ON IMMERSIBLE G-STRUCTURES

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Abstract. The paper revisits and extends the theory of induced G-structures introduced by A. A. Rosly and A. S. Schwarz in [17].
Let N be a n-dimensional smooth manifold endowed with a H-structure, i.e. a reduction $p : Q \to N$ of the principal bundle F_N of all linear frames on N to a Lie subgroup H of $\mathrm{GL}_n(\mathbb{R})$. Any m-dimensional submanifold M of N, satisfying fairly general regularity conditions, inherits a reduction $\pi : P \to M$ of F_M to a Lie subgroup G of $\mathrm{GL}_m(\mathbb{R})$, called the G-structure induced by the ambient geometry (N, Q).
We estabilish necessary and sufficient conditions for a G-structure on a manifold M to be locally equivalent to the G-structure induced by an homogeneous ambient geometry $(N, Q) = (\tilde{H}/\tilde{K}, \tilde{H}/K)$, where K denotes the kernel of the isotropy representation $i : \tilde{K} \to \mathrm{GL}_n(\mathbb{R})$.
In the special case of integrable ambient geometry $(\mathbb{R}^n, \mathbb{R}^n \times H)$, the obstructions to constructing local equivalences are shown to be functions with values in the cohomology groups $H^{p,2}_R(\mathfrak{h})$ of a "restricted" Spencer cochain complex. Several examples are described in detail.

1. Introduction

Let M be a m-dimensional manifold and denote by

$$F_M = \{\text{linear isomorphism } \epsilon_x : \mathbb{R}^m \to T_xM \,|\, x \in M\} \tag{1.1}$$

the $\mathrm{GL}_m(\mathbb{R})$-principal bundle of all linear frames on M. A G-structure on M is a reduction $\pi : P \to M$ of (1.1) to a Lie subgroup G of $\mathrm{GL}_m(\mathbb{R})$. Quite common examples of G-structures are provided by Riemannian metrics, conformal structures, almost complex structures, etc.

The general theory of G-structures is a well-estabilished topic in Mathematics, see e.g. [10, 24] as a starting point to the vast literature on the subject. Of particular interest was the so called *(local) integrability problem*, that is the problem of determining whether there exists, around a fixed point $x_o \in M$, a system of coordinates $\{x^i : U \to \mathbb{R}\}$ with the property that the local frame

$$\epsilon : e_i \mapsto \frac{\partial}{\partial x^i}\,, \qquad \{e_i\} = \text{std. basis of } \mathbb{R}^m$$

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belongs to $P|_x = \pi^{-1}(x)$ at any point $x \in U$. In the above mentioned cases, an integrable G-structure corresponds to a flat Riemannian space, a locally conformally flat manifold and a complex manifold, respectively.

The integrability problem was completely solved (for G-structures of finite type) by Guillemin in [7]. In particular, he showed that the Spencer cohomology group $H^{p,2}(\mathfrak{g})$ associated to the Lie algebra $\mathfrak{g}$ of G describes the space of p-order obstructions to integrability, around each point $x_o \in M$.

This paper deals with weaker notions of integrability for G-structures. They are motivated as follows. Assume $\pi : P \to M$ is induced by an immersion of M into an ambient space N which is endowed with an integrable H-structure, for a Lie subgroup H of $\mathrm{GL}_n(\mathbb{R})$ (see §2 for the precise definition of induced structure and the relation between the groups G and H). We call *H-immersible* any G-structure which is locally of this kind.
Of course, in the special case $\dim M = \dim N$ (i.e. M open in N), one has $G = H$ and H-immersibility coincides with the usual notion of integrability.

The notions of induced G-structure and H-immersibility were first introduced by Rosly and Schwarz in [17]. At that time, the main motivation to study such concepts was related to the interpretation of the torsion and curvature constraints appearing in various $D = 4$ supergravity theories. They estabilished sufficient and necessary conditions for a given G-structure to be H-immersible (see Theorem 4.2 of this paper) and considered the Spencer cohomology groups $H^{p,2}(\mathfrak{h}|_{\mathbb{R}^m})$ to describe part of the obstructions.

