Reliability analysis of systems and lattice polynomial description

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Selected references

A basic reference

- R. E. Barlow and F. Proschan. *Statistical theory of reliability and life testing*. Holt Rinehart and Wilson, New York, 1975.

Personal contributions

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- J.-L. Marichal and P. Mathonet, Computing system signatures through reliability functions, *Stat. Proba. Lett.* 83: 710-717, 2013.
- J.-L. Marichal, Subsignatures of systems, *J. Multivar. Anal.* 124: 226-236, 2014.

Sketch of the presentation

Part I : Semicoherent systems

Part II : Reliability analysis

Part III : Lattice polynomial language

Part IV : Signature and importance indexes

Part V : Additional results in the i.i.d. case

Part I : Semicoherent systems

System

Definition. A *system* is a set of interconnected components

$$C = \{1, \ldots, n\} = [n]$$

Example. Home video system

1. Blu-ray player
2. PlayStation 3
3. LED television
4. Sound amplifier
5. Speaker A
6. Speaker B

Assumptions

- The system and the components are of the crisply *on/off* kind
- The components are nonrepairable

Structure function

State of a component $j \in C = [n]$ $\rightarrow$ Boolean variable

$$x_j = \begin{cases} 1 & \text{if component } j \text{ is functioning} \\ 0 & \text{if component } j \text{ is in a failed state} \end{cases}$$

State of the system $\rightarrow$ Boolean function $\phi : \{0,1\}^n \rightarrow \{0,1\}$

$$\phi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is in a failed state} \end{cases}$$

This function is called the *structure function* of the system

$$S = (C, \phi)$$

Representations of Boolean functions

Boolean function	$\longleftrightarrow$	set function
$\phi : \{0,1\}^n \to \{0,1\}$		$\phi : 2^{[n]} \to \{0,1\}$

$$\phi(\mathbf{1}_A) \;=\; \phi(A) \qquad A \subseteq [n]$$

Polynomial representation of a Boolean function

$$\phi(\mathbf{x}) \;=\; \sum_{A\subseteq[n]} \phi(A) \prod_{j\in A} x_j \prod_{j\in[n]\setminus A} (1-x_j)$$

Representations of Boolean functions

Simple form

$$\phi(\mathbf{x}) = \sum_{A \subseteq [n]} m(A) \prod_{j \in A} x_j$$

where

$$\begin{aligned} m(A) &= \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B) \\ \phi(A) &= \sum_{B \subseteq A} m(B) \end{aligned}$$

(Hammer and Rudeanu 1968)

Coherent and semicoherent systems

The system is said to be *semicoherent* if

- ϕ is nondecreasing : $\mathbf{x} \leqslant \mathbf{x}' \;\Rightarrow\; \phi(\mathbf{x}) \leqslant \phi(\mathbf{x}')$
- $\phi(\mathbf{0}) = 0$, $\phi(\mathbf{1}) = 1$

The system is said to be *coherent* if, in addition

- every component is relevant to ϕ

$$\exists\ \mathbf{x} \in \{0,1\}^n : \phi(1_j, \mathbf{x}) \neq \phi(0_j, \mathbf{x})$$

where

$$\begin{aligned} (1_j, \mathbf{x}) &= (x_1, \ldots, \overset{(j)}{1}, \ldots, x_n) \\ (0_j, \mathbf{x}) &= (x_1, \ldots, \overset{(j)}{0}, \ldots, x_n) \end{aligned}$$

Representations of Boolean functions

$$x_1 \sqcap x_2 = \min(x_1, x_2) = x_1 x_2$$

$$x_1 \sqcup x_2 = \max(x_1, x_2) = 1 - (1 - x_1)(1 - x_2)$$

Since ϕ is nondecreasing and nonconstant

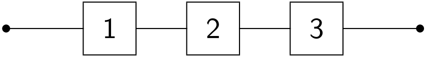
$$\phi(\mathbf{x}) = \coprod_{\substack{A \subseteq [n] \\ \phi(A)=1}} \prod_{j \in A} x_j$$

$$\phi(\mathbf{x}) = \prod_{\substack{A \subseteq [n] \\ \phi([n] \setminus A)=0}} \coprod_{j \in A} x_j$$

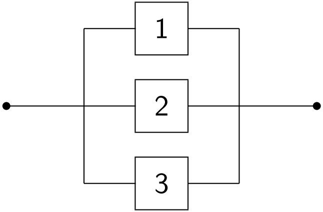
(Hammer and Rudeanu 1968)

Block diagrams

- A serially connected segment of components is functioning if and only if every single component is functioning

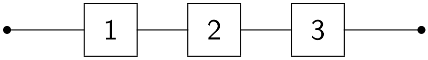


- A system of parallel components is functioning if and only at least one component is functioning



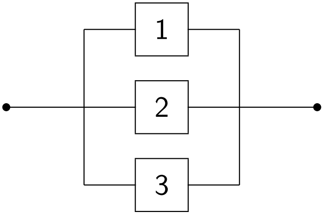
Block diagrams

Series structure



$$\phi(\mathbf{x}) = x_1 \, x_2 \, x_3 = \prod_{i=1}^{3} x_i$$

Parallel structure

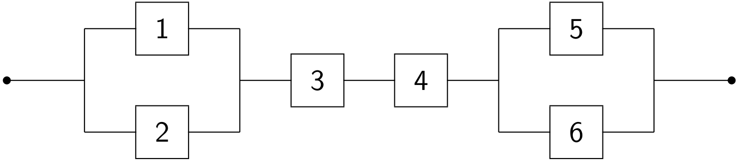


$$\phi(\mathbf{x}) = 1 - (1 - x_1)(1 - x_2)(1 - x_3) = \coprod_{i=1}^{3} x_i$$

Block diagrams

Example. Home video system

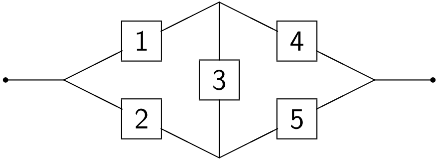
1. Blu-ray player
2. PlayStation 3
3. LED television
4. Sound amplifier
5. Speaker A
6. Speaker B



$$\phi(\mathbf{x}) = (x_1 \amalg x_2)\, x_3\, x_4\, (x_5 \amalg x_6)$$

Block diagrams

Example. Bridge structure



$$\phi(\mathbf{x}) \;=\; x_3\,\phi(1_3,\mathbf{x}) + (1-x_3)\phi(0_3,\mathbf{x})$$

$$\begin{aligned} \phi(1_3,\mathbf{x}) &= (x_1 \coprod x_2)(x_4 \coprod x_5) \\ \phi(0_3,\mathbf{x}) &= (x_1\ x_4) \coprod (x_2\ x_5) \end{aligned}$$

