \to A_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, $\widetilde{\mathrm{z}}_{\mathscr{B}} : \widetilde{A}'_{\mathscr{B}} \to \mathcal{F}_0(\widetilde{A}'_{\mathscr{A}})$ in $\mathbf{W}_{\mathscr{B}}$ and $\widetilde{\mathrm{z}}'_{\mathscr{B}} : \widetilde{A}'_{\mathscr{B}} \to A_{\mathscr{B}}$, a 2-morphism

$$\begin{tikzcd}
 & A_{\mathscr{A}} \arrow[dd, phantom, "\Downarrow\,\widetilde{\alpha}_{\mathscr{A}}"] \arrow[dr, "f^1_{\mathscr{A}}"] & \\
\widetilde{A}'_{\mathscr{A}} \arrow[ur, "\widetilde{\mathrm{v}}_{\mathscr{A}}"] \arrow[dr, "\widetilde{\mathrm{v}}_{\mathscr{A}}"'] & & B_{\mathscr{A}} \\
 & A_{\mathscr{A}} \arrow[ur, "f^2_{\mathscr{A}}"'] &
\end{tikzcd}$$

and an invertible 2-morphism

$$\begin{tikzcd}
 & \mathcal{F}_0(\widetilde{A}'_{\mathscr{A}}) \arrow[dd, phantom, "\Downarrow\,\widetilde{\sigma}_{\mathscr{B}}"] \arrow[dr, "\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}})"] & \\
\widetilde{A}'_{\mathscr{B}} \arrow[ur, "\widetilde{\mathrm{z}}_{\mathscr{B}}"] \arrow[dr, "\widetilde{\mathrm{z}}'_{\mathscr{B}}"'] & & \mathcal{F}_0(A_{\mathscr{A}}), \\
 & A_{\mathscr{B}} \arrow[ur, "\mathrm{v}_{\mathscr{B}}"'] &
\end{tikzcd}$$

such that $\alpha_{\mathscr{B}} * i_{\widetilde{\mathrm{z}}'_{\mathscr{B}}}$ coincides with the following composition:

$$\begin{tikzcd}[column sep=large]
{} & {} & A_{\mathscr{B}} \arrow[ddr, "\mathcal{F}_1(f^1_{\mathscr{A}})\circ \mathrm{v}_{\mathscr{B}}"] \arrow[d, phantom, "\Downarrow\, \theta^{-1}_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathrm{v}_{\mathscr{B}},\widetilde{\mathrm{z}}'_{\mathscr{B}}}"] & {} \\
{} & \mathcal{F}_0(A_{\mathscr{A}}) \arrow[drr, "\mathcal{F}_1(f^1_{\mathscr{A}})"] \arrow[d, phantom, "\Downarrow\, \theta_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}}),\widetilde{\mathrm{z}}_{\mathscr{B}}}"] & {} & {} \\
\widetilde{A}'_{\mathscr{B}} \arrow[uurr, bend left, "\widetilde{\mathrm{z}}'_{\mathscr{B}}"] \arrow[ur, bend left, "\mathrm{v}_{\mathscr{B}}\circ\widetilde{\mathrm{z}}'_{\mathscr{B}}"] \arrow[ur, "\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}})\circ\widetilde{\mathrm{z}}_{\mathscr{B}}"'] \arrow[r, "\widetilde{\mathrm{z}}_{\mathscr{B}}"] \arrow[dr, "\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}})\circ\widetilde{\mathrm{z}}_{\mathscr{B}}"] \arrow[dr, bend right, "\mathrm{v}_{\mathscr{B}}\circ\widetilde{\mathrm{z}}'_{\mathscr{B}}"'] \arrow[ddrr, bend right, "\widetilde{\mathrm{z}}'_{\mathscr{B}}"'] \arrow[u, phantom, "\Downarrow\, \widetilde{\sigma}^{-1}_{\mathscr{B}}"] \arrow[d, phantom, "\Downarrow\, \widetilde{\sigma}_{\mathscr{B}}"] & \mathcal{F}_0(\widetilde{A}'_{\mathscr{A}}) \arrow[rr, bend left, "\mathcal{F}_1(f^1_{\mathscr{A}})\circ\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}})"] \arrow[rr, bend right, "\mathcal{F}_1(f^2_{\mathscr{A}})\circ\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}})"'] \arrow[rr, phantom, "\Downarrow\, \psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\widetilde{\mathrm{v}}_{\mathscr{A}}}\odot\mathcal{F}_2(\widetilde{\alpha}_{\mathscr{A}})\odot\left(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\widetilde{\mathrm{v}}_{\mathscr{A}}}\right)^{-1}"] & {} & \mathcal{F}_0(B_{\mathscr{A}}). \\
{} & \mathcal{F}_0(A_{\mathscr{A}}) \arrow[urr, "\mathcal{F}_1(f^2_{\mathscr{A}})"'] \arrow[u, phantom, "\Downarrow\, \theta^{-1}_{\mathcal{F}_1(f^2_{\mathscr{A}}),\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}}),\widetilde{\mathrm{z}}_{\mathscr{B}}}"] & {} & {} \\
{} & {} & A_{\mathscr{B}} \arrow[uur, "\mathcal{F}_1(f^2_{\mathscr{A}})\circ\mathrm{v}_{\mathscr{B}}"'] \arrow[u, phantom, "\Downarrow\, \theta_{\mathcal{F}_1(f^2_{\mathscr{A}}),\mathrm{v}_{\mathscr{B}},\widetilde{\mathrm{z}}'_{\mathscr{B}}}"] & {}
\end{tikzcd} \tag{3.3}$$

Now we define a 2-morphism $\Gamma_{\mathscr{A}} : (A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, f^1_{\mathscr{A}}) \Rightarrow (A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, f^2_{\mathscr{A}})$ as the 2-morphism represented by the following diagram:

$$\begin{tikzcd}
 & A_{\mathscr{A}} \arrow[dl, "\mathrm{id}_{A_{\mathscr{A}}}"'] \arrow[dr, "f^1_{\mathscr{A}}"] & \\
A_{\mathscr{A}} \arrow[r, phantom, "\Downarrow\, i_{\mathrm{id}_{A_{\mathscr{A}}}\circ\widetilde{\mathrm{v}}_{\mathscr{A}}}"] & \widetilde{A}'_{\mathscr{A}} \arrow[u, "\widetilde{\mathrm{v}}_{\mathscr{A}}"'] \arrow[d, "\widetilde{\mathrm{v}}_{\mathscr{A}}"] \arrow[r, phantom, "\Downarrow\, \widetilde{\alpha}_{\mathscr{A}}"] & B_{\mathscr{A}}. \\
 & A_{\mathscr{A}} \arrow[ul, "\mathrm{id}_{A_{\mathscr{A}}}"] \arrow[ur, "f^2_{\mathscr{A}}"'] &
\end{tikzcd}$$

Using (0.1), we get that $\widetilde{\mathcal{G}}_2(\Gamma_{\mathscr{A}})$ coincides with the class

$$\Big[\mathcal{F}_0(\widetilde{A}'_{\mathscr{A}}), \mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}}), \mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}}), i_{\mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}})\circ\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}})}, \psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\widetilde{\mathrm{v}}_{\mathscr{A}}} \odot \mathcal{F}_2(\widetilde{\alpha}_{\mathscr{A}}) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\widetilde{\mathrm{v}}_{\mathscr{A}}}\Big)^{-1}\Big].$$

Now we consider the 2-morphism in $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$ defined as follows

$$\begin{aligned}\Gamma_{\mathscr{B}} := \Big[A_{\mathscr{B}}, \mathrm{v}_{\mathscr{B}}, \mathrm{v}_{\mathscr{B}}, i_{\mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}})\circ\mathrm{v}_{\mathscr{B}}}, \alpha_{\mathscr{B}}\Big] : \Big(\mathcal{F}_0(A_{\mathscr{A}}), \mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}}), \mathcal{F}_1(f^1_{\mathscr{A}})\Big) \Longrightarrow \\ \Longrightarrow \Big(\mathcal{F}_0(A_{\mathscr{A}}), \mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}}), \mathcal{F}_1(f^2_{\mathscr{A}})\Big);\end{aligned}$$

using the definition of 2-morphism in a bicategory of fractions (see [Pr, § 2.3]) together with the following diagram

$$\begin{tikzcd}
 & \mathcal{F}_0(A_{\mathscr{A}}) \arrow[d, phantom, "\overset{\Rightarrow}{\widetilde{\sigma}^{-1}_{\mathscr{B}}}"] & \\
A_{\mathscr{B}} \arrow[ur, "\mathrm{v}_{\mathscr{B}}"] \arrow[dr, "\mathrm{v}_{\mathscr{B}}"'] & \widetilde{A}'_{\mathscr{B}} \arrow[l, "\widetilde{\mathrm{z}}'_{\mathscr{B}}"'] \arrow[r, "\widetilde{\mathrm{z}}_{\mathscr{B}}"] \arrow[d, phantom, "\underset{\Leftarrow}{\widetilde{\sigma}_{\mathscr{B}}}"] & \mathcal{F}_0(\widetilde{A}'_{\mathscr{A}}) \arrow[ul, "\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}})"'] \arrow[dl, "\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}})"] \\
 & \mathcal{F}_0(A_{\mathscr{A}}) &
\end{tikzcd}$$

and (3.3), we get easily that $\widetilde{\mathcal{G}}_2(\Gamma_{\mathscr{A}}) = \Gamma_{\mathscr{B}}$.

Since $\alpha_{\mathscr{B}}$ is invertible by hypothesis, then it makes sense to consider the inverse for $\Gamma_{\mathscr{B}}$, defined as follows:

$$\Gamma_{\mathscr{B}}^{-1} := \Big[A_{\mathscr{B}}, \mathrm{v}_{\mathscr{B}}, \mathrm{v}_{\mathscr{B}}, i_{\mathcal{F}_1(\mathrm{id}_{A_{\mathscr{A}}})\circ \mathrm{v}_{\mathscr{B}}}, \alpha_{\mathscr{B}}^{-1}\Big].$$

Using again (A5) on the set of data $(A_{\mathscr{A}}, B_{\mathscr{A}}, A_{\mathscr{B}}, f^2_{\mathscr{A}}, f^1_{\mathscr{A}}, \mathrm{v}_{\mathscr{B}}, \alpha_{\mathscr{B}}^{-1})$ and proceeding as above, we get a 2-morphism $\Gamma'_{\mathscr{A}} : (A_{\mathscr{A}}, \mathrm{id}_{\mathscr{A}}, f^2_{\mathscr{A}}) \Rightarrow (A_{\mathscr{A}}, \mathrm{id}_{\mathscr{A}}, f^1_{\mathscr{A}})$ in $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$, such that $\widetilde{\mathcal{G}}_2(\Gamma'_{\mathscr{A}}) = \Gamma_{\mathscr{B}}^{-1}$. Then

$$\widetilde{\mathcal{G}}_2(\Gamma'_{\mathscr{A}} \odot \Gamma_{\mathscr{A}}) = \Gamma_{\mathscr{B}}^{-1} \odot \Gamma_{\mathscr{B}} = i_{\widetilde{\mathcal{G}}_1(A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, f^1_{\mathscr{A}})} \stackrel{(0.1)}{=} \widetilde{\mathcal{G}}_2\left(i_{(A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, f^1_{\mathscr{A}})}\right).$$

Since we are assuming condition (A4), then by Lemma 3.3 we conclude that $\Gamma'_{\mathscr{A}} \odot \Gamma_{\mathscr{A}} = i_{(A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, f^1_{\mathscr{A}})}$. Analogously, we get that $\Gamma_{\mathscr{A}} \odot \Gamma'_{\mathscr{A}} = i_{(A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, f^2_{\mathscr{A}})}$. This proves that $\Gamma_{\mathscr{A}}$ is invertible. So by [T1, Proposition 0.8(iii)] for $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$, there are an object $A'_{\mathscr{A}}$ and a morphism $\mathrm{p}_{\mathscr{A}} : A'_{\mathscr{A}} \to \widetilde{A}'_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, such that $\widetilde{\alpha}_{\mathscr{A}} * i_{\mathrm{p}_{\mathscr{A}}}$ is invertible. Now we use axiom (BF3) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B}})$ in order to get data as in the upper part of the following diagram

$$\begin{tikzcd}
& A'_{\mathscr{B}} \arrow[dl, "\mathrm{z}_{\mathscr{B}}"'] \arrow[dr, "\mathrm{q}_{\mathscr{B}}"] \arrow[d, phantom, "\stackrel{\mu_{\mathscr{B}}}{\Rightarrow}"] & \\
\mathcal{F}_0(A'_{\mathscr{A}}) \arrow[r, "\mathcal{F}_1(\mathrm{p}_{\mathscr{A}})"'] & \mathcal{F}_0(\widetilde{A}'_{\mathscr{A}}) & \widetilde{A}'_{\mathscr{B}}, \arrow[l, "\widetilde{\mathrm{z}}_{\mathscr{B}}"]
\end{tikzcd}$$

with $\mathrm{z}_{\mathscr{B}}$ in $\mathbf{W}_{\mathscr{B}}$ and $\mu_{\mathscr{B}}$ invertible. Then we define a morphism $\mathrm{v}_{\mathscr{A}} := \widetilde{\mathrm{v}}_{\mathscr{A}} \circ \mathrm{p}_{\mathscr{A}} : A'_{\mathscr{A}} \to A_{\mathscr{A}}$ (this morphism belongs to $\mathbf{W}_{\mathscr{A}}$ by (BF2)) and a morphism $\mathrm{z}'_{\mathscr{B}} := \widetilde{\mathrm{z}}'_{\mathscr{B}} \circ \mathrm{q}_{\mathscr{B}} : A'_{\mathscr{B}} \to A_{\mathscr{B}}$. Moreover, we define a 2-morphism

$$\alpha_{\mathscr{A}} := \theta^{-1}_{f^2_{\mathscr{A}}, \widetilde{\mathrm{v}}_{\mathscr{A}}, \mathrm{p}_{\mathscr{A}}} \odot \left(\widetilde{\alpha}_{\mathscr{A}} * i_{\mathrm{p}_{\mathscr{A}}}\right) \odot \theta_{f^1_{\mathscr{A}}, \widetilde{\mathrm{v}}_{\mathscr{A}}, \mathrm{p}_{\mathscr{A}}} : \; f^1_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}} \Longrightarrow f^2_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}}.$$