More common examples of H-immersible G-structures are provided by submanifolds of a flat Riemannian space and immersed CR manifolds. The formal obstructions to H-immersibility correspond to the Gauss-Codazzi-Ricci equations and to the usual CR integrability conditions, respectively.

The main aim of this paper is to revisit and extend the results of [17]. In particular, Theorem 3.1 of this paper extends the above mentioned Theorem 4.2 to the case of an homogeneous ambient space. The result is the following.

Theorem. *Let M be a m-dimensional manifold endowed with a G-structure $\pi : P \to M$ whose soldering form is denoted by $\theta : TP \to \mathbb{R}^m$ and $N = \tilde{H}/\tilde{K}$ a n-dimensional homogeneous space together with its canonically associated H-structure $p : Q \to N$. Assume that the Lie group*

$$G = \{g \in \mathrm{GL}_m(\mathbb{R}) \,|\, \exists\, h \in H \text{ with } h = \begin{pmatrix} g & * \\ 0 & * \end{pmatrix}\} \ .$$

Then there exists a local immersion $\imath : M \to N$ such that P is locally induced by (N, Q) if and only if there exists a 1-form $\omega^{\mathrm{ind}} \in \Omega^1_{loc}(M, \tilde{\mathfrak{h}})$ satisfying the Maurer-Cartan equation and such that $\varphi \circ \omega^{\mathrm{ind}} = \sigma^* \theta$ for some local section $\sigma : M \to P$, where $\varphi : \tilde{\mathfrak{h}} \to \tilde{\mathfrak{h}}/\tilde{\mathfrak{k}}$ denotes the canonical projection.

The above result is suitable to study immersions of G-structures into Klein geometries which are mutant in the sense of [21], moreover the arguments

used in its proof can be easily adapted to deal with G-structures of higher order [20].

In §4.2, the framework of immersible G-structures is used to derive the following.

Theorem. *A m-dimensional Riemannian manifold (M, g) admits a local conformal immersion into the flat Riemannian space $\mathbb{R}^n$ if and only if the following conformal analogue of the Gauss-Codazzi-Ricci equations hold*

$$\boxed{\begin{aligned}
&g(R_{XY}Z,W) - g_E(\alpha(Y,Z),\alpha(X,W)) + g_E(\alpha(X,Z),\alpha(Y,W)) = \\
&= -g \owedge B(X,Y,Z,W) \\
&(\nabla_X\alpha)(Y,Z) - (\nabla_Y\alpha)(X,Z) = D(X)g(Y,Z) - D(Y)g(X,Z) \\
&0 = g_E(R_{XY}\mu,\nu) + \sum_i g_E(\alpha(X,E_i),\mu)g_E(\alpha(Y,E_i),\nu) + \\
&\quad - \sum_i g_E(\alpha(X,E_i),\nu)g_E(\alpha(Y,E_i),\mu)
\end{aligned}}$$

together with the following third order immersibility conditions

$$\boxed{\begin{aligned}
&(\nabla_X B)(Y,Z) + g_E(D(X),\alpha(Y,Z)) && \text{symmetric in } X \text{ and } Y \\
&(\nabla_X D)(Y,\mu) + \sum_i g_E(\alpha(X,E_i),\mu)B(Y,E_i) && \text{symmetric in } X \text{ and } Y
\end{aligned}}$$

The precise definition of all tensors α, B, etc. is in §4.2.

