A CYCLIC EXTENSION OF THE EARTHQUAKE FLOW II

FRANCESCO BONSANTE, GABRIELE MONDELLO, AND JEAN-MARC SCHLENKER

Abstract. The landslide flow, introduced in [5], is a smoother analog of the earthquake flow on Teichmüller space which shares some of its key properties. We show here that further properties of earthquakes apply to landslides. The landslide flow is the Hamiltonian flow of a convex function. The smooth grafting map sgr taking values in Teichmüller space, which is to landslides as grafting is to earthquakes, is proper and surjective with respect to either of its variables. The smooth grafting map SGr taking values in the space of complex projective structures is symplectic (up to a multiplicative constant). The composition of two landslides has a fixed point on Teichmüller space. As a consequence we obtain new results on constant Gauss curvature surfaces in 3-dimensional hyperbolic or AdS manifolds. We also show that the landslide flow has a satisfactory extension to the boundary of Teichmüller space.

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1. Introduction and results

In this paper we consider a closed surface S of genus at least 2. We denote by $\mathcal{T}$ the Teichmüller space of S, considered either as the space of hyperbolic structures on S (considered up to isotopy) or as the space of conformal structures on S (also up to isotopy). We denote by $\mathcal{ML}$ the space of measured laminations on S.

1.1. **Earthquakes and landslides.**

Let γ be a simple closed curve on S, with a weight $w > 0$, and let h be a hyperbolic metric on S. The image of h by the (left) earthquake along the weighted curve $w\gamma$ is obtained by realizing γ as a closed geodesic in (S, h), cutting S open along this geodesic, rotating the right-hand side by a length w in the positive direction, and gluing back. This defines a map $\mathcal{E}(\bullet, w\gamma) : \mathcal{T} \to \mathcal{T}$. Thurston [40] proved that this definition extends from weighted curves to measured laminations, so that we obtain a map:

$$\mathcal{E} : \mathcal{T} \times \mathcal{ML} \to \mathcal{T} .$$

This earthquake map has a number of remarkable properties, of which we can single out, at this stage, the following.

(1) For fixed $\lambda \in \mathcal{ML}$, it defines a flow on $\mathcal{T}$: for all $t_1, t_2 \in \mathbb{R}$, $\mathcal{E}(h, (t_1 + t_2)\lambda) = \mathcal{E}(\mathcal{E}(h, t_1\lambda), t_2\lambda)$.
(2) Thurston's Earthquake Theorem (see [17, 25]): for any $h, h' \in \mathcal{T}$, there exists a unique $\lambda \in \mathcal{ML}$ such that $\mathcal{E}(h, \lambda) = h'$.
(3) McMullen's complex earthquakes [24]: for fixed $\lambda \in \mathcal{ML}$ and $h \in \mathcal{T}$, the map $t \mapsto \mathcal{E}(h, -t\lambda)$ extends to a holomorphic map from the upper half-plane to $\mathcal{T}$.
(4) $\mathcal{E}(h, (t + is)\lambda) = gr(\bullet, s\lambda) \circ \mathcal{E}(\bullet, -t\lambda)$, where $gr(\bullet, s\lambda) : \mathcal{T} \to \mathcal{T}$ is the grafting map.
(5) The grafting map $gr : \mathcal{T} \times \mathcal{ML} \to \mathcal{T}$ can be written as the composition $gr = \Pi \circ Gr$, where $Gr : \mathcal{T} \times \mathcal{ML} \to \mathcal{CP}$ is also called the grafting map but with values in the space $\mathcal{CP}$ of complex projective structures on S, and $\Pi : \mathcal{CP} \to \mathcal{T}$ is the forgetful map sending a complex projective structure to the underlying complex structure.
(6) Thurston proved that the map $Gr : \mathcal{T} \times \mathcal{ML} \to \mathcal{CP}$ is a homeomorphism (see [15] for a proof).

In [5] we introduced the notion of *landslides*, which can be considered as smooth version of earthquakes. The landslide map $\mathcal{L} : S^1 \times \mathcal{T} \times \mathcal{T} \to \mathcal{T} \times \mathcal{T}$, can be defined in different ways, see below. In [5] we showed that properties (1)-(6) above extend from earthquakes to landslides, with the grafting maps gr and Gr replaced by the corresponding smooth grafting maps sgr' and SGr'.

Here we further consider the properties of the earthquake and grafting maps.

(7) For a fixed measured lamination λ, the earthquake flow $(t, h) \mapsto \mathcal{E}(h, t\lambda)$ is the Hamiltonian flow of the length function of $\frac{1}{2}\lambda$, considered as a function on $\mathcal{T}$, with respect to the Weil-Petersson symplectic structure.
(8) The length of a measured lamination is a convex function on $\mathcal{T}$ with respect to the Weil-Petersson metric.
(9) Given two measured laminations $\lambda, \mu \in \mathcal{ML}$ which fill S, the composition $\mathcal{E}(\bullet, \lambda) \circ \mathcal{E}(\bullet, \mu) : \mathcal{T} \to \mathcal{T}$ has a fixed point (conjectured to be unique).
(10) For fixed λ, the map $gr(\bullet, \lambda) : \mathcal{T} \to \mathcal{T}$ is a homeomorphism (see [31]), while, for $h \in \mathcal{T}$ fixed, the map $gr(h, \bullet) : \mathcal{ML} \to \mathcal{T}$ is a homeomorphism (see [7]).
(11) The cotangent space $T^*\mathcal{T}$ can be identified with the product $\mathcal{T} \times \mathcal{ML}$ through the map $d\ell : \mathcal{T} \times \mathcal{ML} \to T^*\mathcal{T}$ which sends (h, λ) to the differential at h of $d\ell_\lambda$ — in particular this map is one-to-one.

(12) The grafting map $Gr : \mathcal{T} \times \mathcal{ML} \to \mathcal{CP}$ can be composed with $(d\ell)^{-1} : T^*\mathcal{T} \to \mathcal{T} \times \mathcal{ML}$ to obtain a map from $T^*\mathcal{T}$ to $\mathcal{CP}$. This map is actually C^1 (although $\mathcal{ML}$ does not have a natural C^1-structure) and symplectic, when one considers on $\mathcal{CP}$ the real symplectic structure equal to the real part of the Goldman symplectic structure on the space of representations of $\pi_1(S)$ to $\mathrm{PSL}(2, \mathbb{C})$.

We will prove that those properties extend to landslides, except for point (10) for which we only prove here that the corresponding maps in the landslide setting are onto. (We also believe that those maps are one-to-one, but could not prove it.)

We will see that points (9) and (10) can be translated in terms of 3-dimensional hyperbolic or anti-de Sitter geometry.

In addition we will show (see Section 1.7 for a more precise statement) that

(13) the landslide map has a satisfactory extension to the space $\mathcal{FML}$ of filling pairs of measured laminations on S, considered as a boundary of $\mathcal{T} \times \mathcal{T}$, and this extension is Hamiltonian for the symplectic structure equal to the sum of the Thurston symplectic forms on the two factors.

1.2. **The landslide flow is Hamiltonian.** We will first define a function F on $\mathcal{T} \times \mathcal{T}$ that plays for landslides the role that the length of a measured lamination plays for earthquakes. Recall that given two hyperbolic metrics h and $h^\star$ on S, there is a unique minimal Lagrangian map m isotopic to the identity from (S, h) to $(S, h^\star)$ (see [23, 34]). This map can be characterized by the existence of a bundle morphism $b : TS \to TS$ which has determinant 1, is self-adjoint for h and satisfies the Codazzi equation $d^\nabla b = 0$, and such that $m^*h^\star = h(b\bullet, b\bullet)$. We call b the *Labourie operator* of the pair $(h, h^\star)$ and c the *center* of $(h, h^\star)$, namely the conformal structure (up to isotopy) underlying the metric $h + m^*h^\star$.

Definition 1.1. Let $F : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ the function defined as

$$F(h, h^\star) = \int_S \mathrm{tr}(b) da_h \ ,$$

where b is the Labourie operator of the pair $(h, h^\star)$ and da_h is the area element associated to h.

Note that $F(h, h^\star) = F(h^\star, h)$: if b is the Labourie operator of the pair $(h, h^\star)$, then the Labourie operator of the pair $(h^\star, h)$ is $b^\star = b^{-1}$, and $\mathrm{tr}(b^\star) = \mathrm{tr}(b)$ since b has determinant 1.

Proposition 1.2. *Let c be the center of $(h, h^\star)$. The functions $E(\bullet, h), F(\bullet, h) : \mathcal{T} \to \mathbb{R}$ are proper and real-analytic, where $E(c, h)$ is the energy of the unique harmonic map from (S, c) to (S, h). Moreover, $2F(h, h^\star) = E(c, h) = E(c, h^\star)$.*

Theorem 1.3. *The landslide flow on $\mathcal{T} \times \mathcal{T}$ is the Hamiltonian flow associated to $\frac{1}{4}F$ for the symplectic form $\omega_{WP,1} + \omega_{WP,2}$.*

As a consequence we see that the landslide flow is certainly *not* the same as the Hamiltonian flow of the length of the Liouville cycle.

1.3. **Convexity of the Hamiltonian.** The following result is an extension to landslides of the convexity of the length function of measured laminations.

Theorem 1.4. *Let $h \in \mathcal{T}$ be fixed. The function $F(h, \bullet) : \mathcal{T} \to \mathbb{R}$ is strictly convex for the Weil-Petersson metric on $\mathcal{T}$. More precisely, at each point $h^\star$, $\mathrm{Hess}(F(h, \bullet)) \geq 2g_{WP}$, with equality exactly when $h = h^\star$.*

1.4. **Landslide symmetries.** There is a simple notion of "symmetry" associated to the notion of landslides — it actually also makes some sense for earthquakes, see below.

Definition 1.5. Let $\theta \in (0, \pi)$, and let $h_0 \in \mathcal{T}$. For all $h \in \mathcal{T}$, there is a unique $h_0^\star \in \mathcal{T}$ such that $\mathcal{L}^1_{e^{i\theta}}(h_0, h_0^\star) = h$ (see [5, Theorem 1.14]). We set $\mathcal{S}_{e^{i\theta}, h_0}(h) = \mathcal{L}^1_{e^{-i\theta}}(h_0, h_0^\star)$. We call $\mathcal{S}$ the symmetry of center h_0 and angle θ.

Note that $\mathcal{S}_{e^{i\theta}, h_0}$ is not an involution, however, by definition, $\mathcal{S}_{e^{i\theta}, h_0} \circ \mathcal{S}_{e^{-i\theta}, h_0} = Id$.

The following statement is an analog for landslides of the main statement in [6].

Theorem 1.6. *Let $\theta_+, \theta_- \in (0, \pi)$ and $h_+, h_- \in \mathcal{T}$ be fixed. The map $\mathcal{S}_{e^{i\theta_+}, h_+} \circ \mathcal{S}_{e^{i\theta_-}, h_-} : \mathcal{T} \to \mathcal{T}$ has a fixed point. If $\theta_+ + \theta_- = \pi$ then this fixed point is unique.*

It would be quite satisfactory to know whether uniqueness holds for other values of $\theta_+ + \theta_-$.

This statement can also be translated in terms of 3-dimensional AdS geometry, see below. The uniqueness question which remains open can then be translated as a natural statement on the uniqueness of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ manifolds with smooth, space-like boundary having a given pair of constant curvature metrics as the induced metric on the boundary, and the analogy with the corresponding hyperbolic situation suggests that it might be true.

In the limit case of earthquakes, one can define a similar notion of symmetry. Given a fixed $h_0 \in \mathcal{T}$, we can define the (left) earthquake symmetry $\mathcal{S}_{h_0}$ as follows. For any $h \in \mathcal{T}$, there is by Thurston's Earthquake Theorem (see [17, 25]) a unique $\lambda \in \mathcal{ML}$ such that $\mathcal{E}(h_0, \lambda) = h$, and we define $\mathcal{S}_{h_0}(h) = \mathcal{E}(h_0, -\lambda)$. One can then ask whether, for $h_+, h_- \in \mathcal{T}$, the composition $\mathcal{S}_{h_+} \circ \mathcal{S}_{h_-}$ has a unique fixed point. A positive answer would be equivalent to a proof of a conjecture of Mess [25] on the existence and uniqueness of a MGH AdS manifold for which the induced metric on the boundary of the convex core is a prescribed pair of hyperbolic metrics. We leave details on this to the reader.

1.5. **Smooth grafting.** In Section 6 we turn to the smooth grafting map, defined in [5] and recalled in Section 2.10, which is a smoother analog of the grafting map. We have the following partial extension/analog of a result of Scannell and Wolf [31] and of a result of Dumas and Wolf [7].

Theorem 1.7. *Let $s > 0$ and $h, h^\star \in \mathcal{T}$. The maps $sgr'_s(h, \bullet) : \mathcal{T} \to \mathcal{T}$ and $sgr'_s(\bullet, h^\star) : \mathcal{T} \to \mathcal{T}$ are proper surjective maps.*

This result can be stated in terms of 3d hyperbolic or de Sitter geometry. Recall that any hyperbolic end has a unique foliation by constant curvature surfaces [23]. The curvature of those surfaces varies between -1 and 0.

Theorem 1.8. *Let $h, h' \in \mathcal{T}$ and let $K \in (-1, 0)$. There is a hyperbolic end M with conformal metric at infinity h' and such that the surface $S^\star$ in M with constant curvature K has an induced metric proportional to h.*

It would be satisfactory to know whether M is unique.

Similarly, any 3-dimensional de Sitter domain of dependence (as defined in [25]) has a unique foliation by constant curvature surfaces, which are actually dual to the constant curvature surfaces in the foliation of the dual hyperbolic end (see [3]).

Theorem 1.9. *Let $h^\star, h' \in \mathcal{T}$ and let $K^\star \in (-\infty, 0)$. There is a de Sitter domain of dependence $M^\star$ with conformal metric at infinity h' and such that the surface S in $M^\star$ with constant curvature $K^\star$ has an induced metric homothetic to $h^\star$.*

The proof of those 3-dimensional theorems, from Theorem 1.7, are in Section 6.4

1.6. **The smooth grafting map is symplectic.** Consider a measured lamination $\lambda \in \mathcal{ML}$. Its length ℓ_λ is a smooth function on $\mathcal{T}$ and, for each $h \in \mathcal{T}$, we can consider its differential $d_h\ell_\lambda \in T^*_h\mathcal{T}$. It is well-known that this defines a one-to-one map between $\mathcal{T} \times \mathcal{ML}$ and $T^*\mathcal{T}$. This has the following counterpart in the context considered here, with $\mathcal{ML}$ replaced by another copy of $\mathcal{T}$ and the length function replaced by the function F defined above.

Proposition 1.10. *For all $h^\star \in \mathcal{T}$, let*

$$\begin{array}{rccl} F_{h^\star}: & \mathcal{T} & \to & \mathbb{R} \\ & h & \mapsto & F(h, h^\star) \ . \end{array}$$

The map $d_1F : \mathcal{T} \times \mathcal{T} \to T^*\mathcal{T}$ sending $(h, h^\star)$ to the differential at h of the function $F_{h^\star}$ is a global diffeomorphism between $\mathcal{T} \times \mathcal{T}$ and $T^*\mathcal{T}$.

Coming back to the familiar setting of measured lamination, we can consider the grafting map $Gr : \mathcal{T} \times \mathcal{ML} \to \mathcal{CP}$, which is known from Thurston's work to be a homeomorphism (see [15]). Composing with the inverse of the map $d\ell : \mathcal{T} \times \mathcal{ML} \to T^*\mathcal{T}$, we obtain a map $Gr \circ (d\ell)^{-1} : T^*\mathcal{T} \to \mathcal{CP}$.

Given a complex projective structure Ξ on a surface, one can consider the underlying complex structure, say c. Riemann uniformization produces another $\mathbb{CP}^1$-structure Ξ_F on S which is Fuchsian with underlying complex structure c. We can then consider the Schwarzian derivative of the identity map from (S, Ξ_F) to (S, Ξ), it is a holomorphic quadratic differential φ for c. Its real part can therefore be considered as a vector in $T^*_c\mathcal{T}$. This classical construction defines a map $Sch : \mathcal{CP} \to T^*\mathcal{T}$. It was proved by Kawai [16] that this map is symplectic (up to a factor), that is, the pull-back by Sch of the (real) cotangent symplectic structure on $T^*\mathcal{T}$ is a multiple of the real part of the Goldman symplectic form ω_G on $\mathcal{CP}$.

We can now consider the map $Sch \circ Gr \circ (d\ell)^{-1} : T^*\mathcal{T} \to T^*\mathcal{T}$. It is proved in [21] that this map is C^1, and that it is symplectic (up to a fixed factor) with respect to the cotangent symplectic form on $T^*\mathcal{T}$. Here we denote by ω_{can} the (real) cotangent symplectic form on $T^*\mathcal{T}$.

Now recall from [5] the definition of the smooth grafting map $SGr' : \mathbb{R} \times \mathcal{T} \times \mathcal{T} \to \mathcal{CP}$. For $s \geq 0$ and $h, h^\star \in \mathcal{T}$, there is a unique equivariant convex immersion of the universal cover $\tilde{S}$ of S into $\mathbb{H}^3$ with induced metric $\cosh^2(s/2)h$ and third fundamental form $\sinh^2(s/2)h^\star$. This equivariant immersion defines on S a complex projective structure, obtained by pulling back on the image surface the complex projective structure at infinity by the "Gauss map" sending a point x of the image to the endpoint at infinity of the geodesic ray from x orthogonal to the surface. This complex projective structure is the image $SGr'_s(h, h^\star)$.

Theorem 1.11. *For all $s \in \mathbb{R}$, the composition map $Sch \circ SGr'_s \circ (dF_1)^{-1} : T^*\mathcal{T} \to T^*\mathcal{T}$ is symplectic up to a constant factor depending on s:*

$$(Sch \circ SGr'_s \circ (dF_1)^{-1})^* \omega_{can} = \sinh(s)\omega_{can} \ .$$

The proof uses a variant of the notion of renormalized volume, generalizing the definition introduced for a similar purpose in [21].

1.7. **Extension at the boundary.** In order to understand how the landslide flow could be extended to the boundary, let's recall the following result by Wolf.

Proposition 1.12 ([42])**.** *Let the center c stay fixed and suppose that $\theta_n h_n \to \lambda$. Then $\theta_n h^\star_n \to \mu$, with the property that the unique holomorphic quadratic differential φ with horizontal and vertical laminations λ and μ has c as underlying complex structure.*

If we consider the space $\mathcal{DY} = \mathcal{T} \times \mathcal{T} \times \mathbb{R}_{<0}$ of couples of metrics $(h, h^\star)$ with the same negative constant curvature K (up to isotopy), then this space can be bordified as $\overline{\mathcal{DY}}$ by adding the space $\mathcal{FML} \subset \mathcal{ML} \times \mathcal{ML}$ of filling couples of measured laminations correspondingly to $K = -\infty$.