Pivotal decomposition of the structure function

$$\phi(\mathbf{x}) \;=\; x_j\,\phi(1_j,\mathbf{x}) + (1-x_j)\phi(0_j,\mathbf{x})$$

Block diagrams

Example. k-out-of-n structure

The system fails upon the kth component failure

i.e., the system is functioning if and only if at least $n - k + 1$ of the n components are functioning

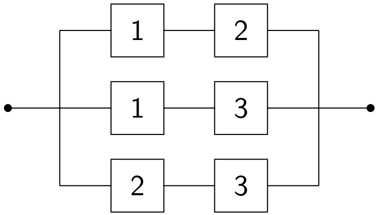
$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{j=1}^{n} x_j \geqslant n - k + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi(\mathbf{x}) = x_{k:n} = \coprod_{|A|=n-k+1} \prod_{j \in A} x_j = \prod_{|A|=k} \coprod_{j \in A} x_j$$

Block diagrams

Example. 2-out-of-3 structure

$$\phi(\mathbf{x}) = x_{2:3} = \coprod_{|A|=2} \prod_{j \in A} x_j = x_1 x_2 \amalg x_1 x_3 \amalg x_2 x_3$$



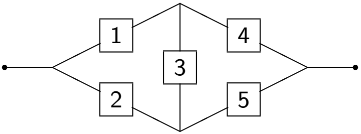
Path and cut sets

Definition. A subset $A \subseteq C$ of components is

- a *path set* of ϕ if $\phi(A) = 1$
- a *cut set* of ϕ if $\phi(C \setminus A) = 0$

A path (cut) set is *minimal* if it does not strictly contain a path (cut) set.

Bridge structure



- Minimal path sets : $\{1,4\}$, $\{2,5\}$, $\{1,3,5\}$, $\{2,3,4\}$
- Minimal cut sets : $\{1,2\}$, $\{4,5\}$, $\{1,3,5\}$, $\{2,3,4\}$

Path and cut sets

If $P_1, \ldots, P_r$ denote the minimal path sets

$$\phi(\mathbf{x}) = \coprod_{j=1}^{r} \prod_{i \in P_j} x_i$$

If $K_1, \ldots, K_s$ denote the minimal cut sets

$$\phi(\mathbf{x}) = \prod_{j=1}^{s} \coprod_{i \in K_j} x_i$$

Bridge structure

$$\begin{aligned}
\phi(\mathbf{x}) &= (x_1 x_4) \amalg (x_2 x_5) \amalg (x_1 x_3 x_5) \amalg (x_2 x_3 x_4) \\
&= (x_1 \amalg x_2)(x_4 \amalg x_5)(x_1 \amalg x_3 \amalg x_5)(x_2 \amalg x_3 \amalg x_4)
\end{aligned}$$

Correspondence Reliability/Game Theory

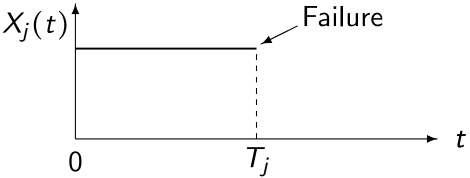
Reliability	**Game Theory**
Component	Player
Semicoherent structure	Simple game
Structure function	Characteristic function
Irrelevant component	Null player
Path set	Winning coalition
Cut set	Blocking coalition
Minimal path set	Minimal winning coalition
Minimal cut set	Minimal blocking coalition
Series structure	Unanimity game
Paralell structure	Decisive game
Module	Committee
Modular set	Committee set

(Ramamurthy 1990)

State variable $\longrightarrow$ Random variable

$$x_j \longrightarrow X_j(t)$$

$$X_j(t) = \begin{cases} 1 & \text{if } j \text{ is functioning at time } t \\ 0 & \text{if } j \text{ is in a failed state at time } t \end{cases}$$



T_j = *random lifetime* of component $j \in C$
$X_j(t) = \text{Ind}(T_j > t)$ = *random state* of j at time $t \geqslant 0$

System lifetime and component lifetimes

$T_S \quad = \quad$ *system lifetime*

$X_S(t) \; = \; \mathrm{Ind}(T_S > t)$ = random *state of the system* at time $t \geqslant 0$

$$\boxed{X_S(t) \; = \; \phi(X_1(t), \ldots, X_n(t)) \qquad t \geqslant 0}$$

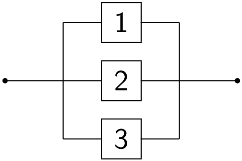
Expression of T_S in terms of $T_1, \ldots, T_n$?