Such a 2-morphism is invertible because $\widetilde{\alpha}_{\mathscr{A}} * i_{\mathrm{p}_{\mathscr{A}}}$ is invertible by construction. In addition, we define an invertible 2-morphism $\sigma_{\mathscr{B}} : \mathcal{F}_1(\mathrm{v}_{\mathscr{A}}) \circ \mathrm{z}_{\mathscr{B}} \Rightarrow \mathrm{v}_{\mathscr{B}} \circ \mathrm{z}'_{\mathscr{B}}$ as the following composition:

$$\begin{tikzcd}[column sep=large, row sep=large]
& \mathcal{F}_0(A'_{\mathscr{A}}) \arrow[rr, bend left, "\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})=\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}}\circ \mathrm{p}_{\mathscr{A}})"] \arrow[drr, "\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}})\circ\mathcal{F}_1(\mathrm{p}_{\mathscr{A}})"'] \arrow[d, phantom, "\Downarrow\,\theta^{-1}_{\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}}),\mathcal{F}_1(\mathrm{p}_{\mathscr{A}}),\mathrm{z}_{\mathscr{B}}}"] \arrow[rr, phantom, shift right=4ex, "\Downarrow\,\psi^{\mathcal{F}}_{\widetilde{\mathrm{v}}_{\mathscr{A}},\mathrm{p}_{\mathscr{A}}}"] & \\
A'_{\mathscr{B}} \arrow[ur, bend left, "\mathrm{z}_{\mathscr{B}}"] \arrow[r, bend left, "\mathcal{F}_1(\mathrm{p}_{\mathscr{A}})\circ \mathrm{z}_{\mathscr{B}}"] \arrow[r, bend right, "\widetilde{\mathrm{z}}_{\mathscr{B}}\circ \mathrm{q}_{\mathscr{B}}"'] \arrow[r, phantom, "\Downarrow\,\mu_{\mathscr{B}}"] \arrow[dr, "\mathrm{q}_{\mathscr{B}}"'] \arrow[ddr, bend right, "\mathrm{z}'_{\mathscr{B}}=\widetilde{\mathrm{z}}'_{\mathscr{B}}\circ \mathrm{q}_{\mathscr{B}}"'] & \mathcal{F}_0(\widetilde{A}'_{\mathscr{A}}) \arrow[r, "\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}})"] \arrow[d, phantom, "\Downarrow\,\theta_{\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}}),\widetilde{\mathrm{z}}_{\mathscr{B}},\mathrm{q}_{\mathscr{B}}}"] & \mathcal{F}_0(A_{\mathscr{A}}). \\
& \widetilde{A}'_{\mathscr{B}} \arrow[ur, bend left, "\mathcal{F}_1(\widetilde{\mathrm{v}}_{\mathscr{A}})\circ\widetilde{\mathrm{z}}_{\mathscr{B}}"] \arrow[ur, "\mathrm{v}_{\mathscr{B}}\circ\widetilde{\mathrm{z}}'_{\mathscr{B}}"'] \arrow[ur, phantom, shift right=3ex, "\Downarrow\,\widetilde{\sigma}_{\mathscr{B}}"] \arrow[d, phantom, "\Downarrow\,\theta^{-1}_{\mathrm{v}_{\mathscr{B}},\widetilde{\mathrm{z}}'_{\mathscr{B}},\mathrm{q}_{\mathscr{B}}}"] & \\
& A_{\mathscr{B}} \arrow[uur, bend right, "\mathrm{v}_{\mathscr{B}}"'] &
\end{tikzcd}$$

Now we recall that $\alpha_{\mathscr{B}} * i_{\widetilde{\mathrm{z}}'_{\mathscr{B}}}$ coincides with diagram (3.3). Then it is not difficult to prove that the set of data $(A'_{\mathscr{A}}, A'_{\mathscr{B}}, \mathrm{v}_{\mathscr{A}}, \mathrm{z}_{\mathscr{B}}, \mathrm{z}'_{\mathscr{B}}, \alpha_{\mathscr{A}}, \sigma_{\mathscr{B}})$ satisfies the claim: basically, one has to consider the composition of (3.3) with $i_{\mathrm{q}_{\mathscr{B}}}$; then one has to insert in the center of such a diagram $\mu_{\mathscr{B}}$ and $\mu_{\mathscr{B}}^{-1}$, together with all the necessary associators of $\mathscr{B}$. $\square$

Lemma 3.5. *If $\mathcal{F}$ satisfies* (A2), (A4) *and* (A5), *then $\widetilde{\mathcal{G}}$ satisfies* (X2c).

Proof. For simplicity of exposition, we are giving this proof only the special case when both $\mathscr{A}$ and $\mathscr{B}$ are 2-categories instead of bicategories. The interested reader can easily fill out the details for the general case.

Let us fix any pair of objects $A_{\mathscr{A}}, B_{\mathscr{A}}$ and any pair of morphisms $(A^m_{\mathscr{A}}, \mathrm{w}^m_{\mathscr{A}}, f^m_{\mathscr{A}}): A_{\mathscr{A}} \to B_{\mathscr{A}}$ in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$ for $m = 1, 2$. Moreover, let us fix any 2-morphism

$$\Gamma_{\mathscr{B}}: \ \widetilde{\mathcal{G}}_1\Big(A^1_{\mathscr{A}}, \mathrm{w}^1_{\mathscr{A}}, f^1_{\mathscr{A}}\Big) \Longrightarrow \widetilde{\mathcal{G}}_1\Big(A^2_{\mathscr{A}}, \mathrm{w}^2_{\mathscr{A}}, f^2_{\mathscr{A}}\Big)$$

in $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$. We need to prove that there is a 2-morphism $\Gamma_{\mathscr{A}}: (A^1_{\mathscr{A}}, \mathrm{w}^1_{\mathscr{A}}, f^1_{\mathscr{A}}) \Rightarrow (A^2_{\mathscr{A}}, \mathrm{w}^2_{\mathscr{A}}, f^2_{\mathscr{A}})$ in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$, such that $\widetilde{\mathcal{G}}_2(\Gamma_{\mathscr{A}}) = \Gamma_{\mathscr{B}}$.

By construction of $\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$, $\Gamma_{\mathscr{B}}$ is represented by a set of data as follows

$$\begin{tikzcd}
 & \mathcal{F}_0(A^1_{\mathscr{A}}) \arrow[dl, "\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})"'] \arrow[dr, "\mathcal{F}_1(f^1_{\mathscr{A}})"] & \\
\mathcal{F}_0(A_{\mathscr{A}}) \arrow[r, phantom, "\Downarrow \gamma_{\mathscr{B}}"] & A_{\mathscr{B}} \arrow[u, "\mathrm{v}^1_{\mathscr{B}}"] \arrow[d, "\mathrm{v}^2_{\mathscr{B}}"'] \arrow[r, phantom, "\Downarrow \alpha_{\mathscr{B}}"] & \mathcal{F}_0(B_{\mathscr{A}}), \\
 & \mathcal{F}_0(A^2_{\mathscr{A}}) \arrow[ul, "\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})"] \arrow[ur, "\mathcal{F}_1(f^2_{\mathscr{A}})"'] &
\end{tikzcd} \tag{3.4}$$

such that $\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}}) \circ \mathrm{v}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ and $\gamma_{\mathscr{B}}$ is invertible. We recall that in all this section we are assuming that $\mathcal{F}_1(\mathbf{W}_{\mathscr{A}}) \subseteq \mathbf{W}_{\mathscr{B},\mathrm{sat}}$, so $\mathcal{F}_1(\mathrm{w}^m_{\mathscr{A}})$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ for each $m = 1, 2$. Therefore using [T2, Proposition 2.11(ii)], $\mathrm{v}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. Hence, using the definition of right saturation there is no loss of generality in assuming that the data in (3.4) are such that $\mathrm{v}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}}$ and $\gamma_{\mathscr{B}}$ is invertible. Moreover, using (BF5) for $(\mathscr{B}, \mathbf{W}_{\mathscr{B}})$ and $\gamma^{-1}_{\mathscr{B}}$, we get that $\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}}) \circ \mathrm{v}^2_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$, hence also $\mathrm{v}^2_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ by [T2, Proposition 2.11(ii)]. If we apply (A2) to the set of data

$$\mathcal{F}_0(A^1_{\mathscr{A}}) \xleftarrow{\ \mathrm{v}^1_{\mathscr{B}}\ } A_{\mathscr{B}} \xrightarrow{\ \mathrm{v}^2_{\mathscr{B}}\ } \mathcal{F}_0(A^2_{\mathscr{A}}),$$

we get an object $A^3_{\mathscr{A}}$, a pair of morphisms $\mathrm{v}^1_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\mathrm{v}^2_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A},\mathrm{sat}}$ as follows

$$A^1_{\mathscr{A}} \xleftarrow{\ \mathrm{v}^1_{\mathscr{A}}\ } A^3_{\mathscr{A}} \xrightarrow{\ \mathrm{v}^2_{\mathscr{A}}\ } A^2_{\mathscr{A}}$$

and a set of data in $\mathscr{B}$ as in the internal part of the following diagram

$$
\begin{tikzcd}
 & & A_{\mathscr{B}} \arrow[dll, "\mathrm{v}^1_{\mathscr{B}}"'] \arrow[drr, "\mathrm{v}^2_{\mathscr{B}}"] & & \\
\mathcal{F}_0(A^1_{\mathscr{A}}) \arrow[rr, phantom, "\Downarrow \phi^1_{\mathscr{B}}"] & & \overline{A}''_{\mathscr{B}} \arrow[u, "\mathrm{u}'_{\mathscr{B}}"'] \arrow[d, "\mathrm{u}_{\mathscr{B}}"] \arrow[rr, phantom, "\Downarrow \phi^2_{\mathscr{B}}"] & & \mathcal{F}_0(A^2_{\mathscr{A}}), \\
 & & \mathcal{F}_0(A^3_{\mathscr{A}}) \arrow[ull, "\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}})"] \arrow[urr, "\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}})"'] & &
\end{tikzcd}
$$

such that $\mathrm{u}'_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}}$ and both $\phi^1_{\mathscr{B}}$ and $\phi^2_{\mathscr{B}}$ are invertible. Since $\phi^1_{\mathscr{B}}$ is invertible, then by (BF2) and (BF5) for $(\mathscr{B},\mathbf{W}_{\mathscr{B}})$ we get that $\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}})\circ\mathrm{u}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}}\subseteq\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. Moreover, $\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}})$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ because $\mathcal{F}_1(\mathbf{W}_{\mathscr{A}})\subseteq\mathbf{W}_{\mathscr{B},\mathrm{sat}}$ by hypothesis. So again by [T2, Proposition 2.11(ii)] we get that $\mathrm{u}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{sat}}$. So by definition of right saturation there are an object $A''_{\mathscr{B}}$ and a morphism $\mathrm{r}_{\mathscr{B}}: A''_{\mathscr{B}}\rightarrow\overline{A}''_{\mathscr{B}}$, such that $\mathrm{u}_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B}}$. Then we set:

$$
\begin{gathered}
\gamma'_{\mathscr{B}} := \Big(i_{\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})} * \phi^2_{\mathscr{B}} * i_{\mathrm{r}_{\mathscr{B}}}\Big) \odot \Big(\gamma_{\mathscr{B}} * i_{\mathrm{u}'_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}}}\Big) \odot \Big(i_{\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})} * \big(\phi^1_{\mathscr{B}}\big)^{-1} * i_{\mathrm{r}_{\mathscr{B}}}\Big): \\
\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})\circ\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}})\circ\mathrm{u}_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}} \Longrightarrow \mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})\circ\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}})\circ\mathrm{u}_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}}
\end{gathered}
$$

and

$$
\begin{gathered}
\alpha'_{\mathscr{B}} := \Big(i_{\mathcal{F}_1(f^2_{\mathscr{A}})} * \phi^2_{\mathscr{B}} * i_{\mathrm{r}_{\mathscr{B}}}\Big) \odot \Big(\alpha_{\mathscr{B}} * i_{\mathrm{u}'_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}}}\Big) \odot \Big(i_{\mathcal{F}_1(f^1_{\mathscr{A}})} * \big(\phi^1_{\mathscr{B}}\big)^{-1} * i_{\mathrm{r}_{\mathscr{B}}}\Big): \\
\mathcal{F}_1(f^1_{\mathscr{A}})\circ\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}})\circ\mathrm{u}_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}} \Longrightarrow \mathcal{F}_1(f^2_{\mathscr{A}})\circ\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}})\circ\mathrm{u}_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}}\,.
\end{gathered}
$$

Then if we use the following diagram

$$
\begin{tikzcd}
 & \mathcal{F}_0(A^1_{\mathscr{A}}) \arrow[d, phantom, "\Rightarrow \; \big(\phi^1_{\mathscr{B}}\big)^{-1} * i_{\mathrm{r}_{\mathscr{B}}}"] & \\
A''_{\mathscr{B}} \arrow[ur, "\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}})\circ\mathrm{u}_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}}"] \arrow[dr, "\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}})\circ\mathrm{u}_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}}"'] & A''_{\mathscr{B}} \arrow[l, "\mathrm{id}_{A''_{\mathscr{B}}}"'] \arrow[r, "\mathrm{u}'_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}}"] \arrow[d, phantom, "\phi^2_{\mathscr{B}} * i_{\mathrm{r}_{\mathscr{B}}} \; \Leftarrow"] & A_{\mathscr{B}} \arrow[ul, "\mathrm{v}^1_{\mathscr{B}}"'] \arrow[dl, "\mathrm{v}^2_{\mathscr{B}}"] \\
 & \mathcal{F}_0(A^2_{\mathscr{A}}) &
\end{tikzcd}
$$

together with (3.4), we get easily that

$$
\Gamma_{\mathscr{B}} = \Big[A''_{\mathscr{B}}, \mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}})\circ\mathrm{u}_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}}, \mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}})\circ\mathrm{u}_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}}, \gamma'_{\mathscr{B}}, \alpha'_{\mathscr{B}}\Big]. \tag{3.5}
$$

Now we define

$$
\begin{gathered}
\widetilde{\alpha}_{\mathscr{B}} := \Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}}} * i_{\mathrm{u}_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}}}\Big)^{-1} \odot \alpha'_{\mathscr{B}} \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}}} * i_{\mathrm{u}_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}}}\Big): \\
\mathcal{F}_1(f^1_{\mathscr{A}}\circ\mathrm{v}^1_{\mathscr{A}})\circ\mathrm{u}_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}} \Longrightarrow \mathcal{F}_1(f^2_{\mathscr{A}}\circ\mathrm{v}^2_{\mathscr{A}})\circ\mathrm{u}_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}}\,.
\end{gathered} \tag{3.6}
$$

