Irreducible decomposition for local representations of quantum Teichmüller space

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Abstract

We give an irreducible decomposition of the so-called local representations [HBL07] of the quantum Teichmüller space $\mathcal{T}_q(\Sigma)$ where Σ is a punctured surface of genus $g > 0$ and q is a N-th root of unity with N odd.

Let Σ be an oriented surface of genus $g > 0$ with s punctures $v_1, ..., v_s$ such that $2g - 2 + s > 0$ (this condition is equivalent to the existence of an ideal triangulation of Σ, ie. a triangulation whose vertices are exactly the v_i). Let $\mathcal{T}(\Sigma)$ be the Teichmüller space of Σ, that is the moduli space of complete hyperbolic metrics on Σ. Given λ an ideal triangulation of Σ, W.P. Thurston [Thu98] constructed a parameterization of $\mathcal{T}(\Sigma)$ by associating a strictly positive real number to each edge λ_i of the ideal triangulation, $i \in \{1, ..., n\}$ (where $n = 6g - 6 + 3s$ is the number of edges of λ). These coordinates are called **shear coordinates** associated to λ. In this coordinates system, the coefficients of the Weil-Petersson form on $\mathcal{T}(\Sigma)$ depend only on the combinatoric of λ and are easy to compute.

For a parameter $q \in \mathbb{C}^*$, L.O. Chekhov, V.V. Fock [FC99] and independently R. Kashaev [Kas98] defined the so-called **quantum Teichmüller space** $\mathcal{T}_q(\Sigma)$ of Σ (the construction of R. Kashaev differs a little from the one of L.O. Chekhov and V.V. Fock), which is a deformation of the Poisson algebra of rational functions over $\mathcal{T}(\Sigma)$. This algebraic object is obtained by gluing together a collection of non-commutative algebra $\mathcal{T}_q(\lambda)$ (called Chekhov-Fock algebra) canonically associated to each ideal triangulation of Σ. A representation of $\mathcal{T}_q(\Sigma)$ is then a family of representation $\{\rho_\lambda : \mathcal{T}_q(\lambda) \to \mathrm{End}(V)\}_{\lambda \in \Lambda(\Sigma)}$, where $\Lambda(\Sigma)$ is the space of all ideal triangulations of Σ, and ρ_λ and $\rho_{\lambda'}$ satisfy compatibility conditions whenever $\lambda \neq \lambda'$. For $\lambda \in \Lambda(\Sigma)$, the representation ρ_λ is an avatar of the representation of $\mathcal{T}_q(\Sigma)$ and carries almost all the information.

When q is a N-th root of unity, $\mathcal{T}_q(\lambda)$ admits finite-dimensional representations. In this paper, we will consider N odd. The irreducible representations of $\mathcal{T}_q(\lambda)$ have been studied in [BL07]. In particular, they show that an irreducible representation of $\mathcal{T}_q(\lambda)$ is classified (up to isomorphism) by a weight $x_i \in \mathbb{C}^*$ assigned to each edge λ_i, a choice of N-th root $p_j = (x_1^{k_{j_1}} ... x_n^{k_{j_n}})^{1/N}$ associated to each puncture v_j (where k_{j_i} is the number of times a small simple loop around

v_j intersects λ_i) and a square root $c = (p_0 ... p_s)^{1/2}$. Such a representation has dimension N^{3g-3+s}.

In [HBL07], the authors introduced another type of representations of $\mathcal{T}_q(\lambda)$, called **local representations**, which are well behaved under cut and paste. A local representation of $\mathcal{T}_q(\lambda)$ is defined by a an embedding into the tensorial product of **triangle algebras** (see definitions below). The local representations of $\mathcal{T}_q(\lambda)$ are classified (up to isomorphism) by a weight $x_i \in \mathbb{C}^*$ associated to each edge λ_i and a choice of N-th root $c = (x_1 ... x_n)^{1/N}$. Such a representation has dimension $N^{4g-4+2s}$.

It follows that a local representation of $\mathcal{T}_q(\lambda)$ is not irreducible. In this paper, we adress the question of the decomposition of a local representation into its irreducible components. In particular, we prove the following result:

Theorem 1. *Let λ be an ideal triangulation of Σ and ρ be a local representation of $\mathcal{T}_q(\lambda)$ classified by weight $x_j \in \mathbb{C}^*$ associated to each edge λ_j and a choice of N-th root $c = (x_1 ... x_n)^{1/N}$. We have the following decomposition:*

$$\rho = \bigoplus_{i \in \mathcal{I}} \rho^{(i)}.$$

Here, $\rho^{(i)}$ is an irreducible representation classified by the same x_j, a N-th root $p_j^{(i)} = (x_1^{k_{j1}} ... x_n^{k_{jn}})^{1/N}$ associated to each puncture, and the same c. Moreover, for each choice of N-th root $p_j = (x_1^{k_{j1}} ... x_n^{k_{jn}})^{1/N}$ for each puncture and $c = (p_0 ... p_s)^{1/2}$, there exists exactly N^g elements $i \in \mathcal{I}$ with $p_j^{(i)} = p_j$ for all $j \in \{0, ..., s\}$.