Proposition 1.13. *Identifying $\mathcal{FML}$ with the space $\mathcal{Q}$ of holomorphic quadratic differentials on S, the action on $\mathcal{DY}$ limits on $\mathcal{FML} = \mathcal{Q}$ to the action $(\theta, \varphi) \mapsto e^{i\theta}\varphi$.*

Notice that the function F extends to $\mathcal{DY}$ as $F(h, h^\star, K) := K^{-2}F(h, h^\star)$. As the Weil-Petersson symplectic form ω_{WP} limits to Thurston's symplectic form ω_{Th} on $\mathcal{ML}$, Theorem 1.3 admits the following extension.

Proposition 1.14. *The function $F : \mathcal{DY} \to \mathbb{R}_+$ extends to $\partial\mathcal{DY} = \mathcal{FML}$ as $F(\lambda, \mu) = i(\lambda, \mu)$. Moreover, the extension of the landslide flow on $\partial\mathcal{DY} = \mathcal{FML}$ is Hamiltonian for the symplectic form $\omega_{Th,1} + \omega_{Th,2}$ with respect to $\frac{1}{4}F$.*

1.8. **Constant curvature surfaces in globally hyperbolic $\mathbb{AdS}^3$ manifolds.** Recall that given a globally hyperbolic $\mathbb{AdS}^3$ manifold N, the complement of its convex core has a unique foliation by constant curvature surfaces [3]. The curvature of those surfaces varies between $-\infty$ and -1.

Theorem 1.15. *Let $h_+, h_- \in \mathcal{T}$, and let $K_+, K_- \in (-\infty, -1)$. There exists a globally hyperbolic $\mathbb{AdS}^3$ manifold N such that the constant curvature K_- surface in the past of the convex core has induced metric homothetic to h_-, while the constant curvature K_+ metric in the future of the convex core of constant curvature K_+ has induced metric homothetic to h_+. If $K_+ = -K_-/(K_- + 1)$, then N is unique.*

It is tempting to conjecture that N is unique for any value of K_+ and K_-. Actually the analogy with quasifuchsian hyperbolic 3-manifolds indicates that Theorem 1.15 could well extend to metrics of non-constant curvature, as follows (see [2, §3.4] for more on this).

Conjecture 1.16. Let h_+, h_- be two smooth metrics on S with curvature $K < -1$. There exists a unique globally hyperbolic $\mathbb{AdS}^3$ manifold homeomorphic to $S \times [-1, 1]$, with smooth, space-like and strictly convex boundary, such that the induced metric on $S \times \{-1\}$ is h_- and the induced metric on $S \times \{1\}$ is h_+.

Theorem 1.15 shows that the existence part of this statement holds when both h_- and h_+ have constant curvature. The analog of Conjecture 1.16 for quasifuchsian manifolds (and more generally convex co-compact manifolds) holds, see [33].

As the limit case of either Conjecture 1.16 or Theorem 1.15 when the curvature of the metrics h_- and h_+ goes to -1, we obtain the following conjecture of Mess.

Conjecture 1.17. Let h_-, h_+ be two hyperbolic metrics on S. There is a unique maximal globally hyperbolic $\mathbb{AdS}^3$ manifold N such that induced metric on the boundary of the convex core of N is given by h_- and h_+.

There is a useful notion of duality in $\mathbb{AdS}^3$, recalled in Section 2. It can be used to translate Theorem 1.15 in terms of the third fundamental form, rather than the induced metric, on surfaces in $\mathbb{AdS}^3$ manifolds.

Theorem 1.18. *Let $h_+, h_- \in \mathcal{T}$, and let $K_+, K_- \in (-\infty, -1)$. There exists a maximal globally hyperbolic $\mathbb{AdS}^3$ manifold N containing a surface S_- with third fundamental form of constant curvature K_- homothetic to h_- in the past of the convex core, and a surface S_+ with constant curvature K_+ and third fundamental form homothetic to h_+ in the future of the convex core. If $K_+ = -K_-/(K_- + 1)$, then N is unique.*

As for Theorem 1.15, the analogy with quasifuchsian manifolds indicates that the statement might hold also for metrics of variable curvature — this would actually follow from Conjecture 1.16 using the same notion of duality.

1.9. **Content of the paper.** Section 2 contains notations and background material, including previous definitions and results on the landslide flow considered here, on smooth grafting, and on their relationships with 3d hyperbolic and $\mathbb{A}\mathrm{d}\mathbb{S}$ geometry.

In Section 3 and 4 we will describe the Hamiltonian interpretation of the landslide flow, Theorem 1.3, and relate its Hamiltonian to the energy of underlying harmonic maps. In Section 5 — probably the most technically involved part of the paper — we show that this Hamiltonian function is convex for the Weil-Petersson metric on $\mathcal{T}$, Theorem 1.4.

In Section 6 we turn to the smooth grafting, and prove Theorem 1.7 and then, as an application, Theorem 1.8. The symplectic properties of the smooth grafting map are investigated in Section 7. In Section 8 we consider the extension of the landslide flow to the boundary. Finally in Section 9 we prove Theorem 1.15 on 3-dimensional $\mathbb{A}\mathrm{d}\mathbb{S}$ geometry and Theorem 1.6 on fixed points of compositions of landslides.

2. Notations and background material

This section collects a number of definitions and results which are used below. It is included here so as to make the paper as self-contained as possible.

2.1. **The space of complex structures and Weil-Petersson product.** We fix an oriented closed surface S of genus $g(S) \geq 2$. The Teichmüller space of S is the quotient of the space of complex structures on S by the action of $\mathrm{Diffeo}_0(S)$.By the uniformization theorem, it can be also regarded as the space of hyperbolic metrics on S up to isotopy.

We will denote by $\mathcal{A}$ the space of almost-complex structures on S: in other words, an element of $\mathcal{A}$ is an operator J on TS such that

- $J^2 = -\mathbb{1}$.
- For every $0 \neq X \in T_pS$, the basis (X, JX) is positively oriented.

Since in dimension 2 every almost-complex structure is integrable, we have a natural map

$$\mathcal{A} \to \mathcal{T},$$

which is a $\mathrm{Diffeo}_0(S)$-principal bundle, where $\mathrm{Diffeo}_0(S)$ acts on $\mathcal{A}$ by pull-back ([10]).

Let $\mathcal{M}_{-1}$ the space of hyperbolic metrics on S. By the Uniformization Theorem, the map $\mathcal{M}_{-1} \to \mathcal{A}$ sending h to the complex structure compatible with h is a $\mathrm{Diffeo}(S)$-equivariant identification. In particular the elements of $\mathcal{T}$ can be also regarded as hyperbolic metrics up to isotopies.

Let h_0 be a hyperbolic metric on S and denote by J_0 the corresponding almost-complex structure.

The tangent space $T_{J_0}\mathcal{A}$ is the set of operators $\dot{J}$ on TS such that

$$\dot{J}J_0 + J_0\dot{J} = 0 \ .$$

Notice that there are two simple characterizations of $\dot{J}$:

- it is a $\mathbb{C}$-anti-linear operator,
- it is traceless and h_0-self-adjoint.

The tangent space of $\mathcal{T}$ at $[J_0]$ turns out to be identified to the quotient of $T_{J_0}\mathcal{A}$ by the vertical subspace. We want to relate this description of $T_{[J_0]}\mathcal{T}$ with the classical description in terms of Beltrami differentials. We will give a description of quadratic differentials and Beltrami differentials as tensors on the surface.

Let us fix a complex atlas $\{(U_j, z_j)\}$ on S compatible with J_0. Namely, putting $z_j = x_j + iy_j$ it results that

$$J_0\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}\ , \qquad J_0\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}\ .$$

2.1.1. *Holomorphic quadratic differentials.* A holomorphic quadratic differential φ on S is a holomorphic section of the square of the canonical bundle $(\Omega^{1,0}S)^{\otimes 2}$: in local coordinates $\varphi|_{U_k} = \varphi_j(z_j)dz_j^2$, where on the intersection $U_j \cap U_k$ we have

$$\varphi_j(z_j) = \varphi_k(z_k)\left(\frac{dz_k}{dz_j}\right)^2(z_k)\ .$$

Quadratic differentials can be regarded as complex bilinear tensors: indeed given $p \in U_j$ and $X, X' \in T_pS$ we can decompose

$$X = u\frac{\partial}{\partial x_j} + v\frac{\partial}{\partial y_j}\,, \qquad X' = u'\frac{\partial}{\partial x_j} + v'\frac{\partial}{\partial y_j}\,.$$

Then, setting

$$\varphi(X, X') = \varphi_j(z_j(p))(u + iv)(u' + iv')$$

one can directly check that this definition does not depend on the complex chart z_j and defines a complex bilinear form at T_pS.

Holomorphic quadratic differentials form a complex vector space — denoted here by $\mathcal{Q}(J_0)$ — of complex dimension $3g(S) - 3$.

There is a holomorphic vector bundle $\pi : \mathcal{Q} \to \mathcal{T}$ whose fiber on the point $[J]$ is canonically identified with $\mathcal{Q}(J)$. This is called the fiber bundle of holomorphic quadratic differentials.

2.1.2. *Beltrami differentials.* A Beltrami differential ν is a section of the differentiable linear bundle $\Omega^{-1,1}S$: locally it can be written as

$$\nu|_{U_j} = \nu_j(z_j)\frac{d\bar{z}_j}{dz_j}\,,$$

and on $U_j \cap U_k$ we have

$$\nu_j(z_j) = \nu_k(z_k)\frac{\overline{\left(\frac{dz_k}{dz_j}\right)}}{\left(\frac{dz_k}{dz_j}\right)}(z_j)\,.$$

Beltrami differentials can be regarded as anti-linear operators of TS. Indeed, given $p \in U_j$ and $X \in T_pS$ with $X = u\frac{\partial}{\partial x_j} + v\frac{\partial}{\partial y_j}$ we can put $\nu(X) = t\frac{\partial}{\partial x_j} + s\frac{\partial}{\partial y_j}$ where $t + is = \nu_j(z_j(p))\overline{(u + iv)}$.

The matrix representative of ν with respect to the real basis $\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\}$ is

$$[\nu]_j = \begin{pmatrix} \Re\nu_j & \Im\nu_j \\ \Im\nu_j & -\Re\nu_j \end{pmatrix}\,,$$

whereas $|\nu|^2 = \frac{1}{2}\mathrm{tr}(\nu^2)$. Finally notice that the multiplication by i of Beltrami differentials corresponds to the composition with the complex structure on TS:

$$(i\nu)(X) = J_0\nu(X)\,.$$

There is a classical open embedding of $\mathcal{A}$ into $Belt(J_0)$, that can be described in the following way. Given a complex structure J, the identity map

$$\mathbb{1} : (T_pS, J_0) \to (T_pS, J)$$

decomposes into a $\mathbb{C}$-linear part $\partial\mathbb{1}$ and an anti-linear part $\bar{\partial}\mathbb{1}$: namely

$$\partial\mathbb{1} = \frac{\mathbb{1} - JJ_0}{2} \qquad \bar{\partial}\mathbb{1} = \frac{\mathbb{1} + JJ_0}{2}\,.$$

In particular, the operator

$$\nu_J = (\partial\mathbb{1})^{-1} \circ (\bar{\partial}\mathbb{1})$$

is an anti-linear operator of (S, J_0) and is thus a Beltrami differential. Locally around p, if z is a local complex coordinate for (S, J_0) and w is a local complex coordinate for (S, J), we have

$$\nu_J = \frac{\frac{\partial w}{\partial \bar{z}}}{\frac{\partial w}{\partial z}}\frac{d\bar{z}}{dz}\,.$$

It is a classical fact that the map

$$\mathcal{A} \ni J \mapsto \nu_J \in Belt(J_0)$$

is an open embedding whose image is the space of Beltrami differential with L^∞-norm less than 1 ([]).

In this way any Beltrami differential corresponds to an infinitesimal deformation of the complex structure of J_0. We say that a Beltrami differential is trivial if it corresponds to a trivial deformation of the complex structure. Trivial Beltrami differentials form a vector space which we denote by $Belt_{tr}(J_0)$.

Classically the tangent space of Teichmüller space is identified to the quotient

$$T_{[J_0]}\mathcal{T} = Belt(J_0)/Belt_{tr}(J_0)\,.$$

Notice that $[\dot{\nu}]$ is a tangent vector to a curve $[J_t]$ of complex structures (starting from J_0) if $\nu_{J_t} = t\dot{\nu} + o(t)$.

In particular, differentiating the relation $\nu_{J_t} = (\mathbb{1} - J_t J_0)^{-1}(\mathbb{1} + J_t J_0)$, we get

$$\dot{\nu} = \frac{1}{2}\dot{J}J_0 = -\frac{1}{2}J_0\dot{J} \tag{1}$$

It turns out that the differential of the natural projection $\pi : \mathcal{A} \to \mathcal{T}$ at J_0 is simply the map

$$d\pi : T_{J_0}\mathcal{A} \ni \dot{J} \mapsto [-\frac{1}{2}J_0\dot{J}] \in Belt(J_0)/Belt_{tr}(J_0)\ .$$

2.1.3. *Pairing between quadratic differentials and Beltrami differentials.* Given a holomorphic quadratic differential φ and a Beltrami differential ν we can consider the complex-bilinear form $\varphi \bullet \nu$ which is defined on U_j as

$$\varphi \bullet \nu|_{U_j} = \varphi_j(z_j)\nu_j(z_j)dx_j \wedge dy_j$$

The form $\varphi \bullet \nu$ can be described explicitly has an alternating 2-form on T_pS in the following way. Given $X, X' \in T_pS$ we have

$$(\varphi \bullet \nu)(X, X') = \frac{\varphi(\nu(X), X') - \varphi(\nu(X'), X)}{2i}$$

It is a classical fact that a Beltrami differential ν is trivial iff

$$\int_S \varphi \bullet \nu = 0$$

for all holomorphic quadratic differentials φ.

As a consequence, the pairing $Belt(J_0) \times \mathcal{Q}(J_0) \to \mathbb{C}$ induces on the quotient a non-degenerate pairing

$$T_{[J_0]}\mathcal{T} \times \mathcal{Q}(J_0) \to \mathbb{C}$$

which allows to identify $\mathcal{Q}(J_0)$ with the cotangent space of $\mathcal{T}$ at $[J_0]$.

2.1.4. *Harmonic Beltrami differentials and Weil-Petersson metric.* Given a holomorphic Beltrami differential φ, its real part is a symmetric 2-form on S, so there exists an h_0-self-adjoint operator ν_φ such that

$$\Re(\varphi)(X, X') = h_0(\nu_\varphi(X), X')\ .$$

Since $\varphi(J_0X, J_0X') = -\varphi(X, X')$ we deduce that $J_0\nu_\varphi J_0 = \nu_\varphi$, that is ν_φ is $\mathbb{C}$-anti-linear. This means that ν_φ is a Beltrami operator. It is called the harmonic Beltrami operator associated to φ.

If $\varphi|_{U_j} = \varphi_j(z_j)dz_j^2$ and $h|_{U_j} = h_j(z_j)|dz_j|^2$ we easily see that

$$(\nu_\varphi)|_{U_j} = \frac{\overline{\varphi}_j(z_j)}{h_j(z_j)}\frac{d\overline{z}_j}{dz_j}\ . \tag{2}$$

Since $\Im(\varphi(X, X')) = -\Re(\varphi(J_0X, X'))$ we get

$$\varphi(X, X') = h_0(\nu_\varphi(X), X') + ih_0(J_0\nu_\varphi(X), X')\ .$$

In particular the map

$$\tilde{wp} : \mathcal{Q}(J_0) \ni \varphi \mapsto \nu_\varphi \in Belt(J_0)$$

is $\mathbb{C}$-anti-linear and injective. Its image can be characterized as the set of self-adjoint traceless operators which satisfy the Codazzi equation $d^\nabla \nu = 0$, where ∇ is the Levi-Civita connection of the hyperbolic metric h_0 ([19]). Let us recall that $d^\nabla \nu$ is in general a 2-form with values in the tangent bundle TS defined by

$$d^\nabla \nu(u, v) = \nabla_u(\nu v) - \nabla_v(\nu u) - \nu([u, v])\ .$$

Regarding ν as a 1-form on S with values in TS, d^∇ coincides with the exterior differential with respect to ∇ on the tangent bundle.

The map $\tilde{wp}$ induces a sequilinear-form

$$\langle \varphi, \varphi' \rangle_{WP} = \int_S \varphi \bullet \nu_{\varphi'}\ .$$

By a local check using (2), one sees that $\langle \bullet, \bullet \rangle_{WP}$ is a positive hermitian form. It is called the Weil-Petersson metric on $\mathcal{Q}(J_0)$.

We will denote by g_{WP} the real part of $\langle \bullet, \bullet \rangle_{WP}$ — which is the Weil Petersson product — whereas ω_{WP} denotes the imaginary part, which is the Weil-Petersson symplectic form.

We deduce:

- The harmonic Beltrami differential ν_φ is trivial iff $\varphi = 0$. Indeed $\int_S \varphi \bullet \nu_\varphi = \|\varphi\|^2_{WP}$.
- The induced map $wp : \mathcal{Q}(J_0) \to T_{[J_0]}\mathcal{T}$ is an anti-linear isomorphism. This means that every element in $T_{[J_0]}\mathcal{T}$ admits a unique harmonic representative.

- Identifying $T_{[J_0]}\mathcal{T}$ with $\mathcal{Q}(J_0)^*$, the map $\varphi \mapsto \langle \bullet, \varphi\rangle_{WP}$ induced by the Weil-Petersson metric coincides with wp.
- In particular we get that the Weil Petersson metric on the tangent space of $\mathcal{T}$ is simply:

$$\langle [\nu_\varphi], [\nu_{\varphi'}]\rangle_{WP} = \langle \varphi, \varphi'\rangle_{WP} \ .$$

A local computation shows that in general

$$\Re(\varphi \bullet \nu) = \frac{1}{2}\int_S \mathrm{tr}(\nu_\varphi \nu) da_{h_0} \ ,$$

where da_{h_0} is the area form of h_0. In particular we deduce that

$$\langle [\nu_\varphi], [\nu_{\varphi'}]\rangle_{WP} = \langle \varphi, \varphi'\rangle_{WP} = \frac{1}{2}\left(\int_S (\mathrm{tr}(\nu_\varphi, \nu_{\varphi'}) + i\mathrm{tr}(J_0\nu_\varphi\nu_{\varphi'}))da_{h_0}\right) \ .$$

2.1.5. *Fischer-Tromba product.* Given a holomorphic quadratic differential φ, we denote by $\dot{J}_\varphi = 2J_0\nu_\varphi$, the infinitesimal deformation of J_0 corresponding to the Beltrami differential ν_φ.

