

Integer Complexity: Breaking the $\Theta(n^2)$ barrier

Srinivas Vivek V., and Shankar B. R.

Abstract—The *integer complexity* of a positive integer n , denoted $f(n)$, is defined as the least number of 1's required to represent n , using only 1's, the addition and multiplication operators, and the parentheses. The running time of the algorithm currently used to compute $f(n)$ is $\Theta(n^2)$. In this paper we present an algorithm with $\Theta(n^{\log_2 3})$ as its running time. We also present a proof of the theorem: the largest solutions of $f(m) = 3k$, $3k \pm 1$ are, respectively, $m = 3^k$, $3^k \pm 3^{k-1}$.

Keywords—Integer complexity, Number theory, Running time

I. INTRODUCTION

THE *integer complexity* of a positive integer n , denoted $f(n)$, is defined as the least number of 1's required to represent n , using only 1's, the addition and multiplication operators, and the parentheses [3]. $f(n)$ can be computed as follows [2]:

$$f(n) = \min \{f(e) + f(n - e), f(d) + f(n/d)\}, \quad (1)$$

where $d|n$, $2 \leq d \leq \sqrt{n}$, and $1 \leq e \leq n/2$.

This is the currently used algorithm to compute $f(n)$. The above algorithm, when implemented in the bottom-up manner i.e. computing the smaller terms of the sequence first and then reusing them later, runs in time $\Theta(n^2)$. We must note that here the running time of the algorithm is measured in terms of the number of comparisons required to determine the value of $f(n)$. The above running time can be arrived at as follows: let the total number of comparisons needed to determine $f(n)$ be C_n . Then $C_n = O\left(\sum_{j=2}^n \left(\frac{j}{2} + \sqrt{j}\right)\right)$ and $C_n = \Omega\left(\sum_{j=2}^n \frac{j}{2}\right)$, where $j \in \mathbb{N}$. Therefore, $C_n = \Theta(n^2)$. We now present an algorithm whose running time is $\Theta(n^{\log_2 3})$.

II. PROPOSED ALGORITHM

We propose that

$$f(n) = \min \{f(e') + f(n - e'), f(d) + f(n/d)\}, \quad (2)$$

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Srinivas Vivek V. was with the Department of Information Technology, National Institute of Technology Karnataka, Surathkal, India. As on October 2014, the author is at the University of Luxembourg. Email: svivekv@gmail.com

Dr. Shankar B. R. is with the Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal, India. Email: shankarbr@gmail.com

where $d|n$, $2 \leq d \leq \sqrt{n}$, $1 \leq e' \leq n(1 - r_n)/2$, $r_n = \sqrt{1 - \frac{4(3)^{1/3}(n-1)^{\log_2 3}}{n^2}}$, and $n \geq 65$.

The inequality $n \geq 65$ follows from the fact that $4(3)^{1/3}(n-1)^{\log_2 3} \leq n^2$. It is interesting to observe that $\lim_{n \rightarrow \infty} r_n = 1$.

Proof of correctness

To prove the correctness of the algorithm (2), we show that

$$f(e') + f(n - e') \geq f(n - 1) + f(1) \quad \forall e' \in \mathbb{N} \cap [n(1 - r_n)/2, n/2]. \quad (3)$$

We arrive at the above result by trying to find a range of values of e for which the inequality (3) is true. In [2] it is shown that $3 \log_3 n \leq f(n) \leq 3 \log_2 n = 3(\log_2 3) \log_3 n$. By using this fact and assigning the individually smallest possible values to $f(e')$ and $f(n - e')$, and the largest possible value to $f(n - 1)$ in the inequality (3), we get

$$\begin{aligned} 3 \log_3 e' + 3 \log_3(n - e') &\geq 3 \log_2(n - 1) + 1 \\ \Rightarrow \log_3 \frac{e'(n - e')}{(n - 1)^{\log_2 3}} &\geq \frac{1}{3} \\ \Rightarrow -(e')^2 + n e' - 3^{1/3}(n - 1)^{\log_2 3} &\geq 0. \quad (4) \end{aligned}$$

The integer values of $e' (\leq n/2)$ which satisfy (4) are $\mathbb{N} \cap \left[\frac{n}{2} \left(1 - \sqrt{1 - \frac{4(3)^{1/3}(n-1)^{\log_2 3}}{n^2}}\right), \frac{n}{2}\right]$, which can be equivalently written as $\mathbb{N} \cap [n(1 - r_n)/2, n/2]$. Clearly, the values of e' which satisfy (4) also satisfy (3). Hence, the proof of correctness of the algorithm.

Running time

Let C_n be as previously defined. Then $C_n = O\left(\sum_{j=65}^n \left(\frac{j(1-r_j)}{2} + \sqrt{j}\right)\right)$ and $C_n = \Omega\left(\sum_{j=65}^n \frac{j(1-r_j)}{2}\right)$. Let $h_j = \frac{4(3)^{1/3}(j-1)^{\log_2 3}}{j^2}$. By binomial expansion,

$$j(1 - r_j) = j \left(1 - \sqrt{1 - h_j}\right) \leq \frac{j h_j}{1 - h_j}$$

$$\Rightarrow j(1 - r_j) = \Theta\left(j^{(\log_2 3) - 1}\right).$$

$$\Rightarrow C_n = \Theta\left(n^{\log_2 3}\right) = O\left(n^{1.59}\right).$$

Hence, the running time of the proposed algorithm is $\Theta(n^{\log_2 3})$.

III. PROOF OF A THEOREM

The following theorem has been mentioned in [2]. It is also mentioned there that the theorem has been proved using induction but the proof was not given. We now give a proof of the theorem. This proof also is based on induction.