System lifetime and component lifetimes

Series structure

$$\phi(\mathbf{x}) = x_1 \; x_2 \; x_3 \qquad \longrightarrow \qquad T_S = T_1 \wedge T_2 \wedge T_3$$

Parallel structure



$$\phi(\mathbf{x}) = x_1 \coprod x_2 \coprod x_3 \qquad \longrightarrow \qquad T_S = T_1 \vee T_2 \vee T_3$$

System lifetime and component lifetimes

General structure (Dukhovny & M. 2012)

$$\phi(\mathbf{x}) = \coprod_{\substack{A\subseteq[n]\\ \phi(A)=1}} \prod_{j\in A} x_j \qquad \longrightarrow \qquad T_S = \bigvee_{\substack{A\subseteq[n]\\ \phi(A)=1}} \bigwedge_{j\in A} T_j$$

Life function

$$p_\phi(t_1,\ldots,t_n) = \bigvee_{\substack{A\subseteq[n]\\ \phi(A)=1}} \bigwedge_{j\in A} t_j \qquad t_j \geqslant 0$$

$\longrightarrow$ lattice polynomial (lattice term)

$$\boxed{T_S = p_\phi(T_1,\ldots,T_n)}$$

System

How to describe $T_1, \ldots, T_n$?

Cumulative distribution function (c.d.f.) of the component lifetimes

$$F(t_1, \ldots, t_n) = \Pr(T_1 \leqslant t_1, \ldots, T_n \leqslant t_n) \qquad t_1, \ldots, t_n \geqslant 0$$

$$\boxed{S = (C, \phi, F)}$$

Classical assumptions

- F absolutely continuous + i.i.d. lifetimes
- F absolutely continuous + exchangeable lifetimes
- F has no ties

$$\Pr(T_i = T_j) = 0 \qquad i \neq j$$

Part II : Reliability analysis

Reliability analysis

Reliability function of component $j \in C$

$$R_j(t) \; = \; \Pr(T_j > t) \qquad t \geqslant 0$$

= probability that component j does not fail in the interval $[0, t]$

$$X_j(t) \; = \; \mathrm{Ind}(T_j > t) \quad \Rightarrow \quad R_j(t) \; = \; \Pr(X_j(t) = 1) \; = \; \mathbb{E}\big[X_j(t)\big]$$

System reliability function

$$R_S(t) \; = \; \Pr(T_S > t) \qquad t \geqslant 0$$

= probability that the system does not fail in the interval $[0, t]$

$$R_S(t) \; = \; \Pr(X_S(t) = 1) \; = \; \mathbb{E}\big[X_S(t)\big]$$

Reliability analysis

We have

$$
\begin{aligned}
R_S(t) &= \mathbb{E}[X_S(t)] = \mathbb{E}\big[\phi(X_1(t),\ldots,X_n(t))\big] \\
&= \sum_{A\subseteq C}\phi(A)\;\underbrace{\mathbb{E}\Bigg[\prod_{j\in A}X_j(t)\prod_{j\in C\setminus A}\big(1-X_j(t)\big)\Bigg]}_{\Pr\left(\forall j\in C\;:\;X_j(t)=1\;\Leftrightarrow\;j\in A\right)}
\end{aligned}
$$

Theorem (Dukhovny 2007)

$$
R_S(t) = \sum_{A\subseteq C}\phi(A)\,\Pr(\mathbf{X}(t)=\mathbf{1}_A)\qquad t\geqslant 0
$$

All the needed information is the distribution of $\mathbf{X}(t)$
(the knowledge of the joint distribution F of the component lifetimes is not necessary)

Reliability analysis

When $T_1, \ldots, T_n$ are independent, we have

$$\begin{aligned} R_S(t) &= \sum_{A \subseteq C} \phi(A) \prod_{j \in A} \mathbb{E}[X_j(t)] \prod_{j \in C \setminus A} \Big(1 - \mathbb{E}[X_j(t)]\Big) \\ &= \sum_{A \subseteq C} \phi(A) \prod_{j \in A} R_j(t) \prod_{j \in C \setminus A} \big(1 - R_j(t)\big) \end{aligned}$$

Corollary

If $T_1, \ldots, T_n$ are independent, then

$$R_S(t) = \overline{\phi}\big(R_1(t), \ldots, R_n(t)\big) \qquad t \geqslant 0$$

Multilinear extension of ϕ $\longrightarrow$ $\overline{\phi} : [0,1]^n \to [0,1]$

$$\overline{\phi}(\mathbf{x}) = \sum_{A \subseteq C} \phi(A) \prod_{j \in A} x_j \prod_{j \in C \setminus A} (1 - x_j)$$

Reliability analysis

Simple form of ϕ

$$\phi(\mathbf{x}) = \sum_{A \subseteq C} m(A) \prod_{j \in A} x_j$$

Corollary (Dukhovny and M. 2008)

We have

$$R_S(t) = \sum_{A \subseteq C} m(A) \Pr(T_j > t \quad \forall j \in A) \qquad t \geqslant 0$$

In case of independence

$$R_S(t) = \sum_{A \subseteq C} m(A) \prod_{j \in A} R_j(t) \qquad t \geqslant 0$$

Mean time-to-failure of the system

Mean time-to-failure of the system

$$\mathrm{MTTF}_S = \mathbb{E}[T_S] = -\int_0^\infty t\, dR_S(t)$$

$$\boxed{\mathrm{MTTF}_S = \int_0^\infty R_S(t)\, dt}$$

In case of independence

$$\mathrm{MTTF}_S = \sum_{A \subseteq C} \phi(A) \int_0^\infty \prod_{j \in A} R_j(t) \prod_{j \in C \setminus A} \big(1 - R_j(t)\big)\, dt$$

$$\mathrm{MTTF}_S = \sum_{A \subseteq C} m(A) \int_0^\infty \prod_{j \in A} R_j(t)\, dt$$

Mean time-to-failure of the system

Example. Assume $R_j(t) = e^{-\lambda_j t}$, $j \in C$

$$
\begin{aligned}
\mathrm{MTTF}_S &= \sum_{A \subseteq C} m(A) \int_0^\infty \prod_{j \in A} e^{-\lambda_j t}\, dt \\
&= \sum_{A \subseteq C} m(A) \int_0^\infty e^{-\lambda_A t}\, dt \qquad \left(\lambda_A = \sum_{j \in A} \lambda_j\right) \\
&= \sum_{\substack{A \subseteq C \\ A \neq \varnothing}} m(A)\, \frac{1}{\lambda_A}
\end{aligned}
$$