This result may be used to define representations of the so-called **Kauffman skein algebra** $\mathcal{S}^A(\overline{\Sigma})$ [Tur91] (where $\overline{\Sigma}$ is the surface Σ without marked points) which corresponds to a quantization by deformation of the character variety

$$\mathcal{R} := \mathrm{Hom}(\pi_1(\overline{\Sigma}), PSL(2, \mathbb{C})) /\!\!/ PSL(2, \mathbb{C}))$$

where $PSL(2, \mathbb{C})$ acts by conjugation on the morphisms and the double slash means that we take the quotient in the sense of Geometric Invariant Theory. In [BW11, Theorem 1], the authors constructed a morphism

$$Tr_\omega(\lambda) : \mathcal{S}_A(\Sigma) \longrightarrow \hat{\mathcal{Z}}_\omega(\lambda),$$

where $q = \omega^4$, $A = \omega^{-2}$ and $\hat{\mathcal{Z}}_\omega(\lambda)$ is an algebra of non-commutative rational fractions such that $\mathcal{T}_q(\lambda)$ consists of rational fractions in $\hat{\mathcal{Z}}_\omega(\lambda)$ involving only even powers of the variables. This morphism, composed with a representation ρ of $\mathcal{T}_q(\lambda)$ is studied in [BW12a] and [BW12b] to define a new kind of representations of $\mathcal{S}^A(\Sigma)$. However, if one wants to define representation of $\mathcal{S}^A(\overline{\Sigma})$ in the same way, one has to consider the direct sum of N^g irreducible components of a local representation $\rho : \mathcal{T}^q(\lambda) \to \mathrm{End}(V)$ arising in the decomposition of Theorem 1 and find a subspace $E \subset V$ stable by $\rho \circ Tr_\omega(\lambda)$ such that $(\rho \circ Tr_\omega(\lambda))_{|E}$ defines a representation of $\mathcal{S}^A(\overline{\Sigma})$ (see [BW14] for the construction). Hopefully, this representation of $\mathcal{S}^A(\overline{\Sigma})$ should be used to define a more intrinsic version of the Kashaev-Baseilhac-Benedetti TQFT (see [Kas95], [Kas99], [BB04], [BB05] and [BB07]).

In the first section, we recall the definition of the Chekhov-Fock algebra, the quantum Teichmüller space, the triangle algebra and the local representations. In the second one, we prove the Theorem 1. The proof is done in two steps: we first prove the result for a special triangulation λ_0 and special weights; we then extend to the general case.

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1 The Chekhov-Fock algebra and the irreducible representations of $\mathcal{T}_q(\Sigma)$

The results of this section come from [BL07] and [HBL07]. From now, for $n \in \mathbb{N}$, define $\mathbb{N}_n := \mathbb{Z}/n\mathbb{Z}$ and denote by $\mathcal{U}(N)$ the group of N-th root of unity.

1.1 The Chekhov-Fock algebra

Let λ be an ideal triangulation of Σ. The **Chekhov-Fock algebra** $\mathcal{T}_q(\lambda)$ associated to λ is the algebra generated by the elements $X_i^{\pm 1}$ associated to each edge λ_i of the triangulation λ. These elements are subjects to the relations:

$$X_i X_j = q^{\sigma_{ij}} X_j X_i,$$

where the coefficients σ_{ij} are the coefficients of the Weil-Petersson form in the shear coordinates associated to λ and depend only on the combinatoric of λ. Namely, we have $\sigma_{ij} = a_{ij} - a_{ji}$ where a_{ij} is the number of angular sector delimited by λ_i and λ_j in the faces of λ with λ_i coming before λ_j counterclockwise. In practice, elements of $\mathcal{T}_q(\lambda)$ are just Laurent polynomials in the variables X_i satisfying non-commutativity conditions. We will sometimes denote $\mathcal{T}_q(\lambda)$ by $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]_q$ to reflect this fact.

Let $\mathbf{k} = (k_1, ..., k_n) \in \mathbb{Z}^n$ be a multi-index; to a monomial X composed of a product of $X_i^{k_i}$, we associate its quantum ordering:

$$[X] := q^{-\sum_{i<j} \sigma_{ij}} X_1^{k_1} ... X_n^{k_n}.$$

It allows us to associate a monomial $X_{\mathbf{k}} \in \mathcal{T}_q(\lambda)$ to each each multi-index $\mathbf{k} \in \mathbb{Z}^n$.

To study finite-dimensional representations of $\mathcal{T}_q(\lambda)$, one needs to determine its center.

Proposition 1 ([BL07], Proposition 15)**.** *The center of $\mathcal{T}_q(\lambda)$ is generated by:*

- *X_i^N for each $i \in \{1, ..., n\}$.*
- *For each puncture v_j, the **puncture invariant** P_j associated to the multi-index $\boldsymbol{k}_j = (k_{j_1}, ..., k_{j_n})$ (where k_{j_i} is the number of intersections of λ_i with a small simple loop around v_j).*

- *The element H associated to the multi index $\boldsymbol{k} = (1, ..., 1)$.*

Note that $[P_1 ... P_s] = H^2$.

1.2 Triangle algebra

Let T be a disk with three punctures $v_1, v_2, v_3 \in \partial T$ endowed with the natural triangulation λ composed of three counterclokwise directed edges λ_1, λ_2 and λ_3 (as in Figure 1).