We consider on $T_{J_0}\mathcal{A}$ the hermitian product

$$(\dot{J}, \dot{J}')_{J_0} = \int_S \mathrm{tr}(\dot{J}\dot{J}')da_{h_0}$$

where da_{h_0} is the area form of h_0.

Lemma 2.1. [10] *The image of the map $\mathcal{Q}(J_0) \ni \varphi \mapsto \dot{J}_\varphi \in T_{J_0}\mathcal{A}$ is the orthogonal complement of the vertical space.*

Proof. Notice that $\dot{J}$ is vertical iff $\nu_{\dot{J}} = -\frac{1}{2}J_0\dot{J}$ is trivial, this being the case exactly iff for all $\varphi \in \mathcal{Q}(J_0)$ we have

$$0 = \Re\left(\int_S \varphi \bullet \nu_{\dot{J}}\right) = \frac{1}{2}\int_S \mathrm{tr}(\nu_\varphi \nu_{\dot{J}}) = \frac{1}{8}\int_S \mathrm{tr}(\dot{J}_\varphi, \dot{J})da_{h_0}.$$

□

In particular given two vectors $\dot{J}$ and $\dot{J}'$ in $T_{J_0}\mathcal{A}$ and denoting by $\dot{J}_H$ and by $\dot{J}'_H$ their projection on the orthogonal complement of the vertical space, there exist two quadratic differentials φ and φ' such that $\dot{J}_H = \dot{J}_\varphi$ and $\dot{J}'_H = \dot{J}_{\varphi'}$. Since $d\pi(\dot{J}) = d\pi(\dot{J}_H) = [\nu_\varphi]$ and $d\pi(\dot{J}') = d\pi(\dot{J}'_H) = [\nu_{\varphi'}]$, we easily deduce that

$$g_{WP}(d\pi(\dot{J}), d\pi(\dot{J}')) = \frac{1}{2}\int_S \mathrm{tr}(\nu_\varphi\nu_{\varphi'})da_{h_0} = \frac{1}{8}\int_S \mathrm{tr}(\dot{J}_H\dot{J}'_H)da_{h_0} = \frac{1}{8}(\dot{J}_H, \dot{J}'_H)_{J_0} \ . \tag{3}$$

Analogously,

$$\omega_{WP}(d\pi(\dot{J}), d\pi(\dot{J}')) = \frac{1}{8}(J_0\dot{J}_H, \dot{J}'_H)_{J_0} \ . \tag{4}$$

2.2. **Harmonic maps vs Minimal Lagrangian map.** We collect here a number of basic facts on harmonic maps and minimal Lagrangian maps between hyperbolic surfaces, and the relation between those two notions.

2.2.1. *Harmonic maps.* Let h_0 and h be two metrics on S. For every C^1 map $f : (S, h_0) \to (S, h)$, we have the following decomposition of $f^*(h)$

$$f^*(h) = \varphi + eh_0 + \bar{\varphi}$$

where φ is a J_0-complex bilinear form on S, called the Hopf differential of f, and e is a positive function on S called the energy density of the map f. The total energy of the map f is defined as

$$E(f) = \int_S e\, da_{h_0} \ ,$$

where da_{h_0} is the area form associated with the metric h_0.

We say that the map f is harmonic if f is a stationary point of the functional E. If f is a diffeomorphism this is equivalent to requiring that φ is a *holomorphic* quadratic differential on (S, J_0) []. Notice that φ and $E(f)$ do not change by changing h_0 in its conformal class (but e does), so the harmonicity of the map f only depends on the complex structure on the source surface.

Let us fix a complex structure J_0 (or equivalently a hyperbolic metric h_0). Given a holomorphic quadratic differential φ, Wolf [42] proved there exists a unique hyperbolic metric h_φ on S such that the identity $id : (S, J_0) \to (S, h_\varphi)$ is an harmonic map with Hopf differential equal to φ. In other words, there is a unique hyperbolic metric on S of the form

$$h_\varphi = \varphi + eh_0 + \bar{\varphi} \ . \tag{5}$$

This allows to construct a map

$$\begin{array}{rccc} \mathcal{W}_{J_0}: & \mathcal{Q}(J_0) & \to & \mathcal{T} \\ & \varphi & \mapsto & [h_\varphi] \end{array}$$

which has been proved to be a homeomorphism [42]. This is called the Wolf parameterization centered at $[J_0]$.

The differential of $\mathcal{W}_{J_0}$ at 0 can be easily computed:

$$d_0(\mathcal{W}_{J_0})(\varphi) = [\nu_\varphi] \ .$$

2.2.2. *Minimal Lagrangian maps.* Given two hyperbolic metrics $h, h^\star$ on S, a map $m : (S,h) \to (S,h^\star)$ is minimal Lagrangian if it is area-preserving and its graph is a minimal surface in $(S \times S, h \oplus h^\star)$.

A map $m : (S,h) \to (S,h^\star)$ is a minimal Lagrangian map iff there exists an operator $b : TS \to TS$ such that

- b is positive and self-adjoint;
- $\det b = 1$;
- b is solution of Codazzi equation $d^\nabla b = 0$ for the Levi-Civita connection ∇ of h;
- $m^*(h^\star) = h(b\bullet, b\bullet)$

Labourie [23] and Schoen [34] proved that there exists a unique such minimal Lagrangian map $m : (S,h) \to (S,h^\star)$ isotopic to the identity. In particular the Labourie operator of the pair $(h,h^\star)$ is the operator b as above, such that the metric $h(b\bullet, b\bullet)$ is isotopic to $h^\star$. We will say that the pair is normalized if $h^\star = h(b\bullet, b\bullet)$, or equivalently if the identity $id : (S,h) \to (S,h^\star)$ is a minimal Lagrangian map.

If $(h,h^\star)$ is a pair of normalized hyperbolic metrics, then the metrics

$$h_c = h((\mathbb{1}+b)\bullet, (\mathbb{1}+b)\bullet)\,, \qquad h'_c = h(b\bullet, \bullet)$$

are conformal, because $(\mathbb{1}+b)^2 = (2+\mathrm{tr}(b))b$ since $\det(b) = 1$.

So they determine a conformal structure, denoted here by c. The identity maps

$$(S,c) \to (S,h)\,, \qquad (S,c) \to (S,h^\star)$$

are harmonic maps with opposite Hopf differential.

The conformal structure is called the *center* of the pair $(h,h^\star)$.

Proof of Proposition 1.2. Real-analyticity of the energy functional $E(\bullet, h) : \mathcal{T} \to \mathbb{R}$ follows mimicking the arguments of [8]. Properness was proven by Tromba [10].

If $\hat{c}$ is a metric in the conformal class of c and $h = \varphi + e\hat{c} + \overline{\varphi}$, then $h^\star = -\varphi + e\hat{c} - \overline{\varphi}$. Thus, $h((\mathbb{1}+b^2)\bullet, \bullet) = h + h^\star = 2e\hat{c}$, which implies

$$2E(c,h) = \int_S 2e da_{\hat{c}} = 2\int_S \sqrt{\det(\mathbb{1}+b^2)} da_h = 2\int_S \mathrm{tr}(b) da_h = 2F(h,h^\star)$$

Finally, fix h and let (c_n) be a divergent sequence in $\mathcal{T}$. This determines a sequence $(h_n^\star)$ such that c_n is the center of $(h,h_n^\star)$. This $(h_n^\star)$ is divergent too, for otherwise a subsequence of $F(h,h_n^\star)$ and so of $E(c_n,h)$ would remain bounded, contradicting the properness of $E(\bullet,h)$. Again up to subsequences, we can then assume that $\theta_n h_n^\star$ converge to a nonzero measured lamination λ, where θ_n is a positive sequence that converges to zero. It follows from Proposition 6.15 of [5] that $\theta_n F(h,h_n^\star) \to \ell_\lambda(h) > 0$ and so $F(h,h_n^\star) \to +\infty$. □

2.3. **The $\mathbb{A}\mathrm{d}\mathbb{S}^3$ space.** The 3-dimensional anti-de Sitter space $\mathbb{A}\mathrm{d}\mathbb{S}^3$ can be defined much like the hyperbolic space. Let $\mathbb{R}^{2,2}$ denote $\mathbb{R}^4$ with the symmetric bilinear form $\langle \bullet, \bullet \rangle_{(2,2)}$of signature $(2,2)$. Then

$$\mathbb{A}\mathrm{d}\mathbb{S}^3 := \{x \in \mathbb{R}^{2,2} \mid \langle x, x\rangle_{(2,2)} = -1\}\ ,$$

with the induced metric.

We refer the reader to [25, 1] for the main properties of $\mathbb{A}\mathrm{d}\mathbb{S}^3$, and only recall here a few key properties, without proof. In many respects $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is reminiscent of $\mathbb{H}^3$, while some of its properties, in particular concerning its isometry group, also relates to the 3-dimensional sphere S^3.

The space $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is Lorentzian, with constant curvature -1. It is not simply connected, however, and its fundamental group is infinite cyclic. As a Lorentz space, it has three types of geodesics: space-like and light-like geodesics are open lines, while time-like geodesics are closed, of length 2π. When considering $\mathbb{A}\mathrm{d}\mathbb{S}^3$ as a quadric in $\mathbb{R}^{2,2}$ as above, the geodesics in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ are the intersections of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with the 2-dimensional planes containing 0 in $\mathbb{R}^{2,2}$. The space-like totally geodesic planes in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ are isometric to the hyperbolic plane.

The multiplication by ± 1 acts on every sphere in $\mathbb{R}^{2,2}$ and so on $\mathbb{A}\mathrm{d}\mathbb{S}^3$ and on the quadric $Q := \{x \in \mathbb{R}^{2,2} \,|\, \langle x, x\rangle_{(2,2)} = 0\}$. The projective model of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ identifies $\mathbb{A}\mathrm{d}\mathbb{S}^3/\{\pm 1\}$ to one connected component of the complement in $\mathbb{RP}^3 \setminus \mathbb{P}Q$. The space-like geodesics in $\mathbb{A}\mathrm{d}\mathbb{S}^3/\{\pm 1\}$ then correspond to the projective lines intersecting the boundary quadric $\mathbb{P}Q$ in two points, while light-like geodesics are tangent to $\mathbb{P}Q$, and time-like geodesics do not intersect $\mathbb{P}Q$. Taking the double cover of $\mathbb{RP}^3$ (namely taking the quotient of $\mathbb{R}^{2,2}$ by $\mathbb{R}_+$) yields a projective model of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ inside S^3. This projective model is one way to define the boundary at infinity of $\mathbb{A}\mathrm{d}\mathbb{S}^3$. It is topologically a torus, with a Lorentz conformal structure.

The isometry group of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is $\mathrm{O}(2,2)$. However, up to finite quotient, this isometry group splits as the product of two copies of $\mathrm{PSL}(2,\mathbb{R})$, more precisely its identity component is isomorphic to $(\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}))/\{\pm1\}$.

2.4. **Space-like surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$.** The local theory of space-like surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is very similar to that of surfaces in the Euclidean or in the hyperbolic 3-dimensional space. Here again we only briefly recall without proof some basic facts which will be useful below (see [] for a treatment of this subject).

Let $\tilde S\subset\mathbb{A}\mathrm{d}\mathbb{S}^3$ be a space-like surface, and let n be a unit normal vector field on $\tilde S$. Given such a surface, we will call I its induced metric (or first fundamental form). The shape operator of $\tilde S$ is an I-self-adjoint bundle morphism $B: T\tilde S\to T\tilde S$ defined by

$$\forall x\in\tilde S\ \forall v\in T_x\tilde S\quad Bv=-D_v n\ ,$$

where D is the restriction to $\tilde S$ of the Levi-Civita connection of $\mathbb{A}\mathrm{d}\mathbb{S}^3$.

The shape operator satisfies two basic equations.

- The Codazzi equation: if ∇ is the Levi-Civita connection of I and u,v are two vector fields on $\tilde S$, then $d^\nabla B=0$.
- The Gauss equation: the curvature of I on $\tilde S$ is equal to $K=-1-\det(B)$.

The second and third fundamental forms of $\tilde S$ are then defined by

$$\forall x\in\tilde S\ \forall u,v\in T_x\tilde S\quad I\!I(u,v)=I(Bu,v),\quad I\!I\!I(u,v)=I(Bu,Bv)\ .$$

2.5. **Globally hyperbolic $\mathbb{A}\mathrm{d}\mathbb{S}^3$ manifolds.** Let N be a Lorentz 3-dimensional manifold locally modeled on $\mathbb{A}\mathrm{d}\mathbb{S}^3$. We say that N is *maximal globally hyperbolic*, or MGH, if:

- it contains a closed space-like surface (*Cauchy surface*),
- any inextendible time-like curve intersects the Cauchy surface exactly once,
- it is maximal (under inclusion) among $\mathbb{A}\mathrm{d}\mathbb{S}^3$ manifolds having those properties, that is, if N' is another 3-dimensional $\mathbb{A}\mathrm{d}\mathbb{S}$ manifold having the previous two properties and $i:N\to N'$ is an isometric embedding, then i is onto.

Mess realized that some properties of MGH $\mathbb{A}\mathrm{d}\mathbb{S}^3$ manifolds are remarkably close of those of quasifuchsian hyperbolic manifolds. Among the analogies are the following points, which will be useful below.

- The space of MGH $\mathbb{A}\mathrm{d}\mathbb{S}^3$ metrics on a fixed manifold $S\times\mathbb{R}$ is parameterized by the product of two copies of $\mathcal{T}$, the Teichmüller space of S, as with the Bers double uniformization theorem for quasifuchsian manifolds. In the $\mathbb{A}\mathrm{d}\mathbb{S}^3$ case this parameterization comes from the holonomy representation ρ of an MGH AdS structure, which takes values in $\mathrm{Isom}_0(\mathbb{A}\mathrm{d}\mathbb{S}^3/\{\pm1\})=\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$, and therefore splits as two representations ρ_l,ρ_r in $\mathrm{PSL}(2,\mathbb{R})$. Mess [25] proves that those two representations have maximal Euler class, so that they are holonomy representations of two hyperbolic metrics h_l,h_r on S. Any pair $(h_l,h_r)\in\mathcal{T}\times\mathcal{T}$ can be obtained from a unique MGH $\mathbb{A}\mathrm{d}\mathbb{S}^3$ structure.
- A MGH $\mathbb{A}\mathrm{d}\mathbb{S}^3$ manifold N contains a smallest closed non-empty convex subset, called its convex core $C(N)$. (A subset C of N is convex if any geodesic segment with endpoints in C is contained in C.) The boundary of $C(N)$ is the disjoint union of two spacelike pleated surfaces, except in the “Fuchsian” case where $C(N)$ is a totally geodesic surface. Each of those pleated surfaces has an induced metric which is hyperbolic, and its pleating is encoded by a measured lamination, as in the quasifuchsian setting.
- The complement of $C(N)$ has a unique foliation by convex, space-like surfaces, with constant curvature varying monotonically from -1 (near the convex core) to $-\infty$ (near the initial/final singularity) on each side of the convex core, see [3]. This is similar to what happens for quasifuchsian manifolds or more generally hyperbolic ends, see [23].

2.6. **The duality between convex surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ manifolds.** There is a well-known “projective” duality (or polarity) between points and hyperplanes in the projective space, or in the sphere. This duality has a hyperbolic version, which associates to a point in $\mathbb{H}^n$ a space-like hyperplane in the de Sitter space $\mathrm{d}\mathbb{S}^n$, and to an oriented hyperplane in $\mathbb{H}^n$ a point in $\mathrm{d}\mathbb{S}^n$, see [13].

A similar duality exists between points and hyperplanes in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ (or more generally in $\mathbb{A}\mathrm{d}\mathbb{S}^n$). We recall here its definition and its main properties. Consider $\mathbb{A}\mathrm{d}\mathbb{S}^3$ as a quadric in $\mathbb{R}^{2,2}$ as in Section 2.4. Every point $x\in\mathbb{A}\mathrm{d}\mathbb{S}^3$ is the intersection in $\mathbb{R}^{2,2}$ of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with a half-line d starting from 0 on which the bilinear form is negative definite. We call $d^\perp$ the oriented hyperplane orthogonal to d in $\mathbb{R}^{2,2}$, so that the induced metric on $d^\perp$ has signature $(1,2)$. The intersection between $d^\perp$ and $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is the disjoint union of two totally geodesic space-like planes, one at distance $\pi/2$ in the future of x, the other at distance $\pi/2$ in the past of x. We define the dual $x^\star$ of x as the oriented space-like plane which is the intersection of $\mathbb{AdS}^3$ with $d^\perp$ at distance $\pi/2$ in the future of x.

Conversely, every totally geodesic space-like plane P in $\mathbb{AdS}^3$ is the intersection of $\mathbb{AdS}^3$ with a hyperplane H of signature $(1,2)$ in $\mathbb{R}^{2,2}$. The orthogonal $H^\perp$ of H then intersects $\mathbb{AdS}^3$ in two antipodal points, and we define the dual $P^\star$ of P as the intersection which is at distance $\pi/2$ in the past of P.

This duality relation has a number of useful properties. It is an involution, and the union of the planes dual to the points of a plane P is the antipodal of the point dual to P.

Consider now a smooth, space-like, strictly convex surface $\tilde{S}$ in $\mathbb{AdS}^3$. Denote by $\tilde{S}^\star$ the set of points which are duals of the support planes of $\tilde{S}$. The relation between $\tilde{S}$ and $\tilde{S}^\star$ is based on the following lemma (see e.g. [32] for the analogous statement concerning the duality between $\mathbb{H}^3$ and $\mathbb{dS}^3$, the proof is the same in the AdS case).

Lemma 2.2. *Let $\tilde{S} \subset \mathbb{AdS}^3$ be a smooth, space-like, locally strictly convex surface of constant curvature $K \in (-\infty, 0)$. Then:*

1. *The dual of $\tilde{S}$ is a smooth, locally strictly convex surface $\tilde{S}^\star$.*
2. *The pull-back of the induced metric on $\tilde{S}^\star$ through the duality map is the third fundamental form of $\tilde{S}$, and vice versa.*
3. *If $\tilde{S}$ is a space-like surface of constant curvature $K \in (-\infty, -1)$ in $\mathbb{AdS}^3$ then its dual $\tilde{S}^\star$ is a space-like surface of constant curvature $K^\star = -K/(K+1)$.*

Consider now a smooth, space-like strictly convex surface S in a MGH $\mathbb{AdS}^3$ manifold N. The lift of S to the universal cover of N can be identified with a surface $\tilde{S}$ in $\mathbb{AdS}^3$, invariant under an action $\rho : \pi_1 S \to \mathrm{Isom}(\mathbb{AdS}^3)$. The dual surface $\tilde{S}^\star$ is then also invariant under ρ so that it corresponds to a surface $S^\star$ in N.