Then we apply (A5) to the set of data

$$
A^3_{\mathscr{A}}, \quad B_{\mathscr{A}}, \quad A''_{\mathscr{B}}, \quad f^1_{\mathscr{A}}\circ\mathrm{v}^1_{\mathscr{A}}, \quad f^2_{\mathscr{A}}\circ\mathrm{v}^2_{\mathscr{A}}, \quad \mathrm{u}_{\mathscr{B}}\circ\mathrm{r}_{\mathscr{B}}, \quad \widetilde{\alpha}_{\mathscr{B}},
$$

so there are a pair of objects $A'_{\mathscr{A}}, A'_{\mathscr{B}}$, a triple of morphisms $\mathrm{u}_{\mathscr{A}} : A'_{\mathscr{A}} \to A^3_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, $\mathrm{z}_{\mathscr{B}} : A'_{\mathscr{B}} \to \mathcal{F}_0(A'_{\mathscr{A}})$ in $\mathbf{W}_{\mathscr{B}}$ and $\mathrm{z}'_{\mathscr{B}} : A'_{\mathscr{B}} \to A''_{\mathscr{B}}$, a 2-morphism

$$\begin{tikzcd}
& A^3_{\mathscr{A}} \arrow[dr, bend left, "f^1_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}}"] \arrow[d, phantom, "\Downarrow \alpha_{\mathscr{A}}"] & \\
A'_{\mathscr{A}} \arrow[ur, bend left, "\mathrm{u}_{\mathscr{A}}"] \arrow[dr, bend right, "\mathrm{u}_{\mathscr{A}}"'] & {} & B_{\mathscr{A}} \\
& A^3_{\mathscr{A}} \arrow[ur, bend right, "f^2_{\mathscr{A}} \circ \mathrm{v}^2_{\mathscr{A}}"'] &
\end{tikzcd}$$

and an invertible 2-morphism

$$\begin{tikzcd}
& \mathcal{F}_0(A'_{\mathscr{A}}) \arrow[dr, bend left, "\mathcal{F}_1(\mathrm{u}_{\mathscr{A}})"] \arrow[d, phantom, "\Downarrow \sigma_{\mathscr{B}}"] & \\
A'_{\mathscr{B}} \arrow[ur, bend left, "\mathrm{z}_{\mathscr{B}}"] \arrow[dr, bend right, "\mathrm{z}'_{\mathscr{B}}"'] & {} & \mathcal{F}_0(A^3_{\mathscr{A}}), \\
& A''_{\mathscr{B}} \arrow[ur, bend right, "\mathrm{u}_{\mathscr{B}} \circ \mathrm{r}_{\mathscr{B}}"'] &
\end{tikzcd}$$

such that:

$$\begin{aligned}\widetilde{\alpha}_{\mathscr{B}} * i_{\mathrm{z}'_{\mathscr{B}}} = \Big(i_{\mathcal{F}_1(f^2_{\mathscr{A}} \circ \mathrm{v}^2_{\mathscr{A}})} * \sigma_{\mathscr{B}}\Big) \odot \Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}} \circ \mathrm{v}^2_{\mathscr{A}}, \mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}}\Big) \odot \Big(\mathcal{F}_2(\alpha_{\mathscr{A}}) * i_{\mathrm{z}_{\mathscr{B}}}\Big) \odot \\ \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}}, \mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}}\Big)^{-1} \odot \Big(i_{\mathcal{F}_1(f^1_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}})} * \sigma^{-1}_{\mathscr{B}}\Big).\end{aligned} \tag{3.7}$$

Using the interchange law on $\mathscr{B}$, we get the following identity:

$$\begin{aligned}
&\alpha'_{\mathscr{B}} * i_{\mathrm{z}'_{\mathscr{B}}} \overset{(3.6)}{=} \\
&\overset{(3.6)}{=} \Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}}, \mathrm{v}^2_{\mathscr{A}}} * i_{\mathrm{u}_{\mathscr{B}} \circ \mathrm{r}_{\mathscr{B}} \circ \mathrm{z}'_{\mathscr{B}}}\Big) \odot \Big(\widetilde{\alpha}_{\mathscr{B}} * i_{\mathrm{z}'_{\mathscr{B}}}\Big) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}}, \mathrm{v}^1_{\mathscr{A}}} * i_{\mathrm{u}_{\mathscr{B}} \circ \mathrm{r}_{\mathscr{B}} \circ \mathrm{z}'_{\mathscr{B}}}\Big)^{-1} \overset{(3.7)}{=} \\
&\overset{(3.7)}{=} \Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}}, \mathrm{v}^2_{\mathscr{A}}} * i_{\mathrm{u}_{\mathscr{B}} \circ \mathrm{r}_{\mathscr{B}} \circ \mathrm{z}'_{\mathscr{B}}}\Big) \odot \Big(i_{\mathcal{F}_1(f^2_{\mathscr{A}} \circ \mathrm{v}^2_{\mathscr{A}})} * \sigma_{\mathscr{B}}\Big) \odot \\
&\odot \Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}} \circ \mathrm{v}^2_{\mathscr{A}}, \mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}}\Big) \odot \Big(\mathcal{F}_2(\alpha_{\mathscr{A}}) * i_{\mathrm{z}_{\mathscr{B}}}\Big) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}}, \mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}}\Big)^{-1} \odot \\
&\odot \Big(i_{\mathcal{F}_1(f^1_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}})} * \sigma^{-1}_{\mathscr{B}}\Big) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}}, \mathrm{v}^1_{\mathscr{A}}} * i_{\mathrm{u}_{\mathscr{B}} \circ \mathrm{r}_{\mathscr{B}} \circ \mathrm{z}'_{\mathscr{B}}}\Big)^{-1} = \\
&= \Big(i_{\mathcal{F}_1(f^2_{\mathscr{A}}) \circ \mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}})} * \sigma_{\mathscr{B}}\Big) \odot \Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}}, \mathrm{v}^2_{\mathscr{A}}} * i_{\mathcal{F}_1(\mathrm{u}_{\mathscr{A}}) \circ \mathrm{z}_{\mathscr{B}}}\Big) \odot \\
&\odot \Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}} \circ \mathrm{v}^2_{\mathscr{A}}, \mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}}\Big) \odot \Big(\mathcal{F}_2(\alpha_{\mathscr{A}}) * i_{\mathrm{z}_{\mathscr{B}}}\Big) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}}, \mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}}\Big)^{-1} \odot \\
&\odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}}, \mathrm{v}^1_{\mathscr{A}}} * i_{\mathcal{F}_1(\mathrm{u}_{\mathscr{A}}) \circ \mathrm{z}_{\mathscr{B}}}\Big)^{-1} \odot \Big(i_{\mathcal{F}_1(f^1_{\mathscr{A}}) \circ \mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}})} * \sigma^{-1}_{\mathscr{B}}\Big).
\end{aligned} \tag{3.8}$$

Since $\mathcal{F}$ is a pseudofunctor and since we are assuming for simplicity that $\mathscr{A}$ and $\mathscr{B}$ are 2-categories, then for each $m = 1, 2$ we have

$$\Big(\psi^{\mathcal{F}}_{f^m_{\mathscr{A}}, \mathrm{v}^m_{\mathscr{A}}} * i_{\mathcal{F}_1(\mathrm{u}_{\mathscr{A}})}\Big) \odot \Big(\psi^{\mathcal{F}}_{f^m_{\mathscr{A}} \circ \mathrm{v}^m_{\mathscr{A}}, \mathrm{u}_{\mathscr{A}}}\Big) = \Big(i_{\mathcal{F}_1(f^m_{\mathscr{A}})} * \psi^{\mathcal{F}}_{\mathrm{v}^m_{\mathscr{A}}, \mathrm{u}_{\mathscr{A}}}\Big) \odot \Big(\psi^{\mathcal{F}}_{f^m_{\mathscr{A}}, \mathrm{v}^m_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}}\Big).$$

So by replacing in (3.8) we get

$$\alpha'_{\mathscr{B}} * i_{\mathrm{z}'_{\mathscr{B}}} = \Big\{i_{\mathcal{F}_1(f^2_{\mathscr{A}})} * \Big[\Big(i_{\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}})} * \sigma_{\mathscr{B}}\Big) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{v}^2_{\mathscr{A}}, \mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}}\Big)\Big]\Big\} \odot \tag{3.9}$$

$$\begin{aligned}&\odot \Big\{\Big[\psi^{\mathcal{F}}_{f^2_{\mathscr{A}}, \mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}} \odot \mathcal{F}_2(\alpha_{\mathscr{A}}) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}}, \mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}}\Big)^{-1}\Big] * i_{\mathrm{z}_{\mathscr{B}}}\Big\} \odot \\ &\odot \Big\{i_{\mathcal{F}_1(f^1_{\mathscr{A}})} * \Big[\Big(\psi^{\mathcal{F}}_{\mathrm{v}^1_{\mathscr{A}}, \mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}}\Big)^{-1} \odot \Big(i_{\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}})} * \sigma^{-1}_{\mathscr{B}}\Big)\Big]\Big\}.\end{aligned} \tag{3.10}$$

Now we define an invertible 2-morphism

$$\begin{aligned}\gamma''_{\mathscr{B}} := \Big\{ i_{\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})} * \Big[\Big(\psi^{\mathcal{F}}_{\mathrm{v}^2_{\mathscr{A}},\mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}} \Big)^{-1} \odot \Big(i_{\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}})} * \sigma^{-1}_{\mathscr{B}} \Big) \odot \Big(\phi^2_{\mathscr{B}} * i_{\mathrm{r}_{\mathscr{B}} \circ \mathrm{z}'_{\mathscr{B}}} \Big) \Big] \Big\} \odot \\ \odot \Big(\gamma_{\mathscr{B}} * i_{\mathrm{u}'_{\mathscr{B}} \circ \mathrm{r}_{\mathscr{B}} \circ \mathrm{z}'_{\mathscr{B}}} \Big) \odot \\ \odot \Big\{ i_{\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})} * \Big[\Big(\big(\phi^1_{\mathscr{B}}\big)^{-1} * i_{\mathrm{r}_{\mathscr{B}} \circ \mathrm{z}'_{\mathscr{B}}} \Big) \odot \Big(i_{\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}})} * \sigma_{\mathscr{B}} \Big) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{v}^1_{\mathscr{A}},\mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}} \Big) \Big] \Big\} : \\ \mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}}) \circ \mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}) \circ \mathrm{z}_{\mathscr{B}} \Longrightarrow \mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}}) \circ \mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}) \circ \mathrm{z}_{\mathscr{B}}.\end{aligned}$$

Then by definition of $\gamma'_{\mathscr{B}}$ and $\gamma''_{\mathscr{B}}$ we have:

$$\begin{aligned}\gamma'_{\mathscr{B}} * i_{\mathrm{z}'_{\mathscr{B}}} = \Big\{ i_{\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})} * \Big[\Big(i_{\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}})} * \sigma_{\mathscr{B}} \Big) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{v}^2_{\mathscr{A}},\mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}} \Big) \Big] \Big\} \odot \\ \odot \gamma''_{\mathscr{B}} \odot \Big\{ i_{\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})} * \Big[\Big(\psi^{\mathcal{F}}_{\mathrm{v}^1_{\mathscr{A}},\mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}} \Big)^{-1} \odot \Big(i_{\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}})} * \sigma^{-1}_{\mathscr{B}} \Big) \Big] \Big\}. \qquad (3.11)\end{aligned}$$

Then let us consider the following set of data

$$\begin{tikzcd}[column sep=large] & \mathcal{F}_0(A^1_{\mathscr{A}}) & \\ A''_{\mathscr{B}} \arrow[ur, "\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}})\circ \mathrm{u}_{\mathscr{B}} \circ \mathrm{r}_{\mathscr{B}}"] \arrow[dr, "\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}})\circ \mathrm{u}_{\mathscr{B}} \circ \mathrm{r}_{\mathscr{B}}"'] & A'_{\mathscr{B}} \arrow[l, "\mathrm{z}'_{\mathscr{B}}"'] \arrow[r, "\mathrm{id}_{A'_{\mathscr{B}}}"] \arrow[u, phantom, "\overset{\Rightarrow}{\xi^1_{\mathscr{B}}}"] \arrow[d, phantom, "\underset{\Leftarrow}{\xi^2_{\mathscr{B}}}"] & A'_{\mathscr{B}}, \arrow[ul, "\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}})\circ \mathrm{z}_{\mathscr{B}}"'] \arrow[dl, "\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}})\circ \mathrm{z}_{\mathscr{B}}"] \\ & \mathcal{F}_0(A^2_{\mathscr{A}}) & \end{tikzcd}$$

where

$$\xi^1_{\mathscr{B}} := \Big(\psi^{\mathcal{F}}_{\mathrm{v}^1_{\mathscr{A}},\mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}} \Big)^{-1} \odot \Big(i_{\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}})} * \sigma^{-1}_{\mathscr{B}} \Big)$$

and

$$\xi^2_{\mathscr{B}} := \Big(i_{\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}})} * \sigma_{\mathscr{B}} \Big) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{v}^2_{\mathscr{A}},\mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}} \Big).$$

Using such data with (3.5), (3.10), (3.11) and the definition of 2-morphisms in [Pr, § 2.3], we conclude that:

$$\begin{aligned}\Gamma_{\mathscr{B}} = \Big[A'_{\mathscr{B}}, \mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}) \circ \mathrm{z}_{\mathscr{B}}, \mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}) \circ \mathrm{z}_{\mathscr{B}}, \gamma''_{\mathscr{B}}, \\ \Big(\psi^{\mathcal{F}}_{\mathrm{f}^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}} \odot \mathcal{F}_2(\alpha_{\mathscr{A}}) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{f}^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}} \Big)^{-1} \Big) * i_{\mathrm{z}_{\mathscr{B}}} \Big]. \qquad (3.12)\end{aligned}$$

Now we consider the invertible 2-morphism

$$\begin{aligned}\widetilde{\gamma}_{\mathscr{B}} := \Big(\psi^{\mathcal{F}}_{\mathrm{w}^2_{\mathscr{A}},\mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}} \Big)^{-1} \odot \gamma''_{\mathscr{B}} \odot \Big(\psi^{\mathcal{F}}_{\mathrm{w}^1_{\mathscr{A}},\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}} * i_{\mathrm{z}_{\mathscr{B}}} \Big) : \\ \mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}) \circ \mathrm{z}_{\mathscr{B}} \Longrightarrow \mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}} \circ \mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}) \circ \mathrm{z}_{\mathscr{B}}.\end{aligned}$$

Since we are assuming (A4) and (A5), then we can apply Lemma 3.4 on the set of data:

$$A'_{\mathscr{A}}, \quad A_{\mathscr{A}}, \quad A'_{\mathscr{B}}, \quad \mathrm{w}^1_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}, \quad \mathrm{w}^2_{\mathscr{A}} \circ \mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}, \quad \mathrm{z}_{\mathscr{B}}, \quad \widetilde{\gamma}_{\mathscr{B}}.$$

Then there are a pair of objects $\widetilde{A}_{\mathscr{A}}, \widetilde{A}_{\mathscr{B}}$, a triple of morphisms $\mathrm{z}_{\mathscr{A}} : \widetilde{A}_{\mathscr{A}} \to A'_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, $\mathrm{t}_{\mathscr{B}} : \widetilde{A}_{\mathscr{B}} \to \mathcal{F}_0(\widetilde{A}_{\mathscr{A}})$ in $\mathbf{W}_{\mathscr{B}}$ and $\mathrm{t}'_{\mathscr{B}} : \widetilde{A}_{\mathscr{B}} \to A'_{\mathscr{B}}$, an *invertible* 2-morphism

$$\begin{tikzcd}[column sep=large]
\widetilde{A}_{\mathscr{A}} \arrow[r, bend left=15, "\mathrm{z}_{\mathscr{A}}"] \arrow[r, bend right=15, "\mathrm{z}_{\mathscr{A}}"'] & A'_{\mathscr{A}} \arrow[r, bend left=15, "\mathrm{w}^1_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}"] \arrow[r, bend right=15, "\mathrm{w}^2_{\mathscr{A}} \circ \mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}"'] & A_{\mathscr{A}}
\end{tikzcd} \quad \Downarrow \gamma_{\mathscr{A}}$$

and an invertible 2-morphism

$$\begin{tikzcd}
& \mathcal{F}_0(\widetilde{A}_{\mathscr{A}}) \arrow[dr, "\mathcal{F}_1(\mathrm{z}_{\mathscr{A}})"] \arrow[d, phantom, "\Downarrow \nu_{\mathscr{B}}"] & \\
\widetilde{A}_{\mathscr{B}} \arrow[ur, "\mathrm{t}_{\mathscr{B}}"] \arrow[dr, "\mathrm{t}'_{\mathscr{B}}"'] & {} & \mathcal{F}_0(A'_{\mathscr{A}}), \\
& A'_{\mathscr{B}} \arrow[ur, "\mathrm{z}_{\mathscr{B}}"'] &
\end{tikzcd}$$

such that

$$\begin{aligned}\widetilde{\gamma}_{\mathscr{B}} * i_{\mathrm{t}'_{\mathscr{B}}} = \Big(i_{\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}} \circ \mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}})} * \nu_{\mathscr{B}}\Big) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{w}^2_{\mathscr{A}} \circ \mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}, \mathrm{z}_{\mathscr{A}}} * i_{\mathrm{t}_{\mathscr{B}}}\Big) \odot \\ \odot \Big(\mathcal{F}_2(\gamma_{\mathscr{A}}) * i_{\mathrm{t}_{\mathscr{B}}}\Big) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{w}^1_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}, \mathrm{z}_{\mathscr{A}}} * i_{\mathrm{t}_{\mathscr{B}}}\Big)^{-1} \odot \Big(i_{\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}} \circ \mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}})} * \nu^{-1}_{\mathscr{B}}\Big).\end{aligned} \tag{3.13}$$