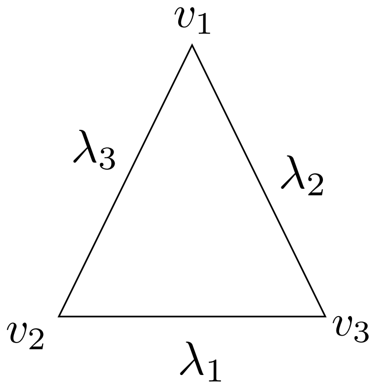

Figure 1: The triangle T

Define the **triangle algebra** as the the Chekhov-Fock algebra $\mathcal{T} := \mathcal{T}_q(\lambda)$. It is generated by $X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}$ with relations $X_i X_{i+1} = q^2 X_{i+1} X_i$ for all $i \in \mathbb{N}_3$. The center of $\mathcal{T}$ is given by X_1^N, X_2^N, X_3^N and $H = q^{-1} X_1 X_2 X_3$. Irreducible finite dimensional representations of $\mathcal{T}$ have dimension N and are classified (up to isomorphism) by a choice of weight $x_i \in \mathbb{C}^*$ associated to each edge λ_i and a central charge, that is a choice of a N-th root $c = (x_1 x_2 x_3)^{1/N}$ (see [HBL07, Lemma 2]).

To be more precise, for V the N-dimensional complex vector space generated by $\{e_1, ..., e_N\}$ and ρ an irreducible repesentation of $\mathcal{T}$ classified by $x_1, x_2, x_3 \in \mathbb{C}^*$ and $c = (x_1 x_2 x_3)^{1/N}$. Up to isomorphism, the action of $\mathcal{T}$ on V defined by ρ is given by:

$$\begin{cases} X_1 e_i = & \widetilde{x}_1 q^{2i} e_i \\ X_2 e_i = & \widetilde{x}_2 e_{i+1} \\ X_3 e_i = & \widetilde{x}_3 q^{1-2i} e_{i-1} \end{cases}$$

where $\widetilde{x}_i$ is an N-th root of x_i such that $\widetilde{x}_1 \widetilde{x}_2 \widetilde{x}_3 = c$. Note that, up to isomorphism, ρ is independent of the choice of the N-th root $\widetilde{x}_i$ with $\widetilde{x}_1 \widetilde{x}_2 \widetilde{x}_3 = c$.

In particular, for the representation ρ classified by $x_1 = x_2 = x_3 = 1$ and $c \in \mathcal{U}(N)$, as $\rho(X_i)^N = Id_V$, the spectrum of $\rho(X_i)$ is a subset of $\mathcal{U}(N)$. For $h \in \mathcal{U}(N)$, denote by $V_h(X_i)$ the eigenspace of $\rho(X_i)$ associated to the eigenvalue h. We have the following lemma which will be usefull in the next section:

Lemma 1. *For each $i \in \{1, 2, 3\}$ and $h \in \mathcal{U}(N)$, $\dim(V_h(X_i)) = 1$.*

Proof. We use the explicit form of the representation ρ in $V = \text{span}\{e_1, ..., e_N\}$. Take $\widetilde{x}_1 = \widetilde{x}_2 = 1$ and $\widetilde{x}_3 = c$.

For $i = 1$, one sees that $V_h(X_1) = \text{span}\{e_k\}$ where $h = q^{2k}$.

For $i = 2$, the vector $\alpha_k := \sum_{i\in\mathbb{N}_N} q^{-2ki} e_i$ satisfies $X_2\alpha_k = q^{2k}\alpha_k$ and $\{\alpha_1, ..., \alpha_k\}$ form a basis of V. Then $V_h(X_2)$ is generated by α_k.

For $i = 3$, we use the fact that $X_1X_2X_3e_i = ce_i$, where c, the central charge of ρ, lies in $\mathcal{U}(N)$. □

1.3 Local representation of $\mathcal{T}_q(\lambda)$

Let λ be an ideal triangulation of Σ. Such a triangulation is composed of m faces $T_1, ..., T_m$ and each face T_j determines a triangle algebra $\mathcal{T}_j$ whose generators are associated to the three edges of T_j. It provides a canonical embedding $\mathfrak{i}$ of $\mathcal{T}_q(\lambda)$ into $\mathcal{T}_1 \otimes ... \otimes \mathcal{T}_m$ defined on the generators as follow:

- $\mathfrak{i}(X_i) = X_{ji} \otimes X_{ki}$ if λ_i belongs to two distinct triangles T_j and T_k and $X_{ji} \in \mathcal{T}_j$, $X_{ki} \in \mathcal{T}_k$ are the generators associated to the edge $\lambda_i \in T_j$ and $\lambda_i \in T_k$ respectively.
- $\mathfrak{i}(X_i) = [X_{ji_1}X_{ji_2}]$ if λ_i corresponds to two sides of the same face T_j and $X_{ji_1}, X_{ij_2} \in \mathcal{T}_j$ are the associated generators.

Now, a local representation of $\mathcal{T}_q(\lambda)$ is a representation which factorizes as $(\rho_1 \otimes ... \otimes \rho_m) \circ \mathfrak{i}$ where $\rho_i : \mathcal{T}_i \to V_i$ is an irreducible representation of the triangle algebra $\mathcal{T}_i$. In particular, such a representation has dimension N^m where $m = 4g - 4 + 2s$ is the number of faces of the triangulation.