The main result of §5 is given by Proposition 5.2 where we show that the cohomology groups $H_R^{p,2}(\mathfrak{h})$, associated to a differential cochain complex which generalizes the usual Spencer complex of a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_n(\mathbb{R})$, describe the spaces of obstructions to H-immersibility.
We decided to call such $H_R^{p,2}(\mathfrak{h})$ the *restricted Spencer cohomology groups*. We will calculate them in [19] in the special case $\mathfrak{h} = \mathfrak{gl}_n(\mathbb{H}) \oplus \mathfrak{sp}(1)$, to study the necessary and sufficient conditions under which a CR quaternionic manifold [13] is immersible into the quaternionic projective space $\mathbb{H}\mathrm{P}^n$ endowed with its canonical quaternionic structure [18].

The paper is organized as follows: in §2 we give the basic definitions and properties of induced G-structures; in §3 we recall how to associate a H-structure to any homogeneous manifold $N = \tilde{H}/\tilde{K}$ and then prove the above mentioned Theorem 3.1 on G-structures induced by a homogeneous ambient geometry; in §4 we regain the main result of [17] by specializing Theorem 3.1 to the case of an integrable ambient geometry, and consider in detail two examples; in §5 we give the basic definitions and properties of restricted Spencer cohomology groups and present few more examples.

Notations. Given a manifold M, we denote by T_xM its tangent space at a point $x \in M$ and by $TM := \cup_{x\in M} T_xM$ its tangent bundle. The class of all local smooth sections of TM is denoted by $\mathfrak{X}_{loc}(M)$ while that of all local smooth p-forms with values in a fixed vector space V by $\Omega^p_{loc}(M, V)$.

Conventions. We mainly stick to the conventions adopted in [5]: in particular the exterior differential d acting on $\Omega^\bullet_{loc}(M,V)$ is defined accordingly. The curvature of a linear connection ∇ is given by the following expression:

$$R_{XY}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \ . \tag{1.2}$$

Note that (1.2) is the opposite of that considered in some texts, as e.g. [2]. Finally, the bracket $[\alpha,\beta]$ between forms $\alpha,\beta\in\Omega^\bullet_{loc}(M,\mathfrak{h})$ with values in a Lie algebra $\mathfrak{h}$ is defined as in [21].

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2. Preliminaries

Let N be a n-dimensional manifold endowed with a H-structure, that is a reduction

$$p : Q \to N \ , \tag{2.1}$$

to a Lie subgroup H of $\mathrm{GL}_n(\mathbb{R})$, of the principal bundle

$$\begin{aligned} F_N &= \{\text{linear isomorphism } \epsilon_x : \mathbb{R}^n \to T_xN \,|\, x\in N\} \\ &\simeq \{\text{frame } (v_1,\dots,v_n) \text{ of } T_xN \,|\, x\in N\} \end{aligned}$$

of all linear frames on N. Consider a m-dimensional submanifold M of N, where $m\leq n$ (we do not discard the case M open in N). We will now see that, under fairly general assumptions, the H-structure (2.1) induces a G-structure on M, the Lie subgroup G of $\mathrm{GL}_m(\mathbb{R})$ being an appropriate subquotient of H.

Definition 2.1. A submanifold M of N is called *regular* if at any point $x\in M$ there is a frame $(v_1,\dots,v_m)$ of T_xM which can be completed to a frame $(v_1,\dots,v_m,v_{m+1},\dots,v_n)$ of T_xN belonging to $Q|_x = p^{-1}(x)$.

Any frame $(v_1,\dots,v_n)$ of T_xN as in Definition 2.1 is called *adapted* and the set of all adapted frames is denoted by

$$P^{adp} = \{\text{adapted frame } (v_1,\dots,v_n) \text{ of } T_xN \,|\, x\in M\} \ . \tag{2.2}$$

On the other hand, any frame $(v_1,\dots,v_m)$ of T_xM which can be completed to an adapted frame is called *induced* and the set of all induced frames is denoted by

$$P^{ind} = \{\text{induced frame } (v_1,\dots,v_m) \text{ of } T_xM \,|\, x\in M\} \ . \tag{2.3}$$

