On the use of power indexes in reliability theory

Jean-Luc Marichal

University of Luxembourg
Luxembourg

Part I : Semicoherent systems

System

Definition. A *system* is a set of interconnected components

$$C = \{1, \ldots, n\} = [n]$$

Example. Home video system

1. Blu-ray player
2. PlayStation 3
3. LED television
4. Sound amplifier
5. Speaker A
6. Speaker B

Assumptions

- The system and the components are of the crisply on/off kind
- The components are nonrepairable

Structure function

State of a component $j \in C = [n]$ $\rightarrow$ Boolean variable

$$x_j = \begin{cases} 1 & \text{if component } j \text{ is functioning} \\ 0 & \text{if component } j \text{ is in a failed state} \end{cases}$$

State of the system $\rightarrow$ Boolean function $\phi : \{0,1\}^n \rightarrow \{0,1\}$

$$\phi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is in a failed state} \end{cases}$$

This function is called the *structure function* of the system

$$\boxed{S = (C, \phi)}$$

Representations of Boolean functions

Boolean function $\longleftrightarrow$ set function

$\phi : \{0,1\}^n \to \{0,1\}$ $\qquad$ $\phi : 2^{[n]} \to \{0,1\}$

$$\phi(\mathbf{1}_A) \;=\; \phi(A) \qquad A \subseteq [n]$$

Polynomial representation of a Boolean function

$$\phi(\mathbf{x}) \;=\; \sum_{A \subseteq [n]} \phi(A) \prod_{j \in A} x_j \prod_{j \in [n] \setminus A} (1 - x_j)$$

Semicoherent systems

The system is said to be *semicoherent* if

- ϕ is nondecreasing : $\mathbf{x} \leqslant \mathbf{x}' \;\Rightarrow\; \phi(\mathbf{x}) \leqslant \phi(\mathbf{x}')$
- $\phi(\mathbf{0}) = 0$, $\phi(\mathbf{1}) = 1$

Representations of Boolean functions

$$x_1 \sqcap x_2 = \min(x_1, x_2) = x_1 x_2$$
$$x_1 \sqcup x_2 = \max(x_1, x_2) = 1 - (1 - x_1)(1 - x_2)$$

Since ϕ is nondecreasing and nonconstant

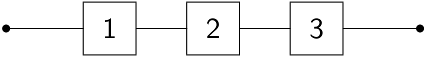
$$\phi(\mathbf{x}) = \coprod_{\substack{A \subseteq [n] \\ \phi(A)=1}} \prod_{j \in A} x_j$$

$$\phi(\mathbf{x}) = \prod_{\substack{A \subseteq [n] \\ \phi([n] \setminus A)=0}} \coprod_{j \in A} x_j$$

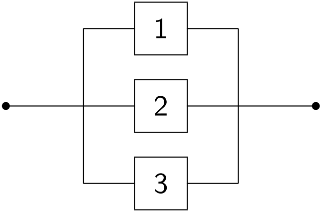
(Hammer and Rudeanu 1968)

Block diagrams

- A serially connected segment of components is functioning if and only if every single component is functioning

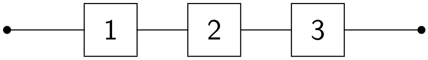


- A system of parallel components is functioning if and only at least one component is functioning



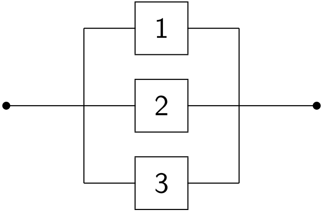
Block diagrams

Series structure



$$\phi(\mathbf{x}) = x_1 \; x_2 \; x_3 = \prod_{i=1}^{3} x_i$$

Parallel structure

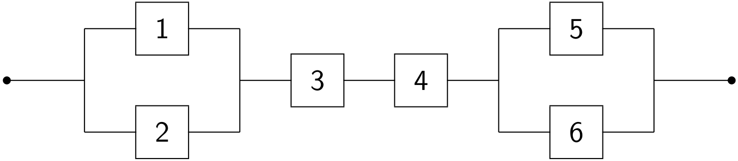


$$\phi(\mathbf{x}) = 1 - (1 - x_1)(1 - x_2)(1 - x_3) = \coprod_{i=1}^{3} x_i$$

Block diagrams

Example. Home video system

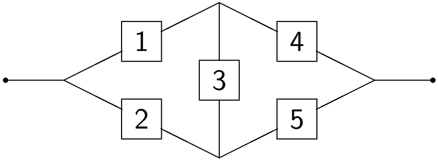
1. Blu-ray player
2. PlayStation 3
3. LED television
4. Sound amplifier
5. Speaker A
6. Speaker B