Now we replace in (3.13) the definition of $\widetilde{\gamma}_{\mathscr{B}}$ and we do a series of computations analogous to those leading from (3.7) to (3.10). So we get that

$$\begin{aligned}&\gamma''_{\mathscr{B}} * i_{\mathrm{t}'_{\mathscr{B}}} = \\ &= \Big\{i_{\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})} * \Big[\Big(i_{\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}})} * \nu_{\mathscr{B}}\Big) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}, \mathrm{z}_{\mathscr{A}}} * i_{\mathrm{t}_{\mathscr{B}}}\Big)\Big]\Big\} \odot \\ &\odot \Big\{\Big[\psi^{\mathcal{F}}_{\mathrm{w}^2_{\mathscr{A}}, \mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}} \odot \mathcal{F}_2(\gamma_{\mathscr{A}}) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{w}^1_{\mathscr{A}}, \mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}}\Big)^{-1}\Big] * i_{\mathrm{t}_{\mathscr{B}}}\Big\} \odot \\ &\odot \Big\{i_{\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})} * \Big[\Big(\psi^{\mathcal{F}}_{\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}, \mathrm{z}_{\mathscr{A}}} * i_{\mathrm{t}_{\mathscr{B}}}\Big)^{-1} \odot \Big(i_{\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}})} * \nu^{-1}_{\mathscr{B}}\Big)\Big]\Big\}.\end{aligned} \tag{3.14}$$

Moreover, using the interchange law on the 2-category $\mathscr{B}$, we have:

$$\begin{aligned}&\Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}}, \mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}} \odot \mathcal{F}_2(\alpha_{\mathscr{A}}) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}}, \mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}}\Big)^{-1}\Big) * i_{\mathrm{z}_{\mathscr{B}} \circ \mathrm{t}'_{\mathscr{B}}} = \\ &= \Big\{i_{\mathcal{F}_1(f^2_{\mathscr{A}})} * \Big[\Big(i_{\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}})} * \nu_{\mathscr{B}}\Big) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}, \mathrm{z}_{\mathscr{A}}} * i_{\mathrm{t}_{\mathscr{B}}}\Big)\Big]\Big\} \odot \\ &\odot \Big\{\Big[\psi^{\mathcal{F}}_{f^2_{\mathscr{A}}, \mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}} \odot \mathcal{F}_2(\alpha_{\mathscr{A}} * i_{\mathrm{z}_{\mathscr{A}}}) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}}, \mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}}\Big)^{-1}\Big] * i_{\mathrm{t}_{\mathscr{B}}}\Big\} \odot \\ &\odot \Big\{i_{\mathcal{F}_1(f^1_{\mathscr{A}})} * \Big[\Big(\psi^{\mathcal{F}}_{\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}, \mathrm{z}_{\mathscr{A}}} * i_{\mathrm{t}_{\mathscr{B}}}\Big)^{-1} \odot \Big(i_{\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}})} * \nu^{-1}_{\mathscr{B}}\Big)\Big]\Big\}.\end{aligned} \tag{3.15}$$

Then we consider the following diagram

$$\begin{tikzcd}
& \mathcal{F}_0(A^1_{\mathscr{A}}) & \\
A'_{\mathscr{B}} \arrow[ur, "\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}) \circ \mathrm{z}_{\mathscr{B}}"] \arrow[dr, "\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}) \circ \mathrm{z}_{\mathscr{B}}"'] & \widetilde{A}_{\mathscr{B}} \arrow[l, "\mathrm{t}'_{\mathscr{B}}"'] \arrow[r, "\mathrm{t}_{\mathscr{B}}"] \arrow[u, phantom, "\overset{\Rightarrow}{\eta^1_{\mathscr{B}}}"] \arrow[d, phantom, "\underset{\Leftarrow}{\eta^2_{\mathscr{B}}}"] & \mathcal{F}_0(\widetilde{A}_{\mathscr{A}}), \arrow[ul, "\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}})"'] \arrow[dl, "\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}})"] \\
& \mathcal{F}_0(A^2_{\mathscr{A}}) &
\end{tikzcd}$$

where

$$\eta^1_{\mathscr{B}} := \Big(\psi^{\mathcal{F}}_{\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}, \mathrm{z}_{\mathscr{A}}} * i_{\mathrm{t}_{\mathscr{B}}}\Big)^{-1} \odot \Big(i_{\mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}})} * \nu^{-1}_{\mathscr{B}}\Big)$$

and

$$\eta^2_{\mathscr{B}} := \Big(i_{\mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}})} * \nu_{\mathscr{B}}\Big) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}}, \mathrm{z}_{\mathscr{A}}} * i_{\mathrm{t}_{\mathscr{B}}}\Big).$$

Using (3.12), (3.14), (3.15) and the previous diagram, we conclude that:

$$\begin{aligned}\Gamma_{\mathscr{B}} = \Big[&\mathcal{F}_0(\widetilde{A}_{\mathscr{A}}), \mathcal{F}_1(\mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}), \mathcal{F}_1(\mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}), \\ &\psi^{\mathcal{F}}_{\mathrm{w}^2_{\mathscr{A}}, \mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}} \odot \mathcal{F}_2(\gamma_{\mathscr{A}}) \odot \Big(\psi^{\mathcal{F}}_{\mathrm{w}^1_{\mathscr{A}}, \mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}}\Big)^{-1}, \\ &\psi^{\mathcal{F}}_{f^2_{\mathscr{A}}, \mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}} \odot \mathcal{F}_2(\alpha_{\mathscr{A}} * i_{\mathrm{z}_{\mathscr{A}}}) \odot \Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}}, \mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}}\Big)^{-1}\Big].\end{aligned} \tag{3.16}$$

Then we define a 2-morphism in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$ as follows

$$\begin{gathered}\Gamma_{\mathscr{A}} := \Big[\widetilde{A}_{\mathscr{A}}, \mathrm{v}^1_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}, \mathrm{v}^2_{\mathscr{A}} \circ \mathrm{u}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}, \gamma_{\mathscr{A}}, \alpha_{\mathscr{A}} * i_{\mathrm{z}_{\mathscr{A}}}\Big] : \\ \Big(A^1_{\mathscr{A}}, \mathrm{w}^1_{\mathscr{A}}, f^1_{\mathscr{A}}\Big) \Longrightarrow \Big(A^2_{\mathscr{A}}, \mathrm{w}^2_{\mathscr{A}}, f^2_{\mathscr{A}}\Big).\end{gathered}$$

Using (3.16) and (0.1), we conclude that $\Gamma_{\mathscr{B}} = \widetilde{\mathcal{G}}_2(\Gamma_{\mathscr{A}})$; this proves that $\widetilde{\mathcal{G}}$ satisfies condition (X2c). □

Note that in the proof above the 2-morphism $\Gamma_{\mathscr{A}}$ is well-defined because $\gamma_{\mathscr{A}}$ is an invertible 2-morphism thanks to Lemma 3.4 (we recall that by [Pr, § 2.3] the data defining a 2-morphism in a bicategory of fractions must satisfy such a technical condition). This explains why we needed to prove such a result before Lemma 3.5.

4. Proofs of the main results

Proof of Theorem 0.2. Let us fix any pair $(\mathcal{G}, \kappa)$ as in Theorem 0.1(iv). By that Theorem, $\mathcal{G} : \mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right] \to \mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B}}\right]$ is an equivalence of bicategories if and only if $\widetilde{\mathcal{G}} : \mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right] \to \mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\mathrm{sat}}\right]$ is an equivalence of bicategories. Using Lemmas from 2.1 to 2.5, if $\widetilde{\mathcal{G}}$ is an equivalence of bicategories, then $\mathcal{F}$ satisfies conditions (A1) – (A5). Conversely, if $\mathcal{F}$ satisfies such conditions, then $\widetilde{\mathcal{G}}$ is an equivalence of bicategories by Lemmas from 3.1 to 3.5. This is enough to conclude. □

In the remaining part of this section we are going to prove Theorem 0.4 and Corollary 0.5.

We recall (see [PP, Definition 3.3]) that a *quasi-unit* of any given bicategory $\mathscr{B}$ is any morphism of the form $f_{\mathscr{B}} : A_{\mathscr{B}} \to A_{\mathscr{B}}$, admitting an invertible 2-morphism to $\mathrm{id}_{A_{\mathscr{B}}}$. We denote by $\mathbf{W}_{\mathscr{B},\mathrm{min}}$ the class of quasi-units of $\mathscr{B}$; it is easy to see that $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{min}})$ satisfies conditions (BF), so it makes sense to consider the associated bicategory of fractions. Then we have:

Proposition 4.1. *Let us fix any pair $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ satisfying conditions* (BF)*, and any pseudofunctor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, such that $\mathcal{F}_1(\mathbf{W}_{\mathscr{A}}) \subseteq \mathbf{W}_{\mathscr{B},\mathrm{equiv}}$. Then for each $i = 1, \cdots, 5$ the following facts are equivalent:*

- *$\mathcal{F}$ satisfies condition* (A i) *when $\mathbf{W}_{\mathscr{B}} := \mathbf{W}_{\mathscr{B},\mathrm{min}}$;*
- *$\mathcal{F}$ satisfies condition* (B i).

Proof. We recall (see [T2, Lemma 2.5(ii)]) that the right saturation of $\mathbf{W}_{\mathscr{B},\mathrm{min}}$ is the class $\mathbf{W}_{\mathscr{B},\mathrm{equiv}}$ of internal equivalences of $\mathscr{B}$.

Clearly (B1) implies (A1) when $\mathbf{W}_{\mathscr{B}} = \mathbf{W}_{\mathscr{B},\mathrm{min}}$: it suffices to take $A'_{\mathscr{B}} := \mathcal{F}_0(A_{\mathscr{A}})$, $\mathrm{w}^1_{\mathscr{B}} := \mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})}$ and $\mathrm{w}^2_{\mathscr{B}} := e_{\mathscr{B}}$ (this makes sense since $\mathbf{W}_{\mathscr{B},\mathrm{equiv}}$ is the right saturation of $\mathbf{W}_{\mathscr{B},\mathrm{min}}$). Conversely, let us assume that (A1) holds for $\mathbf{W}_{\mathscr{B},\mathrm{min}}$; since $\mathrm{w}^1_{\mathscr{B}}$ and $\mathrm{w}^2_{\mathscr{B}}$ belong to $\mathbf{W}_{\mathscr{B},\mathrm{min}}$ and $\mathbf{W}_{\mathscr{B},\mathrm{equiv}}$ respectively, then $A'_{\mathscr{B}} = \mathcal{F}_0(A_{\mathscr{A}})$ and $\mathrm{w}^2_{\mathscr{B}}$ is an internal equivalence from $\mathcal{F}_0(A_{\mathscr{A}})$ to $A_{\mathscr{B}}$, so (B1) holds.

Let us suppose that (A2) holds for $\mathbf{W}_{\mathscr{B},\mathrm{min}}$, let us fix any pair of objects $A^1_{\mathscr{A}}, A^2_{\mathscr{A}}$ and any internal equivalence $e_{\mathscr{B}} : \mathcal{F}_0(A^1_{\mathscr{A}}) \rightarrow \mathcal{F}_0(A^2_{\mathscr{A}})$ (i.e. any element of the right saturated of $\mathbf{W}_{\mathscr{B},\mathrm{min}}$) and let us apply (A2) to the following set of data:

$$\mathcal{F}_0(A^1_{\mathscr{A}}) \xleftarrow{\mathrm{id}_{\mathcal{F}_0(A^1_{\mathscr{A}})}} \mathcal{F}_0(A^1_{\mathscr{A}}) \xrightarrow{\quad e_{\mathscr{B}}\quad} \mathcal{F}_0(A^2_{\mathscr{A}}).$$

Then there are an object $A^3_{\mathscr{A}}$, a pair of morphisms $\mathrm{w}^1_{\mathscr{A}} : A^3_{\mathscr{A}} \rightarrow A^1_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\mathrm{w}^2_{\mathscr{A}} : A^3_{\mathscr{A}} \rightarrow A^2_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A},\mathrm{sat}}$ and a set of data in $\mathscr{B}$ as in the internal part of the following diagram

$$\begin{tikzcd}
 & \mathcal{F}_0(A^1_{\mathscr{A}}) \arrow[dl, "\mathrm{id}_{\mathcal{F}_0(A^1_{\mathscr{A}})}"'] \arrow[dr, "e_{\mathscr{B}}"] & \\
\mathcal{F}_0(A^1_{\mathscr{A}}) \arrow[r, phantom, "\Downarrow\, \gamma^1_{\mathscr{B}}"] & A'_{\mathscr{B}} \arrow[u, "\mathrm{z}^1_{\mathscr{B}}"'] \arrow[d, "e'_{\mathscr{B}}"] \arrow[r, phantom, "\Downarrow\, \gamma^2_{\mathscr{B}}"] & \mathcal{F}_0(A^2_{\mathscr{A}}), \\
 & \mathcal{F}_0(A^3_{\mathscr{A}}) \arrow[ul, "\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})"] \arrow[ur, "\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})"'] &
\end{tikzcd}$$

such that $\mathrm{z}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{min}}$ and both $\gamma^1_{\mathscr{B}}$ and $\gamma^2_{\mathscr{B}}$ are invertible. Since conditions (BF1), (BF2) and (BF5) hold for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{min}})$, then the morphism $\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}}) \circ e'_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{min}} \subseteq \mathbf{W}_{\mathscr{B},\mathrm{equiv}}$. Moreover, $\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{equiv}}$ by hypothesis. So by [T2, Proposition 2.11(ii)] for $(\mathscr{B}, \mathbf{W}_{\mathscr{B},\mathrm{min}})$ we get that $e'_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{min},\mathrm{sat}} = \mathbf{W}_{\mathscr{B},\mathrm{equiv}}$, i.e. it is an internal equivalence of $\mathscr{B}$.