Notation. Matrices in $\mathfrak{gl}_n(\mathbb{R})$ are represented in block form w.r.t. the decomposition $\mathbb{R}^n = \mathbb{R}^m\oplus\mathbb{R}^{n-m}$, i.e. any $h\in\mathfrak{gl}_n(\mathbb{R})$ is denoted by

$$h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \ , \tag{2.4}$$

for blocks $h_{11} \in \mathfrak{gl}_m(\mathbb{R})$, $h_{22} \in \mathfrak{gl}_{n-m}(\mathbb{R})$, $h_{12} \in \mathrm{Mat}(m, n-m; \mathbb{R})$ and $h_{22} \in \mathrm{Mat}(n-m, m; \mathbb{R})$.

The main result of this section is the following.

Proposition 2.2. *Let M be a regular submanifold of a manifold N which is endowed with a H-structure $p : Q \to N$. If we denote by $\imath : M \hookrightarrow N$ the defining embedding, then:*

i) *The set P^{adp} of adapted frames is a reduction of the pull-back bundle $\imath^*Q \to M$ to the subgroup*

$$H' = \{h \in H \,|\, h = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$$

made up of elements of H preserving the subspace $\mathbb{R}^m$ of $\mathbb{R}^n$.

ii) *The set P^{ind} of induced frames is endowed with a natural projection $\pi : P^{\mathrm{ind}} \to M$ together with a right action of the Lie group*

$$G = \{g \in \mathrm{GL}_m(\mathbb{R}) \,|\, \exists\, h \in H' \text{ with } h = \begin{pmatrix} g & * \\ 0 & * \end{pmatrix}\}$$

which makes $\pi : P^{ind} \to M$ a G-structure on M.

Proof. First note that the set P^{adp} inherits from $\imath^*Q$ a natural projection

$$\pi : P^{adp} \to M\,, \qquad (v_1, \ldots, v_n) \to x \quad \text{if} \quad (v_1, \ldots, v_n) \in P^{adp} \cap Q|_x \quad (2.5)$$

and a natural right H'-action which is transitive on the fibers of (2.5). Part (i) of the proposition is then true if one shows the existence of local smooth sections of $\imath^*Q \to M$ taking values in P^{adp} (see pag. 84 of [11]). This is now accomplished by exhibiting a suitable set of local coordinates of $\imath^*Q$.
Let $\{x^i\}_{i=1}^m$ be a set of coordinates defined on an open set U of M which is completed to a set $\{x^i\}_{i=1}^n$ of coordinates of N and consider also the local identification $\imath^*Q \simeq U \times H$ determined by a fixed smooth local section

$$x \to (\sum_{i=1}^n \psi_1^i(x)\frac{\partial}{\partial x^i}, \ldots, \sum_{i=1}^n \psi_n^i(x)\frac{\partial}{\partial x^i}) \in Q|_x\,, \quad x \in U$$

of the bundle $\imath^*Q$. Denote by $\psi : U \to \mathrm{GL}_n(\mathbb{R})$ the function given by $\psi(x) = (\psi_j^i(x))$ and by P the parabolic subgroup of $\mathrm{GL}_n(\mathbb{R})$ made up of invertible matrices preserving the subspace $\mathbb{R}^m$. One can then easily check that the (locally defined) map

$$\Psi : \imath^*Q \to \mathbb{R}^m \times \mathrm{GL}_n(\mathbb{R})/P$$

given by

$$(x, h) \to (x^1(x), \ldots, x^m(x), [\psi(x)h] \bmod P)\,, \quad x \in U\,,\ h \in H$$

satisfies $\Psi^{-1}(\mathbb{R}^m \times [\mathrm{Id}]) = \pi^{-1}(U) \subset P^{adp}$ and has constant rank equal to $m + (\dim H - \dim H')$ at all points of $\imath^*Q$. By Theorem 15.5 of [9], one gets