$$\phi(\mathbf{x}) = (x_1 \amalg x_2)\, x_3\, x_4\, (x_5 \amalg x_6)$$

Block diagrams

Example. Bridge structure



$$\phi(\mathbf{x}) \;=\; x_3\,\phi(1_3,\mathbf{x}) + (1 - x_3)\phi(0_3,\mathbf{x})$$

$$\begin{aligned}
\phi(1_3,\mathbf{x}) &= (x_1 \textstyle\coprod x_2)(x_4 \textstyle\coprod x_5) \\
\phi(0_3,\mathbf{x}) &= (x_1\ x_4) \textstyle\coprod (x_2\ x_5)
\end{aligned}$$

Pivotal decomposition of the structure function

$$\phi(\mathbf{x}) \;=\; x_j\,\phi(1_j,\mathbf{x}) + (1 - x_j)\phi(0_j,\mathbf{x})$$

Correspondence Reliability/Game Theory

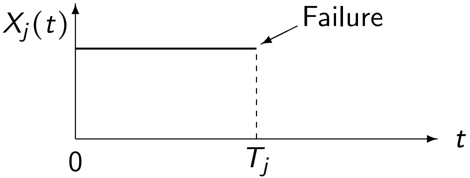
Reliability	Game Theory
Component	Player
Semicoherent structure	Simple game
Structure function	Characteristic function
Irrelevant component	Null player
Path set	Winning coalition
Cut set	Blocking coalition
Minimal path set	Minimal winning coalition
Minimal cut set	Minimal blocking coalition
Series structure	Unanimity game
Paralell structure	Decisive game
Module	Committee
Modular set	Committee set

(Ramamurthy 1990)

State variable $\longrightarrow$ Random variable

$$x_j \quad \longrightarrow \quad X_j(t)$$

$$X_j(t) = \begin{cases} 1 & \text{if } j \text{ is functioning at time } t \\ 0 & \text{if } j \text{ is in a failed state at time } t \end{cases}$$



T_j = *random lifetime* of component $j \in C$

$X_j(t) = \mathrm{Ind}(T_j > t)$ = *random state* of j at time $t \geqslant 0$

System lifetime and component lifetimes

T_S = *system lifetime*

$X_S(t) = \mathrm{Ind}(T_S > t)$ = random *state of the system* at time $t \geqslant 0$

$$X_S(t) = \phi(X_1(t), \ldots, X_n(t)) \qquad t \geqslant 0$$

How to describe $T_1, \ldots, T_n$?

System lifetime and component lifetimes

Cumulative distribution function (c.d.f.) of the component lifetimes

$$F(t_1, \ldots, t_n) = \Pr(T_1 \leqslant t_1, \ldots, T_n \leqslant t_n) \qquad t_1, \ldots, t_n \geqslant 0$$

$$\boxed{S = (C, \phi, F)}$$

Classical assumptions

- F absolutely continuous + i.i.d. lifetimes
- F absolutely continuous + exchangeable lifetimes
- F has no ties

$$\Pr(T_i = T_j) = 0 \qquad i \neq j$$

Part II : Signature and importance indexes

Simple game

Let $N = \{1, \ldots, n\}$ be the set of *players*

Characteristic function of the game
= set function $v : 2^N \to \mathbb{R}$ which assigns to each coalition $S \subseteq N$ of players a real number $v(S)$ which represents the *worth* of S

The game is said to be *simple* if v takes on its values in $\{0, 1\}$

The set function v can be regarded as a Boolean function $v : \{0, 1\}^n \to \{0, 1\}$

Power indexes

Let $v : 2^N \to \{0,1\}$ be a simple game on a set N of n players
Let $j \in N$ be a player

Banzhaf power index (Banzhaf 1965)

$$\psi_{\mathrm{B}}(v,j) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{j\}} \big(v(S \cup \{j\}) - v(S)\big)$$

Shapley power index (Shapley 1953)

$$\psi_{\mathrm{Sh}}(v,j) = \sum_{S \subseteq N \setminus \{j\}} \frac{1}{n\binom{n-1}{|S|}} \big(v(S \cup \{j\}) - v(S)\big)$$