Series structure: $\mathrm{MTTF}_S = \dfrac{1}{\lambda_C}$

Parallel structure: $\mathrm{MTTF}_S = \displaystyle\sum_{\substack{A \subseteq C \\ A \neq \varnothing}} (-1)^{|A|-1}\, \frac{1}{\lambda_A}$

Part III : Lattice polynomial language

Life function

$$p_\phi(t_1,\ldots,t_n) \;=\; \bigvee_{\substack{A\subseteq[n]\\ \phi(A)=1}} \; \bigwedge_{j\in A} t_j \qquad t_j \geqslant 0$$

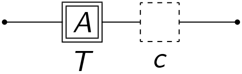
$\longrightarrow$ lattice polynomial (lattice term)

$$\boxed{T_S \;=\; p_\phi(T_1,\ldots,T_n)}$$

Advantage of the lattice polynomial language

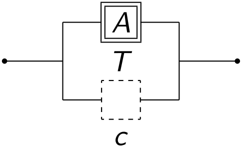
Suppose there is

(i) an upper bound on lifetimes of a subset A of components (imposed by the physical properties of the assembly)



subset lifetime = $T \wedge c$

(ii) a lower bound (imposed by a back-up block with a constant lifetime)



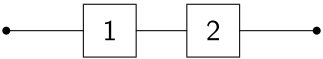
subset lifetime = $T \vee c$

Advantage of the lattice polynomial language

The lifetime of a general system with upper and/or lower bounds can be described through a lattice polynomial function

$$T_S = p(T_1, \ldots, T_n)$$

Example.



Suppose that the lifetime of component #2 must lie in the time interval $[c, d]$

$$\begin{aligned} T_S &= T_1 \wedge \operatorname{median}(c, T_2, d) \\ &= T_1 \wedge \big(c \vee (T_2 \wedge d)\big) \\ &= (c \wedge T_1) \vee (d \wedge T_1 \wedge T_2) \end{aligned}$$

Lattice polynomial functions

Representations of a l.p. function (Goodstein 1967)

$$p(t_1,\ldots,t_n) \;=\; \bigvee_{A\subseteq[n]}\Big(\alpha(A)\wedge\bigwedge_{j\in A} t_j\Big) \qquad t_1,\ldots,t_n \geqslant 0$$

$$\alpha(A) \;=\; p(\mathbf{e}_A)$$

$$(\mathbf{e}_A)_j \;=\; \begin{cases}\infty & \text{if } j\in A\\ 0 & \text{otherwise}\end{cases}$$

Lattice polynomial functions

$$p(t_1,\ldots,t_n) \;=\; \bigvee_{A\subseteq[n]}\Big(\alpha(A)\wedge\bigwedge_{j\in A}t_j\Big)\qquad t_1,\ldots,t_n\geqslant 0$$

Theorem (Dukhovny & M. 2008)

If $T_S = p(T_1,\ldots,T_n)$ then

$$X_S(t) \;=\; \phi_t(X_1(t),\ldots,X_n(t))\qquad t\geqslant 0$$

where

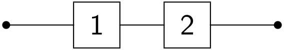
$$\phi_t(\mathbf{x}) \;=\; \sum_{A\subseteq[n]}\mathrm{Ind}(\alpha(A)>t)\prod_{j\in A}x_j\prod_{j\in[n]\setminus A}(1-x_j)$$

This extends the classical formula

$$X_S(t) \;=\; \phi(X_1(t),\ldots,X_n(t))\qquad t\geqslant 0$$

Lattice polynomial functions

Example (cont'd)



$$T_S = (c \wedge T_1) \vee (d \wedge T_1 \wedge T_2)$$

$$p(t_1, t_2) = (c \wedge t_1) \vee (d \wedge t_1 \wedge t_2)$$

Then we have

$$X_S(t) = \big(\text{Ind}(c > t)\ X_1(t)\big) \coprod \big(\text{Ind}(d > t)\ X_1(t)\ X_2(t)\big)$$

Reliability analysis

Exact reliability formulas (Dukhovny & M. 2008)

$$\begin{aligned} R_S(t) &= \sum_{A \subseteq C} \phi_t(A) \Pr(\mathbf{X}(t) = \mathbf{1}_A) \\ R_S(t) &= \sum_{A \subseteq C} m_t(A) \Pr(T_j > t \;\forall j \in A) \end{aligned}$$

In case of independence

$$\begin{aligned} R_S(t) &= \sum_{A \subseteq C} \phi_t(A) \prod_{j \in A} R_j(t) \prod_{j \in C \setminus A} (1 - R_j(t)) \\ R_S(t) &= \sum_{A \subseteq C} m_t(A) \prod_{j \in A} R_j(t) \end{aligned}$$

Mean time-to-failure of the system

$$\begin{aligned}
\mathrm{MTTF}_S &= \int_0^\infty R_S(t)\,dt \\
&= \sum_{A\subseteq C}\int_0^\infty m_t(A)\prod_{j\in A}R_j(t)\,dt \\
&= \sum_{A\subseteq C}\int_0^\infty \left(\sum_{B\subseteq A}(-1)^{|A|-|B|}\phi_t(B)\right)\prod_{j\in A}R_j(t)\,dt \\
&= \sum_{A\subseteq C}\sum_{B\subseteq A}(-1)^{|A|-|B|}\int_0^\infty \mathrm{Ind}(\alpha(B)>t)\prod_{j\in A}R_j(t)\,dt \\
&= \sum_{A\subseteq C}\sum_{B\subseteq A}(-1)^{|A|-|B|}\int_0^{\alpha(B)}\prod_{j\in A}R_j(t)\,dt
\end{aligned}$$