Since $\mathrm{z}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{min}}$, then $A'_{\mathscr{B}} = \mathcal{F}_0(A^1_{\mathscr{A}})$ and there is an invertible 2-morphism $\xi_{\mathscr{B}} : \mathrm{z}^1_{\mathscr{B}} \Rightarrow \mathrm{id}_{\mathcal{F}_0(A^1_{\mathscr{A}})}$. Then we define a pair of invertible 2-morphisms:

$$\delta^1_{\mathscr{B}} := \xi_{\mathscr{B}} \odot \upsilon_{\mathrm{z}^1_{\mathscr{B}}} \odot \left(\gamma^1_{\mathscr{B}}\right)^{-1} : \; \mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}}) \circ e'_{\mathscr{B}} \Longrightarrow \mathrm{id}_{\mathcal{F}_0(A^1_{\mathscr{A}})}$$

and

$$\delta^2_{\mathscr{B}} := \gamma^2_{\mathscr{B}} \odot \left(i_{e_{\mathscr{B}}} * \xi^{-1}_{\mathscr{B}}\right) \odot \pi^{-1}_{e_{\mathscr{B}}} : \; e_{\mathscr{B}} \Longrightarrow \mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}}) \circ e'_{\mathscr{B}}.$$

This proves that (B2) holds.

Conversely, let us suppose that (B2) holds and let us fix any triple of objects $A^1_{\mathscr{A}}, A^2_{\mathscr{A}}, A_{\mathscr{B}}$ and any pair of morphisms $\mathrm{w}^1_{\mathscr{B}} : A_{\mathscr{B}} \rightarrow \mathcal{F}_0(A^1_{\mathscr{A}})$ in $\mathbf{W}_{\mathscr{B},\mathrm{min}}$ and $\mathrm{w}^2_{\mathscr{B}} : A_{\mathscr{B}} \rightarrow \mathcal{F}_0(A^2_{\mathscr{A}})$ in $\mathbf{W}_{\mathscr{B},\mathrm{equiv}}$. Since $\mathrm{w}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\mathrm{min}}$, then $A_{\mathscr{B}} = \mathcal{F}_0(A^1_{\mathscr{A}})$ and there is an invertible 2-morphism $\xi_{\mathscr{B}} : \mathrm{w}^1_{\mathscr{B}} \Rightarrow \mathrm{id}_{\mathcal{F}_0(A^1_{\mathscr{A}})}$. Now let us apply (B2) to the internal equivalence $\mathrm{w}^2_{\mathscr{B}} : \mathcal{F}_0(A^1_{\mathscr{A}}) \rightarrow \mathcal{F}_0(A^2_{\mathscr{A}})$. Then there are an object $A^3_{\mathscr{A}}$, a pair of morphisms $\mathrm{w}^1_{\mathscr{A}} : A^3_{\mathscr{A}} \rightarrow A^1_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\mathrm{w}^2_{\mathscr{A}} : A^3_{\mathscr{A}} \rightarrow A^2_{\mathscr{A}}$

in $\mathbf{W}_{\mathscr{A},\mathrm{sat}}$, an internal equivalence $e'_{\mathscr{B}}:\mathcal{F}_0(A^1_{\mathscr{A}})\rightarrow\mathcal{F}_0(A^3_{\mathscr{A}})$ and invertible 2-morphisms $\delta^1_{\mathscr{B}}:\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})\circ e'_{\mathscr{B}}\Rightarrow\mathrm{id}_{\mathcal{F}_0(A^1_{\mathscr{A}})}$ and $\delta^2_{\mathscr{B}}:\mathrm{w}^2_{\mathscr{B}}\Rightarrow\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})\circ e'_{\mathscr{B}}$. Then the following diagram proves that (A2) holds.

$$\begin{tikzcd}
 & \mathcal{F}_0(A^1_{\mathscr{A}})=A_{\mathscr{B}} \arrow[dl, "\mathrm{w}^1_{\mathscr{B}}"'] \arrow[dr, "\mathrm{w}^2_{\mathscr{B}}"] & \\
\mathcal{F}_0(A^1_{\mathscr{A}}) \arrow[r, phantom, "\Downarrow\left(\delta^1_{\mathscr{B}}\right)^{-1}\odot\xi_{\mathscr{B}}\odot\pi_{\mathrm{w}^1_{\mathscr{B}}}"] & \mathcal{F}_0(A^1_{\mathscr{A}}) \arrow[u, "\mathrm{id}_{\mathcal{F}_0(A^1_{\mathscr{A}})}"'] \arrow[d, "e'_{\mathscr{B}}"] \arrow[r, phantom, "\Downarrow\delta^2_{\mathscr{B}}\odot\pi_{\mathrm{w}^2_{\mathscr{B}}}"] & \mathcal{F}_0(A^2_{\mathscr{A}}). \\
 & \mathcal{F}_0(A^3_{\mathscr{A}}) \arrow[ul, "\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})"] \arrow[ur, "\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})"'] &
\end{tikzcd}$$

Now let us suppose that (B3) holds and let us fix any pair of objects $B_{\mathscr{A}},A_{\mathscr{B}}$ and any morphism $f_{\mathscr{B}}:A_{\mathscr{B}}\rightarrow\mathcal{F}_0(B_{\mathscr{A}})$. Then there are data $(A_{\mathscr{A}},f_{\mathscr{A}},e_{\mathscr{B}},\alpha_{\mathscr{B}})$ as in (B3). So (A3) for $\mathbf{W}_{\mathscr{B},\min}$ is verified by the data $(A_{\mathscr{A}},f_{\mathscr{A}},A_{\mathscr{B}},\mathrm{id}_{A_{\mathscr{B}}},e_{\mathscr{B}},\alpha_{\mathscr{B}}\odot\pi_{f_{\mathscr{B}}})$.

Conversely, let us suppose that (A3) holds for $\mathbf{W}_{\mathscr{B},\min}$ and let us fix any set of data $B_{\mathscr{A}},A_{\mathscr{B}}$ and $f_{\mathscr{B}}$ as before. Then there is a set of data $(A_{\mathscr{A}},f_{\mathscr{A}},A'_{\mathscr{B}},\mathrm{v}^1_{\mathscr{B}},\mathrm{v}^2_{\mathscr{B}},\alpha_{\mathscr{B}})$ as in (A3). In particular, $\mathrm{v}^1_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\min}$, hence $A'_{\mathscr{B}}=A_{\mathscr{B}}$ and there is an invertible 2-morphism $\xi_{\mathscr{B}}:\mathrm{v}^1_{\mathscr{B}}\Rightarrow\mathrm{id}_{A_{\mathscr{B}}}$; moreover $\mathrm{v}^2_{\mathscr{B}}$ belongs to the saturation of $\mathbf{W}_{\mathscr{B},\min}$, i.e. it belongs to $\mathbf{W}_{\mathscr{B},\mathrm{equiv}}$. Then the set of data $(A_{\mathscr{A}},f_{\mathscr{A}},\mathrm{v}^2_{\mathscr{B}},\alpha_{\mathscr{B}}\odot(i_{f_{\mathscr{B}}}*\xi^{-1}_{\mathscr{B}})\odot\pi^{-1}_{f_{\mathscr{B}}})$ proves that (B3) holds.

Clearly (A4) for $\mathbf{W}_{\mathscr{B},\min}$ implies (B4) in the special case when $A'_{\mathscr{B}}=\mathcal{F}_0(A_{\mathscr{A}})$ and $\mathrm{z}_{\mathscr{B}}=\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})}$ (since this morphism belongs to $\mathbf{W}_{\mathscr{B},\min}$). Conversely, let us assume (B4) and let us fix any set of data $(A_{\mathscr{A}},f^1_{\mathscr{A}},f^2_{\mathscr{A}},\gamma^1_{\mathscr{A}},\gamma^2_{\mathscr{A}},A'_{\mathscr{B}},\mathrm{z}_{\mathscr{B}})$ as in (A4). In particular, $\mathrm{z}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\min}$, hence $A'_{\mathscr{B}}=\mathcal{F}_0(A_{\mathscr{A}})$ and there is an invertible 2-morphism $\xi_{\mathscr{B}}:\mathrm{z}_{\mathscr{B}}\Rightarrow\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})}$. Since $\mathcal{F}_2(\gamma^1_{\mathscr{A}})*i_{\mathrm{z}_{\mathscr{B}}}=\mathcal{F}_2(\gamma^2_{\mathscr{A}})*i_{\mathrm{z}_{\mathscr{B}}}$, then using $\xi_{\mathscr{B}}$ we get that $\mathcal{F}_2(\gamma^1_{\mathscr{A}})=\mathcal{F}_2(\gamma^2_{\mathscr{A}})$. So we can use (B4) in order to conclude that (A4) holds for $\mathbf{W}_{\mathscr{B},\min}$.

Now let us suppose that (B5) holds and let us fix any set of data $(A_{\mathscr{A}},B_{\mathscr{A}},A_{\mathscr{B}},f^1_{\mathscr{A}},f^2_{\mathscr{A}},\mathrm{v}_{\mathscr{B}},\alpha_{\mathscr{B}})$ as in (A5) for $\mathbf{W}_{\mathscr{B},\min}$. In particular, $\mathrm{v}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\min}$, hence $A_{\mathscr{B}}=\mathcal{F}_0(A_{\mathscr{A}})$ and there is an invertible 2-morphism $\xi_{\mathscr{B}}:\mathrm{v}_{\mathscr{B}}\Rightarrow\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})}$. Hence we can define a 2-morphism

$$\overline{\alpha}_{\mathscr{B}}:=\pi_{\mathcal{F}_1(f^2_{\mathscr{A}})}\odot\Big(i_{\mathcal{F}_1(f^2_{\mathscr{A}})}*\xi_{\mathscr{B}}\Big)\odot\alpha_{\mathscr{B}}\odot\Big(i_{\mathcal{F}_1(f^1_{\mathscr{A}})}*\xi^{-1}_{\mathscr{B}}\Big)\odot\pi^{-1}_{\mathcal{F}_1(f^1_{\mathscr{A}})}:\ \mathcal{F}_1(f^1_{\mathscr{A}})\Rightarrow\mathcal{F}_1(f^2_{\mathscr{A}}).$$

Then by (B5) for $\overline{\alpha}_{\mathscr{B}}$, there are an object $A'_{\mathscr{A}}$, a morphism $\mathrm{v}_{\mathscr{A}}:A'_{\mathscr{A}}\rightarrow A_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, a 2-morphism $\alpha_{\mathscr{A}}:f^1_{\mathscr{A}}\circ\mathrm{v}_{\mathscr{A}}\Rightarrow f^2_{\mathscr{A}}\circ\mathrm{v}_{\mathscr{A}}$, such that

$$\overline{\alpha}_{\mathscr{B}}*i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}=\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\odot\mathcal{F}_2(\alpha_{\mathscr{A}})\odot\left(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\right)^{-1}.$$

Using the definition of $\overline{\alpha}_{\mathscr{B}}$ and the coherence axioms on the bicategory $\mathscr{B}$, this implies that

$$\alpha_{\mathscr{B}}*i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}=\theta_{\mathcal{F}_1(f^2_{\mathscr{A}}),\mathrm{v}_{\mathscr{B}},\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\odot\Big\{i_{\mathcal{F}_1(f^2_{\mathscr{A}})}*\Big[\Big(\xi^{-1}_{\mathscr{B}}*i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\Big)\odot$$
$$\odot\upsilon^{-1}_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\odot\pi_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\Big]\Big\}\odot\theta^{-1}_{\mathcal{F}_1(f^2_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{id}_{\mathcal{F}_0(A'_{\mathscr{A}})}}\odot$$

$$\odot\Big\{\Big[\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\odot\mathcal{F}_2(\alpha_{\mathscr{A}})\odot\Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\Big)^{-1}\Big]*i_{\mathrm{id}_{\mathcal{F}_0(A'_{\mathscr{A}})}}\Big\}\odot\theta_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{id}_{\mathcal{F}_0(A'_{\mathscr{A}})}}\odot$$
$$\odot\Big\{i_{\mathcal{F}_1(f^1_{\mathscr{A}})}*\Big[\pi^{-1}_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\odot\upsilon_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\odot\Big(\xi_{\mathscr{B}}*i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\Big)\Big]\Big\}\odot\theta^{-1}_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathrm{v}_{\mathscr{B}},\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}.$$

Then (A5) for $\mathbf{W}_{\mathscr{B},\min}$ is satisfied if we set $A'_{\mathscr{B}}:=\mathcal{F}_0(A'_{\mathscr{A}})$, $\mathrm{z}_{\mathscr{B}}:=\mathrm{id}_{\mathcal{F}_0(A'_{\mathscr{A}})}$, $\mathrm{z}'_{\mathscr{B}}:=\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})$ and

$$\sigma_{\mathscr{B}}:=\Big(\xi^{-1}_{\mathscr{B}}*i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\Big)\odot\upsilon^{-1}_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\odot\pi_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}.$$

Conversely, let us suppose that (A5) holds for $\mathbf{W}_{\mathscr{B},\min}$ and let us fix any set of data $(A_{\mathscr{A}},B_{\mathscr{A}},f^1_{\mathscr{A}},\,f^2_{\mathscr{A}},\alpha_{\mathscr{B}})$ as in (B5). Then let us apply (A5) in the case when we fix $A_{\mathscr{B}}:=\mathcal{F}_0(A_{\mathscr{A}})$, $\mathrm{v}_{\mathscr{B}}:=\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})}$ and we replace $\alpha_{\mathscr{B}}$ with $\pi^{-1}_{\mathcal{F}_1(f^2_{\mathscr{A}})}\odot\alpha_{\mathscr{B}}\odot\pi_{\mathcal{F}_1(f^1_{\mathscr{A}})}$. Then there are a pair of objects $A'_{\mathscr{A}},A'_{\mathscr{B}}$, a triple of morphisms $\mathrm{v}_{\mathscr{A}}:A'_{\mathscr{A}}\to A_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, $\mathrm{z}_{\mathscr{B}}:A'_{\mathscr{B}}\to\mathcal{F}_0(A'_{\mathscr{A}})$ in $\mathbf{W}_{\mathscr{B},\min}$ and $\mathrm{z}'_{\mathscr{B}}:A'_{\mathscr{B}}\to\mathcal{F}_0(A_{\mathscr{A}})$, a 2-morphism $\alpha_{\mathscr{A}}:f^1_{\mathscr{A}}\circ\mathrm{v}_{\mathscr{A}}\Rightarrow f^2_{\mathscr{A}}\circ\mathrm{v}_{\mathscr{A}}$ and an invertible 2-morphism $\sigma_{\mathscr{B}}:\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})\circ\mathrm{z}_{\mathscr{B}}\Rightarrow\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})}\circ\mathrm{z}'_{\mathscr{B}}$, such that

$$\begin{aligned}\Big(\pi^{-1}_{\mathcal{F}_1(f^2_{\mathscr{A}})}\odot\alpha_{\mathscr{B}}\odot\pi_{\mathcal{F}_1(f^1_{\mathscr{A}})}\Big)*i_{\mathrm{z}'_{\mathscr{B}}}=\theta_{\mathcal{F}_1(f^2_{\mathscr{A}}),\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})},\mathrm{z}'_{\mathscr{B}}}\odot\Big(i_{\mathcal{F}_1(f^2_{\mathscr{A}})}*\sigma_{\mathscr{B}}\Big)\odot\\ \odot\theta^{-1}_{\mathcal{F}_1(f^2_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{z}_{\mathscr{B}}}\odot\Big(\Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\odot\mathcal{F}_2(\alpha_{\mathscr{A}})\odot\Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\Big)^{-1}\Big)*i_{\mathrm{z}_{\mathscr{B}}}\Big)\odot\\ \odot\theta_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{z}_{\mathscr{B}}}\odot\Big(i_{\mathcal{F}_1(f^1_{\mathscr{A}})}*\sigma^{-1}_{\mathscr{B}}\Big)\odot\theta^{-1}_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})},\mathrm{z}'_{\mathscr{B}}}.\end{aligned}\tag{4.1}$$