1.4 Classification of these representations

Here we recall [BL07, Theorem 21] and [HBL07, Proposition 6] respectively:

Theorem 2. *(F. Bonahon, X. Liu) An irreducible representation of $\mathcal{T}_q(\lambda)$ is determined by its restriction to the center of $\mathcal{T}_q(\lambda)$ and is classified by a non-zero complex number x_i associated to each edges λ_i, for each puncture v_j, a choice of a N-th root $p_j = (x_1^{k_{j1}}...x_n^{k_{jn}})^{1/N}$ and a choice of a square root $c = (p_0...p_s)^{1/2}$.*

Such a representation satisfies:

- $\rho(X_i^N) = x_i Id$,
- $\rho(P_j) = p_j Id$,
- $\rho(H) = c Id$.

Theorem 3. *(H. Bai, F. Bonahon, X. Liu) Up to isomorphism, a local representation of $\mathcal{T}_q(\lambda)$ is classified by a non-zero complex number x_i associated to the edge λ_i and a choice of a N-th root $c = (x_1...x_n)^{1/N}$. Such a representation satisfies:*

- $\rho(X_i^N) = x_i Id$,
- $\rho(H) = c Id$.

1.5 The quantum Teichmüller spaces and its representations

If one wants to quantize the Teichmüller space, he has to do it in a canonical way. The definition of the Chekhov-Fock algebra $\mathcal{T}_q(\lambda)$ involves the choice of an ideal triangulation. So we have to understand the behavior when one changes from an ideal triangulation λ to another one λ'. Set $\mathcal{T}_q(\lambda) = \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]_q$ and $\mathcal{T}_q(\lambda') = \mathbb{C}[X_1'^{\pm 1}, ..., X_n'^{\pm 1}]_q$. These algebras admit a division algebras, denoted by $\hat{\mathcal{T}}_q(\lambda)$ and $\hat{\mathcal{T}}_q(\lambda')$ respectively, consisting of rational fractions in the variables X_i (respectively X_i') satisfying some non-commutativity relations.

For each pair of ideal triangulation λ and λ', L.O. Chekhov and V.V. Fock constructed coordinates change isomorphisms

$$\Psi^q_{\lambda\lambda'} : \hat{\mathcal{T}}_q(\lambda') \longrightarrow \hat{\mathcal{T}}_q(\lambda),$$

which are the unique isomorphism satisfying naturals conditions (as for example $\Psi^q_{\lambda\lambda''} = \Psi^q_{\lambda\lambda'} \circ \Psi^q_{\lambda'\lambda''}$ for each λ, λ' and λ'' ideal triangulations of Σ). See [Liu09] for more details and explicit formulae of $\Psi^q_{\lambda\lambda'}$.

Now, the **quantum Teichmüller space** $\mathcal{T}_q(\Sigma)$ is defined by:

$$\mathcal{T}_q(\Sigma) := \bigsqcup_{\lambda \in \Lambda(\Sigma)} \hat{\mathcal{T}}_q(\lambda)/\sim,$$

where $\Lambda(\Sigma)$ is the set of ideal triangulation of Σ, and the equivalence relation $\sim$ identifies each pair of $\hat{\mathcal{T}}_q(\lambda)$ and $\hat{\mathcal{T}}_q(\lambda')$ by the isomorphism $\Psi^q_{\lambda\lambda'}$. Note that, as each coordinates change $\Psi^q_{\lambda\lambda'}$ is an algebra isomorphism, $\mathcal{T}_q(\Sigma)$ inherits an algebra structure, and the $\hat{\mathcal{T}}_q(\lambda)$ can be thought as "global coordinates" on $\mathcal{T}_q(\Sigma)$.

A natural definition for a finite dimensional representation of $\mathcal{T}_q(\Sigma)$ would be a family of finite dimensional representation $\{\rho_\lambda : \hat{\mathcal{T}}_q(\lambda) \longrightarrow \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(\Sigma)}$ such that for each pair of ideal triangulation λ and λ', $\rho_{\lambda'}$ is isomorphic to $\rho_\lambda \circ \Psi^q_{\lambda,\lambda'}$. Note that, as pointed out in [HBL07, Section 4.2], there exists no algebra homomorphism $\rho_\lambda : \hat{\mathcal{T}}_q(\lambda) \longrightarrow \mathrm{End}(V_\lambda)$ for V_λ finite dimensional. In fact, as $\hat{\mathcal{T}}_q(\lambda)$ is infinite dimensional as a vector space and $\mathrm{End}(V_\lambda)$ is finite dimensional, such a homomorphism ρ_λ would have non-zero kernel. Hence, there would exists elements $x \in \hat{\mathcal{T}}_q(\lambda)$ such that $\rho_\lambda(x) = 0$ and so, $\rho_\lambda(x^{-1})$ would make no sense.