systems of local coordinates $\{\xi^\alpha\}$ of H and $\{\eta^\beta\}$ of $\mathrm{GL}_n(\mathbb{R})/P$ such that, for indices $1 \leq i \leq m$, $1 \leq \alpha \leq \dim H'$ and $1 \leq \beta \leq \dim H - \dim H'$, the set

$$\{x^i, \xi^\alpha, \eta^\beta \circ f\} \tag{2.6}$$

is a set of local coordinates of $\imath^*Q$ satisfying

$$\Psi^{-1}(\mathbb{R}^m \times [\mathrm{Id}]) = \{\eta^\beta \circ f = 0\} \ .$$

The required smooth section of $\imath^*Q \to M$ is then given in coordinates (2.6) simply by $(x^1, \dots, x^m) \mapsto (x^1, \dots, x^m, 0, \dots, 0)$.
Part (ii) of the proposition is now a direct consequence of the following two identifications

$$P^{ind} \simeq P^{adp}/H'' \,, \qquad G \simeq H'/H'' \,, \tag{2.7}$$

where $H'' = \{h \in H' \,|\, h = \begin{pmatrix} \mathrm{Id} & * \\ 0 & * \end{pmatrix}\}$ is a closed normal subgroup of H'.
In particular, the bundle projection $\pi : P^{ind} \to M$ is nothing else that the map induced on the quotient by the projection (2.5). □

The following definition is well-posed due to Proposition 2.2.

Definition 2.3. The G-structure $\pi : P^{\mathrm{ind}} \to M$ canonically associated to a regular submanifold M of N as in Proposition 2.2 is called the *G-structure induced by the ambient geometry* (N, Q).

3. G-structures induced by a homogeneous ambient geometry

We now restrict ourselves to the case of an ambient geometry which determined by a homogeneous manifold $N = \tilde{H}/\tilde{K}$, where $\tilde{H}$ denotes a Lie group acting transitively and effectively on N and $\tilde{K}$ is the stabilizer at a fixed point $o \in N$. We first recall how to naturally associate a H-structure $p : Q \to N$ to any homogeneous manifold $N = \tilde{H}/\tilde{K}$. After that, we prove a result giving necessary and sufficient conditions for a G-structure $\pi : P \to M$ on a manifold M to be locally equivalent to the G-structure $\pi : P^{\mathrm{ind}} \to M$ induced by an homogeneous ambient geometry (N, Q).

Convention. We always make use of the canonical identification $T_oN \simeq \tilde{\mathfrak{h}}/\tilde{\mathfrak{k}}$, where $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{k}}$ denote the Lie algebras of $\tilde{H}$ and $\tilde{K}$ respectively. We also assume that a complement $\tilde{\mathfrak{h}} = \tilde{\mathfrak{k}} \oplus \mathbb{R}^n$ has been fixed once and for all (the examples in §4 and §5 always admit a natural choice of it). In particular, an identification $T_oN \simeq \mathbb{R}^n$ has also been fixed once and for all.

Consider the isotropy representation of the stabilizer $\tilde{K}$:

$$i : \tilde{K} \to \mathrm{Aut}\,(T_oN) \simeq \mathrm{GL}_n(\mathbb{R}) \,, \qquad \tilde{k} \mapsto dL_{\tilde{k}}|_o : T_oN \to T_oN \,, \tag{3.1}$$

where $L_{\tilde{k}} : N \to N$ is the left action of $\tilde{k} \in \tilde{K}$ on $N = \tilde{H}/\tilde{K}$. Denoting the kernel of (3.1) by $K = \operatorname{Ker} i$, one can always consider the principal bundle

$$p : \tilde{H}/K \to \tilde{H}/\tilde{K} \tag{3.2}$$

whose structure group is $\tilde{K}/K$, together with the following commutative diagram

$$\begin{array}{ccccc} & & \tilde{H}/K & \longrightarrow & F_N & & \\ & & & & & & \\ \tilde{K}/K & & \downarrow & & \downarrow & \mathrm{GL}_n(\mathbb{R}) & . \\ & & & & & & \\ & & \tilde{H}/\tilde{K} & = & N & & \end{array}$$