Cardinality index

Cardinality index (Yager 2002)

$$C_k = \frac{1}{(n-k)\binom{n}{k}} \sum_{|S|=k} \sum_{j \in N \setminus S} \big(v(S \cup \{j\}) - v(S)\big) \quad (k = 0, \ldots, n-1)$$

$$C_k = \frac{1}{\binom{n}{k+1}} \sum_{|S|=k+1} v(S) - \frac{1}{\binom{n}{k}} \sum_{|S|=k} v(S)$$

Interpretation:
C_k is the average gain that we obtain by adding an arbitrary player to an arbitrary k-player coalition

Barlow-Proschan importance index

System $S = (C, \phi, F)$
Assume that the components have independent lifetimes

Importance index (Barlow-Proschan 1975)

$$I_{\mathrm{BP}}^{(j)} = \Pr(T_S = T_j) \qquad j \in C$$

$$\mathbf{I}_{\mathrm{BP}} = (I_{\mathrm{BP}}^{(1)}, \ldots, I_{\mathrm{BP}}^{(n)}) \qquad \sum_j I_{\mathrm{BP}}^{(j)} = 1$$

$I_{\mathrm{BP}}^{(j)}$ is an measure of importance of component j

Barlow-Proschan importance index

In the i.i.d. case:

$$\mathbf{I}_{\mathrm{BP}} = (I_{\mathrm{BP}}^{(1)}, \ldots, I_{\mathrm{BP}}^{(n)}) \qquad \longrightarrow \qquad \mathbf{b} = (b_1, \ldots, b_n)$$

$$b_j = \sum_{A \subseteq C \setminus \{j\}} \frac{1}{n\binom{n-1}{|A|}} \big(\phi(A \cup \{j\}) - \phi(A)\big)$$

$$b_j = \psi_{\mathrm{Sh}}(\phi, j)$$

b_j is independent of F !

$\Rightarrow$ **b** defines a *structure importance index*

System signature

Assume that F is absolutely continuous and the components have i.i.d. lifetimes

Order statistics

$$T_1, \ldots, T_n \qquad \longrightarrow \qquad T_{1:n} \leqslant \cdots \leqslant T_{n:n}$$

System signature (Samaniego 1985)

$$\boxed{s_k \;=\; \Pr(T_S = T_{k:n}) \qquad k = 1, \ldots, n}$$

$$\mathbf{s} \;=\; (s_1, \ldots, s_n) \qquad \textstyle\sum_k s_k = 1$$

System signature

Explicit expression (Boland 2001)

$$s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{\substack{A \subseteq C \\ |A|=n-k+1}} \phi(A) - \frac{1}{\binom{n}{n-k}} \sum_{\substack{A \subseteq C \\ |A|=n-k}} \phi(A)$$

$$C_k = \frac{1}{\binom{n}{k+1}} \sum_{|S|=k+1} v(S) - \frac{1}{\binom{n}{k}} \sum_{|S|=k} v(S)$$

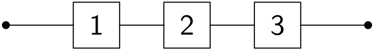
$$\boxed{s_k = C_{n-k}}$$

s_k is independent of F !

$\Rightarrow$ $\mathbf{s}$ defines the *structure signature*

Barlow-Proschan importance index and system signature

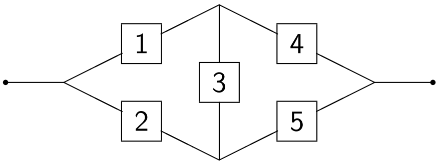
Series structure



$$\mathbf{I}_{\mathrm{BP}} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \qquad \mathbf{s} = (1,0,0)$$

Barlow-Proschan importance index and system signature

Bridge structure

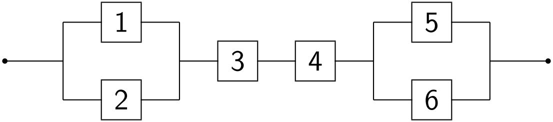


$$\mathbf{I}_{\mathrm{BP}} = \left(\frac{7}{30}, \frac{7}{30}, \frac{2}{30}, \frac{7}{30}, \frac{7}{30}\right)$$

$$\mathbf{s} = \left(0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0\right)$$

Barlow-Proschan importance index and system signature

Home video system



$$\mathbf{I}_{\mathrm{BP}} = \left(\frac{2}{30}, \frac{2}{30}, \frac{11}{30}, \frac{11}{30}, \frac{2}{30}, \frac{2}{30}\right)$$

$$\mathbf{s} = \left(\frac{5}{15}, \frac{6}{15}, \frac{4}{15}, 0, 0, 0\right)$$

Correspondence Reliability/Game Theory

Reliability	**Game Theory**
Component	Player
Importance of a component	Power of a player
Barlow-Proschan importance index	Shapley power index
Birnbaum importance index	Banzhaf power index
Signature	Cardinality index