So one defines a **local representation (respectively irreducible representation) of** $\mathcal{T}_q(\Sigma)$ as a family of representation $\{\rho_\lambda : \mathcal{T}_q(\lambda) \longrightarrow \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(\Sigma)}$ such that for each $\lambda, \lambda' \in \Lambda(\Sigma)$, ρ_λ is a local representation (respectively irreducible representation) of $\mathcal{T}_q(\lambda)$, and $\rho_{\lambda'}$ is isomorphic (as representation) to $\rho_\lambda \circ \Psi^q_{\lambda\lambda'}$ whenever $\rho_\lambda \circ \Psi^q_{\lambda\lambda'}$ makes sense. We say that $\rho_\lambda \circ \Psi^q_{\lambda\lambda'}$ makes sense, if for each Laurent polynomial $X' \in \mathcal{T}_q(\lambda')$, there exists P, P', Q and $Q' \in \mathcal{T}_q(\lambda)$ such that:

$$\Psi_{\lambda\lambda'}(X') = PQ^{-1} = Q'^{-1}P' \in \hat{\mathcal{T}}_q(\lambda);$$

now, as $\rho_\lambda(\mathcal{T}_q(\lambda)) \subset GL(V_\lambda)$, $\rho_\lambda(Q)$ and $\rho_\lambda(Q')$ are invertibles, so we can define:

$$\rho_\lambda \circ \Psi_{\lambda\lambda'}(X') := \rho_\lambda(P)\rho_\lambda(Q)^{-1} = \rho_\lambda(Q')^{-1}\rho_\lambda(P').$$

A fundamental result in [BL07] and [HBL07, Proposition 10] is that for each pair of ideal triangulations λ and λ', there exists a rational map

$$\varphi_{\lambda\lambda'} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

such that a local representation $\rho_{\lambda'}$ of $\mathcal{T}_q(\lambda')$ classified by $x'_i \in \mathbb{C}^*$ associated to λ'_i and $c' = (x'_1...x'_n)^{1/N}$ is isomorphic to $\rho_\lambda \circ \Psi_{\lambda\lambda'}$ (whenever it makes sense) for a representation ρ_λ of $\mathcal{T}_q(\lambda)$ classified by $x_i \in \mathbb{C}^*$ associated to λ_i and $c = (x_1...x_n)^{1/N}$ if and only if $c = c'$ and

$$(x'_1, ..., x'_n) = \varphi_{\lambda\lambda'}(x_1, ..., x_n).$$

2 Proof of Theorem 1

2.1 Special case

Here we prove Theorem 1 for a local representation $\rho : \mathcal{T}_q(\lambda_0) \longrightarrow \mathrm{End}(V)$ where λ_0 is special triangulation of Σ and ρ is classified by weights $x_i = 1$ and $c \in \mathcal{U}(N)$. Here, Σ is a genus $g > 0$ surface with $s+1$ punctures $v_0, ..., v_s$. Recall that $m = 4g - 4 + 2(s+1)$ and $n = 6g - 6 + 3(s+1)$ are respectively the number of faces and edges of λ_0. Moreover, we denote by Xu the action of $X \in \mathcal{T}_q(\lambda_0)$ on $u \in V$ defined by ρ.

To decompose ρ into irreducible factors, one has to look at the eigenspaces of $\rho(P_j)$ for each puncture invariant P_j associated to the puncture v_j. Note that, as $\rho(P_j)^N = Id$, the spectrum of P_j is contained in $\mathcal{U}(N)$.

The idea of the proof is to look at the action of the P_j on each factor of a nice decomposition of V into a tensorial product of vector spaces. It is based on the following remark:

Remark 1. *For a decomposition $V = E_1 \otimes E_2$, if $x_j \in E_j$ satisfies $Px_j = h_j x_j$ for $j = 1, 2$ where $P \in \{P_0, ..., P_s\}$ and $h_j \in \mathcal{U}(N)$, then $P(x_1 \otimes x_2) = h_1 h_2 x_1 \otimes x_2$. That is, the eigenspace of P in V associated to the eigenvalue $h \in \mathcal{U}(N)$ contains the tensorial product of eigenspaces of P in E_j associated to the eigenvalues h_j, for $j = 1, 2$, whenever $h = h_1 h_2$.*

For $\mathbf{h} = (h_1, ..., h_s) \in \mathcal{U}(N)^s$, set

$$V_{\mathbf{h}} := \{u \in V,\ P_i u = h_i u,\ i = 1, ..., s\}.$$

Proposition 2. *For each $\boldsymbol{h} \in \mathcal{U}(N)^s$, $\dim V_{\boldsymbol{h}} = N^{m-s}$.*

Proof. Take an ideal triangulation $\widetilde{\lambda}$ of $\Sigma \setminus \{v_1, ..., v_s\}$ (which is a one punctured surface), and for a triangle T of $\widetilde{\lambda}$, consider the triangulation of $T \cup \{v_1, ..., v_s\}$ as in Picture 2.

The union of these two triangulations gives an ideal triangulation λ_0 of Σ. Denote by $\widetilde{V}$ the tensorial product of all the vector spaces associated to the triangles of $\widetilde{\lambda} \setminus T$. As the triangulation $\widetilde{\lambda}$ contains $3g - 1$ triangles, $\dim(\widetilde{V}) = N^{3g-2}$ (because we do not consider the vector space associated to T). Denote by V^j and V'^k the j^{th} (resp. k^{th}) vector space associated to the triangle T_j (resp. T'_k) as in Figure 2 (here, $j \in \{0, ..., s\}$ and $k \in \{1, ..., s\}$).