Since $\mathrm{z}_{\mathscr{B}}$ belongs to $\mathbf{W}_{\mathscr{B},\min}$, then $A'_{\mathscr{B}}=\mathcal{F}_0(A'_{\mathscr{A}})$ and there is an invertible 2-morphism $\xi_{\mathscr{B}}:\mathrm{z}_{\mathscr{B}}\Rightarrow\mathrm{id}_{\mathcal{F}_0(A'_{\mathscr{A}})}$. Using the coherence axioms for the bicategory $\mathscr{B}$ several times, we have:

$$\begin{aligned}&\alpha_{\mathscr{B}}*i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}=\\ &=\Big(i_{\mathcal{F}_1(f^2_{\mathscr{A}})}*\Big(\pi_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\odot\Big(i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}*\xi_{\mathscr{B}}\Big)\Big)\Big)\odot\Big(i_{\mathcal{F}_1(f^2_{\mathscr{A}})}*\sigma^{-1}_{\mathscr{B}}\Big)\odot\\ &\odot\Big(\alpha_{\mathscr{B}}*i_{\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})}\circ\mathrm{z}'_{\mathscr{B}}}\Big)\odot\\ &\odot\Big(i_{\mathcal{F}_1(f^1_{\mathscr{A}})}*\sigma_{\mathscr{B}}\Big)\odot\Big(i_{\mathcal{F}_1(f^1_{\mathscr{A}})}*\Big(\Big(i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}*\xi^{-1}_{\mathscr{B}}\Big)\odot\pi^{-1}_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\Big)\Big)=\\ &=\Big(i_{\mathcal{F}_1(f^2_{\mathscr{A}})}*\Big(\pi_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\odot\Big(i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}*\xi_{\mathscr{B}}\Big)\Big)\Big)\odot\Big(i_{\mathcal{F}_1(f^2_{\mathscr{A}})}*\sigma^{-1}_{\mathscr{B}}\Big)\odot\\ &\odot\theta^{-1}_{\mathcal{F}_1(f^2_{\mathscr{A}}),\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})},\mathrm{z}'_{\mathscr{B}}}\odot\Big(\Big(\pi^{-1}_{\mathcal{F}_1(f^2_{\mathscr{A}})}\odot\alpha_{\mathscr{B}}\odot\pi_{\mathcal{F}_1(f^1_{\mathscr{A}})}\Big)*i_{\mathrm{z}'_{\mathscr{B}}}\Big)\odot\theta_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})},\mathrm{z}'_{\mathscr{B}}}\odot\\ &\odot\Big(i_{\mathcal{F}_1(f^1_{\mathscr{A}})}*\sigma_{\mathscr{B}}\Big)\odot\Big(i_{\mathcal{F}_1(f^1_{\mathscr{A}})}*\Big(\Big(i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}*\xi^{-1}_{\mathscr{B}}\Big)\odot\pi^{-1}_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\Big)\Big)\overset{(4.1)}{=}\\ &\overset{(4.1)}{=}\Big(i_{\mathcal{F}_1(f^2_{\mathscr{A}})}*\Big(\pi_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\odot\Big(i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}*\xi_{\mathscr{B}}\Big)\Big)\Big)\odot\\ &\odot\theta^{-1}_{\mathcal{F}_1(f^2_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{z}_{\mathscr{B}}}\odot\Big(\Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\odot\mathcal{F}_2(\alpha_{\mathscr{A}})\odot\Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\Big)^{-1}\Big)*i_{\mathrm{z}_{\mathscr{B}}}\Big)\odot\\ &\odot\theta_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{z}_{\mathscr{B}}}\odot\Big(i_{\mathcal{F}_1(f^1_{\mathscr{A}})}*\Big(\Big(i_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}*\xi^{-1}_{\mathscr{B}}\Big)\odot\pi^{-1}_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\Big)\Big)=\\ &=\Big(i_{\mathcal{F}_1(f^2_{\mathscr{A}})}*\pi_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\Big)\odot\theta^{-1}_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})}}\odot\Big(\Big(\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\odot\mathcal{F}_2(\alpha_{\mathscr{A}})\odot\\ &\odot\Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\Big)^{-1}\Big)*i_{\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})}}\Big)\odot\theta_{\mathcal{F}_1(f^1_{\mathscr{A}}),\mathcal{F}_1(\mathrm{v}_{\mathscr{A}}),\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})}}\odot\Big(i_{\mathcal{F}_1(f^1_{\mathscr{A}})}*\pi^{-1}_{\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})}\Big)=\\ &=\psi^{\mathcal{F}}_{f^2_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\odot\mathcal{F}_2(\alpha_{\mathscr{A}})\odot\Big(\psi^{\mathcal{F}}_{f^1_{\mathscr{A}},\mathrm{v}_{\mathscr{A}}}\Big)^{-1}.\end{aligned}$$

So we have proved that (B5) holds. □

Then we are ready to give the following:

Proof of Theorem 0.4. By [T2, Lemma 2.5(iii)] for $\mathscr{C} := \mathscr{B}$, the pseudofunctor

$$\mathcal{U}_{\mathbf{W}_{\mathscr{B},\min}} : \mathscr{B} \longrightarrow \mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\min}\right]$$

is an equivalence of bicategories. Let us fix any pair $(\overline{\mathcal{G}}, \overline{\kappa})$ as in Theorem 0.3(2); then it makes sense to set

$$\mathcal{L} := \mathcal{U}_{\mathbf{W}_{\mathscr{B},\min}} \circ \overline{\mathcal{G}} : \ \mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right] \longrightarrow \mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\min}\right]$$

and

$$\rho := \theta_{\mathcal{U}_{\mathbf{W}_{\mathscr{B},\min}}, \overline{\mathcal{G}}, \mathcal{U}_{\mathbf{W}_{\mathscr{A}}}} \odot \left(i_{\mathcal{U}_{\mathbf{W}_{\mathscr{B},\min}}} * \overline{\kappa}\right) : \ \mathcal{U}_{\mathbf{W}_{\mathscr{B},\min}} \circ \mathcal{F} \Longrightarrow \mathcal{L} \circ \mathcal{U}_{\mathbf{W}_{\mathscr{A}}}.$$

By hypothesis the pseudofunctor $\mathscr{A} \to \mathrm{Cyl}(\mathscr{B})$ associated to $\overline{\kappa}$ sends each morphism of $\mathbf{W}_{\mathscr{A}}$ to an internal equivalence, hence so does the pseudofunctor $\mathscr{A} \to \mathrm{Cyl}\left(\mathscr{B}\left[\mathbf{W}^{-1}_{\mathscr{B},\min}\right]\right)$ associated to $i_{\mathcal{U}_{\mathbf{W}_{\mathscr{B},\min}}} * \overline{\kappa}$, hence so does the pseudofunctor associated to ρ. So the pair $(\mathcal{L}, \rho)$ satisfies Theorem 0.1(iv) when $\mathbf{W}_{\mathscr{B}} := \mathbf{W}_{\mathscr{B},\min}$. Therefore, by Theorem 0.2, $\mathcal{L}$ is an equivalence of bicategories if and only if $\mathcal{F}$ satisfies conditions (A1) – (A5) when $\mathbf{W}_{\mathscr{B}} := \mathbf{W}_{\mathscr{B},\min}$. So by Proposition 4.1, $\mathcal{L}$ is an equivalence of bicategories if and only if $\mathcal{F}$ satisfies conditions (B1) – (B5).

Moreover, since $\mathcal{U}_{\mathbf{W}_{\mathscr{B},\min}}$ is an equivalence of bicategories, then $\overline{\mathcal{G}}$ is an equivalence of bicategories if and only if $\mathcal{L}$ is an equivalence of bicategories, i.e. if and only if and only if $\mathcal{F}$ satisfies conditions (B1) – (B5). $\square$

Lastly, we are able to give also the following proof:

Proof of Corollary 0.5. Let us suppose that (a) holds, i.e. let us suppose that there is an equivalence of bicategories $\overline{\mathcal{G}} : \mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right] \to \mathscr{B}$. Then we define $\mathcal{F} := \overline{\mathcal{G}} \circ \mathcal{U}_{\mathbf{W}_{\mathscr{A}}} : \mathscr{A} \to \mathscr{B}$. Since $\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}$ sends each morphism of $\mathbf{W}_{\mathscr{A}}$ to an internal equivalence, so does $\mathcal{F}$. Moreover, we set

$$\overline{\kappa} := i_{\overline{\mathcal{G}} \circ \mathcal{U}_{\mathbf{W}_{\mathscr{A}}}} : \ \mathcal{F} \Longrightarrow \overline{\mathcal{G}} \circ \mathcal{U}_{\mathbf{W}_{\mathscr{A}}}.$$

So the pair $(\overline{\mathcal{G}}, \overline{\kappa})$ satisfies the conditions of Theorem 0.3(2), hence by Theorem 0.4 we get that $\mathcal{F}$ satisfies conditions (B1) – (B5). So we have proved that (a) implies (b).

Conversely, let us suppose that there is a pseudofunctor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$, such that $\mathcal{F}_1(\mathbf{W}_{\mathscr{A}}) \subseteq \mathbf{W}_{\mathscr{B},\mathrm{equiv}}$. Therefore, there are a pseudofunctor $\overline{\mathcal{G}} : \mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right] \to \mathscr{B}$ and a pseudonatural equivalence $\overline{\kappa}$ as in Theorem 0.3(2). If we further assume that $\mathcal{F}$ satisfies conditions (B1) – (B5), then by Theorem 0.4 we conclude that $\overline{\mathcal{G}}$ is an equivalence of bicategories, so we have proved that (b) implies (a). $\square$

Appendix

Let us fix any pseudofunctor $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ sending each morphism of $\mathbf{W}_{\mathscr{A}}$ to an internal equivalence. As we mentioned in the introduction, Theorem 0.4 fixed some problems appearing in [Pr, Proposition 24]. As Theorem 0.4, also [Pr, Proposition 24] deals with necessary and sufficient conditions such that $\mathcal{F}$ induces an equivalence of bicategories from $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$ to $\mathscr{B}$. However, such a statement is partially incorrect. The existence a problem in such a statement was already mentioned in [R1, Remark before Proposition 1.8.2] and [R2, Remark after Proposition 6.3], but with no further details. For clarity of exposition, in the next lines we describe where the problem lies exactly. We recall that [Pr, Proposition 24] states the following: having fixed any $\mathcal{F}:\mathscr{A}\rightarrow\mathscr{B}$ such that $\mathcal{F}_1(\mathbf{W}_{\mathscr{A}})\subseteq\mathbf{W}_{\mathscr{B},\mathrm{equiv}}$, one should have an induced equivalence from $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$ to $\mathscr{B}$ if and only if $\mathcal{F}$ satisfies the following 3 conditions.

(EF1) $\mathcal{F}$ is essentially surjective, i.e. for each object $A_{\mathscr{B}}$ there are an object $A_{\mathscr{A}}$ and an *isomorphism* $t_{\mathscr{B}}:\mathcal{F}_0(A_{\mathscr{A}})\rightarrow A_{\mathscr{B}}$.

(EF2) For each pair of objects $A_{\mathscr{A}},B_{\mathscr{A}}$ and for each morphism $f_{\mathscr{B}}:\mathcal{F}_0(A_{\mathscr{A}})\rightarrow\mathcal{F}_0(B_{\mathscr{A}})$, there are an object $A'_{\mathscr{A}}$, a morphism $f_{\mathscr{A}}:A'_{\mathscr{A}}\rightarrow B_{\mathscr{A}}$, a morphism $\mathrm{w}_{\mathscr{A}}:A'_{\mathscr{A}}\rightarrow A_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and an invertible 2-morphism $\alpha_{\mathscr{B}}:\mathcal{F}_1(f_{\mathscr{A}})\Rightarrow f_{\mathscr{B}}\circ\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})$;.

(EF3) For each pair of objects $A_{\mathscr{A}},B_{\mathscr{A}}$, for each pair of morphisms $f^1_{\mathscr{A}},f^2_{\mathscr{A}}:A_{\mathscr{A}}\rightarrow B_{\mathscr{A}}$ and for each 2-morphism $\alpha_{\mathscr{B}}:\mathcal{F}_1(f^1_{\mathscr{A}})\Rightarrow\mathcal{F}_1(f^2_{\mathscr{A}})$, there is a *unique* 2-morphism $\alpha_{\mathscr{A}}:f^1_{\mathscr{A}}\Rightarrow f^2_{\mathscr{A}}$, such that $\mathcal{F}_2(\alpha_{\mathscr{A}})=\alpha_{\mathscr{B}}$.

One can easily see that such conditions are *too strong than necessary*: clearly (EF1) must be relaxed by allowing not only isomorphisms but also internal equivalences. But most importantly, (EF3) is definitely too much restrictive. This can be seen easily with a toy example as follows: let us consider a 2-category $\mathscr{C}$ with only 2 objects A,B, only a non-trivial morphism $\mathrm{v}:A\rightarrow B$ and only a non-trivial 2-morphism $\gamma:\mathrm{id}_B\Rightarrow\mathrm{id}_B$ (together with all the necessary identities and 2-identities). This clearly satisfies all the axioms of a 2-category and we have $\gamma\neq i_{\mathrm{id}_B}$ but $\gamma*i_{\mathrm{v}}=i_{\mathrm{v}}$, since there are no non-trivial 2-morphisms over v. Then we set $\mathbf{W}:=\{\mathrm{v},\mathrm{id}_A,\mathrm{id}_B\}$ and it is easy to prove that $(\mathscr{C},\mathbf{W})$ satisfies axioms (BF). Then following the constructions in [Pr, § 2.3 and 2.4], we have

$$\begin{aligned}\mathcal{U}_{\mathbf{W},2}(\gamma)&=\Big[B,\mathrm{id}_B,\mathrm{id}_B,i_{\mathrm{id}_B},\gamma\Big]=\Big[A,\mathrm{v},\mathrm{v},i_{\mathrm{v}},\gamma*i_{\mathrm{v}}\Big]=\\&=\Big[A,\mathrm{v},\mathrm{v},i_{\mathrm{v}},i_{\mathrm{v}}\Big]=\Big[B,\mathrm{id}_B,\mathrm{id}_B,i_{\mathrm{id}_B},i_{\mathrm{id}_B}\Big]=\mathcal{U}_{\mathbf{W},2}(i_{\mathrm{id}_B}).\end{aligned}$$

So the pseudofunctor $\mathcal{F}:=\mathcal{U}_{\mathbf{W}}:\mathscr{C}\rightarrow\mathscr{C}\left[\mathbf{W}^{-1}\right]$ does not satisfy the uniqueness condition of (EF3). However, if we consider the pair $(\overline{\mathcal{G}},\overline{\kappa})$ given by $\overline{\mathcal{G}}:=\mathrm{id}_{\mathscr{C}[\mathbf{W}^{-1}]}$ and $\overline{\kappa}:=i_{\mathcal{U}_{\mathbf{W}}}:\mathcal{U}_{\mathbf{W}}\Rightarrow\mathrm{id}_{\mathscr{C}[\mathbf{W}^{-1}]}\circ\mathcal{U}_{\mathbf{W}}$, then such a pair satisfies Theorem 0.3(2) and is such that $\overline{\mathcal{G}}$ is an equivalence of bicategories. This proves that *condition* (EF3) *is not necessary*, hence that [Pr, Proposition 24] has some flaw.