Extension of signature to dependent lifetimes

General dependent case : we only assume that F has no ties

Probability signature (Navarro-Spizzichino-Balakrishnan 2010)

$$p_k \; = \; \Pr(T_S = T_{k:n}) \qquad k = 1, \ldots, n$$

$$\mathbf{p} = (p_1, \ldots, p_n) \qquad \sum_k p_k = 1$$

Can we provide an explicit expression for p_k in terms of ϕ and F ?

$$S \; = \; (C, \phi, F)$$

Extension of signature to dependent lifetimes

Relative quality function $q : 2^C \to [0,1]$

$$\begin{aligned} q(A) &= \Pr\left(T_i < T_j \ : \ i \notin A,\ j \in A\right) \\ &= \Pr\left(\max_{i \notin A} T_i < \min_{j \in A} T_j\right) \end{aligned}$$

(M. & Mathonet 2011)

$q(A)$ = probability that the best $|A|$ components (those having the longest lifetimes) are exactly A

$\to$ $q(A)$ measures the overall *quality* of the components A *when compared with* the components $C \setminus A$

Remark: q is independent of ϕ (q depends only on C and F)

Extension of signature to dependent lifetimes

Theorem (M. & Mathonet 2011)

$$p_k = \sum_{\substack{A \subseteq C \\ |A|=n-k+1}} q(A)\,\phi(A) - \sum_{\substack{A \subseteq C \\ |A|=n-k}} q(A)\,\phi(A)$$

$\longrightarrow$ extends Boland's formula

$$s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{\substack{A \subseteq C \\ |A|=n-k+1}} \phi(A) - \frac{1}{\binom{n}{n-k}} \sum_{\substack{A \subseteq C \\ |A|=n-k}} \phi(A)$$

Extension of signature to dependent lifetimes

Proposition

If $T_1, \ldots, T_n$ are exchangeable, then q is symmetric

$$q(A) = \frac{1}{\binom{n}{|A|}}$$

$$\Rightarrow \quad p_k = s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{\substack{A \subseteq C \\ |A|=n-k+1}} \phi(A) - \frac{1}{\binom{n}{n-k}} \sum_{\substack{A \subseteq C \\ |A|=n-k}} \phi(A)$$

$$\mathbf{p} = \mathbf{s}$$

Extension of BP index to dependent lifetimes

Relative quality function of component j

$$q_j : 2^{C \setminus \{j\}} \to [0,1]$$

$$q_j(A) \;=\; \Pr\Big(\max_{i \in C \setminus A} T_i = T_j < \min_{i \in A} T_i\Big)$$

(M. & Mathonet 2013)

$q_j(A)$ = probability that the components that are better than component j are precisely A.

Extension of BP index to dependent lifetimes

We have

$$\sum_{A \subseteq C \setminus \{j\}} q_j(A) = 1 \qquad (j \in C)$$

Theorem (M. & Mathonet 2013)

$$I_{\mathrm{BP}}^{(j)} = \sum_{A \subseteq C \setminus \{j\}} q_j(A) \big(\phi(A \cup \{j\}) - \phi(A)\big)$$

In the i.i.d. case:

$$I_{\mathrm{BP}}^{(j)} = b_j = \sum_{A \subseteq C \setminus \{j\}} \frac{1}{n\binom{n-1}{|A|}} \big(\phi(A \cup \{j\}) - \phi(A)\big)$$

Extension of BP index to dependent lifetimes

Proposition

If $T_1, \ldots, T_n$ are exchangeable, then

$$q_j(A) = \frac{1}{n\binom{n-1}{|A|}}$$

$$I_{\mathrm{BP}}^{(j)} = b_j = \sum_{A \subseteq C \setminus \{j\}} \frac{1}{n\binom{n-1}{|A|}} \big(\phi(A \cup \{j\}) - \phi(A)\big)$$

$$\mathbf{I}_{\mathrm{BP}} = \mathbf{b}$$

Part III : Additional results in the exchangeable case

Manual computation of the Barlow-Proschan index

$$b_j \;=\; \psi_{\mathrm{Sh}}(\phi,j) \;=\; \sum_{A\subseteq C\setminus\{j\}} \frac{1}{n\binom{n-1}{|S|}}\,\big(\phi(A\cup\{j\})-\phi(A)\big)$$

$$\overline{\phi}(\mathbf{x}) \;=\; \text{multilinear extension of } \phi(\mathbf{x})$$