For $\mathbf{h} = (h_1, ..., h_s) \in \mathcal{U}(N)^s$ and $j \in \{1, ..., s\}$, define:

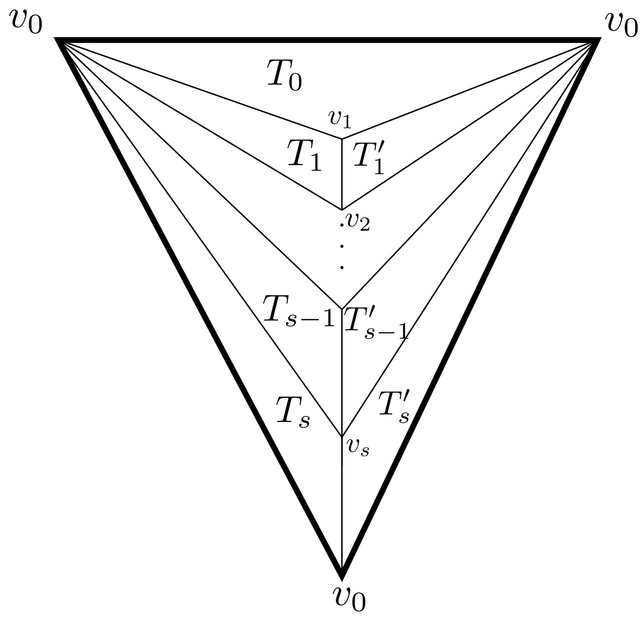

Figure 2: Triangulation of $T \cup \{v_1, ..., v_s\}$

. $\mathcal{V}^j_{\mathbf{h}} = \{x \in V^j \otimes V'^j,\ P_k x = h_k x,\ k = 1, ..., s\}$.

. $\mathcal{V}^0_{\mathbf{h}} = \{x \in V^0,\ P_k x = h_k x,\ k = 1, ..., s\}$.

We have the following lemma:

Lemma 2.

i. $\dim \mathcal{V}^0_{\boldsymbol{h}} = \begin{cases} 1 & \textit{if } h_k = 1\ \forall k \neq 1 \\ 0 & \textit{otherwise.} \end{cases}$

ii. $\forall j \in \{1, ..., s-1\}$ $\dim \mathcal{V}^j_{\boldsymbol{h}} = \begin{cases} 1 & \textit{if } h_k = 1\ \forall k \notin \{j, j+1\} \\ 0 & \textit{otherwise.} \end{cases}$

iii. $\dim \mathcal{V}^s_{\boldsymbol{h}} = \begin{cases} N & \textit{if } h_k = 1\ \forall k \neq s \\ 0 & \textit{otherwise.} \end{cases}$

Proof. *i.* If $k \neq 1$, v_k is not a vertex of T_0. It follows that P_k acts on V^0 by the identity; so if $h_k \neq 1$, $\mathcal{V}^0_{\mathbf{h}} = \{0\}$.

Now, if $h_k = 1$ for all $k \neq 1$, then $\mathcal{V}^0_{\mathbf{h}} = V^0_{h_1}(P_1)$ (as defined in Lemma 1) which is one dimensional.

ii. Fix $j \in \{1, ..., s-1\}$. For $k \notin \{j, j+1\}$, v_k is neither a vertex of T_j nor of T_j'. So P_j acts on $V^j \otimes V'^j$ as the identity. Hence, if $h_k \neq 1$, then $\mathcal{V}^j_{\mathbf{h}} = \{0\}$.

Take $h_k = 1$ for all $k \notin \{j, j+1\}$ and denote by $\mathcal{T}_j = \mathbb{C}[X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}]_q$, $\mathcal{T}_j' = \mathbb{C}[X'^{\pm 1}, Y'^{\pm 1}, Z'^{\pm 1}]_q$ the triangle algebras associated to the triangles T_j and T_j' respectively (as in Figure 3).

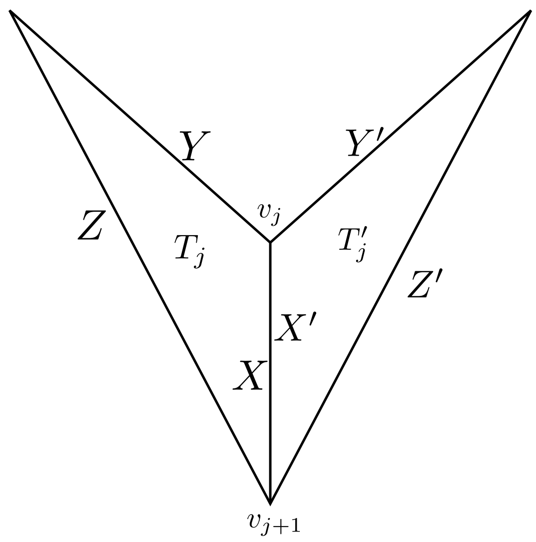

Figure 3: The generators of $\mathcal{T}_j$ and $\mathcal{T}'_j$

For $c_j, c'_j \in \mathcal{U}(N)$ the central charges of the restriction of the representation to $\mathcal{T}_j$ and $\mathcal{T}'_j$ respectively, P_j acts on $V^j := \text{span}\{e_0, ..., e_{N-1}\}$ like $c_j Z^{-1}$, on $V'^j = \text{span}\{e'_0, ..., e'_{N-1}\}$ like $c'_j Z'^{-1}$ and P_{j+1} acts on V_j like $c_j Y^{-1}$, on V'_j like $c'_j Y'^{-1}$. Set $c_j = q^p$ and $c'_j = q^{p'}$, we get the following:

$$\begin{aligned} P_j e_k &= q^{2k-1+p} e_{k+1} \\ P_j e'_l &= q^{1-2l+p'} e_{l+1} \end{aligned}$$