Mean time-to-failure of the system

Example. Assume $R_j(t) = e^{-\lambda_j t}$, $j \in C$

$$
\begin{aligned}
\mathrm{MTTF}_S &= \sum_{A \subseteq C} \sum_{B \subseteq A} (-1)^{|A|-|B|} \int_0^{\alpha(B)} \prod_{j \in A} e^{-\lambda_j t}\, dt \\
&= \sum_{A \subseteq C} \sum_{B \subseteq A} (-1)^{|A|-|B|} \int_0^{\alpha(B)} e^{-\lambda_A t}\, dt \\
&= \alpha(\varnothing) + \sum_{\substack{A \subseteq [n] \\ A \neq \varnothing}} \sum_{B \subseteq A} (-1)^{|A|-|B|} \, \frac{1 - e^{-\lambda_A \alpha(B)}}{\lambda_A}
\end{aligned}
$$

Part IV : Signature and importance indexes

Simple game

Let $N = \{1, \ldots, n\}$ be the set of *players*

Characteristic function of the game
= set function $v : 2^N \to \mathbb{R}$ which assigns to each coalition $S \subseteq N$ of players a real number $v(S)$ which represents the *worth* of S

The game is said to be *simple* if v takes on its values in $\{0, 1\}$

The set function v can be regarded as a Boolean function
$v : \{0, 1\}^n \to \{0, 1\}$

Power indexes

Let $v : 2^N \to \{0,1\}$ be a simple game on a set N of n players
Let $j \in N$ be a player

Banzhaf power index (Banzhaf 1965)

$$\psi_{\mathrm{B}}(v,j) \;=\; \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{j\}} \big(v(S \cup \{j\}) - v(S) \big)$$

Shapley power index (Shapley 1953)

$$\psi_{\mathrm{Sh}}(v,j) \;=\; \sum_{S \subseteq N \setminus \{j\}} \frac{1}{n \binom{n-1}{|S|}} \big(v(S \cup \{j\}) - v(S) \big)$$

Cardinality index

Cardinality index (Yager 2002)

$$C_k = \frac{1}{(n-k)\binom{n}{k}} \sum_{|S|=k} \sum_{j \in N \setminus S} \big(v(S \cup \{j\}) - v(S)\big) \quad (k = 0, \ldots, n-1)$$

$$C_k = \frac{1}{\binom{n}{k+1}} \sum_{|S|=k+1} v(S) - \frac{1}{\binom{n}{k}} \sum_{|S|=k} v(S)$$

Interpretation:
C_k is the average gain that we obtain by adding an arbitrary player to an arbitrary k-player coalition

Barlow-Proschan importance index

System $S = (C, \phi, F)$
Assume that the components have independent lifetimes

Importance index (Barlow-Proschan 1975)

$$I_{\mathrm{BP}}^{(j)} = \Pr(T_S = T_j) \qquad j \in C$$

$$\mathbf{I}_{\mathrm{BP}} = (I_{\mathrm{BP}}^{(1)}, \ldots, I_{\mathrm{BP}}^{(n)}) \qquad \sum_j I_{\mathrm{BP}}^{(j)} = 1$$

$I_{\mathrm{BP}}^{(j)}$ is an measure of importance of component j

Barlow-Proschan importance index

In the i.i.d. case:

$$\mathbf{I}_{\mathrm{BP}} = (I_{\mathrm{BP}}^{(1)}, \ldots, I_{\mathrm{BP}}^{(n)}) \qquad \longrightarrow \qquad \mathbf{b} = (b_1, \ldots, b_n)$$

$$b_j = \sum_{A \subseteq C \setminus \{j\}} \frac{1}{n \binom{n-1}{|A|}} \big(\phi(A \cup \{j\}) - \phi(A)\big)$$

$$\boxed{b_j = \psi_{\mathrm{Sh}}(\phi, j)}$$

b_j is independent of F !

$\Rightarrow$ **b** defines a *structure importance index*

System signature

Assume that F is absolutely continuous and the components have i.i.d. lifetimes

Order statistics

$$T_1, \ldots, T_n \qquad \longrightarrow \qquad T_{1:n} \leqslant \cdots \leqslant T_{n:n}$$

System signature (Samaniego 1985)

$$\boxed{s_k \;=\; \Pr(T_S = T_{k:n}) \qquad k = 1, \ldots, n}$$

$$\mathbf{s} \;=\; (s_1, \ldots, s_n) \qquad \textstyle\sum_k s_k = 1$$

System signature

Explicit expression (Boland 2001)

$$s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{\substack{A\subseteq C \\ |A|=n-k+1}} \phi(A) - \frac{1}{\binom{n}{n-k}} \sum_{\substack{A\subseteq C \\ |A|=n-k}} \phi(A)$$

$$C_k = \frac{1}{\binom{n}{k+1}} \sum_{|S|=k+1} v(S) - \frac{1}{\binom{n}{k}} \sum_{|S|=k} v(S)$$

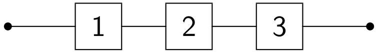
$$\boxed{s_k = C_{n-k}}$$

s_k is independent of F !

$\Rightarrow$ **s** defines the *structure signature*

Barlow-Proschan importance index and system signature

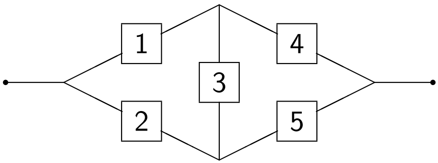
Series structure



$$\mathbf{I}_{\mathrm{BP}} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \qquad \mathbf{s} = (1, 0, 0)$$

Barlow-Proschan importance index and system signature

Bridge structure

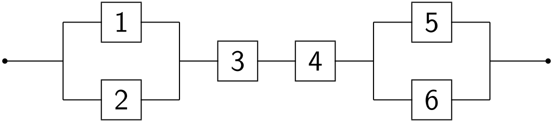


$$\mathbf{I}_{\mathrm{BP}} = \left(\frac{7}{30}, \frac{7}{30}, \frac{2}{30}, \frac{7}{30}, \frac{7}{30}\right)$$

$$\mathbf{s} = \left(0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0\right)$$