Here the injective bundle morphism $\tilde{H}/K \to F_N$ is the one induced by the map

$$\tilde{H} \to F_N\,, \quad \tilde{h} \mapsto dL_{\tilde{h}}|_o : \mathbb{R}^n \simeq T_oN \to T_{[\tilde{h}]}N\ ,$$

while the Lie group monomorphism $\tilde{K}/K \to \mathrm{GL}_n(\mathbb{R})$ is induced by (3.1). It follows that the bundle (3.2) is identifiable with a reduction of F_N, i.e. with a H-structure $p : Q \to N$ where $H \simeq \tilde{K}/K$. We conclude by noticing that the soldering form

$$\vartheta \in \Omega^1(\tilde{H}/K, \tilde{\mathfrak{h}}/\tilde{\mathfrak{k}})\,, \quad w \to (dL_{\tilde{h}}|_o)^{-1}(dp|_{[\tilde{h}]}(w)) \quad \text{if} \quad w \in T_{[\tilde{h}]}\tilde{H}/K$$

of the H-structure (3.2) is uniquely determined by the Maurer-Cartan form $\omega_{MC} : T\tilde{H} \to \tilde{\mathfrak{h}}$ of the Lie group $\tilde{H}$ via the equation

$$\phi^*\vartheta = \varphi \circ \omega_{MC}\ , \tag{3.3}$$

where $\phi : \tilde{H} \to \tilde{H}/K$ and $\varphi : \tilde{\mathfrak{h}} \to \tilde{\mathfrak{h}}/\tilde{\mathfrak{k}}$ are the canonical projections.

We are now ready to state and prove the following.

Theorem 3.1. *Let M be a m-dimensional manifold endowed with a G-structure $\pi : P \to M$ and $N = \tilde{H}/\tilde{K}$ a n-dimensional homogeneous space together with its canonically associated H-structure $p : Q \to N$, where $Q \simeq \tilde{H}/K$, $H \simeq \tilde{K}/K$ and K is the kernel of the isotropy representation* (3.1)*. Assume that the Lie group*

$$H' = \{h \in H \,|\, h = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$$

made up of elements of $H \subset \mathrm{GL}_n(\mathbb{R})$ preserving the subspace $\mathbb{R}^m$ of $\mathbb{R}^n$ satisfies

$$G = \{g \in \mathrm{GL}_m(\mathbb{R}) \,|\, \exists\, h \in H' \text{ with } h = \begin{pmatrix} g & * \\ 0 & * \end{pmatrix}\}\ .$$

Fix a point $x_o \in M$. Then there exists a local immersion $\imath : M \to N$ with $\imath(x_o) = o$ and such that P is locally equivalent to the G-structure P^{ind} induced by the ambient geometry (N, Q) if and only if there exists a 1-form $\omega^{\mathrm{ind}} \in \Omega^1_{loc}(M, \tilde{\mathfrak{h}})$ defined around x_o and satisfying

i) $\varphi \circ \omega^{\mathrm{ind}} : TM \to \tilde{\mathfrak{h}}/\tilde{\mathfrak{k}} \simeq \mathbb{R}^n$ *takes values into* $\mathbb{R}^m$ *and*

$$\varphi \circ \omega^{\mathrm{ind}} = \sigma^* \theta$$

for some local section $\sigma : M \to P$ *of the* G*-structure* $\pi : P \to M$*, whose soldering form has been denoted by* $\theta : TP \to \mathbb{R}^m$*;*

ii) the Maurer-Cartan equation $d\omega^{ind} + \frac{1}{2}[\omega^{ind}, \omega^{ind}] = 0$*.*