It follows that the action of P_j on $V^j \otimes V'^j$ is given by:

$$P_j \epsilon_{k,l} = q^{2(k-l)+p+p'} \epsilon_{k+1,l+1} \text{ where } \epsilon_{k,l} := e_k \otimes e'_l.$$

In the same way, one sees that the action of P_{j+1} on $V^j \otimes V'^j$ is given by:

$$P_{j+1} \epsilon_{k,l} = q^{p+p'} \epsilon_{k-1,l-1}.$$

Now, for $m, n \in \mathbb{N}$, set $\alpha_{m,n} := \sum_{k=0}^{N-1} q^{2km} \epsilon_{k,k+n}$, an easy calculation shows that:

$$\left\{ \begin{array}{ll} P_j \alpha_{m,n} = & q^{-2(m+n)+p+p'} \alpha_{m,n} \\ P_{j+1} \alpha_{m,n} = & q^{2m+p+p'} \alpha_{m,n}. \end{array} \right.$$

It follows that $\{\alpha_{n,m},\ n, m \in \mathbb{N}\}$ is a base of $V^j \otimes V'^j$ and, for all $h_j, h_{j+1} \in \mathcal{U}(N)$, there exists a unique couple $(m, n) \in \mathbb{N}_N^2$ with $h_j = q^{-2(m+n)+p+p'}$ and $h_{j+1} = q^{2m+p+p'}$. So $\dim \mathcal{V}_{\mathbf{h}}^j = 1$ if and only if $h_k = 1$ for all $k \notin \{j, j+1\}$.

iii. If $k \neq s$, v_k is neither a vertex of T_s nor T'_s, so if $h_k \neq 1$, $\mathcal{V}^s_{\mathbf{h}} = \{0\}$. Suppose that $h_k = 1$ for all $k \in \{1, ..., s-1\}$, then

$$\mathcal{V}^s_{\mathbf{h}} \supset \bigoplus_{h_a h_b = h_s} V^s_{h_a}(P_s) \otimes V'^s_{h_b}(P_s),$$

(where $V^s_{h_a}(P_s)$ and $\mathcal{V}^s_{h_b}$ are defined as in Lemma 1). The direct sum contains N terms of dimension one, hence $\dim \mathcal{V}^s_{\mathbf{h}} \geq N$. But, we have

$$\dim(V^s \otimes V'^s) = N^2 = \sum_{\mathbf{h} \in \mathcal{U}(N)^s} \dim(\mathcal{V}^s_{\mathbf{h}}) \geq N \times N.$$

So $\mathcal{V}^s_{\mathbf{h}}$ is N-dimensional.

□

Now, the proof of Proposition 2 is straightforward: from Remark 1, we have

$$\bigoplus_{\mathbf{h}^0 \mathbf{h}^1 ... \mathbf{h}^s = \mathbf{h}} \mathcal{V}^0_{\mathbf{h}^0} \otimes ... \otimes \mathcal{V}^s_{\mathbf{h}^s} \otimes \widetilde{V} \subset V_{\mathbf{h}}.$$

Writing $\mathbf{h}^j = (h^j_1, ..., h^j_s)$ and $\mathbf{h} = (h_1, ..., h_s)$, one notes that the only non-zero terms in the direct sum are those who satisfy:

$$\begin{cases} h^0_1 h^1_1 = h_1 \\ h^1_2 h^2_2 = h_2 \\ \vdots \\ h^{s-1}_s h^s_s = h_s \end{cases}$$

There exists exactly N^s different choices for $\mathbf{h}^0, ..., \mathbf{h}^s \in \mathcal{U}(N)^s$ satisfying the above relations, and each non-zero vector space of the direct sum has dimension N^{m-2s}. So $\dim V_h \geq N^{m-s}$. Now, we have

$$\dim V = N^m = \sum_{\mathbf{h} \in \mathcal{U}(N)^s} \dim V_{\mathbf{h}} \geq N^s \times N^{m-s},$$

and so $\dim V_{\mathbf{h}} = N^{m-s}$ for each $\mathbf{h} \in \mathcal{U}(N)$. □

In particular, it proves the decomposition of Theorem 1 for ρ. In fact, let $\rho^{(i)} : \mathcal{T}_q(\lambda_0) \longrightarrow \mathrm{End}(V^{(i)})$ be an irreducible representation in the decomposition of ρ. It must satisfies $\rho^{(i)}(X_i)^N = \mathrm{Id}_{V^{(i)}}$ and $\rho^{(i)}(H) = c\mathrm{Id}_{V^{(i)}}$, in other word, $\rho^{(i)}$ must be associated to the same weights $x_i = 1$ and global charge $c \in \mathcal{U}(N)$ than ρ.