Lemma 2.3. *If N is a MGH $\mathbb{AdS}^3$ manifold and S is a space-like, past-convex surface in the future of the convex core of N, then its dual $S^\star$ is space-like, future-convex surface in the past of the convex core.*

2.7. **Constant curvature foliations in MGH $\mathbb{AdS}^3$ manifolds.** An important fact used below is the existence of a foliation of the complement of the convex core of a MGH $\mathbb{AdS}^3$ manifold by surfaces of constant curvature. This is described in the following result obtained by Barbot, Béguin and Zeghib.

Theorem 2.4 (Barbot, Béguin, Zeghib [3])**.** *Let N be a MGH $\mathbb{AdS}^3$ manifold, and let $K \in (-\infty, -1)$. There is a unique past-convex (resp. future-convex) closed space-like surface in the future (resp. past) of the convex core, with constant curvature K. Those surfaces form a foliation of the complement in N of the convex core.*

Similar statements hold in the de Sitter and the Minkowski case, see [3].

2.8. **The landslide flow.** The landslide flow can be defined in at least three related ways, each of which can be convenient in some cases:

- in terms of harmonic maps and holomorphic quadratic differentials,
- using minimal Lagrangian maps,
- in terms of 3-dimensional globally hyperbolic $\mathbb{AdS}$ manifolds.

We briefly recall here two definitions, one in terms of minimal Lagrangian maps, the other in terms of 3-dimensional $\mathbb{AdS}$ manifolds. More details can be found in [5].

Let $h, h^\star$ be two hyperbolic metrics on S. We have recalled above that there is a unique minimal Lagrangian map $m : (S,h) \to (S,h^\star)$ isotopic to the identity. This map can be decomposed as $m = f^\star \circ f^{-1}$, where $f : (S,c) \to (S,h)$ and $f^\star : (S,c) \to (S,h^\star)$ are harmonic maps isotopic to the identity, for the conformal structure c on S, and f and $f^\star$ have opposite Hopf differentials φ_f and $\varphi_{f^\star} = -\varphi_f$. For each $e^i\theta \in S^1$, there is a unique hyperbolic metric h_θ on S such that $e^{i\theta}\varphi_f$ is the Hopf differential of the harmonic map isotopic to the identity from (S,c) to (S,h_θ), and a unique hyperbolic metric $h^\star_\theta$ such that $-e^{i\theta}\varphi_f$ is the Hopf differential of the harmonic map isotopic to the identity from (S,c) to $(S,h^\star_\theta)$. Then $(h_\theta, h^\star_\theta) = \mathcal{L}_{e^{i\theta}}(h,h^\star)$ is the image of $(h,h^\star)$ by the landslide flow with parameter $e^{i\theta}$.

In terms of AdS geometry, the definition is the following. Let again $h, h^\star$ be hyperbolic metrics on S and let $e^{i\theta} \in S^1$. There is a unique equivariant embedding of $\tilde{S}$ in $\mathbb{AdS}^3$ with induced metric $\cos^2(\theta/2)h$ and third fundamental form $\sin^2(\theta/2)h^\star$. The corresponding representation $\rho : \pi_1 S \to \mathrm{Isom}(\mathbb{AdS}^3)$ is the holonomy representation of a MGH $\mathbb{AdS}^3$ manifold N (so that N contains a space-like surface isometric to $\cos^2(\theta/2)h$ with third fundamental form equal to $\sin^2(\theta/2)h^\star$). Then $(h_\theta, h^\star_\theta)$ are the left and right hyperbolic metrics of N.

2.9. **Hyperbolic ends.** An example of a hyperbolic end is an end of a quasifuchsian hyperbolic manifold, that is, a connected component of the complement of the convex core in a quasifuchsian manifold. More generally, a hyperbolic end M is a 3-dimensional manifold homeomorphic to $S \times \mathbb{R}_{>0}$, with a non-complete hyperbolic metric g such that:

- g is complete on the end of $S \times \mathbb{R}_{>0}$ corresponding to infinity,
- (M, g) has a metric completion for which the boundary corresponding to $S \times \{0\}$ is a concave pleated surface.

Hyperbolic ends will appear in relation to the smooth grafting map, which is to landslides as the grafting map is to earthquakes.

A key result that we will use is that any hyperbolic end has a unique foliation by constant curvature surfaces, with the curvature varying between -1 (close to the pleated surface boundary) to 0 (near the complete boundary), see [23].

Given a hyperbolic end M, its boundary at infinity is the connected component of its boundary corresponding to $S \times \{\infty\}$. This boundary at infinity $\partial_\infty M$ is the quotient of a domain in $\mathbb{CP}^1 = \partial_\infty \mathbb{H}^3$ by an action of $\pi_1 S$ by complex projective transformations, so that $\partial_\infty M$ is endowed with a complex projective structure.

2.10. **The smooth grafting maps.** We will consider two versions of smooth grafting:

- the map $SGr' : \mathbb{R}_{>0} \times \mathcal{T} \times \mathcal{T} \to \mathcal{CP}$ sending two hyperbolic metrics and a real parameter to a complex projective structure on S,
- the map $sgr' : \mathbb{R}_{>0} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ which is the composition of SGr' with the projection from $\mathcal{CP}$ to $\mathcal{T}$ sending a complex projective structure to the underlying complex structure.

The map SGr' can be defined using 3-dimensional geometry as follows. Let again $h, h^\star \in \mathcal{T}$ be two hyperbolic metrics on S, and let $s > 0$. Up to global isometry there is a unique equivariant convex embedding of $\tilde{S}$ in $\mathbb{H}^3$ such that the induced metric is $\cosh^2(s/2)\tilde{h}$ and the third fundamental form is $\sinh^2(s/2)\tilde{h}^\star$. The corresponding representation is the holonomy representation of a hyperbolic end M (so that M contains a convex surface with induced metric $\cosh^2(s/2)h$ and third fundamental form $\sinh^2(s/2)h^\star$). We define $SGr'_s(h, h^\star)$ as the complex projective structure on $\partial_\infty M$.

3. First order computations

Let us fix a normalized pair of hyperbolic metrics $(h, h^\star)$ and denote by J, $J^\star$ the corresponding complex structures. Let us fix also a holomorphic quadratic differential $\varphi \in \mathcal{Q}(J)$ and let us consider the family of hyperbolic metrics

$$h_t = t\varphi + e(t)h + t\bar{\varphi} \tag{6}$$

given by (5).

We will denote by α_t the positive self-adjoint operator such that

$$h_t = h(\alpha_t \bullet, \alpha_t \bullet) \ .$$

Notice that

$$\alpha_t^2 = e(t)\mathbb{1} + 2t\nu_\varphi \ . \tag{7}$$

Moreover we will denote by b_t the Labourie operator of the pair $(h_t, h^\star)$. Let us stress that in general $(h_t, h^\star)$ is not a normalized pair of hyperbolic metrics, so $h_t^\star = h_t(b_t \bullet, b_t \bullet)$ does not coincide with $h^\star$, but there is a continuous family of diffeomorphisms $m_t : S \to S$ such that

$$h_t^\star = m_t^*(h^\star) \ . \tag{8}$$

In this section we will point out some relations between the first variation α_t and the first variation of b_t. This technical computation will be the key tool to prove Theorems 1.3 and 1.4.

3.1. **First order variation of α_t.** Denote by $\dot{\alpha}$ the derivative of α_t at $t = 0$. By (7) we simply see that

$$\dot{\alpha} = \nu_\varphi \ . \tag{9}$$

In particular we deduce that $\dot{\alpha}$ is a self-adjoint operator such that

$$\mathrm{tr}\dot{\alpha} = 0 \ , \qquad d^\nabla \dot{\alpha} = 0 \ . \tag{10}$$

Remark 3.1. The computation we will make in this section only depend on the properties above of $\dot\alpha$. That is, the results of this section are valid for any family of hyperbolic metrics $h_t = h(\alpha'_t, \alpha'_t)$ supposing that $\dot\alpha'$ verifies (10).

On the other hand, in order to compute the Hessian of F with respect to the Weil-Petersson metric, it will be necessary to use the deformation given by (6).

3.2. **First order variation of the area form of h_t.** Let $da_t = da_{h_t}$ denote the area form of h_t and let $\frac{d}{dt}(da_t)$ be its time-derivative. Since we have

$$da_t = \det(\alpha_t)da_h \ ,$$

we easily deduce that

Lemma 3.2. $\frac{d}{dt}(da_t) = \mathrm{tr}(\alpha_t^{-1}\dot\alpha_t)da_t$.

Notice in particular that $\frac{d}{dt}(da_t)|_{t=0} = 0$.

3.3. **First order variation of b.** In order to get information about $\dot b$ at $t = 0$ we will differentiate the identities satisfied by b_t. In particular we have

- $\det b_t = 1$,
- b_t is h_t-self-adjoint,
- $d^{\nabla^t} b_t = 0$, where ∇^t is the Levi-Civita connection for h_t,
- $h_t(b_t\bullet, b_t\bullet) = m_t^*(h^\star)$, where m_t is a smooth family of diffeomorphisms of S such that $m_0 = \mathbb{1}$.

Differentiating the first identity, we get

$$\mathrm{tr}(b^{-1}\dot b) = 0 \ . \tag{11}$$

About the second property, notice that the fact that b_t is h_t-self-adjoint is equivalent to requiring that $\mathrm{tr}(J_t b_t) = 0$, where J_t is the complex structure compatible with h_t. Differentiating this identity and using that $J_t = \alpha_t^{-1} J \alpha_t$, and that $J\dot\alpha = -\dot\alpha J$ one finds:

Lemma 3.3. $\mathrm{tr}(J\dot b) = -2\mathrm{tr}(J\dot\alpha b)$.

In order to get the infinitesimal information by the last properties it is convenient to introduce the operator $\psi_t = \alpha_t b_t$. Notice that we have

$$h_t^\star = h(\psi_t, \psi_t) \ , \tag{12}$$

and the infinitesimal deformation of ψ at $t = 0$ is simply $\dot\psi = \dot\alpha b + \dot b$.

By the fact that $h_t^\star$ is a trivial family of hyperbolic metrics we deduce the following relation.

Lemma 3.4. *There exist a family of vector fields Z_t on S and a family of functions r_t on S such that*

$$\nabla^{\star,t} Z_t + r_t J_t^\star = \psi_t^{-1}\dot\psi_t \tag{13}$$

where $J_t^\star$ is the complex structure for $h_t^\star$ and $\nabla^{\star,t}$ is the Levi-Civita connection of $h_t^\star$.

Proof. By (12),

$$\dot h_t^\star = h(\dot\psi_t\bullet, \psi_t\bullet) + h(\psi_t\bullet, \dot\psi_t) = h^\star(\psi_t^{-1}\dot\psi_t\bullet, \bullet) + h^\star(\bullet, \psi_t^{-1}\dot\psi_t\bullet) \ .$$

On the other hand, let us consider the field

$$Z_t(p) = d_p(m_t^{-1})\left(\frac{\partial m_t(p)}{\partial t}\right) \ .$$

By (8), we have

$$\dot h_t^\star = h_t^\star(\nabla^{\star,t} Z_t, \bullet) + h_t^\star(\bullet, \nabla^{\star,t} Z_t)$$

from which we obtain that the difference $\nabla^{\star,t} Z_t - \psi_t^{-1}\dot\psi_t$ is $h_t^\star$-skew-symmetric, and the conclusion follows. □

Notice that at $t = 0$ we have $\psi_0 = b$ so we deduce that $\nabla^\star Z_0 + r_0 J^\star = b^{-1}\dot\psi$. It can be shown that $\nabla^\star = b^{-1}\nabla b$ (see [5, Lemma 3.3] in the case $\theta = \pi$), whereas $J^\star = b^{-1}Jb$. So we can rewrite the identity above in the form

$$\dot\psi = \nabla Y + rJb \ , \tag{14}$$

where we have put $Y = bZ_0$ and $r = r_0$.

Finally differentiating the identity $d^{\nabla^t} b_t = 0$ at $t = 0$, we get:

Lemma 3.5. $d^\nabla(\dot\psi) = 0$.

The proof of Lemma 3.5 given below is based on the computation of d^{∇^t}, which relies on the following two results.

Lemma 3.6. *There exists a family of vector fields V_t on S such that for $v, w \in TS$ we have $(d^\nabla \alpha_t)(v, w) = da_h(v, w)V_t$ with $V_0 = \dot{V}_0 = 0$*

Proof. On a point $p \in S$ take any h-orthonormal basis e_1, e_2 of T_pS. Then putting $V_t(p) = (d^\nabla \alpha_t)(e_1, e_2)$, it follows that

$$(d^\nabla \alpha_t)(v, w) = da_h(v, w)V_t(p)$$

for every $v, w \in T_pS$. Clearly V_t smoothly depends on p and t. Since $\alpha_0 = \mathbb{1}$, V_0 vanishes everywhere. On the other hand, by the linearity of d^∇ we have

$$d^\nabla \dot{\alpha} = da_h \otimes \dot{V}_0 \ ,$$

and by (10) we deduce that $\dot{V}_0$ vanishes everywhere. □

Lemma 3.7. *Let ∇^t the Levi-Civita connection of the metric h_t. If v, w are vector fields on S we have*

$$\nabla^t_v w = \alpha_t^{-1} \nabla^t_v(\alpha w) + h_t(W_t, v)J_t(w)$$

where J_t is the complex structure compatible with h_t, $W_t = \det(\alpha_t^{-1})\alpha_t^{-1}V_t$, and V_t is the field defined in Lemma 3.6.

Proof. Notice that the connection $\alpha_t^{-1}\nabla(\alpha_t \bullet)$ is compatible with the metric h_t but is not symmetric (since we are not assuming that α_t is a solution of the Codazzi equation for h).

In particular, the difference

$$T(v, w) = \nabla^t_v w - \alpha_t^{-1}\nabla_v(\alpha_t w)$$

is a vector-valued 2-form such that $T(v, \bullet)$ is h_t-skew-symmetric. That is, there exists a 1-form ζ such that $T(v, w) = \zeta(v)J_t w$. On the other hand an explicit computation shows that

$$T(v, w) - T(w, v) = -\alpha_t^{-1}(d^\nabla \alpha_t)(v, w) = -da_h(v, w)\alpha_t^{-1}V_t \ .$$

If (e_1, e_2) is a positive h_t-orthonormal basis, we have $\zeta(e_1)J_t(e_2) - \zeta(e_2)J_t(e_1) = -da_h(e_1, e_2)\alpha_t^{-1}V_t$, that is

$$\zeta(e_1)e_1 + \zeta(e_2)e_2 = W_t \ ,$$

so $\zeta(e_i) = h_t(e_i, W_t)$ and the result follows. □

Proof of Lemma 3.5. Take two vector fields v, w on S, and consider the identity

$$\nabla^t_v b_t(w) - \nabla^t_w b_t(v) - b_t([v, w]) = 0.$$

By Lemma 3.7, we can rewrite this identity as

$$\begin{aligned} 0 &= \alpha_t^{-1}(\nabla_v(\alpha_t b_t(w)) - \nabla_w(\alpha_t b_t(v)) - \alpha_t b_t([v, w])) + h_t(W_t, v)J_t(w) - h_t(W_t, w)J_t(v) \\ &= \alpha_t^{-1} d^\nabla(\alpha_t b_t)(v, w) + h_t(W_t, v)J_t(w) - h_t(W_t, w)J_t(v) \end{aligned}$$

Since $V_0 = \dot{V}_0 = d^\nabla \psi_0 = 0$ we have $W_0 = \dot{W}_0 = 0$, so differentiating the last identity at $t = 0$ we have

$$d^\nabla(\dot{\psi}) = 0 \ .$$

□

Lemma 3.5 implies the following interesting relation between the field Y and the function r appearing in (14).

Lemma 3.8. *We have*

$$JY = b^{-1} \operatorname{grad} r \ .$$

Proof. Take a positive h-orthonormal basis (e_1, e_2) of T_pS. By Lemma 3.5 we have that $d^\nabla(\nabla Y + rJb) = 0$. It follows that

$$d^\nabla(\nabla Y)(e_1, e_2) + d^\nabla(rJb)(e_1, e_2) = 0 \ .$$

We have $(d^\nabla \nabla Y)(e_1, e_2) = R(e_1, e_2)Y = JY$. On the other hand, since $d^\nabla(Jb) = Jd^\nabla b = 0$,

$$d^\nabla(rJb)(e_1, e_2) = dr(e_1)Jbe_2 - dr(e_2)Jbe_1 = JbJ(dr(e_1)e_1 + dr(e_2)e_2) = JbJ \operatorname{grad} r = -b^{-1} \operatorname{grad} r \ ,$$

and the conclusion follows. □

3.4. **The function F and its variation.** We consider on $\mathcal{M}_{-1}\times\mathcal{M}_{-1}$ the function

$$\tilde{F}(h,h^\star)=\int \mathrm{tr}(b)da_h\ ,$$

where b is the Labourie operator of the pair $(h,h^\star)$. Clearly F is invariant by the action of $\mathrm{Diffeo}_0\times\mathrm{Diffeo}_0$, so it induces a smooth function

$$F:\mathcal{T}\times\mathcal{T}\to\mathbb{R}\ .$$

In this section we will compute the derivative of the function $F_t:=F([h_t],[h^\star])$ with respect to t.