Actually, conditions (EF1) – (EF3) are *sufficient* in order to have that $\overline{\mathcal{G}}$ is an equivalence of bicategories. We are going to prove this in the remaining part of this Appendix.

Lemma 4.2. *Let us fix any pair $(\mathscr{A},\mathbf{W}_{\mathscr{A}})$ satisfying conditions* (BF)*, any bicategory $\mathscr{B}$ and any pseudofunctor $\mathcal{F}:\mathscr{A}\rightarrow\mathscr{B}$, such that:*

- *$\mathcal{F}$ sends each morphism of $\mathbf{W}_{\mathscr{A}}$ to an internal equivalence of $\mathscr{B}$;*
- *$\mathcal{F}$ satisfies conditions* (EF2) *and* (EF3).

Moreover, let us fix any morphism $f_{\mathscr{A}}:B_{\mathscr{A}}\rightarrow A_{\mathscr{A}}$, such that $\mathcal{F}_1(f_{\mathscr{A}})$ is an internal equivalence in $\mathscr{B}$. Then there are an object $C_{\mathscr{A}}$ and a morphism $g_{\mathscr{A}}:C_{\mathscr{A}}\rightarrow B_{\mathscr{A}}$, such that $f_{\mathscr{A}}\circ g_{\mathscr{A}}$ belongs to $\mathbf{W}_{\mathscr{A}}$ and $\mathcal{F}_1(g_{\mathscr{A}})$ is an internal equivalence.

Proof. Since $\mathcal{F}_1(f_{\mathscr{A}})$ is an internal equivalence, then there are an internal equivalence $\overline{e}_{\mathscr{B}}:\mathcal{F}_0(A_{\mathscr{A}})\rightarrow\mathcal{F}_0(B_{\mathscr{A}})$ and an invertible 2-morphism $\xi_{\mathscr{B}}:\mathcal{F}_1(f_{\mathscr{A}})\circ\overline{e}_{\mathscr{B}}\Rightarrow\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})}$. By applying (EF2) to $\mathcal{F}_1(f_{\mathscr{A}})\circ\overline{e}_{\mathscr{B}}:\mathcal{F}_0(A_{\mathscr{A}})\rightarrow\mathcal{F}_0(A_{\mathscr{A}})$, we get an object $A'_{\mathscr{A}}$, a morphism $m_{\mathscr{A}}:A'_{\mathscr{A}}\rightarrow A_{\mathscr{A}}$, a morphism $\mathrm{u}_{\mathscr{A}}:A'_{\mathscr{A}}\rightarrow A_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and an invertible 2-morphism $\gamma_{\mathscr{B}}:\mathcal{F}_1(m_{\mathscr{A}})\Rightarrow(\mathcal{F}_1(f_{\mathscr{A}})\circ\overline{e}_{\mathscr{B}})\circ\mathcal{F}_1(\mathrm{u}_{\mathscr{A}})$. Then we consider the following invertible 2-morphism

$$\eta_{\mathscr{B}} := \upsilon_{\mathcal{F}_1(\mathrm{u}_{\mathscr{A}})} \odot \Big(\xi_{\mathscr{B}} * i_{\mathcal{F}_1(\mathrm{u}_{\mathscr{A}})}\Big) \odot \gamma_{\mathscr{B}} : \ \mathcal{F}_1(\mathrm{m}_{\mathscr{A}}) \Longrightarrow \mathcal{F}_1(u_{\mathscr{A}}).$$

By (EF3), there is a unique 2-morphism $\eta_{\mathscr{A}} : \mathrm{m}_{\mathscr{A}} \Rightarrow u_{\mathscr{A}}$, such that $\mathcal{F}_2(\eta_{\mathscr{A}}) = \eta_{\mathscr{B}}$. Again by (EF3) we get that $\eta_{\mathscr{A}}$ is invertible since $\eta_{\mathscr{B}}$ is so. Since $\mathrm{u}_{\mathscr{A}}$ belongs to $\mathbf{W}_{\mathscr{A}}$ by construction, then by (BF5) for $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ we conclude that also $m_{\mathscr{A}}$ belongs to $\mathbf{W}_{\mathscr{A}}$.

Now we apply again (EF2) to the morphism $\overline{e}_{\mathscr{B}} \circ \mathcal{F}_1(\mathrm{u}_{\mathscr{A}}) : \mathcal{F}_0(A'_{\mathscr{A}}) \rightarrow \mathcal{F}_0(B_{\mathscr{A}})$. So there are an object $C_{\mathscr{A}}$, a morphism $g_{\mathscr{A}} : C_{\mathscr{A}} \rightarrow B_{\mathscr{A}}$, a morphism $\mathrm{z}_{\mathscr{A}} : C_{\mathscr{A}} \rightarrow A'_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and an invertible 2-morphism $\delta_{\mathscr{B}} : \mathcal{F}_1(g_{\mathscr{A}}) \Rightarrow (\overline{e}_{\mathscr{B}} \circ \mathcal{F}_1(\mathrm{u}_{\mathscr{A}})) \circ \mathcal{F}_1(\mathrm{z}_{\mathscr{A}})$. Then we consider the following invertible 2-morphism:

$$\begin{gathered}\sigma_{\mathscr{B}} := \Big(\psi^{\mathcal{F}}_{\mathrm{m}_{\mathscr{A}},\mathrm{z}_{\mathscr{A}}}\Big)^{-1} \odot \Big(\gamma_{\mathscr{B}}^{-1} * i_{\mathcal{F}_1(\mathrm{z}_{\mathscr{A}})}\Big) \odot \Big(\theta_{\mathcal{F}_1(f_{\mathscr{A}}),\overline{e}_{\mathscr{B}},\mathcal{F}_1(\mathrm{u}_{\mathscr{A}})} * i_{\mathcal{F}_1(\mathrm{z}_{\mathscr{A}})}\Big) \odot \\ \odot\, \theta_{\mathcal{F}_1(f_{\mathscr{A}}),\overline{e}_{\mathscr{B}}\circ\mathcal{F}_1(\mathrm{u}_{\mathscr{A}}),\mathcal{F}_1(\mathrm{z}_{\mathscr{A}})} \odot \Big(i_{\mathcal{F}_1(f_{\mathscr{A}})} * \delta_{\mathscr{B}}\Big) \odot \psi^{\mathcal{F}}_{f_{\mathscr{A}},g_{\mathscr{A}}} : \\ \mathcal{F}_1(f_{\mathscr{A}} \circ g_{\mathscr{A}}) \Longrightarrow \mathcal{F}_1(m_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}).\end{gathered}$$

By (EF3), there is a unique 2-morphism $\sigma_{\mathscr{A}} : f_{\mathscr{A}} \circ g_{\mathscr{A}} \Rightarrow m_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}$, such that $\mathcal{F}_2(\sigma_{\mathscr{A}}) = \sigma_{\mathscr{B}}$. Moreover, again by (EF3) we get that $\sigma_{\mathscr{A}}$ is invertible. Now both $m_{\mathscr{A}}$ and $\mathrm{z}_{\mathscr{A}}$ belong to $\mathbf{W}_{\mathscr{A}}$, hence by (BF2) for $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$, so does their composition. Then by (BF5) applied to $\sigma_{\mathscr{A}}$, also the morphism $f_{\mathscr{A}} \circ g_{\mathscr{A}}$ belongs to $\mathbf{W}_{\mathscr{A}}$.

Since $f_{\mathscr{A}} \circ g_{\mathscr{A}}$ belongs to $\mathbf{W}_{\mathscr{A}}$, then by hypothesis we get that $\mathcal{F}_1(f_{\mathscr{A}} \circ g_{\mathscr{A}})$ is an internal equivalence of $\mathscr{B}$. Therefore we get easily that also $\mathcal{F}_1(f_{\mathscr{A}}) \circ \mathcal{F}_1(g_{\mathscr{A}})$ is an internal equivalence. Since $\mathcal{F}_1(f_{\mathscr{A}})$ is an internal equivalence by hypothesis, then by [T2, Lemma 1.2] we conclude that $\mathcal{F}_1(g_{\mathscr{A}})$ is an internal equivalence. □

Proposition 4.3. *Let us fix any pair* $(\mathscr{A}, \mathbf{W}_{\mathscr{A}})$ *satisfying conditions* (BF), *any bicategory* $\mathscr{B}$ *and any pseudofunctor* $\mathcal{F} : \mathscr{A} \rightarrow \mathscr{B}$, *such that:*

- $\mathcal{F}$ *sends each morphism in* $\mathbf{W}_{\mathscr{A}}$ *to an internal equivalence of* $\mathscr{B}$;
- $\mathcal{F}$ *satisfies conditions* (EF1), (EF2) *and* (EF3).

Then $\mathcal{F}$ *satisfies conditions* (B1) – (B5), *hence by Corollary 0.5 there is an equivalence of bicategories* $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right] \rightarrow \mathscr{B}$. *In other terms,* [Pr, Proposition 24] *lists sufficient* (but not necessary) *conditions for having an induced equivalence from* $\mathscr{A}\left[\mathbf{W}_{\mathscr{A}}^{-1}\right]$ *to* $\mathscr{B}$.

Proof. Clearly (EF1) implies (B1).

Let us prove (B2), so let us fix any pair of objects $A^1_{\mathscr{A}}, A^2_{\mathscr{A}}$ and any internal equivalence $e_{\mathscr{B}} : \mathcal{F}_0(A^1_{\mathscr{A}}) \rightarrow \mathcal{F}_0(A^2_{\mathscr{A}})$. Then by (EF2) applied to $e_{\mathscr{B}}$, there are an object $A^3_{\mathscr{A}}$, a morphism $\mathrm{w}^1_{\mathscr{A}} : A^3_{\mathscr{A}} \rightarrow A^1_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$, a morphism $\mathrm{w}^2_{\mathscr{A}} : A^3_{\mathscr{A}} \rightarrow A^2_{\mathscr{A}}$ and an invertible 2-morphism $\alpha_{\mathscr{B}} : \mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}}) \Rightarrow e_{\mathscr{B}} \circ \mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})$. Since $\mathcal{F}$ sends each morphism of $\mathbf{W}_{\mathscr{A}}$ to an internal equivalence, then $\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})$ is an internal equivalence. Then by [T2, Lemmas 1.1 and 1.2] we conclude that also $\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})$ is an internal equivalence in $\mathscr{B}$. Then by Lemma 4.2 applied to $\mathrm{w}^2_{\mathscr{A}}$, there are an object $A^4_{\mathscr{A}}$ and a morphism $\mathrm{v}_{\mathscr{A}} : A^4_{\mathscr{A}} \rightarrow A^3_{\mathscr{A}}$, such that $\mathrm{w}^2_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}}$ belongs to $\mathbf{W}_{\mathscr{A}}$ and $\mathcal{F}_1(\mathrm{v}_{\mathscr{A}})$ is an internal equivalence. Again by Lemma 4.2 applied to $\mathrm{v}_{\mathscr{A}}$, there are an object $A^5_{\mathscr{A}}$ and a morphism $\mathrm{z}_{\mathscr{A}} : A^5_{\mathscr{A}} \rightarrow A^4_{\mathscr{A}}$, such that $\mathrm{v}_{\mathscr{A}} \circ \mathrm{z}_{\mathscr{A}}$ belongs to $\mathbf{W}_{\mathscr{A}}$. Using the definition of right saturated, this proves that $\mathrm{w}^2_{\mathscr{A}}$ belongs to $\mathbf{W}_{\mathscr{A},\mathrm{sat}}$.

Since $\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})$ is an internal equivalence, then there are an internal equivalence $e'_{\mathscr{B}}:\mathcal{F}_0(A^1_{\mathscr{A}})\rightarrow\mathcal{F}_0(A^3_{\mathscr{A}})$ and an invertible 2-morphism $\delta^1_{\mathscr{B}}:\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}})\circ e'_{\mathscr{B}}\Rightarrow\mathrm{id}_{\mathcal{F}_0(A^1_{\mathscr{A}})}$. Then we define an invertible 2-morphism as follows:

$$\delta^2_{\mathscr{B}}:=\Big(\alpha^{-1}_{\mathscr{B}}*i_{e'_{\mathscr{B}}}\Big)\odot\theta_{e_{\mathscr{B}},\mathcal{F}_1(\mathrm{w}^1_{\mathscr{A}}),e'_{\mathscr{B}}}\odot\Big(i_{e_{\mathscr{B}}}*(\delta^1_{\mathscr{B}})^{-1}\Big)\odot\pi^{-1}_{e_{\mathscr{B}}}:\ e_{\mathscr{B}}\Longrightarrow\mathcal{F}_1(\mathrm{w}^2_{\mathscr{A}})\circ e'_{\mathscr{B}}.$$

This proves that (B2) holds.