Set $h^{(i)}_j \in \mathcal{U}(N)$ the weight of $\rho^{(i)}$ associated to the each puncture v_j, that is, $\rho^{(i)}(P_j) = h^{(i)}_j Id_{V^{(i)}}$. Note that, as $\rho^{(i)}([P_0...P_s]) = \rho^{(i)}([H^2]) = h^{(i)}_0 h^{(i)}_1 ... h^{(i)}_s \mathrm{Id}_{V^{(i)}} = c^2 \mathrm{Id}_{V^{(i)}}$, a necessary condition for $\rho^{(i)}$ to be in the decomposition of ρ is to satisfy $h^{(i)}_0 ... h^{(i)}_s = c^2$. Hence, if $\rho^{(i)}$ is in the decomposition of ρ, knowing $h^{(i)}_j$ for each $j = 1, ..., s$ uniquely determine $h^{(i)}_0$ and so fully determine $\rho^{(i)}$.

Now, as for each $\mathbf{h} = (h_1, ..., h_s) \in \mathcal{U}(N)^s$, $V_{\mathbf{h}}$ has dimension $N^{m-s} = N^{4g-3+(s+1)}$ and as an irreducible representation of $\mathcal{T}_q(\lambda_0)$ has dimension $N^{3g-3+(s+1)}$, then each space V_h contains exactly N^g times the representation $\rho^{(i)}$, classified by $p_0 = c^2 h_1^{-1} ... h_s^{-1},\ p_1 = h_1, ..., p_s = h_s$.

2.2 Proof in the global case

Now, to complete the proof of Theorem 1, one remarks that the decomposition of ρ into irreducible factors only depends on the decomposition of $\rho(P_j)$ into eigenspaces (for each puncture v_j), that is on the possible choices of N-th root of $x_1^{k_{k_1}} ... x_n^{k_{jn}}$ (where P_j is associated to the multi-index $\mathbf{k}_j = (k_{j_1}, ..., k_{j_n})$). But this choice is discrete and depends continuously on the weights x_i associated to the edge λ_i, hence does not depend on the choice of $x_i \in \mathbb{C}^*$. It proves Theorem 1 for the triangulation λ_0 and every weight $x_i \in \mathbb{C}^*$.

Note that the map $\varphi_{\lambda_0 \lambda}$ defined in Subsection 1.5 is rational, hence defined on a Zariski dense open set of $\mathbb{C}^n$. As we extended the decomposition for all weights x_i associated to each edge of the triangulation λ_0, there exists a local representation $\{\rho_\lambda : \mathcal{T}_q(\lambda) \longrightarrow \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(\Sigma)}$ of $\mathcal{T}_q(\Sigma)$ as defined in Subsection 1.5. So, for each $\lambda \in \Lambda(\Sigma)$, $\rho_{\lambda_0} \circ \Psi^q_{\lambda_0 \lambda} : \mathcal{T}_q(\lambda) \longrightarrow \mathrm{End}(V_{\lambda_0})$ makes sense and is isomorphic to $\rho_\lambda : \mathcal{T}_q(\lambda) \longrightarrow \mathrm{End}(V_\lambda)$. That is, there exists a vector space isomorphism $L_{\lambda_0 \lambda} : V_\lambda \longrightarrow V_{\lambda_0}$ such that, for each $X \in \mathcal{T}_q(\lambda)$,

$$\rho_{\lambda_0}(\Psi^q_{\lambda_0 \lambda}(X)) = L_{\lambda_0 \lambda} \circ \rho_\lambda(X) \circ L^{-1}_{\lambda_0 \lambda}.$$

However, ρ_{λ_0} is a local representation of $\mathcal{T}_q(\lambda_0)$, hence there exists an irreducible decomposition of ρ_{λ_0} given by the decomposition $V_{\lambda_0} = \bigoplus_{i \in \mathcal{I}} V^i_{\lambda_0}$ as in Theorem 1. That is, for each $i \in \mathcal{I}$, $V^i_{\lambda_0}$ is stable by ρ_{λ_0} and has dimension $N^{3g-3+s+1}$.

Using the isomorphism $\Psi_{\lambda \lambda_0}$, one gets that for each $X \in \mathcal{T}_q(\lambda)$, $\rho_{\lambda_0}(\Psi_{\lambda_0 \lambda}(X)) V^i_{\lambda_0} = V^i_{\lambda_0}$. Set $V^i_\lambda := L^{-1}_{\lambda_0 \lambda}(V^i_{\lambda_0})$, we have $\dim V^i_\lambda = \dim V^i_{\lambda_0} = 3g - 3 + s + 1$ (because $L_{\lambda_0 \lambda}$ is an isomorphism) so for each $X \in \mathcal{T}_q(\lambda)$, $\rho_\lambda(X) V^i_\lambda = V^i_\lambda$. In other words, we have a decomposition

$$\rho_\lambda = \bigoplus_{i \in \mathcal{I}} \rho^{(i)}_\lambda,$$

where $\rho^{(i)}_\lambda : \mathcal{T}_q(\lambda) \longrightarrow \mathrm{End}(V^i_\lambda)$. As each V^i_λ as the dimension of an irreducible representation, we get an irreducible decomposition of ρ_λ. One easily checks that it satisfies the conditions of Theorem 1. Now we extend this decomposition by continuity for all weight $x_i \in \mathbb{C}^*$ associated to the edge λ_i of λ.

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