Proposition 3.9. *The first-order derivative of F_t is*

$$\dot{F}_t=\int_S\left[\mathrm{tr}(b_t)\mathrm{tr}(\alpha_t^{-1}\dot{\alpha}_t)-\mathrm{tr}(\alpha_t^{-1}\dot{\alpha}_t b_t)\right]da_t\ . \tag{15}$$

In particular, at $t=0$,

$$\dot{F}=-\int_S\mathrm{tr}(\dot{\alpha}b)da_h\ .$$

Proof of Proposition 3.9. We want to compute

$$\dot{F}_t=\int_S\mathrm{tr}(\dot{b}_t)da_t+\mathrm{tr}(b_t)\tfrac{d}{dt}(da_t)\ . \tag{16}$$

Since $\psi_t=\alpha_t b_t$, Equation (13) can be rearranged as

$$\dot{\psi}_t=\alpha_t\nabla^t(b_tZ_t)+r_t\alpha_tJ_tb_t\ .$$

In particular,

$$\dot{b}_t=-\alpha_t^{-1}\dot{\alpha}_tb_t+\nabla^t(b_tZ_t)+r_tJ_tb_t$$

Since b_t is h_t-self-adjoint $\mathrm{tr}(J_tb_t)=0$, and so $\mathrm{tr}(r_tJ_tb_t)=0$.
Moreover, $\mathrm{tr}(\nabla^t(b_tZ_t))=\mathrm{div}_t(b_tZ_t)$ and so

$$\int_S\mathrm{tr}(\nabla^t(b_tZ_t))da_t=\int_S\mathrm{div}_t(b_tZ_t)da_t=0\ .$$

But $\frac{d}{dt}(da_t)=\mathrm{tr}(\alpha_t^{-1}\dot{\alpha}_t)da_t$ by Lemma 3.2, so we obtain that

$$\dot{F}_t=\int_S\left[\mathrm{tr}(b_t)\mathrm{tr}(\alpha_t^{-1}\dot{\alpha}_t)-\mathrm{tr}(\alpha_t^{-1}\dot{\alpha}_t b_t)\right]da_t\ . \tag{17}$$

By (10), at time $t=0$ we obtain

$$\dot{F}=-\int_S\mathrm{tr}(\dot{\alpha}b)da_h\ . \tag{18}$$

□

Remark 3.10. Formula (15) holds for any family of deformations h_t of the metric h, even without assuming (10). Notice indeed that both the proofs of Proposition 3.9 and Lemma 3.4 do not make use of this hypothesis.

4. The landslide flow is Hamiltonian

Let F be the function on $\mathcal{T}\times\mathcal{T}$ defined in Section 3. The goal of this section is to show that $\frac{1}{4}F$ is the Hamiltonian of the landslide flow with respect to the product symplectic form $\omega_{WP,1}+\omega_{WP,2}$.

If $(X,X^\star)\in T_h\mathcal{T}\oplus T_{h^\star}\mathcal{T}$ is the generator of the landslide flow at the point $(h,h^\star)$, we need to prove that it coincides with the symplectic gradient of F at $([h],[h^\star])$.

This is equivalent to showing that X coincides with the symplectic gradient of $F(\bullet,[h^\star])$ at the point h for ω_{WP}, and analogously that $X^\star$ is the symplectic gradient of $F([h],\bullet)$ at $h^\star$.

By a simple symmetry argument, it is sufficient to check the first point. In particular, given any tangent vector $v\in T_{[h]}\mathcal{T}$ we need to show that

$$\omega_{WP}(X,v)=\frac{1}{4}d(F(\bullet,[h^\star]))(v)\ . \tag{19}$$

Now, there exists a holomorphic quadratic differential φ such that $v=[\nu_\varphi]$. Let $\dot{J}_\varphi$ be the first order variation associated with this Beltrami differential ν_φ, and let $\dot{J}_X\in T_J\mathcal{A}$ be the first order variation of the complex structure corresponding to the landslide deformation of the metric $h_t = h(\beta_t \bullet, \beta_t \bullet)$ with $\beta_t = \cos(t/2)\mathbb{1} + \sin(t/2)Jb$. Notice that $d\pi(\dot{J}_\varphi) = v$ and $d\pi(\dot{J}_X) = X$ so by (4) we have that

$$\omega_{WP}(X, v) = \frac{1}{8}\int_S \mathrm{tr}(J\dot{J}_X^H \dot{J}_\varphi^H) da_h \ .$$

Now by (1), $\dot{J}_\varphi = 2J\nu_\varphi = -2\nu_\varphi J$, whereas $\dot{J}_X = \frac{1}{2}(JJb - JbJ)$. In particular, since $\dot{J}_\varphi$ is horizontal, we get

$$\omega_{WP}(X, v) = \frac{1}{8}\int_S \mathrm{tr}(J\dot{J}_X \dot{J}_\varphi) da_h \ = -\frac{1}{4}\int_S \mathrm{tr}(b\nu_\varphi) da_h \ . \tag{20}$$

To compute the right-hand side of (19), we can consider the path of metrics $h_t = t\varphi + e(t)h + t\bar{\varphi}$ as in (6). Then we have

$$d(F(\bullet, [h^\star]))(v) = \frac{dF([h_t], [h^\star])}{dt}(0) \ .$$

With the notations of Section 3, Proposition 3.9 implies that

$$d(F(\bullet, [h^\star]))(v) = -\int_S \mathrm{tr}(\dot{\alpha} b) da_h \ .$$

By (9), comparing this identity with (20), we get (19).

5. Convexity of F

The aim of this section is to show that the function $F : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ is convex on each factor with respect to the Weil-Petersson metric.

In [42], it has been shown that the family of metrics h_t introduced in (6) determines a path in $\mathcal{T}$ which is WP-geodesic at $t = 0$. So fixing $h^\star$, the Hessian of the function $F(\bullet, h^\star)$ at $[h]$ is determined by

$$\mathrm{Hess}(F(\bullet, [h^\star]))([\nu_\varphi], [\nu_\varphi]) = \frac{d^2 F_t}{dt^2}(0) \ ,$$

where $F_t = F([h_t], [h^\star])$.

Using the notations of Section 3, by differentiating Equation (17), we have

$$\ddot{F}_t = \int_S \Big[\mathrm{tr}(\dot{b}_t)\mathrm{tr}(\alpha_t^{-1}\dot{\alpha}_t) + \mathrm{tr}(b_t)\mathrm{tr}[\alpha_t^{-1}\ddot{\alpha}_t - (\alpha_t^{-1}\dot{\alpha}_t)^2] + \mathrm{tr}[(\alpha_t^{-1}\dot{\alpha}_t)^2 - \alpha_t^{-1}\ddot{\alpha}_t)b_t] - \mathrm{tr}(\alpha_t^{-1}\dot{\alpha}_t\dot{b}_t)\Big] da_t +$$
$$+ \int_S (\mathrm{tr}(b_t)\mathrm{tr}(\alpha_t^{-1}\dot{\alpha}_t) - \mathrm{tr}(\alpha_t^{-1}\dot{\alpha}_t b_t))\tfrac{d}{dt}(da_t) \ .$$

We have already seen that $\frac{d}{dt}(da_t) = 0$ at time $t = 0$. So, at time $t = 0$, we obtain

$$\ddot{F} = \int_S \Big[\mathrm{tr}(b)\mathrm{tr}(\xi) - \mathrm{tr}(\xi b) - \mathrm{tr}(\dot{\alpha}\dot{b})\Big] da_h \ , \tag{21}$$

where $\xi := \ddot{\alpha} - \dot{\alpha}^2$.

Since $\dot{\alpha}$ is self-adjoint and traceless we deduce that $\dot{\alpha}^2$ is a non-negative multiple of the identity. On the other hand, by comparing the relation $h_t = h(\alpha_t, \alpha_t)$ with (6) we get

$$\ddot{e}(0)h = \ddot{h} = 2h((\dot{\alpha}^2 + \ddot{\alpha})\bullet, \bullet) \ .$$

so we deduce that

$$\dot{\alpha}^2 + \ddot{\alpha} = \frac{1}{2}\ddot{e}(0)\mathbb{1} \ . \tag{22}$$

In [42], the function $e(t)$ has been computed up to the second order. More precisely, if in local conformal coordinates $\varphi = \varphi(z)dz^2$ and $h = h(z)|dz|^2$, then we have

$$e(t) = 1 + t^2\left(\frac{|\varphi(z)|^2}{h(z)^2} + 2(2-\Delta)^{-1}\frac{|\varphi(z)|^2}{h(z)^2}\right) + O(t^3) \ . \tag{23}$$

With the real notation, $\dfrac{|\varphi(z)|^2}{h(z)^2} = |\nu_\varphi|^2 = \mathrm{tr}(\dot{\alpha}^2)/2$. So we can rewrite (23) as

$$\frac{1}{2}\ddot{e}(0) = |\nu_\varphi|^2 + 2(2-\Delta)^{-1}(|\nu_\varphi|^2) \ . \tag{24}$$

Using (24) in (22) we have

$$\ddot{\alpha} = 2(2-\Delta)^{-1}(|\nu_\varphi|^2)\mathbb{1} \ .$$

In particular the operator ξ in (21) is equal to $\xi = [2(2-\Delta)^{-1}(|\nu_\varphi|^2) - |\nu_\varphi|^2]\mathbb{1}$. Since ξ is a multiple of the identity, we have that

$$\mathrm{tr}(b)\mathrm{tr}(\xi) - \mathrm{tr}(b\xi) = \mathrm{tr}(b\xi) = \mathrm{tr}(b(\ddot\alpha - \dot\alpha^2)) \ . \tag{25}$$

In order to conclude the computation we need to estimate the integral of the term $\mathrm{tr}(\dot\alpha\dot b)$ appearing in (21).

Proposition 5.1. *We have*

$$\int_S \mathrm{tr}(\dot\alpha\dot b)da_h = -\int_S \mathrm{tr}(\dot\alpha^2 b)da_h - \int_S h(\mathrm{grad}\, r, b^{-1}\,\mathrm{grad}\, r)da_h \int_S r^2\mathrm{tr}(b)da_h \ . \tag{26}$$

In particular

$$\int_S \mathrm{tr}(\dot\alpha\dot b)da_h \leq -\int_S \mathrm{tr}(\dot\alpha^2 b)da_h \ . \tag{27}$$

Proof. Multiplying the identity (14) by $\dot\alpha$ and taking the trace, we get

$$\mathrm{tr}(\dot\alpha\dot b) = -\mathrm{tr}(\dot\alpha^2 b) + \mathrm{tr}(\dot\alpha\nabla Y) + \mathrm{tr}(r\dot\alpha J b) \ . \tag{28}$$

Since $\dot\alpha$ solves the Codazzi equation $d^\nabla\dot\alpha = 0$ we have $\dot\alpha\nabla Y = \nabla(\dot\alpha Y) - \nabla_Y\dot\alpha$. Taking the trace and considering that $\mathrm{tr}\dot\alpha = 0$, it results that

$$\mathrm{tr}(\dot\alpha\nabla Y) = \mathrm{div}(\dot\alpha Y) \ .$$

On the other hand, multiplying the identity (14) by J we get

$$J\dot b + J\dot\alpha b = \nabla(JY) - rb \ .$$

Taking the trace and using Lemma 3.3 we have $\mathrm{tr}(J\dot\alpha b) = -\mathrm{div}(JY) + r\mathrm{tr}(b)$. Using again that $\dot\alpha J = -J\dot\alpha$ we get

$$\mathrm{tr}(\dot\alpha J b) = \mathrm{div}(JY) - r\mathrm{tr}(b) \ .$$

In particular replacing this identity in (28) we obtain that

$$\mathrm{tr}(\dot\alpha\dot b) = -\mathrm{tr}(\dot\alpha^2 b) + \mathrm{div}(\dot\alpha Y) + r\mathrm{div}(JY) - r^2\mathrm{tr}b \ .$$

Integrating, it results that

$$\int_S \mathrm{tr}(\dot\alpha\dot b)da_h = -\int_S \mathrm{tr}(\dot\alpha^2 b)da_h - \int_S h(\mathrm{grad}\, r, JY)da_h - \int_S r^2\mathrm{tr}(b)da_h \ .$$

By Lemma 3.8, $h(\mathrm{grad}\, r, JY) = h(\mathrm{grad}\, r, b^{-1}\,\mathrm{grad}\, r) > 0$, so the result easily follows. □

Proof of Theorem 1.4. Using (26) in (21) and taking into account (25) we get

$$\ddot F = \int_S [\mathrm{tr}(b(\ddot\alpha - \dot\alpha^2)) + \mathrm{tr}(b\dot\alpha^2)]da_h + \int_S h(\mathrm{grad}\, r, b^{-1}\,\mathrm{grad}\, r)da_h + \int_S r^2\mathrm{tr}(b)da_h =$$
$$\int_S \mathrm{tr}(b\ddot\alpha)da_h + \int_S h(\mathrm{grad}\, r, b^{-1}\,\mathrm{grad}\, r)da_h + \int_S r^2\mathrm{tr}(b)da_h \ .$$

In particular, we deduce that

$$\ddot F \geq \int_S \mathrm{tr}(b\ddot\alpha)da_h = \int_S 2(2-\Delta)^{-1}(|\nu_\varphi|^2)\mathrm{tr}(b)da_h \ .$$

Now, let $\varphi \neq 0$ and put $u := (2-\Delta)^{-1}(|\nu_\varphi|^2)$. We have that $2u - \Delta u = |\nu_\varphi|^2$ and, by the maximum principle, $u > 0$ (since $|\nu_\varphi|^2$ is positive). Hence, $\ddot F$ is positive.

A simple case is where $h = h^\star$. In this a case it is not difficult to check that $r = 0$ and $b = \mathbb{1}$. Then we simply get

$$\ddot F = \int_S 4u\, da_h = \int_S 2(|\nu_\varphi|^2 + \Delta u)da_h = \int_S |\nu_\varphi|^2 da_h = 2g_{WP}(\varphi,\varphi) \ .$$

□

6. Smooth grafting

We now turn to the smooth grafting map, and to the proof of Theorem 1.7.

6.1. **Notation and hypotheses.** Fix a point $[h]$ in Teichmüller space, where h is a hyperbolic metric on S. Fix also $s > 0$ and consider a one-parameter family $t \mapsto b_t$ of h-Labourie operators, that gives a family of hyperbolic metrics $h^\star_t = h(b_t\bullet, b_t\bullet)$.

The smooth grafting $sgr'_s(h, h^\star_t)$ is represented by the metric $h^\#_t = h(\beta_t, \beta_t)$, where

$$\beta_t = \cosh(s/2)\mathbb{1} + \sinh(s/2)b_t\,.$$

We will show that the map

$$sgr'_s(h, \bullet) : \mathcal{T} \longrightarrow \mathcal{T}$$

is surjective.

More precisely we will show the following results:

(a) $sgr'_s(h, \bullet)$ is proper;
(b) the degree of the map $sgr'_s(h, \bullet) : \mathcal{T} \to \mathcal{T}$ is 1.

Surjectivity is an immediate consequence of (b). Notice that (a) is needed to define the degree of the map $sgr'_s(h, \bullet)$.

6.2. **The map $sgr'_s(h, \bullet)$ is proper.** Let $[h^\star_n]$ be a divergent sequence in $\mathcal{T}$ and let $J^\#_n$ be the complex structure associated to $h^\#_n$. We want to show that $J^\#_n$ is a diverging sequence in $\mathcal{T}$. The proof is based on the following lemma.

Notation. Let $f : S \to S$ be any smooth map. For any two metrics g, g' on S, denote by $E(f; g, g')$ the energy of f regarded as a map $f : (S, g) \to (S, g')$. Since $E(f; g, g')$ is invariant by conformal deformations of g, we sometimes replace g by its underlying conformal structure.

Lemma 6.1. *Let h be a hyperbolic metric and let $s > 0$. Given a h-Labourie operator b, call $\beta_c = \mathbb{1} + b$ and $\beta = \cosh(s/2)\mathbb{1} + \sinh(s/2)b$, and consider the metrics $h_c = h(\beta_c\bullet, \beta_c\bullet)$ and $h^\# = h(\beta\bullet, \beta\bullet)$. Then*

$$E(f; h_c, h) < \tau^{-1}E(f; h^\#, h)$$

where $\tau = \tanh(s/2)$, for any smooth map $f : S \to S$.

Proof. Notice that $h = h_c(\gamma, \gamma)$ with $\gamma = \beta_c^{-1}\beta$ and that

$$da_{h_c} \leq \tau^{-1}\cosh^{-2}(s/2)da_{h^\#} \tag{29}$$

as $\tau \det(\beta_c) = \tau(2 + \mathrm{tr}(b)) \leq 1 + \tau^2 + \mathrm{tr}(b) = \cosh^{-2}(s/2)\det(\beta)$.

Now if α is the $h^\#$-self-adjoint operator such that

$$f^*(h) = h^\#(\alpha\bullet, \alpha\bullet)$$

we have that $f^*(h) = h_c(\gamma\alpha\bullet, \gamma\alpha\bullet)$, and so

$$e(f; h^\#, h) = \frac{1}{2}\mathrm{tr}(\alpha^2)\,, \qquad e(f; h_c, h) = \frac{1}{2}\mathrm{tr}((\gamma\alpha)^\dagger\gamma\alpha)\;.$$

where † denotes the adjoint with respect to h_c. Notice that the eigenvalues of γ are less than $\cosh(s/2)$ everywhere and so $h_c(\gamma v, \gamma v) < \cosh^2(s/2)h_c(v, v)$ for every nonzero tangent vector v. Indeed if k is the biggest eigenvalue of b, then the eigenvalues of γ are $\cosh(s/2)\frac{1+\tau k}{1+k}$ and $\cosh(s/2)\frac{\tau+k}{1+k}$. It easily follows that

$$e(f; h_c, h) < e(f; h^\#, h)\,.$$

This inequality with (29) implies the statement. □

Now, let $\beta_{c_n} = \mathbb{1} + b_n$ and $h_{c_n} = h(\beta_{c_n}\bullet, \beta_{c_n}\bullet)$, and consider the smoothly grafted metric $h(\beta^\#_n\bullet, \beta^\#_n\bullet)$, where $\beta^\#_n = \cosh(s/2)\mathbb{1} + \sinh(s/2)b_n$, defining the conformal class $J^\#_n$.

By Lemma 6.1 applied to the unique harmonic map $f_n : (S, J^\#_n) \to (S, h)$ isotopic to the identity, we have

$$E(id; c_n, h) \leq E(f_n; c_n, h) \leq \tau^{-1}E(f_n; h^\#_n, h)\;.$$

So, as $h^\star_n$ is diverging, c_n is diverging too and $E(id; c_n, h) \to \infty$. As a consequence, $E(f_n; h^\#_n, h) \to \infty$, which implies that the isotopy class of the underlying complex structure $[J^\#_n]$ is diverging in $\mathcal{T}$ (see [42]).

6.3. **The degree of $sgr'_s(h, \bullet)$.** In this section we will compute the topological degree of the map

$$\mathcal{G} := sgr'_s(h, \bullet) : \mathcal{T} \to \mathcal{T}$$

and we will prove that it is equal to 1.

In fact we will prove that $\mathcal{G}^{-1}(h) = \{h\}$ and that the map $\mathcal{G}$ is locally invertible around h.

Lemma 6.2. *If $h^\# = sgr'_s(h, h^\star)$ represents the same point in $\mathcal{T}$ as h, then $h^\star = h$.*

This lemma is a simple consequence of the following statement.