Proof. We first prove the necessary implication: Assume there exists a local immersion $\imath : M \to N$ such that P is the G-structure induced by the homogeneous ambient geometry (N, Q). Let $\varsigma : M \to P^{\mathrm{adp}}$ be a local section of the bundle $P^{\mathrm{adp}} \subset \imath^* Q$ of adapted frames and denote by $\sigma : M \to P^{\mathrm{ind}}$ the section induced via the identifications (2.7). By local triviality of the fibration $\phi : \tilde{H} \to Q$, one also gets a local section $\Sigma : M \to \imath^* \tilde{H}$ satisfying $\phi \circ \Sigma = \varsigma$. Consider the following commutative diagram

$$\begin{array}{ccccccccc} & & & & & & \imath^*\tilde{H} & \longrightarrow & \tilde{H} & & \\ & & & & & & \downarrow & & \downarrow & \phi & \\ & & P^{ind} & \leftarrow & P^{adp} & \subset & \imath^* Q & \longrightarrow & Q & & \\ \pi & & \downarrow & & \downarrow & & \downarrow & & \downarrow & p & \\ & & M & = & M & = & M & \longrightarrow & N & & \\ & & & & & & & \imath & & & \end{array}$$

and define $\omega^{\mathrm{ind}} = \Sigma^* \omega_{MC} : TM \to \tilde{\mathfrak{h}}$. The Maurer-Cartan equation $d\omega^{\mathrm{ind}} + \frac{1}{2}[\omega^{\mathrm{ind}}, \omega^{\mathrm{ind}}] = 0$ obviously holds while (i) follows from the equation $\varphi \circ \omega^{ind} = \Sigma^*(\varphi \circ \omega_{MC}) = \Sigma^*(\phi^* \vartheta) = \varsigma^* \vartheta = \sigma^* \theta$. Note that the last equality involving the soldering forms of the bundles Q and P exactly amounts to the fact that P is the G-structure induced by the ambient geometry (N, Q).

We now prove the other implication. Consider the local one-form $\omega^{\mathrm{ind}} - \omega_{MC}$ on the product $M \times \tilde{H}$ and denote its kernel at $(x, \tilde{h}) \in M \times \tilde{H}$ by $\mathcal{D}|_{(x,\tilde{h})}$. The distribution $\mathcal{D} = \mathrm{Ker}(\omega^{\mathrm{ind}} - \omega_{MC})$ satisfies the following two properties:

- it has constant rank $\mathrm{rk}\, \mathcal{D} = \dim M$,
- it is involutive (this follows from hypothesis (ii)).

By standard arguments, the existence of a local map $\Sigma : M \to \tilde{H}$ satisfying

$$\omega^{\mathrm{ind}} = \Sigma^* \omega_{MC} \quad \text{and} \quad \Sigma(x_o) = e$$

is guaranteed. Using (i), one easily sees that the differential of the map

$$\imath := p \circ \phi \circ \Sigma : M \to N \tag{3.4}$$

is injective i.e. (3.4) is a local immersion around $x_o \in M$; what we need to show is that P is the G-structure induced by the ambient geometry (N, Q) through (3.4). Consider the commutative diagram

$$\begin{array}{ccc} \imath^* Q & \longrightarrow & Q \\ \downarrow & & \downarrow \; p \\ M & \longrightarrow & N \\ & \imath & \end{array} \qquad \imath^* Q = \{(x, q) \,|\, x \in M, q \in Q \,\text{with}\, \imath(x) = p(q)\}$$

and define a H'-reduction P^{adp} of the principal H-bundle $\imath^*Q$ by

$$P^{\mathrm{adp}} := \{(x, \phi(\Sigma(x))h) \,|\, x \in M, h \in H'\} \; ;$$

a local section of this bundle is provided by $\varsigma := \phi \circ \Sigma : M \to P^{\mathrm{adp}}$. Hypothesis (i) implies that