Now let us prove that condition (B3) is satisfied, so let us fix any pair of objects $B_{\mathscr{A}},A_{\mathscr{B}}$ and any morphism $f_{\mathscr{B}}:A_{\mathscr{B}}\rightarrow\mathcal{F}_0(B_{\mathscr{A}})$. By (EF1) there are an object $A_{\mathscr{A}}$ and an isomorphism $\mathrm{t}_{\mathscr{B}}:\mathcal{F}_0(A_{\mathscr{A}})\rightarrow A_{\mathscr{B}}$. Let us apply (EF2) to the morphism $f_{\mathscr{B}}\circ t_{\mathscr{B}}:\mathcal{F}_0(A_{\mathscr{A}})\rightarrow\mathcal{F}_0(B_{\mathscr{A}})$. Then there are an object $A'_{\mathscr{A}}$, a morphism $f_{\mathscr{A}}:A'_{\mathscr{A}}\rightarrow B_{\mathscr{A}}$, a morphism $\mathrm{w}_{\mathscr{A}}:A'_{\mathscr{A}}\rightarrow A_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and an invertible 2-morphism $\delta_{\mathscr{B}}:\mathcal{F}_1(f_{\mathscr{A}})\Rightarrow(f_{\mathscr{B}}\circ t_{\mathscr{B}})\circ\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})$. Since $\mathcal{F}$ sends each morphism of $\mathbf{W}_{\mathscr{A}}$ to an internal equivalence, then there are an internal equivalence $\mathrm{w}_{\mathscr{B}}:\mathcal{F}_0(A_{\mathscr{A}})\rightarrow\mathcal{F}_0(A'_{\mathscr{A}})$ and an invertible 2-morphism $\xi_{\mathscr{B}}:\mathrm{id}_{\mathcal{F}_0(A_{\mathscr{A}})}\Rightarrow\mathcal{F}_1(\mathrm{w}_{\mathscr{A}})\circ\mathrm{w}_{\mathscr{B}}$. Then we define an internal equivalence

$$e_{\mathscr{B}}:=\mathrm{w}_{\mathscr{B}}\circ\mathrm{t}^{-1}_{\mathscr{B}}:\ A_{\mathscr{B}}\longrightarrow\mathcal{F}_0(A'_{\mathscr{A}})$$

and an invertible 2-morphism as follows

$$\begin{gathered}\alpha_{\mathscr{B}}:=\theta^{-1}_{\mathcal{F}_1(f_{\mathscr{A}}),\mathrm{w}_{\mathscr{B}},\mathrm{t}^{-1}_{\mathscr{B}}}\odot\Big(\Big(\delta^{-1}_{\mathscr{B}}*i_{\mathrm{w}_{\mathscr{B}}}\Big)*i_{\mathrm{t}^{-1}_{\mathscr{B}}}\Big)\odot\Big(\theta_{f_{\mathscr{B}}\circ t_{\mathscr{B}},\mathcal{F}_1(\mathrm{w}_{\mathscr{A}}),\mathrm{w}_{\mathscr{B}}}*i_{\mathrm{t}^{-1}_{\mathscr{B}}}\Big)\odot\\ \odot\Big(\Big(i_{f_{\mathscr{B}}\circ t_{\mathscr{B}}}*\xi_{\mathscr{B}}\Big)*i_{\mathrm{t}^{-1}_{\mathscr{B}}}\Big)\odot\Big(\pi^{-1}_{f_{\mathscr{B}}\circ t_{\mathscr{B}}}*i_{\mathrm{t}^{-1}_{\mathscr{B}}}\Big)\odot\theta_{f_{\mathscr{B}},t_{\mathscr{B}},\mathrm{t}^{-1}_{\mathscr{B}}}\odot\pi^{-1}_{f_{\mathscr{B}}}:\\ f_{\mathscr{B}}\Longrightarrow\mathcal{F}_1(f_{\mathscr{A}})\circ e_{\mathscr{B}}.\end{gathered}$$

This proves that (B3) holds. The proof that (B4) and (B5) hold is a direct consequence of (EF3). □

Remark 4.4. The already mentioned [Pr, Proposition 24] is applied in [Pr] only for what concerns the *sufficiency* of conditions (EF1) – (EF3). Hence even if such conditions are only sufficient but not necessary, Proposition 4.3 shows that the error in [Pr, Proposition 24] does not affect the rest of the computations in [Pr].

Remark 4.5. Let us consider the special case when $\mathscr{B}:=\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$ and $\mathcal{F}:=\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}$. In this case, one can consider the pair $(\overline{\mathcal{G}},\overline{\kappa})$ given by $\overline{\mathcal{G}}:=\mathrm{id}_{\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]}$ and $\overline{\kappa}:=i_{\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}}:\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}\Rightarrow\overline{\mathcal{G}}\circ\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}$. This pair satisfies Theorem 0.3(2) and $\overline{\mathcal{G}}$ is an equivalence of bicategories, so Theorem 0.4 tells us that $\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}$ satisfies conditions (B). As a check for the correctness of Theorem 0.4, we want to verify by hand that result.

In this case, clearly (B1) holds since $\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}$ is a bijection on objects. Now let us prove (B2), so let us fix any pair of objects $A^1_{\mathscr{A}},A^2_{\mathscr{A}}$ and any internal equivalence $e_{\mathscr{B}}$ from $A^1_{\mathscr{A}}$ to $A^2_{\mathscr{A}}$ in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$. By [T2, Corollary 2.7] for $(\mathscr{A},\mathbf{W}_{\mathscr{A}})$, $e_{\mathscr{B}}$ is necessarily of the form

$$e_{\mathscr{B}}=\Big(A^1_{\mathscr{A}}\xleftarrow{\ \mathrm{w}^1_{\mathscr{A}}\ }A^3_{\mathscr{A}}\xrightarrow{\ \mathrm{w}^2_{\mathscr{A}}\ }A^2_{\mathscr{A}}\Big),$$

with $\mathrm{w}^1_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and $\mathrm{w}^2_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A},\mathrm{sat}}$. Then we define an internal equivalence $e'_{\mathscr{B}}:A^1_{\mathscr{A}}\rightarrow A^3_{\mathscr{A}}$ in $\mathscr{B}:=\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$ as follows:

$$e'_{\mathscr{B}} := \Big(A^1_{\mathscr{A}} \xleftarrow{\mathrm{w}^1_{\mathscr{A}}} A^3_{\mathscr{A}} \xrightarrow{\mathrm{id}_{A^3_{\mathscr{A}}}} A^3_{\mathscr{A}}\Big).$$

Moreover, we define a pair of invertible 2-morphisms in $\mathscr{B} = \mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$:

$$\delta^1_{\mathscr{B}} := \Big[A^3_{\mathscr{A}}, \mathrm{id}_{A^3_{\mathscr{A}}}, \mathrm{w}^1_{\mathscr{A}}, \upsilon^{-1}_{\mathrm{w}^1_{\mathscr{A}}} \odot \pi_{\mathrm{w}^1_{\mathscr{A}}} \odot \pi_{\mathrm{w}^1_{\mathscr{A}} \circ \mathrm{id}_{A^3_{\mathscr{A}}}}, \upsilon^{-1}_{\mathrm{w}^1_{\mathscr{A}}} \odot \pi_{\mathrm{w}^1_{\mathscr{A}}} \odot \pi_{\mathrm{w}^1_{\mathscr{A}} \circ \mathrm{id}_{A^3_{\mathscr{A}}}}\Big]:$$
$$\mathcal{U}_{\mathbf{W}_{\mathscr{A}},1}(\mathrm{w}^1_{\mathscr{A}}) \circ e'_{\mathscr{B}} = \Big(A^3_{\mathscr{A}}, \mathrm{w}^1_{\mathscr{A}} \circ \mathrm{id}_{A^3_{\mathscr{A}}}, \mathrm{w}^1_{\mathscr{A}} \circ \mathrm{id}_{A^3_{\mathscr{A}}}\Big) \Longrightarrow \Big(A^1_{\mathscr{A}}, \mathrm{id}_{A^1_{\mathscr{A}}}, \mathrm{id}_{A^1_{\mathscr{A}}}\Big) = \mathrm{id}_{A^1_{\mathscr{A}}}$$

and

$$\delta^2_{\mathscr{B}} := \Big[A^3_{\mathscr{A}}, \mathrm{id}_{A^3_{\mathscr{A}}}, \mathrm{id}_{A^3_{\mathscr{A}}}, \pi^{-1}_{\mathrm{w}^1_{\mathscr{A}} \circ \mathrm{id}_{A^3_{\mathscr{A}}}}, \pi^{-1}_{\mathrm{w}^2_{\mathscr{A}} \circ \mathrm{id}_{A^3_{\mathscr{A}}}}\Big]:$$
$$e_{\mathscr{B}} \Longrightarrow \Big(A^3_{\mathscr{A}}, \mathrm{w}^1_{\mathscr{A}} \circ \mathrm{id}_{A^3_{\mathscr{A}}}, \mathrm{w}^2_{\mathscr{A}} \circ \mathrm{id}_{A^3_{\mathscr{A}}}\Big) = \mathcal{U}_{\mathbf{W},1}(\mathrm{w}^2_{\mathscr{A}}) \circ e'_{\mathscr{B}}.$$

This suffices to prove that (B2) holds for $\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}$. Condition (B3) is a direct condition of the definition of morphisms in a bicategory of fractions.

In order to prove (B4), it suffices to use [T1, Proposition 0.7] in the special case when $(\mathscr{C}, \mathbf{W}) := (\mathscr{A}, \mathbf{W}_{\mathscr{A}})$, $A^1 = A^2 =: A_{\mathscr{A}}$, $\mathrm{w}^1 = \mathrm{w}^2 := \mathrm{id}_{A_{\mathscr{A}}}$, $\alpha := i_{\mathrm{id}_{A_{\mathscr{A}}} \circ \mathrm{id}_{A_{\mathscr{A}}}}$ and $\gamma^m := \gamma^m_{\mathscr{A}} * i_{\mathrm{id}_{A_{\mathscr{A}}}}$, so that $\Gamma^m = \mathcal{U}_{\mathbf{W}_{\mathscr{A}},2}(\gamma^m_{\mathscr{A}})$ for $m = 1, 2$.

Lastly, let us prove (B5), so let us fix any pair of objects $A_{\mathscr{A}}, B_{\mathscr{A}}$, any pair of morphisms $f^1_{\mathscr{A}}, f^2_{\mathscr{A}} : A_{\mathscr{A}} \to B_{\mathscr{A}}$ and any 2-morphism $\Gamma_{\mathscr{A}}$ from $(A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, f^1_{\mathscr{A}}) = \mathcal{U}_{\mathbf{W}_{\mathscr{A}},1}(f^1_{\mathscr{A}})$ to $(A_{\mathscr{A}}, \mathrm{id}_{A_{\mathscr{A}}}, f^2_{\mathscr{A}}) = \mathcal{U}_{\mathbf{W}_{\mathscr{A}},1}(f^2_{\mathscr{A}})$ in $\mathscr{A}\left[\mathbf{W}^{-1}_{\mathscr{A}}\right]$. By [T2, Lemma 6.1] applied to $\alpha := i_{\mathrm{id}_{A_{\mathscr{A}}} \circ \mathrm{id}_{A_{\mathscr{A}}}}$, there are an object $A'_{\mathscr{A}}$ and a morphism $\mathrm{v}_{\mathscr{A}} : A'_{\mathscr{A}} \to A_{\mathscr{A}}$ in $\mathbf{W}_{\mathscr{A}}$ and a 2-morphism $\gamma_{\mathscr{A}} : f^1_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}} \Rightarrow f^2_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}}$, such that

$$\Gamma_{\mathscr{A}} = \Big[A'_{\mathscr{A}}, \mathrm{v}_{\mathscr{A}}, \mathrm{v}_{\mathscr{A}}, i_{\mathrm{id}_{A_{\mathscr{A}}} \circ \mathrm{v}_{\mathscr{A}}}, \gamma_{\mathscr{A}}\Big].$$

Hence, using [Pr, pagg. 259–261] and the description of $\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}$ we have that

$$\Gamma_{\mathscr{A}} * i_{\mathcal{U}_{\mathbf{W}_{\mathscr{A}},1}(\mathrm{v}_{\mathscr{A}})} = \Gamma_{\mathscr{A}} * i_{\left(A'_{\mathscr{A}}, \mathrm{id}_{A'_{\mathscr{A}}}, \mathrm{v}_{\mathscr{A}}\right)}$$

is represented by the following diagram

$$\begin{tikzcd}
 & A'_{\mathscr{A}} \arrow[dl, "\mathrm{id}_{A'_{\mathscr{A}}} \circ \mathrm{id}_{A'_{\mathscr{A}}}"'] \arrow[dr, "f^1_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}}"] & \\
A'_{\mathscr{A}} \arrow[r, phantom, "\Downarrow i_{(\mathrm{id}_{A'_{\mathscr{A}}} \circ \mathrm{id}_{A'_{\mathscr{A}}}) \circ \mathrm{id}_{A'_{\mathscr{A}}}}"] & A'_{\mathscr{A}} \arrow[u, "\mathrm{id}_{A'_{\mathscr{A}}}"'] \arrow[d, "\mathrm{id}_{A'_{\mathscr{A}}}"] \arrow[r, phantom, "\Downarrow \gamma_{\mathscr{A}} * i_{\mathrm{id}_{A'_{\mathscr{A}}}}"] & B_{\mathscr{A}}. \\
 & A'_{\mathscr{A}} \arrow[ul, "\mathrm{id}_{A'_{\mathscr{A}}} \circ \mathrm{id}_{A'_{\mathscr{A}}}"] \arrow[ur, "f^2_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}}"'] &
\end{tikzcd}$$

Moreover, for each $m = 1, 2$, a direct check proves that the associator $\psi^{\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}}_{f^m_{\mathscr{A}}, \mathrm{v}_{\mathscr{A}}}$ for $\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}$ is represented by the following diagram:

$$\begin{tikzcd}
 & & A'_{\mathscr{A}} \arrow[dll, "\mathrm{id}_{A'_{\mathscr{A}}}"'] \arrow[drr, "f^m_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}}"] & & \\
A'_{\mathscr{A}} \arrow[rr, phantom, "\Downarrow \pi^{-1}_{\mathrm{id}_{A'_{\mathscr{A}}} \circ \mathrm{id}_{A'_{\mathscr{A}}}}"] & & A'_{\mathscr{A}} \arrow[u, "\mathrm{id}_{A'_{\mathscr{A}}}"'] \arrow[d, "\mathrm{id}_{A'_{\mathscr{A}}}"] \arrow[rr, phantom, "\Downarrow i_{(f^m_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}}) \circ \mathrm{id}_{A'_{\mathscr{A}}}}"] & & B_{\mathscr{A}}. \\
 & & A'_{\mathscr{A}} \arrow[ull, "\mathrm{id}_{A'_{\mathscr{A}}} \circ \mathrm{id}_{A'_{\mathscr{A}}}"] \arrow[urr, "f^m_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}}"'] & &
\end{tikzcd}$$

Therefore, it is easy to see that the composition

$$\psi^{\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}}_{f^2_{\mathscr{A}}, \mathrm{v}_{\mathscr{A}}} \odot \Gamma_{\mathscr{A}} * i_{\mathcal{U}_{\mathbf{W}_{\mathscr{A}},1}(\mathrm{v}_{\mathscr{A}})} \odot \left(\psi^{\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}}_{f^1_{\mathscr{A}}, \mathrm{v}_{\mathscr{A}}}\right)^{-1} \tag{4.2}$$

is represented by the diagram

$$\begin{tikzcd}
 & & A'_{\mathscr{A}} \arrow[dll, "\mathrm{id}_{A'_{\mathscr{A}}}"'] \arrow[drr, "f^1_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}}"] & & \\
A'_{\mathscr{A}} \arrow[rr, phantom, "\Downarrow i_{\mathrm{id}_{A'_{\mathscr{A}}} \circ \mathrm{id}_{A'_{\mathscr{A}}}}"] & & A'_{\mathscr{A}} \arrow[u, "\mathrm{id}_{A'_{\mathscr{A}}}"'] \arrow[d, "\mathrm{id}_{A'_{\mathscr{A}}}"] \arrow[rr, phantom, "\Downarrow \gamma_{\mathscr{A}} * i_{\mathrm{id}_{A'_{\mathscr{A}}}}"] & & B_{\mathscr{A}}, \\
 & & A'_{\mathscr{A}} \arrow[ull, "\mathrm{id}_{A'_{\mathscr{A}}}"] \arrow[urr, "f^2_{\mathscr{A}} \circ \mathrm{v}_{\mathscr{A}}"'] & &
\end{tikzcd}$$

i.e. (4.2) coincides with $\mathcal{U}_{\mathbf{W}_{\mathscr{A}},2}(\gamma_{\mathscr{A}})$ (see [Pr, § 2.4]), so (B5) holds for $\mathcal{U}_{\mathbf{W}_{\mathscr{A}}}$.

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