Theorem: The largest solutions of $f(m) = 3k$, $3k \pm 1$ are, respectively, $m = 3^k$, $3^k \pm 3^{k-1}$.

Proof: Define $g(s)$ as the largest number which can be formed using s number of 1's, the addition and multiplication operators, and the parentheses. Clearly, the largest solution of $f(m) = s$ is $m = g(s)$. $g(s)$ can be recursively obtained by the following formula:

$$g(s) = \max \{g(e) + g(s - e), g(e) \times g(s - e)\},$$

where $1 \leq e \leq s/2$.

Let the above theorem be true for $k = 1, 2 \dots n - 1$. First, we prove the case of $f(m) = 3n - 1$, and in a similar manner we subsequently prove the cases of $f(m) = 3n$ and $f(m) = 3n + 1$. It can be easily verified that when $k = 1$, the theorem is true since $g(2) = 2$, $g(3) = 3$, and $g(4) = 4$. Define $g'(s, e) = \max \{g(e) + g(s - e), g(e) \times g(s - e)\}$. It is easy to see that $g'(s, e) = g(e) \times g(s - e) \forall e \geq 2, s \geq 4$.

First, consider $f(m) = 3n - 1$.

$$g'(3n - 1, 1) = 1 + g(3(n - 1) + 1) = 1 + 3^{n-1} + 3^{n-2}.$$

$$\text{Let } 2 \leq e = 3k' - 1 \leq (3n - 1)/2, \text{ then } g'(3n - 1, e) = g(3k' - 1) \times g(3(n - k')) = (3^{k'} - 3^{k'-1}) \times 3^{n-k'} = 3^n - 3^{n-1}.$$

$$\text{Let } 2 \leq e = 3k' \leq (3n - 1)/2, \text{ then } g'(3n - 1, e) = g(3k') \times g(3(n - k') - 1) = 3^{k'} \times (3^{n-k'} - 3^{n-k'-1}) = 3^n - 3^{n-1}.$$

$$\text{Let } 2 \leq e = 3k' + 1 \leq (3n - 1)/2, \text{ then } g'(3n - 1, e) = g(3k' + 1) \times g(3(n - k') - 1) = (3^{k'} + 3^{k'-1}) \times (3^{n-k'-1} + 3^{n-k'-2}) = 16 \times 3^{n-3}.$$

Therefore, the maximum possible value of $g'(3n - 1, e)$ is $3^n - 3^{n-1}$. Hence, the theorem is proved for the case of $f(m) = 3n - 1$.

Next, consider $f(m) = 3n$.

$$g'(3n, 1) = 1 + g(3n - 1) = 1 + 3^n - 3^{n-1}.$$

$$\text{Let } 2 \leq e = 3k' - 1 \leq 3n/2, \text{ then } g'(3n, e) = g(3k' - 1) \times g(3(n - k') + 1) = (3^{k'} - 3^{k'-1}) \times (3^{n-k'} + 3^{n-k'-1}) = 8 \times 3^{n-2}.$$

$$\text{Let } 2 \leq e = 3k' \leq 3n/2, \text{ then } g'(3n, e) = g(3k') \times g(3(n - k')) = 3^{k'} \times 3^{n-k'} = 3^n.$$

$$\text{Let } 2 \leq e = 3k' + 1 \leq 3n/2, \text{ then } g'(3n, e) = g(3k' + 1) \times g(3(n - k') - 1) = (3^{k'} + 3^{k'-1}) \times (3^{n-k'} - 3^{n-k'-1}) = 8 \times 3^{n-2}.$$

Therefore, the maximum possible value of $g'(3n, e)$ is 3^n . Hence, the theorem is proved for the case of $f(m) = 3n$.

Finally, consider $f(m) = 3n + 1$.

$$g'(3n + 1, 1) = 1 + g(3n) = 1 + 3^n.$$

$$\text{Let } 2 \leq e = 3k' - 1 \leq (3n + 1)/2, \text{ then } g'(3n + 1, e) =$$

$$g(3k' - 1) \times g(3(n - k') + 1) = (3^{k'} - 3^{k'-1}) \times (3^{n-k'+1} - 3^{n-k'}) = 3^n + 3^{n-1}.$$

$$\text{Let } 2 \leq e = 3k' \leq (3n + 1)/2, \text{ then } g'(3n + 1, e) = g(3k') \times g(3(n - k') + 1) = 3^{k'} \times (3^{n-k'} + 3^{n-k'-1}) = 3^n + 3^{n-1}.$$

$$\text{Let } 2 \leq e = 3k' + 1 \leq (3n + 1)/2, \text{ then } g'(3n + 1, e) = g(3k' + 1) \times g(3(n - k')) = (3^{k'} + 3^{k'-1}) \times 3^{n-k'} = 3^n + 3^{n-1}.$$

Therefore, the maximum possible value of $g'(3n + 1, e)$ is $3^n + 3^{n-1}$. Hence, the theorem is proved for the case of $f(m) = 3n + 1$, and this completes the proof of the theorem. ■

IV. CONCLUSION

In this paper we have presented a more efficient algorithm to compute the *integer complexity* that runs in $\Theta(n^{\log_2 3})$ time when compared with the existing algorithm that runs in $\Theta(n^2)$ time. Working on the same lines, a better algorithm can be provided if one can give a tighter upper bound on the value of $f(n)$ than the existing $3 \log_2 n$ bound. Also, we have given a proof of the theorem: the largest solutions of $f(m) = 3k$, $3k \pm 1$ are, respectively, $m = 3^k$, $3^k \pm 3^{k-1}$, using induction.

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