Proposition 6.3. *Let $\hat{h}$ be the unique hyperbolic metric in the conformal class of $h^\#$. Then $F([\hat{h}], [h^\star]) \leq F([h], [h^\star])$ and the equality holds iff $h = h^\star$.*

Proof. By [5] we have that

$$F([\hat{h}], [h^\star]) = \inf_{[h'] \in \mathcal{T}} E([h'], [\hat{h}]) + E([h'], [h^\star]) \leq E([c], [\hat{h}]) + E([c], [h^\star])$$

so we only need to show that $E([c], [\hat{h}]) \leq E([c], [h]) = E([c], [h^\star])$ and that the equality holds only if $h = h^\star$.

Let us set $\hat{h} = e^{2u} h^\#$ for some function u on S. Since the operator $\beta_\tau = \mathbb{1} + \tau b$ is a self-adjoint solution of the Codazzi equation, the curvature of $h^\#$ is $K^\# = -\det(\beta_\tau)^{-1}$. Notice that $\det \beta_\tau = 1 + \tau^2 + \tau \mathrm{tr} b \geq (1+\tau)^2$ so $K^\# \geq -(1+\tau)^{-2}$ and the equality holds only at points where $b = \mathbb{1}$.

The Liouville equation reads

$$\Delta_{h^\#} u = e^{2u} + K^\# \geq e^{2u} - (1+\tau)^{-2} \ .$$

By the maximum principle we deduce that

$$e^{2u} \leq (1+\tau)^{-2} \ , \tag{30}$$

and if the equality holds at some points, then $b = \mathbb{1}$ everywhere.

Now we have $\hat{h} = e^{2u} h_c((\mathbb{1}+b)^{-2}(\mathbb{1}+\tau b)^2 \bullet, \bullet)$, so

$$E(id; c, \hat{h}) = \frac{1}{2} \int_S e^{2u} \mathrm{tr}[(\mathbb{1}+\tau b)^2 (\mathbb{1}+b)^{-2}] \det(\mathbb{1}+b) da_h \ .$$

On the other hand

$$\det(\mathbb{1}+b) \mathrm{tr}[(\mathbb{1}+\tau b)^2 (\mathbb{1}+b)^{-2}] = \det(\mathbb{1}+b)^{-1} \mathrm{tr}[(\mathbb{1}+\tau b)^2 (\mathbb{1}+b^{-1})^2]$$

where the last equality holds since $\det b = 1$. But we have

$$\begin{aligned}(\mathbb{1}+\tau b)^2(\mathbb{1}+b^{-1})^2 &= [(1+\tau)\mathbb{1} + \tau b + b^{-1}]^2 = \\ &= (1+\tau)^2 \mathbb{1} + (\tau b)^2 + b^{-2} + 2(1+\tau)\tau b + 2(1+\tau) b^{-1} + 2\tau \mathbb{1} \ .\end{aligned}$$

Taking the trace we deduce that

$$\mathrm{tr}[(\mathbb{1}+\tau b)^2(\mathbb{1}+b^{-1})^2] = (1+\tau)^2[2 + 2\mathrm{tr}(b) + \mathrm{tr}(b^2)] - \tau[\mathrm{tr}(b^2) - 2] \ .$$

Using (30) in this identity, we obtain

$$\begin{aligned}E([c], [\hat{h}]) \leq E(id, c, \hat{h}) &= \frac{1}{2} \int_S e^{2u} \mathrm{tr}[(\mathbb{1}+\tau b)^2(\mathbb{1}+b)^{-2}] \det(\mathbb{1}+b) da_h \leq \\ &\leq \frac{1}{2} \int_S \left\{[2 + 2\mathrm{tr}(b) + \mathrm{tr}(b^2)] - 2\tau(1+\tau)^{-2}[\mathrm{tr}(b^2) - 2]\right\} \det(\mathbb{1}+b)^{-1} da_h \ .\end{aligned} \tag{31}$$

On the other hand,

$$\begin{aligned}E([c], [h]) &= E(id; c, h) = \\ &= \frac{1}{2} \int_S \mathrm{tr}[(\mathbb{1}+b)^{-2}] \det(\mathbb{1}+b) da_h = \\ &= \frac{1}{2} \int_S \mathrm{tr}[(\mathbb{1}+b^{-1})^2] \det(\mathbb{1}+b)^{-1} da_h = \\ &= \frac{1}{2} \int_S \mathrm{tr}[\mathbb{1} + 2b^{-1} + b^{-2}] \det(\mathbb{1}+b)^{-1} da_h = \\ &= \frac{1}{2} \int_S [2 + 2\mathrm{tr}(b) + \mathrm{tr}(b^2)] \det(\mathbb{1}+b)^{-1} da_h \ .\end{aligned}$$

Comparing this identity with (31), we get that

$$E([c],[\hat{h}]) \leq E([c],[h]) - \tau(1+\tau)^{-2}\int [\mathrm{tr}(b^2)-2]\det(\mathbb{1}+b)^{-1}da_h \ ,$$

and this completes the proof. □

In order to conclude that the topological degree of $\mathcal{G} = sgr'_s(h,\bullet)$ is 1, it is sufficient to prove that $d\mathcal{G}$ at $[h]$ is non-degenerate.

Consider a one-parameter family of Labourie operators $t \mapsto b_t$ such that $b_0 = \mathbb{1}$ and $\dot{b}$ is non-zero. Notice that $\dot{b}$ is a traceless self-adjoint solution of the Codazzi equation. Now consider the path of complex structure $\widetilde{J}_t$ compatible with $\widetilde{h}^{\#}_t = h(\beta_t, \beta_t)$ and $\beta_t = \cosh(s/2)\mathbb{1} + \sinh(s/2)b_t$.

Since $\widetilde{J}_t = \beta_t^{-1}J\beta_t$, the derivative of $\widetilde{J}_t$ at $t=0$ is the traceless h-self-adjoint operator $\dot{\widetilde{J}} = [\widetilde{J}, \dot{\beta}] = 2J\dot{b}$.

The derivative of the path $[\widetilde{J}_t] \in \mathcal{T}$ at $t=0$ is the projection of $\dot{\widetilde{J}}$ to $T_{[J]}\mathcal{T}$, through the natural map $\mathcal{A} \to \mathcal{T}$. By [10], Codazzi solutions in $T_J\mathcal{A}$ form a complement of the kernel of the projection $T_J\mathcal{A} \to T_{[J]}\mathcal{T}$. Since $\dot{\widetilde{J}}$ lies in this subspace, then it projects to a non-zero vector.

6.4. **Hyperbolic ends.** In this section we use the parameterization of landslide and smooth grafting by the upper half-plane, so that we use the notations SGr' and sgr' as in [5]. Recall from [5, Section 5] that two smooth grafting maps can be considered. One, SGr', takes its values in $\mathcal{CP}$, the space of complex projective structures on S, while the other, sgr', goes to the Teichmüller space of S.

Given two hyperbolic metrics $h, h^\star \in \mathcal{T}$ and $s > 0$, there is a unique equivariant embedding σ of the universal cover $\tilde{S}$ of S inside $\mathbb{H}^3$ with induced metric $\cosh^2(s/2)\tilde{h}$ and third fundamental form $\sinh^2(s/2)\tilde{h}^\star$. Then $SGr'_s(h,h^\star)$ is the complex projective structure induced on S from the complex projective structure on $\partial_\infty\mathbb{H}^3$ by the hyperbolic Gauss map. The complex structure $sgr'_s(h,h^\star)$ is the complex structure underlying $SGr'_s(h,h^\star)$.

The equivariant embedding σ is locally convex by the Gauss formula (the Gaussian curvature of the induced metric is $-1/\cosh^2(s/2) > -1$) so that the quotient of the image of σ by the image of its associated representation of $\pi_1 S$ is a convex surface in a hyperbolic end M. This hyperbolic end is uniquely determined by $h, h^\star$ and s, and its conformal structure at infinity is equal to $sgr'_s(h,h^\star)$.

Proof of Theorem 1.8. According to Theorem 1.7, the map $sgr'_s(h,\bullet) : \mathcal{T} \to \mathcal{T}$ is surjective. This means precisely that, given h and $c \in \mathcal{T}$, there is a $h^\star \in \mathcal{T}$ such that $sgr'_s(h,h^\star) = c$, so that there is a hyperbolic end with complex structure at infinity c containing a surface of constant curvature $-1/\cosh^2(s/2)$ with induced metric homothetic to h. □

We now recall briefly some key points concerning de Sitter domains of dependence, so as to be able to prove Theorem 1.9. A de Sitter domain of dependence is a (non-complete) 3-dimensional manifold locally modelled on the de Sitter space, which is future-complete and globally hyperbolic.

De Sitter domains of dependence are in one-to-one correspondence with hyperbolic ends. One way to see this correspondence is that, given a hyperbolic end, there is a unique de Sitter domain of dependence with the same fundamental group and the same representation of the fundamental group into $\mathrm{PSL}(2,\mathbb{C})$.

However it is perhaps simpler here to characterize this correspondence in terms of convex embedded surfaces. Let M be a hyperbolic end, and let S be a locally strictly convex surface in M which bounds a convex domain. The universal cover $\tilde{S}$ of S is then a complete, locally convex surface in $\mathbb{H}^3$ invariant under the action of the fundamental group of M. The dual surface $\tilde{S}^\star$ is then a strictly future-convex, space-like surface in the de Sitter space $\mathbb{dS}^3$, also invariant under the action of the fundamental group of M but now considered as acting on the de Sitter space. The action of $\pi_1 S$ is free and properly discontinuous on a convex domain $\tilde{C}$ in $\mathbb{dS}^3$ containing $\tilde{S}^\star$, and the quotient is the de Sitter domain of dependence C corresponding to M.

The conformal structure at infinity of C is the same as the conformal structure at infinity of M. It can be defined in terms of the conformal structure at future infinity of C, or in terms of the quotient by $\pi_1 S$ of the boundary at infinity of $\tilde{C} \subset \mathbb{dS}^3$.

Proof of Theorem 1.9. Let $h^\star, h' \in \mathcal{T}$, and let $K^\star \in (-\infty, 0)$. Let $K := K^\star/(1-K^\star)$, so that $K \in (-1,0)$ — thus, K is the curvature of a surface in $\mathbb{H}^3$ dual to a surface of curvature $K^\star$ in $\mathbb{dS}^3$. The second part of Theorem 1.7 implies that there exists $h \in \mathcal{T}$ such that $sgr'_s(h,h^\star) = h'$, where s is chosen so that $-1/\cosh^2(s/2) = K$.

This means precisely that there exists a hyperbolic end M containing a surface S with constant curvature K, with induced metric homothetic to h, third fundamental form homothetic to $h^\star$, and conformal structure at infinity equal to h'.

But then the de Sitter domain of dependence corresponding to M contains a surface $S^\star$ — dual to S — with constant curvature $K^\star$, induced metric proportional to $h^\star$ and third fudamental form proportional to h. This proves the theorem. □

7. The smooth grafting map is symplectic

In this section we consider symplectic properties of the smooth grafting map, and prove Proposition 1.10 and Theorem 1.11. A key point in the proof of Proposition 1.10 will actually be a consequence of the symplectic arguments occuring in the proof of Theorem 1.11.

7.1. **The renormalized volume beyond a K-surface.** A Poincaré-Einstein manifold is a manifold M diffeomorphic to the interior of a compact manifold with boundary $\overline{M}$, with a Riemannian metric g which is Einstein and can be written near the boundary as

$$g = \frac{\overline{g}}{\rho^2} ~,$$

where $\overline{g}$ is a smooth metric on $\overline{M}$ and ρ is a smooth function on $\overline{M}$ vanishing on the boundary and with $\|d\rho\|_{\overline{g}} = 1$ on ∂M. In dimension 3, Poincaré-Einstein manifolds are the same as convex co-compact hyperbolic manifolds.

The volume of a Poincaré-Einstein manifold is always infinite. However it is possible to define a "renormalized volume" which is finite and has interesting properties, see [12]. In even total dimension, this renormalized volume is well-defined, while in odd total dimension it depends on the choice of a metric in the conformal class at infinity.

For quasifuchsian manifolds, in total dimension 3, it makes sense to choose as the metric at infinity the (unique) hyperbolic metric in the conformal class at infinity. The renormalized volume which is then obtained is intimately related to the Liouville functional introduced by Takhtajan and Zograf [37, 38] for the Schottky uniformization and for the punctured sphere, later extended to higher genus surfaces [39].

Here we follow the analysis of the renormalized volume of hyperbolic 3-manifolds developed in [20, 22]. The argument we use is strongly related to that used in [21], so we only sketch the main points. We consider a hyperbolic end M containing a convex surface S of constant curvature, isotopic to the boundary at infinity.

Consider a foliation of a neighborhood of infinity in M by equidistant surfaces $(\Sigma_t)_{t\geq t_0}$, with all leaves between S and the boundary at infinity of M. Let $I_t, I\!I_t, I\!I\!I_t$ and da_t, respectively, be the induced metric, second fundamental form, third fundamental form, and area form of Σ_t, and by $I, I\!I, I\!I\!I$ and da the corresponding quantities on S. For both S and Σ_t we use the unit normal pointing towards infinity in M when defining $I\!I$. We also call H (resp. H_t) the mean curvature of S (resp. Σ_t), that is, $H = \mathrm{tr}_I I\!I$.

Definition 7.1. For all $t \geq t_0$ we denote by V_t the volume of the domain of M bounded by S and Σ_t, and set

$$W_t = V_t - \frac{1}{4}\int_{\Sigma_t} H_t da_t + \frac{1}{2}\int_S H da ~.$$

The following proposition is a direct consequence of the main result of [29, 28].

Proposition 7.2. *In a first-order deformation of M, the first-order variation of W_t is given by:*

$$\frac{dW_t}{dt} = \frac{1}{4}\int_{\Sigma_t}\left(\frac{dH_t}{dt} + \langle\frac{dI_t}{dt}, I\!I_t - \frac{H_t}{2}I_t\rangle\right)da_t - \frac{1}{2}\int_S \langle\frac{dI}{dt}, I\!I - HI\rangle da ~. \quad (32)$$

Proof. According to [29, Theorem 1], in any first-order deformation of M,

$$\frac{dV_t}{dt} = \frac{1}{2}\int_{\Sigma_t}\left(\frac{dH_t}{dt} + \frac{1}{2}\langle\frac{dI_t}{dt}, I\!I_t\rangle\right)da_t - \frac{1}{2}\int_S \left(\frac{dH}{dt} + \frac{1}{2}\langle\frac{dI}{dt}, I\!I\rangle\right)da ~.$$

However an elementary computation shows that

$$\frac{d}{dt}\int_S H da = \int_S \left(\frac{dH}{dt} + \frac{H}{2}\langle\frac{dI}{dt}, I\rangle\right)da ~,$$

and similarly for Σ_t. The result follows by a simple computation. □

Corollary 7.3. *The derivative of W_t with respect to t is given by*

$$\frac{dW_t}{dt} = -\pi\chi(S) ~.$$

Proof. Since the surfaces Σ_t are equidistant, we have

$$\frac{dI_t}{dt} = 2I\!I_t \ , \ \frac{dB_t}{dt} = \mathbb{1} - B_t^2 \ , \ \frac{dH_t}{dt} = 2 - \mathrm{tr}(B_t^2) \ .$$

Replacing this in Equation (32) leads to

$$\begin{aligned} \frac{dW_t}{dt} &= \frac{1}{4}\left(\int_{\Sigma_t} 2 - \mathrm{tr}(B_t^2) + 2\mathrm{tr}(B_t^2) - H_t^2\right) da_t \\ &= \frac{1}{2}\int_{\Sigma_t} (1 - \det(B_t))\, da_t \\ &= \frac{1}{2}\int_{\Sigma_t} (-K_t) da_t \ , \end{aligned}$$

where K_t is the curvature of I_t. The result follows by the Gauss-Bonnet formula. □

Definition 7.4. We define the renormalized volume above S by

$$W := W_t + \pi\chi(S)t \ ,$$

which is clearly independent of the choice of $t \geq t_0$.

Note that W can be defined simply as W_0 if $t_0 \leq 0$, however this is not always the case.
This quantity W depends only on the hyperbolic end M, on S, and on the equidistant foliation of M near infinity. Below we defined another quantity $\mathcal{W}$, depending only on M and on S, obtained by taking a special, canonically defined foliation near infinity.

7.2. **The data at infinity of a hyperbolic end.** Recall that if Σ is a surface in hyperbolic 3-space, and if Σ_t is a surface at constant distance t from Σ, the induced metric on Σ_t can be expressed in terms of the induced metric I and the shape operator B of Σ as:

$$I_t(x, y) = I\big((\cosh(t)\mathbb{1} + \sinh(t)B)x, (\cosh(t)\mathbb{1} + \sinh(t)B)y\big) \ .$$

It follows directly that the induced metrics I_t have a simple asymptotic development as $t \to \infty$, which can be written as:

$$I_t = e^{2t} I_\infty + 2I\!I_\infty + e^{-2t} I\!I\!I_\infty \ ,$$

where $I_\infty, I\!I_\infty$ and $I\!I\!I_\infty$ are bilinear symmetric forms on S which can be expressed quite simply in terms of I_t and B_t for any given value of t. We call ∇^∞ the Levi-Civita connection of I_∞, K_∞ its curvature, $B_\infty : TS \to TS$ the linear map which is self-adjoint for I_∞ and such that

$$I\!I_\infty(x, y) = I_\infty(B_\infty x, y) \quad \forall p \in S, \ \forall x, y \in T_pS$$

and $H_\infty = \mathrm{tr}(B_\infty)$.
The following lemma recalls some properties of this asymptotic expansion, details can be found in [20, 22].

Lemma 7.5. (1) *I_∞ is in the conformal class at infinity of M,*
(2) *I_∞ and B_∞ satisfy the Codazzi equation, $d^{\nabla^\infty} B_\infty = 0$, and a modified version of the Gauss equation, $K_\infty = -H_\infty$,*
(3) *I_∞ and $I\!I_\infty$ together determine uniquely M,*
(4) *any metric I_∞ in the conformal class at infinity of M is obtained from a unique foliation of a neighborhood of infinity in M by equidistant surfaces.*

A key point is that there are simple formulas relating the data $I_t, B_t, I\!I_t, I\!I\!I_t$ on a surface Σ_t to the corresponding data at infinity, see [20, Section 5] or [22]. This leads in particular to the following analog of Proposition 7.2, see [20, Lemma 6.1].