Theorem (Owen 1972)

$$b_j \;=\; \psi_{\mathrm{Sh}}(\phi,j) \;=\; \int_0^1 \Big(\frac{\partial}{\partial x_j}\,\overline{\phi}\Big)(x,\ldots,x)\,dx$$

Manual computation of the Barlow-Proschan index

Example. Home video system

$$\phi(x_1, \ldots, x_6) = (x_1 \amalg x_2)\; x_3\; x_4\; (x_5 \amalg x_6)$$

$$\begin{aligned}\overline{\phi}(x_1, \ldots, x_6) = {} & x_1x_3x_4x_5 + x_2x_3x_4x_5 + x_1x_3x_4x_6 + x_2x_3x_4x_6 \\ & - x_1x_2x_3x_4x_5 - x_1x_2x_3x_4x_6 - x_1x_3x_4x_5x_6 - x_2x_3x_4x_5x_6 \\ & + x_1x_2x_3x_4x_5x_6\end{aligned}$$

Example: $b_2 = ?$

$$\Big(\frac{\partial}{\partial x_2}\, \overline{\phi}\Big)(x, \ldots, x) = 2x^3 - 3x^4 + x^5$$

$$b_2 = \int_0^1 \big(2x^3 - 3x^4 + x^5\big)\, dx = \frac{2}{30}$$

Manual computation of the signature

How can we efficiently compute the system signature

$$s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{\substack{A \subseteq C \\ |A|=n-k+1}} \phi(A) - \frac{1}{\binom{n}{n-k}} \sum_{\substack{A \subseteq C \\ |A|=n-k}} \phi(A) \qquad ?$$

Manual computation of the signature

With any n-degree polynomial $p : \mathbb{R} \to \mathbb{R}$ we associate the *reflected* polynomial $R^n p : \mathbb{R} \to \mathbb{R}$ defined by

$$(R^n p)(x) = x^n \, p\left(\tfrac{1}{x}\right)$$

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \Rightarrow \quad (R^n p)(x) = a_n + a_{n-1} x + \cdots + a_0 x^n$$

(M. 2014)

Setting $p(x) = \frac{d}{dx}\,\overline{\phi}(x,\ldots,x)$, we have

$$\int_0^x (R^{n-1}p)(t+1)\,dt = \sum_{k=1}^{n} \binom{n}{k} s_k \, x^k$$

Manual computation of the signature

Example. Home video system

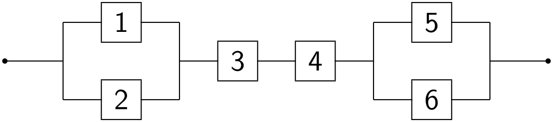
$$\overline{\phi}(x_1,\dots,x_6) = x_1x_3x_4x_5 + x_2x_3x_4x_5 + x_1x_3x_4x_6 + x_2x_3x_4x_6$$
$$-x_1x_2x_3x_4x_5 - x_1x_2x_3x_4x_6 - x_1x_3x_4x_5x_6 - x_2x_3x_4x_5x_6$$
$$+x_1x_2x_3x_4x_5x_6$$

$$\overline{\phi}(x,\dots,x) = 4x^4 - 4x^5 + x^6$$
$$p(x) = \tfrac{d}{dx}\,\overline{\phi}(x,\dots,x) = 16x^3 - 20x^4 + 6x^5$$
$$(R^5p)(x) = 6 - 20x + 16x^2$$

$$\int_0^x (R^5p)(t+1)\,dt = 2\,x + 6\,x^2 + \frac{16}{3}\,x^3$$
$$= \binom{6}{1}s_1\,x + \binom{6}{2}s_2\,x^2 + \cdots + \binom{6}{6}s_6\,x^6$$

Barlow-Proschan importance index and system signature

Home video system



$$\mathbf{s} = \left(\frac{5}{15}, \frac{6}{15}, \frac{4}{15}, 0, 0, 0\right)$$

$$\mathbf{C} = \left(0, 0, 0, \frac{4}{15}, \frac{6}{15}, \frac{5}{15}\right)$$

$$\mathbf{I}_{\mathrm{BP}} = \left(\frac{2}{30}, \frac{2}{30}, \frac{11}{30}, \frac{11}{30}, \frac{2}{30}, \frac{2}{30}\right)$$

Thank you for your attention!