Barlow-Proschan importance index and system signature

Home video system



$$\mathbf{I}_{\mathrm{BP}} = \left(\frac{2}{30}, \frac{2}{30}, \frac{11}{30}, \frac{11}{30}, \frac{2}{30}, \frac{2}{30}\right)$$

$$\mathbf{s} = \left(\frac{5}{15}, \frac{6}{15}, \frac{4}{15}, 0, 0, 0\right)$$

Correspondence Reliability/Game Theory

Reliability	**Game Theory**
Component	Player
Importance of a component	Power of a player
Barlow-Proschan importance index	Shapley power index
Birnbaum importance index	Banzhaf power index
Signature	Cardinality index

Extension of signature to dependent lifetimes

General dependent case : we only assume that F has no ties

Probability signature (Navarro-Spizzichino-Balakrishnan 2010)

$$p_k \;=\; \Pr(T_S = T_{k:n}) \qquad k = 1, \ldots, n$$

$$\mathbf{p} = (p_1, \ldots, p_n) \qquad \textstyle\sum_k p_k = 1$$

Can we provide an explicit expression for p_k in terms of ϕ and F ?

$$S \;=\; (C, \phi, F)$$

Extension of signature to dependent lifetimes

Relative quality function $q : 2^C \to [0,1]$

$$\begin{aligned} q(A) &= \Pr\left(T_i < T_j \ : \ i \notin A,\ j \in A\right) \\ &= \Pr\left(\max_{i \notin A} T_i < \min_{j \in A} T_j\right) \end{aligned}$$

(M. & Mathonet 2011)

$q(A)$ = probability that the best $|A|$ components (those having the longest lifetimes) are exactly A

$\to$ $q(A)$ measures the overall *quality* of the components A *when compared with* the components $C \setminus A$

Remark: q is independent of ϕ (q depends only on C and F)

Extension of signature to dependent lifetimes

Theorem (M. & Mathonet 2011)

$$p_k = \sum_{\substack{A\subseteq C\\ |A|=n-k+1}} q(A)\,\phi(A) - \sum_{\substack{A\subseteq C\\ |A|=n-k}} q(A)\,\phi(A)$$

$\longrightarrow$ extends Boland's formula

$$s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{\substack{A\subseteq C\\ |A|=n-k+1}} \phi(A) - \frac{1}{\binom{n}{n-k}} \sum_{\substack{A\subseteq C\\ |A|=n-k}} \phi(A)$$

Open problem

Find necessary and sufficient conditions under which a set function on C is the relative quality function of a system $S = (C, \phi, F)$

Extension of signature to dependent lifetimes

Proposition

If $T_1, \ldots, T_n$ are exchangeable, then q is symmetric

$$q(A) = \frac{1}{\binom{n}{|A|}}$$

$$\Rightarrow \quad p_k = s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{\substack{A \subseteq C \\ |A|=n-k+1}} \phi(A) - \frac{1}{\binom{n}{n-k}} \sum_{\substack{A \subseteq C \\ |A|=n-k}} \phi(A)$$

$$\mathbf{p} = \mathbf{s}$$

Extension of signature to dependent lifetimes

Theorem (M. & Mathonet & Waldhauser 2011)

The identity $\mathbf{p} = \mathbf{s}$ holds for every n-component semicoherent system *if and only if* q is symmetric

Extension of BP index to dependent lifetimes

Relative quality function of component j

$$q_j : 2^{C \setminus \{j\}} \to [0,1]$$

$$q_j(A) \;=\; \Pr\Big(\max_{i \in C \setminus A} T_i = T_j < \min_{i \in A} T_i \Big)$$

(M. & Mathonet 2013)

$q_j(A)$ = probability that the components that are better than component j are precisely A.

Extension of BP index to dependent lifetimes

We have

$$\sum_{A\subseteq C\setminus\{j\}} q_j(A) = 1 \qquad (j\in C)$$

Theorem (M. & Mathonet 2013)

$$I_{\mathrm{BP}}^{(j)} = \sum_{A\subseteq C\setminus\{j\}} q_j(A)\big(\phi(A\cup\{j\})-\phi(A)\big)$$

In the i.i.d. case:

$$I_{\mathrm{BP}}^{(j)} = b_j = \sum_{A\subseteq C\setminus\{j\}} \frac{1}{n\binom{n-1}{|A|}}\big(\phi(A\cup\{j\})-\phi(A)\big)$$

Extension of BP index to dependent lifetimes

Proposition

If $T_1, \ldots, T_n$ are exchangeable, then

$$q_j(A) = \frac{1}{n\binom{n-1}{|A|}}$$

$$I_{\mathrm{BP}}^{(j)} = b_j = \sum_{A \subseteq C \setminus \{j\}} \frac{1}{n\binom{n-1}{|A|}} \big(\phi(A \cup \{j\}) - \phi(A)\big)$$

$$\mathbf{I}_{\mathrm{BP}} = \mathbf{b}$$

Extension of BP index to dependent lifetimes

Theorem (M. & Mathonet 2013)