Proposition 7.6. *In a first-order deformation of M and of the foliation $(\Sigma_t)_{t\geq t_0}$, the first-order variation of W is given by:*

$$\frac{dW}{dt} = -\frac{1}{4}\int_{\partial_\infty S}\left(\frac{dH_\infty}{dt} + \langle\frac{dI_\infty}{dt}, I\!I_\infty - \frac{H_\infty}{2} I_\infty\rangle\right) da_\infty - \frac{1}{2}\int_S \langle\frac{dI}{dt}, I\!I - HI\rangle da \ . \tag{33}$$

7.3. **Smooth grafting is symplectic.** Point (4) of Lemma 7.5 in particular is used in the next definition.

Definition 7.7. We let $\mathcal{W}$ be the value of W when the foliation $(\Sigma_t)_{t\geq t_0}$ is the unique foliation such that I_∞ is the hyperbolic metric at infinity.

With this definition, Proposition 7.6 has a direct consequence. Let $(\mathit{II}_\infty)_0$ be the traceless part of II_∞ (with respect to I_∞).

Proposition 7.8. *In a first-order deformation of M, the first-order variation of $\mathcal{W}$ is given by*

$$\frac{d\mathcal{W}}{dt} = -\frac{1}{4}\int_{\partial_\infty S}\langle \frac{dI_\infty}{dt}, (\mathit{II}_\infty)_0\rangle_{I_\infty} da_\infty - \frac{1}{2}\int_S \langle \frac{dI}{dt}, \mathit{II} - HI\rangle_I da_I \ . \tag{34}$$

Proof. This follows directly from Proposition 7.6 using the fact that $H_\infty = -K_\infty = 1$, that $\mathrm{tr}_{I_\infty}\mathit{II}_\infty = H_\infty$, and that $\mathrm{tr}_I \mathit{II} = H$. □

Both terms occuring in (34) can be interpreted in an interesting way.
We need to identify the image by d_1F of a point $(h, h^\star) \in \mathcal{T}\times\mathcal{T}$.

Lemma 7.9. *Let $(h, h^\star) \in \mathcal{T}\times\mathcal{T}$, and let b be the Labourie operator of $(h, h^\star)$. Then $d_1F(h, h^\star) = (h, \beta) \in T^*\mathcal{T}$, where $\beta \in T^*_h\mathcal{T}$ is defined, for any first-order variation $\dot h$ of h, by*

$$\beta(\dot h) = -\frac{1}{2}\int_S \langle \dot h, h(b\bullet, \bullet) - \mathrm{tr}(b)h\rangle_h da_h \ . \tag{35}$$

Proof. If we put $h_t = h(\alpha_t, \alpha_t)$ with α_t positive self-adjoint, then $\dot h = 2h(\dot\alpha\bullet, \bullet)$.
In particular we have

$$\frac{1}{2}\langle \dot h, h(b\bullet, \bullet) - \mathrm{tr}(b)h\rangle_h = \mathrm{tr}(\dot\alpha b) - \mathrm{tr}(\dot\alpha)\mathrm{tr}(b) \ .$$

The result then follows from (15) and Remark 3.10. □

To give a geometric interpretation of this fact, we note that the smooth grafting map $SGr'_s : \mathcal{T}_h \times \mathcal{T}_{h^\star} \to \mathcal{CP}$ can be decomposed as follows. Let $\mathfrak{E}$ be the space of hyperbolic ends. There is a natural homeomorphism $\partial_\infty : \mathfrak{E} \to \mathcal{CP}$ sending a hyperbolic end to its complex projective structure at infinity. Moreover, each hyperbolic end $M \in \mathfrak{E}$ contains a unique convex surface S_M with constant curvature $-1/\cosh^2(s/2)$, and we can consider the map $\kappa_s : \mathfrak{E} \to \mathcal{T}_h \times \mathcal{T}_{h^\star}$ sending M to $(h, h^\star)$, where h and $h^\star$ are the hyperbolic metrics homothetic respectively to the induced metric and to the third fundamental form of S_M. By construction, the following diagram commutes

$$\begin{array}{ccccc} \mathfrak{E} & \xrightarrow{\kappa_s} & \mathcal{T}_h \times \mathcal{T}_{h^\star} & \xrightarrow{d_1F} & T^*\mathcal{T}_h \\ & \searrow^{\partial_\infty} & \downarrow{\scriptstyle SGr'_s} & & \\ & & \mathcal{CP} & \xrightarrow{Sch} & T^*\mathcal{T}_\infty \end{array}$$

where Sch is the Schwarzian derivative with respect to the Fuchsian section.

Definition 7.10. We denote by λ the Liouville form on $T^*\mathcal{T}$.

We can now identify the second integral in (34).

Corollary 7.11. *The pull-back of the Liouville form on $T^*\mathcal{T}_h$ through $d_1F \circ \kappa_s : \mathfrak{E} \to T^*\mathcal{T}_h$ is given by*

$$(d_1F\circ\kappa_s)^*\lambda = \frac{1}{\sinh(s)}\int_S \langle \delta I, \mathit{II} - HI\rangle_I da_I \ .$$

Proof. This follows directly from Lemma 7.9, taking into account the homothetic factors: $I = \cosh^2(s/2)h$, $\mathit{II} = \cosh(s/2)\sinh(s/2)h(b\bullet, \bullet)$ and $\mathit{III} = \sinh^2(s/2)h(b\bullet, b\bullet)$. □

Finally we can identify the first integral in (34). The following lemma is another way to state Lemma 8.3 in [20].

Lemma 7.12. *The pull-back of the Liouville form of $T^*\mathcal{T}_\infty$ through the map $Sch\circ\partial_\infty : \mathfrak{E} \to T^*\mathcal{T}_\infty$ is the 1-form given by*

$$(Sch\circ\partial_\infty)^*\lambda = -\int_{\partial_\infty S}\langle \delta I_\infty, (\mathit{II}_\infty)_0\rangle_{I_\infty} da_\infty \ .$$

Proof of Theorem 1.11. Consider the two 1-forms defined on $\mathfrak{E}$ by

$$\eta(M) = \int_S \langle \delta I, I\!I - HI \rangle_I da_I \ , \qquad \eta_\infty(M) = \int_{\partial_\infty M} \langle \delta I_\infty, (I\!I_\infty)_0 \rangle_{I_\infty} da_{I_\infty} \ .$$

According to Proposition 7.8, we have on $\mathfrak{E}$

$$d\mathcal{W} = -\frac{1}{4}\eta_\infty - \frac{1}{2}\eta \ ,$$

so that

$$d\eta_\infty + 2d\eta = 0 \ .$$

However Corollary 7.11 shows that

$$2\eta = 2\sinh(s)(d_1 F \circ \kappa_s)^* \lambda \ ,$$

while Lemma 7.12 indicates that

$$\eta_\infty = -(Sch \circ \partial_\infty)^* \lambda \ .$$

So

$$2\sinh(s)(d_1 F \circ \kappa_s)^* \lambda = (Sch \circ \partial_\infty)^* \lambda \ ,$$

and, calling $\omega_{can} = d\lambda$ the cotangent symplectic form on $T^*\mathcal{T}$, we have

$$2\sinh(s)(d_1 F \circ \kappa_s)^* \omega_{can} = (Sch \circ \partial_\infty)^* \omega_{can} \ .$$

This proves the result. □

7.4. **Proof of Proposition 1.10.** The proof of Proposition 1.10 is based on the following two lemmas.

Lemma 7.13. *The differential of the map $d_1 F : \mathcal{T} \times \mathcal{T} \to T^*\mathcal{T}$ is an isomorphism at each point.*

Lemma 7.14. *The map $d_1 F : \mathcal{T} \times \mathcal{T} \to T^*\mathcal{T}$ is proper.*

It follows from Lemma 7.13 that $d_1 F$ is a local homeomorphism. Since it is proper, it is a covering. But $T^*\mathcal{T}$ is simply connected and $\mathcal{T} \times \mathcal{T}$ is connected, so $d_1 F$ is a homeomorphism. This concludes the proof of the proposition.

We now turn to the proofs of those lemmas.

Proof of Lemma 7.13. We have seen above that

$$d\eta_\infty = -(Sch \circ \partial_\infty)^* \omega_{can} \ .$$

Since both Sch and ∂_∞ are diffeomorphisms, it follows that $d\eta_\infty$ is non-degenerate.

However we have also seen that $d\eta_\infty + 2d\eta = 0$, so that $d\eta$ is also non-degenerate. Since

$$2d\eta = 2\sinh(s)(d_1 F \circ \kappa_s)^* \omega_{can} \ ,$$

and κ_s is onto, both $d_1 F$ and κ_s have differentials of maximal rank at each point. □

Proof of Lemma 7.14. Let $(h_n, h_n^\star)_{n\in\mathbb{N}}$ be a diverging sequence in $\mathcal{T} \times \mathcal{T}$. If h_n diverges, so does $(d_1 F)_{(h_n, h_n^\star)}$. Hence, up to extracting a subsequence, we can assume that $h_n \to h \in \mathcal{T}$ and that $\theta_n \ell_{h_n^\star} \to \iota(\lambda, \bullet)$, where $\theta_n \to 0$ and λ is a nonzero measured lamination.

Now, the functions $F_n = F_{h_n^\star} : \mathcal{T} \to \mathbb{R}$ determine a sequence $(\theta_n F_n)_{n\in\mathbb{N}}$ that converges to ℓ_λ, uniformly on the compact subsets of $\mathcal{T}$. As the functions F_n and ℓ_λ are real-analytic, the convergence is also C^∞ on the compact subsets of $\mathcal{T}$. Hence, $\theta_n d_{h_n} F_n \to d_h \ell_\lambda$ and so $(d_1 F)_{(h_n, h_n^\star)} = d_{h_n} F_n$ diverges. □

8. Extension to the boundary

Fix $c \in \mathcal{T}$ and let $\mathcal{Q}_c$ be the space of holomorphic quadratic differentials on (S, c). Then consider the Sampson-Wolf map

$$SW_c : \mathcal{Q}_c \longrightarrow \mathcal{T}$$

that assigns to q the class of the unique hyperbolic metric $h = SW_c(\varphi)$ such that the identity $id : (S, c) \to (S, h)$ is harmonic with Hopf differential equal to $\frac{1}{4}\varphi$.

Theorem 8.1 (Sampson [30], Wolf [42])**.** *The map SW_c is a real-analytic diffeomorphism.*

It is well-known since Thurston [9] that it is possible to produce a compactification $\overline{\mathcal{T}}$ of $\mathcal{T}$ by adding the space of projectively measured laminations $\mathbb{P}\mathcal{ML}$ at infinity.

Here we recall that to every nonzero holomorphic quadratic differential φ on a Riemann surface S we can attach a horizontal foliation $\mathcal{F}_+(\varphi)$ (resp. vertical foliation $\mathcal{F}_-(\varphi)$) along which φ restricts as a positive-definite (resp. negative-definite) real quadratic form, which is singular at the points where φ vanishes. Moreover, $\mathcal{F}_+(\varphi)$ (resp. $\mathcal{F}_-(\varphi)$) comes endowed with a measure $|\mathrm{Im}\sqrt{\varphi}|$ (resp. $|\mathrm{Re}\sqrt{\varphi}|$) transverse to its leaves. (A more extensive discussion can be found in [36].)

To every measured foliation $\mathcal{F}$ one can associate a measured lamination, intuitively by "straightening" the leaves of $\mathcal{F}$ to geodesics with respect to some hyperbolic metric. We notice that the measured laminations $\lambda_\pm(\varphi)$ associated to $\mathcal{F}_\pm(\varphi)$ fill the surface in the following sense.

Definition 8.2. A couple (λ_+, λ_-) of measured laminations on S is filling if $i(\lambda_+, \mu) + i(\lambda_-, \mu) > 0$ for every lamination $\mu \neq 0$. We denote by $\mathcal{FML} \subset \mathcal{ML} \times \mathcal{ML}$ the open locus of filling laminations.

The exact correspondence between holomorphic quadratic differentials and filling measured laminations relies on the following result.

Theorem 8.3 (Hubbard-Masur [14]). *The map $\mathcal{Q} \to \mathcal{FML} \cup \{0\}$ defined as $\varphi \mapsto (\lambda_+(\varphi), \lambda_-(\varphi))$ is a homeomorphism.*

In order to extend our construction to some boundary at infinity, the following result will play a key role.

Theorem 8.4 (Wolf [42] [43]). *Let $\overline{\mathcal{Q}}_c$ be the compactification of $\mathcal{Q}_c$ obtained by adding the sphere at infinity $\partial\mathcal{Q}_c = (\mathcal{Q}_c \setminus \{0\})/\mathbb{R}_+$. Then SW_c extends as a homeomorphism*

$$\overline{SW}_c : \overline{\mathcal{Q}}_c \longrightarrow \overline{\mathcal{T}}$$

by defining $\overline{SW}_c([q]) = [\mathcal{F}_-(q)]$ for every $[q] \in \partial\mathcal{Q}_c$.

It will be more practical to work with a de-homogeneized version of the above result. Indeed, in Thurston's picture the space $\mathcal{Y} = \mathcal{T} \times \mathbb{R}_{<0}$ of metrics of constant negative curvature on S (up to isotopy) can be completed as $\overline{\mathcal{Y}} = (\mathcal{ML} \times \{-\infty\}) \cup \mathcal{Y}$ by adding a copy of $\mathcal{ML}$.

Corollary 8.5. *The following map*

$$\begin{aligned} \widehat{SW}_c :\ \mathcal{Q}_c \times [-\infty, 0) &\longrightarrow \overline{\mathcal{Y}} \\ (\varphi, K) &\longmapsto (SW_c(|K|\varphi), K) && \text{if } K \in (-\infty, 0) \\ (\varphi, -\infty) &\longmapsto (\mathcal{F}_-(\varphi), -\infty) && \text{if } K = -\infty \end{aligned}$$

is a homeomorphism.

In order to study the behavior of the landslide flow as the metrics degenerate, we consider the space $\mathcal{DY} = \mathcal{T} \times \mathcal{T} \times \mathbb{R}_{<0}$ of couple of metrics with the same constant negative curvature on S (up to isotopy) and the partial completion $\overline{\mathcal{DY}} = \mathcal{DY} \cup (\mathcal{FML} \times \{-\infty\})$.

Now consider

$$\begin{aligned} \widehat{SW} :\ \mathcal{Q} \times [-\infty, 0) &\longrightarrow \overline{\mathcal{DY}} \\ (c, \varphi, K) &\longmapsto (SW_c(|K|\varphi), SW_c(-|K|\varphi), K) && \text{if } K \in (-\infty, 0) \\ (c, \varphi, -\infty) &\longmapsto (\mathcal{F}_-(\varphi), \mathcal{F}_+(\varphi), -\infty) && \text{if } K = -\infty\ . \end{aligned}$$

Proposition 8.6. *The map $\widehat{SW}$ is a homeomorphism.*

We recall that the extremal length of c with respect to λ depends real-analytically on $c \in \mathcal{T}$ and it satisfies $\mathrm{Ext}_\lambda(c) = \|\varphi\| = 2E(c, \lambda)$, where $E(c, \lambda)$ is the energy of the harmonic map f from c to the $\mathbb{R}$-tree dual to λ, and $\frac{1}{4}\varphi$ is the Hopf differential of f and also the unique holomorphic quadratic differential on c with $\mathcal{F}_-(\varphi) = \lambda$ (see for instance [43]).

Lemma 8.7. *Given $h', h \in \mathcal{T}$ and let $E(h', h)$ be the energy of the unique hamornic map $(S, h') \to (S, h)$ isotopic to the identity. For every $h, h^\star \in \mathcal{T}$, the function $E(\bullet, h) + E(\bullet, h^\star) : \mathcal{T} \to \mathbb{R}_+$ is proper and achieves a unique minimum at the center c of the couple $(h, h^\star)$. Similarly, if $(\lambda, \mu) \in \mathcal{FML}$, then the function $\mathrm{Ext}_\lambda + \mathrm{Ext}_\mu : \mathcal{T} \to \mathbb{R}_+$ is proper and achieves a unique minimum at c, where c is the conformal structure underlying the Hubbard-Masur quadratic differential φ associated to (λ, μ).*

Proof. Properness of the energy function is proven in Proposition 1.2 and the remaining part of the first claim can be found in Theorem 1.10(iv) of [5].

For the second statement, we have $\mathrm{Ext}_\lambda \geq \ell_\lambda^2/(2\pi|\chi(S)|)$ by the definition of extremal length. As $\ell_\lambda + \ell_\mu$ is proper [18], the same holds for $\mathrm{Ext}_\lambda + \mathrm{Ext}_\mu$. Moreover, Gardiner's formula [11] gives

$$d\mathrm{Ext}_\lambda|_{\bullet=c} = -\frac{\varphi}{2}$$

where $\frac{1}{4}\varphi$ is the Hopf differential of the harmonic map from c to the $\mathbb{R}$-tree dual to λ; in other words, φ is also the unique holomorphic quadratic differential on c whose vertical foliation corresponds to λ. Thus, if $d\mathrm{Ext}_\mu|_{\bullet=c} = -\frac{1}{2}\psi$, then c is a minimum if and only if $\varphi = -\psi$ and so μ corresponds to the horizontal foliation of φ. We conclude by Theorem 8.3. □

Remark 8.8. The previous argument also shows that the functions $E(\bullet, h) + E(\bullet, h^\star)$ and $\mathrm{Ext}_\lambda + \mathrm{Ext}_\mu$ have a unique *local* minimum.

Proof of Proposition 8.6. By the above corollary, $\widehat{SW}$ is continuous. Moreover, $\widehat{SW}$ is bijective and its inverse can be described as follows.

Given $(h, h^\star, K)$ with $K \in (-\infty, 0)$, we can assume that $(h, h^\star)$ are normalized representatives. Then we let c be the conformal structure underlying the metric $h + h^\star$, so that $id : (S, c) \to (S, h)$ and $id : (S, c) \to (S, h^\star)$ have Hopf differentials $\frac{1}{4}\varphi$ and $-\frac{1}{4}\varphi$. Finally, $\widehat{SW}^{-1}(h, h^\star, K) = (c, |K|^{-1}\varphi, K)$.

On the other hand, $\widehat{SW}^{-1}(\lambda, \mu, -\infty) = (c, \varphi, -\infty)$, where φ is the Hubbard-Masur c-holomorphic quadratic differential with $\mathcal{F}_-(\varphi) = \lambda$ and $\mathcal{F}_+(\varphi) = \mu$.

In order to show that $\widehat{SW}$ is closed, we consider a sequence $\{(c_n, \varphi_n, K_n)\}$ in $\mathcal{Q} \times [-\infty, 0)$ such that $(h_n, h_n^\star, K_n) = \widehat{SW}(c_n, \varphi_n, K_n)$ converges and we want to show that $\{(c_n, \varphi_n, K_n)\}$ has an accumulation point. Let $K = \lim_{n\to\infty} K_n \in [-\infty, 0)$.