The identity $\mathbf{I}_{\mathrm{BP}} = \mathbf{b}$ holds for every n-component semicoherent system *if and only if*

$$q_j(A) = \frac{1}{n\binom{n-1}{|A|}}$$

Case of independent lifetimes

We now assume that $T_1, \ldots, T_n$ are *independent* lifetimes

Every T_j has a
- a p.d.f. f_j
- a c.d.f. F_j with $F_j(0) = 0$

Theorem

$$q(A) = \sum_{j \in A} \int_0^\infty f_j(t) \prod_{i \notin A} F_i(t) \prod_{i \in A \setminus \{j\}} \overline{F}_i(t)\, dt \qquad (A \neq \varnothing)$$

where $\overline{F}_j(t) = 1 - F_j(t)$

→ provides an explicit expression for the signature in the independent case

Case of independent lifetimes

Example: *independent exponential* lifetimes

$$F_j(t) = 1 - e^{-\lambda_j t} \qquad \lambda_j > 0$$

Corollary

$$q(A) \;=\; \sum_{B \subseteq C \setminus A} (-1)^{|B|} \frac{\lambda_A}{\lambda_{A \cup B}} \qquad (A \neq \varnothing)$$

where $\lambda_A = \sum_{j \in A} \lambda_j$

Case of independent lifetimes

The ratio

$$\frac{\lambda_{\{j\}}}{\lambda_C} = q(C \setminus \{j\})$$

is the probability that j is the worst component

More generally,

$$\frac{\lambda_A}{\lambda_C} = \sum_{j \in A} q(C \setminus \{j\})$$

is the probability that the worst component is in A

Case of independent lifetimes

Theorem

$$q_j(A) \;=\; \int_0^\infty f_j(t) \prod_{i \notin A \cup \{j\}} F_i(t) \prod_{i \in A} \overline{F}_i(t)\, dt$$

→ provides an explicit expression for Barlow–Proschan index in the independent case

Corollary

For independent exponential lifetimes

$$q_j(A) \;=\; \sum_{B \subseteq C \setminus (A \cup \{j\})} (-1)^{|B|} \frac{\lambda_{\{j\}}}{\lambda_{A \cup B \cup \{j\}}}$$

Interpretation in game theory

Is there an interpretation in game theory of the formula

$$\Pr(T_S = T_j) = \sum_{A \subseteq C \setminus \{j\}} q_j(A) \big(\phi(A \cup \{j\}) - \phi(A) \big) \quad ?$$

Yes : based on the derivation of the Shapley power index from a bargaining procedure (Shapley 1953)

Interpretation in game theory

The players agree to play the game v in a grand coalition

- The coalition adds one player at a time until everyone has been admitted
- The order in which the players are to join is determined by chance, with all arrangements equally probable
- Each player, on his admission, is promised the amount corresponding to his marginal contribution

Let $S \subseteq N \setminus \{j\}$ be the set of players preceding j

$\longrightarrow$ payment to j : $v(S \cup \{j\}) - v(S)$

$\longrightarrow$ probability of that contingency is $\frac{1}{n\binom{n-1}{|S|}}$

$\longrightarrow$ total expectation of player j

$$\psi_{\mathrm{Sh}}(v,j) = \sum_{S \subseteq N \setminus \{j\}} \frac{1}{n\binom{n-1}{|S|}} \big(v(S \cup \{j\}) - v(S)\big)$$

Interpretation in game theory

General case : T_j = time at which player j is admitted in the coalition

⟶ probability that S is the set of players preceding j

$$p_j(S) = \Pr\left(\max_{i \in S} T_i < T_j = \min_{i \in N \setminus S} T_i \right)$$

⟶ total expectation of player j

$$\sum_{S \subseteq N \setminus \{j\}} p_j(S) \left(v(S \cup \{j\}) - v(S) \right)$$

If the game is monotone and simple

$$\Pr(T_N = T_j) = \sum_{S \subseteq N \setminus \{j\}} p_j(S) \left(v(S \cup \{j\}) - v(S) \right)$$

T_N = time at which the forming coalition turns from losing to winning

Interpretation in game theory

Let $S \subseteq N$, $|S| = k$, be the set of the first k players $(k = 0, \ldots, n-1)$

$\longrightarrow$ probability that this coalition forms is

$$p(S) \;=\; \Pr\Big(\max_{i \in S} T_i < \min_{i \in N \setminus S} T_i\Big)$$

$\longrightarrow$ average marginal contribution of an additional arbitrary player

$$\sum_{\substack{S \subseteq N \\ |S|=k+1}} p(S)\, v(S) \;-\; \sum_{\substack{S \subseteq N \\ |S|=k}} p(S)\, v(S)$$

If the game is monotone and simple

$$\Pr(T_N = T_{k+1:n}) \;=\; \sum_{\substack{S \subseteq N \\ |S|=k+1}} p(S)\, v(S) \;-\; \sum_{\substack{S \subseteq N \\ |S|=k}} p(S)\, v(S)$$

Subsignature

Let $M \subseteq C$

Subsignature (M. 2014)

$$p_M^{(k)} = \Pr(T_S = T_{k:M}) \qquad k = 1, \ldots, |M|$$

Explicit formula

$$p_M^{(k)} = \sum_{\substack{A \subseteq C \\ |M \setminus A| = k}} \sum_{j \in M \setminus A} q_j(A) \big(\phi(A \cup \{j\}) - \phi(A)\big)$$

+ interpretation in game theory

Decomposition of reliability

Recall that

$$R_S(t) \;=\; \Pr(T_S > t) \quad \text{and} \quad R_{k:n}(t) \;=\; \Pr(T_{k:n} > t)$$

Proposition (Samaniego 1985)

If F is absolutely continuous with i.i.d. lifetimes, we have

$$R_S(t) \;=\; \sum_{k=1}^{n} s_k R_{k:n}(t)$$

for every $t \geqslant 0$ and every n-component coherent system

Decomposition of reliability

Theorem (M. & Mathonet & Waldhauser 2011)

For any $t \geqslant 0$, we have

$$R_S(t) \;=\; \sum_{k=1}^{n} s_k R_{k:n}(t)$$

for every n-component coherent system if and only if the state variables $X_1(t), \ldots, X_n(t)$ are exchangeable

Remark. This condition is weaker than exchangeability of the component lifetimes $T_1, \ldots, T_n$

Part V : Additional results in the i.i.d. case

Manual computation of the Barlow-Proschan index

$$b_j = \psi_{\mathrm{Sh}}(\phi, j) = \sum_{A \subseteq C \setminus \{j\}} \frac{1}{n\binom{n-1}{|S|}} \big(\phi(A \cup \{j\}) - \phi(A)\big)$$

$$\overline{\phi}(\mathbf{x}) = \text{multilinear extension of } \phi(\mathbf{x})$$