Suppose that $K \in (-\infty, 0)$ and $(h_n, h_n^\star) \to (h, h^\star) \in \mathcal{T} \times \mathcal{T}$. Because harmonic maps and minimal Lagrangian maps depend regularly on the metrics, $c_n = [h_n + h_n^\star] \to c = [h + h^\star]$ and $\varphi_n \to \varphi$, where $\frac{\varphi}{4|K|}$ is the Hopf differential of the harmonic map $(S, c) \to (S, h)$.

Suppose now that $K = -\infty$ and that $|K_n|^{-1}(h_n, h_n^\star) \to (\lambda, \mu) \in \mathcal{FML}$. It follows from [43] that the function $E(\bullet, |K_n|^{-1}h_n)$ converges C^∞ on the compact subsets of $\mathcal{T}$ to $\frac{1}{2}\mathrm{Ext}_\lambda(\bullet)$. By Lemma 8.7, the function $\mathrm{Ext}_\lambda(\bullet) + \mathrm{Ext}_\mu(\bullet)$ achieves a unique minimum at the conformal structure c underlying the quadratic differential φ with foliations (λ, μ). By the above remark, the minima c_n of $E(\bullet, |K_n|^{-1}h_n) + E(\bullet, |K_n|^{-1}h_n^\star)$ converge to c and by [43] we conclude that $\varphi_n \to \varphi$. □

Proof of Proposition 1.13. The landslide flow on $\mathcal{T} \times \mathcal{T}$ can be extended to $\mathcal{DY}$ as $\mathcal{L}_{e^{i\theta}}(h, h^\star, K) = (h_\theta, h_\theta^\star, K)$. It is immediate to see that $\widehat{SW}$ conjugates this landslide flow on $\mathcal{DY}$ with the flow $e^{i\theta} \cdot (c_n, \varphi_n, K) = (c_n, e^{i\theta}\varphi_n, K)$ on $\mathcal{Q} \times (-\infty, 0)$ and so it extends to $\mathcal{Q} \times \{-\infty\} \cong \mathcal{FML} \times \{-\infty\}$. □

Proof of Proposition 1.14. The function F on $\partial\mathcal{DY}$ is given by $F(\lambda, \mu) = 2E(c, \lambda) = 2E(c, \mu) = \|\varphi\|$, where $\frac{1}{4}\varphi$ is the Hopf differential of the harmonic map from c to the $\mathbb{R}$-tree dual to λ and φ is the quadratic differential corresponding to (λ, μ) and c is conformal structure underlying φ, and so $F(\lambda, \mu) = i(\lambda, \mu)$.

Using charts of $\mathcal{FML}$ given by couples of maximal recurrent (and transversely recurrent) train tracks transverse to each other, the symplectic form $\omega_{Th,1} + \omega_{Th,2}$ and the 1-form dF have constant coefficients and so define a local Hamiltonian flow (in charts). We want to show that this local flow is exactly the limit of the landslide flow.

Notice that F is real-analytic on $\mathcal{T} \times \mathcal{T}$ and so it extends as a C^1 function to those points $\mathcal{FML}_{max}$ of $\mathcal{FML}$ that have a tangent space, namely to couples (λ, μ) of maximal measured laminations, which represent a dense subset of full measure.

From [26] and [35] it follows that $K^{-2}\omega_{WP}$ on $\mathcal{Y} = \mathcal{T} \times (-\infty, 0)$ continuously extends as Thurston's symplectic form ω_{Th} at those points of $\mathcal{ML} \times \{-\infty\}$ that represent maximal measured laminations.

Thus, the vector field $\omega_{WP}^{-1}(\frac{1}{4}dF, -)$ that generates the landslide flow converges almost everywhere to $\omega_{Th}^{-1}(\frac{1}{4}dF, -)$. This implies that the landslide flow converges locally uniformly to the flow locally determined by $(\omega_{Th,1} + \omega_{Th,2})^{-1}(\frac{1}{4}dF, -)$ on $\mathcal{FML}$. □

9. AdS geometry and composition of earthquakes

9.1. **Dual constant curvature surfaces in AdS manifolds.**

Definition 9.1. For any $K < -1$, set $K^\star = -K/(K + 1)$.

Our first goal is to prove the special case of Theorem 1.15 when the curvatures of the future and past surfaces satisfy the relation $K_+ = K_-^\star$.

Lemma 9.2. *Let N be a MGH AdS manifold, and let S_+ and S_- be the surfaces in N with curvature K_+ and K_-, respectively, in the future and in the past of the convex core of N. If $K_+ = -K_-^\star$ then S_- is dual to S_+ (and conversely). If we identify S_+ to S_- by the natural duality map, then the third fundamental form of S_+ is equal to the induced metric on S_-, and conversely.*

Proof. It follows from Lemma 2.3 that the surface dual to S_+ is a future convex surface $S_+^\star$ in the past of the convex core of N. Point (3) of Lemma 2.2 then shows that $S_+^\star$ has constant curvature $K_- = K_+^\star$. But according to the main result of [3], there is a unique such space-like surface of constant curvature K_- in the past of the convex core of N, so $S_- = S_+^\star$. Lemma 2.2 then shows that, under the identification of S_+ with S_- by the duality map, the induced metric on S_+ corresponds to the third fundamental form of S_-, and conversely. □

The special case of Theorem 1.15 directly follows.

Lemma 9.3. *Let $h_+, h_- \in \mathcal{T}$, and let $K_+, K_- < -1$ with $K_+ = K_-^\star$. There exists a unique MGH AdS manifold N such that the past-convex surface of constant curvature K_+ in N is homothetic to h_+ while the future-convex surface of constant curvature K_- in N is homothetic to h_-.*

Proof. Given two hyperbolic metrics $h_+, h_- \in \mathcal{T}$ and two constants $K_-, K_+ < -1$ such that $K_- = K_+^\star$, let $I = (-1/K_+)h_+, I\!I\!I = (-1/K_-)h_-$. Consider the identification between (S, h_-) and (S, h_+) by the unique minimal Lagrangian map isotopic to the identity, and let b be the Labourie operator such that $h_- = h_+(b\bullet, b\bullet)$.

Let $k = \sqrt{-1 - K_+}$, and set $B = kb$. Then B is self-adjoint for I, solution of the Codazzi equation for I, and of the AdS Gauss equation $\det(B) = -1 - K_+$. So there exists an equivariant embedding of the universal cover of (S, I) as a space-like, locally strictly convex surface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with shape operator equal to the lift of B to $\tilde{S}$. This implies that there is an isometric embedding of (S, I) in a MGH AdS manifold N, with shape operator equal also to B.

The properties of the duality map in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ then imply that the surface $S^\star$ dual to S in N has induced metric equal to $I\!I\!I$, in particular it has constant curvature K_- and is homothetic to h_-. This already shows the existence of N containing the required surfaces.

The uniqueness of N follows from the same arguments, and from the fact that any MGH AdS manifolds contains a unique past-convex and a unique future-convex surface of any given curvature in $(-\infty, -1)$, see [3], so that for any $K \in (-\infty, -1)$, the past-convex surface of constant curvature $K^\star$ is always dual to the future-convex surface of constant curvature K. □

9.2. **AdS manifolds with constant curvature boundary.** The more general part of Theorem 1.15 will follow from a compactness argument. We will need the following elementary statement on the Teichmüller distance. Given a hyperbolic metric h and a closed curve γ on S, we denote by $\ell_\gamma(h)$ the length of the geodesic for h homotopic to γ.

Lemma 9.4. *Let $R > 1$ and $h \in \mathcal{T}$.*

1. *The set of hyperbolic metrics h' on S such that, for all closed curve γ on S, $\ell_\gamma(h') \leq R\,\ell_\gamma(h)$, is compact.*
2. *Similarly, the set of metrics h' on S such that, for all closed curves γ, $\ell_\gamma(h) \leq R\,\ell_\gamma(h')$, is compact.*

Proof. Recall that Thurston's asymetric distance $d_{Th}(h, h')$ between h and h' is defined as the log of the infimum of the Lipschitz constants over all smooth maps from (S, h) to (S, h') isotopic to the identity (see [41]). It can also be defined as the supremum of the ratio of length for h and for h' of closed curves on S, see [41]. It is known (see [27]) that, if h is fixed, then $d_{Th}(h, h'_n) \to \infty$ as $h'_n \to \infty$. This proves the first point. Similarly, if h' is fixed and $h_n \to \infty$, then $d_{Th}(h_n, h') \to \infty$, and this proves the second point. □

Corollary 9.5. *Let $R > 1$ and $\mathcal{C} \subset \mathcal{T}$ be compact. Let $\mathcal{C}'$ be the set of all metrics $h' \in \mathcal{T}$ such that $\ell_\gamma(h') \leq R\,\ell_\gamma(h)$ (resp. $\ell_\gamma(h) \leq R\,\ell_\gamma(h')$) for some $h \in \mathcal{C}$ and for all closed curve γ on S. Then $\mathcal{C}'$ is compact.*

This corollary will be useful in conjunction which the following basic estimate from AdS geometry.

Lemma 9.6. *Let N be a MGH AdS manifold, let $K < K' < -1$, and let S, S' be the future-convex surfaces of constant curvature K and K', respectively, in N. Let γ be a closed geodesic in S. Then the length of γ is smaller than the length of the closed geodesic γ' in S' homotopic to γ.*

Proof. This follows from the elementary fact that, in a foliation of an AdS manifold by future-convex surfaces (identified by the normal flow), the metric is decreasing when moving towards the past, see e.g. [4]. □

Corollary 9.7. *Let $K_-, K_+ < -1$ with $K_- < K_+^\star$. Let N be a MGH AdS manifold containing a past-convex surface S_+ with induced metric $(-1/K_+)h_+$ and third fundamental form $(-1/K_+^\star)h_+^\star$, so that h_+ and $h_+^\star$ are hyperbolic metrics. Let h_- (resp. $h_-^\star$) be the hyperbolic metric homothetic to the induced metric (resp. third fundamental form) of the future-convex surface S_- of constant curvature K_-. Then*

$$h_- \leq \left(\frac{K_-}{K_+^\star}\right) h_+^\star \ , \qquad h_+ \leq \left(\frac{K_+}{K_-^\star}\right) h_-^\star \ ,$$

where the inequalities are understood in the sense of the length spectrum. Similarly if $K_+ < K_-^\star$ then

$$h_+ \leq \left(\frac{K_+}{K_-^\star}\right) h_-^\star \ , \qquad h_- \leq \left(\frac{K_-}{K_+^\star}\right) h_+^\star \ .$$

Definition 9.8. Let $K_-, K_+ < -1$. We denote by $\Phi_{K_-,K_+} : \mathcal{T} \times \mathcal{T} \to \mathcal{T} \times \mathcal{T}$ the map sending (h_l, h_r) to the hyperbolic metrics h_-, h_+ such that the MGH AdS manifold N with left and right metrics h_l and h_r contains a past-convex surface of constant curvature K_+ with induced metric $(-1/K_+)h_+$, and a future-convex surface of constant curvature K_- with induced metric $(-1/K_-)h_-$.

It follows from Lemma 9.3 that $\Phi_{K_-,K_-^\star}$ is a homeomorphism for all $K_- < -1$. To prove Theorem 1.15, we will show that Φ_{K_-,K_+} is "bounded" in a suitable sense by $\Phi_{K_-,K_-^\star}$ or $\Phi_{K_+^\star,K_+}$.

Corollary 9.9. *For all $K_-, K_+ < -1$, Φ_{K_-,K_+} is proper.*

Proof. We consider two cases, depending on whether K_- is smaller or larger than $K_+^\star$. Assume first that $K_- < K_+^\star$. Let $\mathcal{DC} \subset \mathcal{T} \times \mathcal{T}$ be compact, and let $\mathcal{C}_-, \mathcal{C}_+$ be two compact subsets of $\mathcal{T}$ such that $\mathcal{DC} \subset \mathcal{C}_- \times \mathcal{C}_+$. Suppose that $\Phi_{K_-,K_+}(h_l, h_r) = (h_-, h_+) \in \mathcal{DC}$. Then $\Phi_{K_-,K_-^\star}(h_l, h_r) = (h_-, h_-^\star)$ with $h_+ \leq \left(K_+/K_-^\star\right) h_-^\star$ by Corollary 9.7. It follows that $h_-^\star$ is in a compact set $\mathcal{C}_-^\star$ which depends only on $\mathcal{C}_+$ and on $K_+/K_-^\star$ by Corollary 9.5.

Since $\Phi_{K_-,K_-^\star}$ is a homeomorphism, $\Phi^{-1}_{K_-,K_-^\star}(\mathcal{C}_- \times \mathcal{C}_-^\star)$ is a compact subset $\mathcal{DC}'$ of $\mathcal{T} \times \mathcal{T}$. By construction, $(h_l, h_r) \in \mathcal{DC}'$ whenever $(h_-, h_+) \in \mathcal{DC}$. This shows that Φ_{K_-,K_+} is proper.

The same argument proves the same result when $K_- > K_+^\star$, except that now $h_+ \leq \left(K_+/K_-^\star\right) h_-^\star$ and the other inequality has to be used in Corollary 9.5. □

Proof of Theorem 1.15. As Φ_{K_-,K_+} is proper, its degree is well-defined for all $K_-, K_+ < -1$. Moreover, it easily follows from the above corollary that, for every $K_{min} < K_{max} < -1$, the map $\Phi : [K_{min}, K_{max}]^2 \times \mathcal{T} \times \mathcal{T} \to \mathcal{T} \times \mathcal{T}$ defined as $\Phi(K_-, K_+, h_l, h_r) := \Phi_{K_-,K_+}(h_l, h_r)$ is proper. Hence, the degree of Φ_{K_-,K_+} does not depend on the chosen (K_-, K_+), and in particular it coincides with the degree of $\Phi_{K_-,K_-^\star}$. But we already know that $\Phi_{K_-,K_-^\star}$ is a homeomorphism. Hence, for all $K_-, K_+ < -1$ the map Φ_{K_-,K_+} has degree 1 and so it is onto. This proves the theorem. □

9.3. **Prescribed third fundamental forms.** Theorem 1.18 follows from Theorem 1.15 through the duality between constant curvature surfaces in MGH AdS manifolds (see Lemma 2.3). In particular, if N is a MGH AdS manifold containing a past-convex surface S_+ of constant curvature K_+ with induced metric homothetic to h_+ and a future-convex surface S_- of constant curvature K_- and induced metric homothetic to h_-, then the surface $S_+^\star$ dual to S_+ is future-convex, has constant curvature $K_+^\star$ and third fundamental form homothetic to h_+, while the surface $S_-^\star$ dual to S_- has constant curvature $K_-^\star$ and third fundamental form homothetic to h_-.

9.4. **Fixed points of compositions of landslides.** We now turn to the proof of Theorem 1.6.

The relationship between constant curvature surfaces in MGH AdS manifolds and landslides is captured in the following statement, strongly analoguous to a well-known statement for earthquakes, see [25, 1, 6].

Lemma 9.10. *Let N be a MGH AdS manifold, with left and right hyperbolic metrics h_l, h_r. Let $K_-, K_+ < -1$, and let S_-, S_+ be the future-convex and past-convex surfaces with constant curvature K_- and K_+, respectively. Let h_-, h_+ (resp. $h_-^\star, h_+^\star$) be the hyperbolic metrics homothetic to the induced metrics (resp. third fundamental forms) on S_- and S_+, respectively. Then*

$$h_l = \mathcal{L}^1_{e^{it_+}}(h_+, h_+^\star) \ , \qquad h_r = \mathcal{L}^1_{e^{-it_+}}(h_+, h_+^\star) \ , \tag{36}$$

$$h_l = \mathcal{L}^1_{e^{-it_-}}(h_-, h_-^\star) \ , \qquad h_r = \mathcal{L}^1_{e^{it_-}}(h_-, h_-^\star) \ , \tag{37}$$

where $K_+ = -1/\cos^2(t_+/2), K_- = -1/\cos^2(t_-/2)$.

Conversely, if (36) and (37) are satisfied then there exists a MGH AdS manifold N with left and right hyperbolic metrics h_l, h_r, containing a past-convex surface S_+ with constant curvature K_+ and induced metric *and third fundamental form homothetic to* h_+ *and* $h_+^\star$*, and a future convex surface* S_- *with constant curvature* K_- *and induced metric and third fundamental form homothetic to* h_- *and* $h_-^\star$*.*

Proof. The first point follows directly from [5, Lemma 1.9]. The converse also follows from the same lemma, because a MGH AdS manifold is uniquely determined by its left and right hyperbolic metrics (see [25]) so that the MGH AdS manifold containing a past-convex space-like surface of curvature K_+ with I and $I\!I\!I$ respectively homothetic to h_+ and $h_+^\star$ is the same as the MGH AdS manifold containing a future-convex space-like surface of curvature K_- with I and $I\!I\!I$ respectively homothetic to h_- and $h_-^\star$. □

Proof of Theorem 1.6. Let $\theta_-, \theta_+ \in (0,\pi)$, and let $h_-, h_+ \in \mathcal{T}$. Set $K_+ = -1/\cos^2(\theta_+/2)$ and $K_- = -1/\cos^2(\theta_-/2)$. Theorem 1.15 indicates that there exists a MGH AdS manifold N containing a past-convex space-like surface of constant curvature K_+ proportional to h_+, and a future-convex space-like surface of constant curvature K_- proportional to h_-. Moreover, if $\theta_- + \theta_+ = \pi$, then N is unique.

Let h_l, h_r be the left and right hyperbolic metrics of N. Lemma 9.10 then shows that $h_r = \mathcal{S}_{e^{i\theta_+},h_+}(h_l)$, while $h_l = \mathcal{S}_{e^{i\theta_-},h_-}(h_r)$. Thus h_r is a fixed point of $\mathcal{S}_{e^{i\theta_+},h_+} \circ \mathcal{S}_{e^{i\theta_-},h_-}$. This proves the existence part of the statement.

The uniqueness part when $\theta_- + \theta_+ = \pi$ follows from the uniqueness of N in this case, together with the converse part of Lemma 9.10. □

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Università degli Studi di Pavia, Via Ferrata, 1, 27100 Pavia, Italy
E-mail address: `francesco.bonsante@unipv.it`

Università di Roma "La Sapienza" - Dipartimento di Matematica "Guido Castelnuovo", piazzale Aldo Moro 5, 00185 Roma, Italy
E-mail address: `mondello@mat.uniroma1.it`

Institut de Mathématiques de Toulouse, UMR CNRS 5219, Université Toulouse III, 31062 Toulouse cedex 9, France
E-mail address: `schlenker@math.univ-toulouse.fr`