Theorem (Owen 1972)

$$b_j = \psi_{\mathrm{Sh}}(\phi, j) = \int_0^1 \Big(\frac{\partial}{\partial x_j}\, \overline{\phi}\Big)(x, \dots, x)\, dx$$

Manual computation of the Barlow-Proschan index

Example. Home video system

$$\phi(x_1,\ldots,x_6) = (x_1 \amalg x_2)\ x_3\ x_4\ (x_5 \amalg x_6)$$

$$\begin{aligned}\overline{\phi}(x_1,\ldots,x_6) = {} & x_1x_3x_4x_5 + x_2x_3x_4x_5 + x_1x_3x_4x_6 + x_2x_3x_4x_6 \\ & -x_1x_2x_3x_4x_5 - x_1x_2x_3x_4x_6 - x_1x_3x_4x_5x_6 - x_2x_3x_4x_5x_6 \\ & +x_1x_2x_3x_4x_5x_6\end{aligned}$$

Example: $b_2 = ?$

$$\Big(\frac{\partial}{\partial x_2}\,\overline{\phi}\Big)(x,\ldots,x) = 2x^3 - 3x^4 + x^5$$

$$b_2 = \int_0^1 \big(2x^3 - 3x^4 + x^5\big)\,dx = \frac{2}{30}$$

Manual computation of the signature

How can we efficiently compute the system signature

$$s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{\substack{A\subseteq C \\ |A|=n-k+1}} \phi(A) - \frac{1}{\binom{n}{n-k}} \sum_{\substack{A\subseteq C \\ |A|=n-k}} \phi(A) \qquad ?$$

Manual computation of the signature

With any n-degree polynomial $p : \mathbb{R} \to \mathbb{R}$ we associate the *reflected* polynomial $R^n p : \mathbb{R} \to \mathbb{R}$ defined by

$$(R^n p)(x) = x^n \, p\left(\frac{1}{x}\right)$$

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \Rightarrow \quad (R^n p)(x) = a_n + a_{n-1} x + \cdots + a_0 x^n$$

(M. 2014)

Setting $p(x) = \frac{d}{dx} \, \overline{\phi}(x, \ldots, x)$, we have

$$\int_0^x (R^{n-1} p)(t+1) \, dt = \sum_{k=1}^{n} \binom{n}{k} s_k \, x^k$$

Manual computation of the signature

Example. Home video system

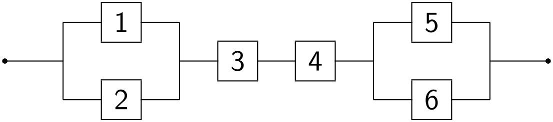
$$\begin{aligned}\overline{\phi}(x_1,\ldots,x_6) &= x_1x_3x_4x_5 + x_2x_3x_4x_5 + x_1x_3x_4x_6 + x_2x_3x_4x_6 \\ &\quad -x_1x_2x_3x_4x_5 - x_1x_2x_3x_4x_6 - x_1x_3x_4x_5x_6 - x_2x_3x_4x_5x_6 \\ &\quad +x_1x_2x_3x_4x_5x_6\end{aligned}$$

$$\begin{aligned}\overline{\phi}(x,\ldots,x) &= 4x^4 - 4x^5 + x^6 \\ p(x) = \tfrac{d}{dx}\,\overline{\phi}(x,\ldots,x) &= 16x^3 - 20x^4 + 6x^5 \\ (R^5p)(x) &= 6 - 20x + 16x^2\end{aligned}$$

$$\begin{aligned}\int_0^x (R^5p)(t+1)\,dt &= 2\,x + 6\,x^2 + \frac{16}{3}\,x^3 \\ &= \binom{6}{1} s_1\,x + \binom{6}{2} s_2\,x^2 + \cdots + \binom{6}{6} s_6\,x^6\end{aligned}$$

Barlow-Proschan importance index and system signature

Home video system



$$\mathbf{s} = \left(\frac{5}{15}, \frac{6}{15}, \frac{4}{15}, 0, 0, 0\right)$$

$$\mathbf{C} = \left(0, 0, 0, \frac{4}{15}, \frac{6}{15}, \frac{5}{15}\right)$$

$$\mathbf{I}_{\mathrm{BP}} = \left(\frac{2}{30}, \frac{2}{30}, \frac{11}{30}, \frac{11}{30}, \frac{2}{30}, \frac{2}{30}\right)$$

Computation of the signature from the minimal path sets

The multilinear extension $\overline{\phi}(\mathbf{x})$ can be obtained from the minimal path sets $P_1, \ldots, P_r$ simply by

(i) expressing the structure function, e.g., as a coproduct over the minimal path sets

$$\phi(\mathbf{x}) = \coprod_{j=1}^{r} \prod_{i \in P_j} x_i$$

(ii) expanding the coproduct

(iii) simplifying the resulting algebraic expression (using $x_j^2 = x_j$) until it becomes multilinear.

Then we compute $p(x) = \frac{d}{dx}\,\overline{\phi}(x, \ldots, x)$ and

$$\int_0^x (R^{n-1}p)(t+1)\,dt = \sum_{k=1}^{n} \binom{n}{k} s_k\, x^k$$

Open problems

One can show that there is a linear bijection between the signature $\mathbf{s}$ and the polynomial function $\overline{\phi}(x, \ldots, x)$

- Find necessary and sufficient conditions under which an n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ is the signature of a semicoherent system
- Find necessary and sufficient conditions under which an n-degree polynomial function $P(x)$ is the function $\overline{\phi}(x, \ldots, x)$ of a semicoherent system
- Enumerate all the possible semicoherent systems having a prescribed $\overline{\phi}(x, \ldots, x)$

Thank you for your attention!