$$\sigma^*\theta = \varphi \circ \omega^{\mathrm{ind}} = \varphi \circ \Sigma^*\omega_{MC} : TM \to \mathbb{R}^m$$

and, exploiting (3.3), one gets

$$\vartheta(d\varsigma(v)) = \vartheta(d\phi \circ d\Sigma(v)) = \varphi \circ \omega_{MC}(d\Sigma(v)) = \theta(d\sigma(v)) \,, \tag{3.5}$$

for any $v \in T_xM$. Equation (3.5) says exactly that $\varsigma^*\vartheta = \sigma^*\theta$, i.e.

- the submanifold (3.4) is regular,
- the bundle P^{adp} is the bundle of adapted frames to M and
- the G-structure P is induced by (N, Q) through (3.4).

This concludes the proof. □

In §4, we specialize Theorem 3.1 to the case of an ambient geometry given by the integrable H-structure $(N, Q) = (\mathbb{R}^n, \mathbb{R}^n \times H)$. As a direct corollary of Theorem 3.1 we obtain Theorem 4.2, which was first estabilished by Rosly and Schwarz in [17]; after that we consider two examples in detail.

4. The Theorem of Rosly and Schwarz

For any Lie subgroup H of $\mathrm{GL}_n(\mathbb{R})$, the *integrable H-structure* $p : Q \to \mathbb{R}^n$ is defined as follows: It is the unique reduction to the group H of the bundle $F_{\mathbb{R}^n}$ of all linear frames on $\mathbb{R}^n$ such that the global section of $F_{\mathbb{R}^n}$ given by partial derivatives

$$x \to (\frac{\partial}{\partial x^1}|_x, \ldots, \frac{\partial}{\partial x^1}|_x)$$

takes values in $Q|_x$ for any $x \in \mathbb{R}^n$. From now on the integrable H-structure is denoted by the symbol $p : \mathbb{R}^n \times H \to \mathbb{R}^n$.

There is also the notion of *maximal transitive prolongation* of the Lie algebra $\mathfrak{h}$ of H [24]: It is the unique (possibly infinite-dimensional) $\mathbb{Z}$-graded Lie algebra

$$\mathfrak{h}^\infty = \sum_{p \in \mathbb{Z}} \mathfrak{h}^p \tag{4.1}$$

which is maximal between the class of $\mathbb{Z}$-graded Lie algebras satisfying

- $\mathfrak{h}^p$ is finite dimensional for every $p \in \mathbb{Z}$,
- $\mathfrak{h}^0 = \mathfrak{h}$, $\mathfrak{h}^{-1} = \mathbb{R}^n$ and $\mathfrak{h}^p = 0$ for $p < -1$,
- the adjoint action of $\mathfrak{h}^0$ on $\mathfrak{h}^{-1}$ is the standard action of $\mathfrak{h}$ on $\mathbb{R}^n$,
- for all $p \geq 0$, if $X \in \mathfrak{h}^p$ is such that $[X, \mathfrak{h}^{-1}] = 0$, then $X = 0$.

It is well known that $\mathfrak{h}^p$ is identifiable as a vector space with the intersection $\mathbb{R}^n \otimes S^{p+1}\mathbb{R}^{n*} \cap \mathfrak{h} \otimes S^p\mathbb{R}^{n*}$ and that the maximal transitive prolongation (4.1) describes the Lie algebra of all infinitesimal automorphisms of the H-structure $p : \mathbb{R}^n \times H \to \mathbb{R}^n$ if $\dim \mathfrak{h}^\infty < \infty$ (in that case $\mathfrak{h}$ is of *finite type*).

It is convenient to consider the following.

Definition 4.1. The G-structure $\pi : P^{\rm ind} \to M$ induced by the ambient geometry $(\mathbb{R}^n, \mathbb{R}^n \times H)$ onto a regular submanifold M of $\mathbb{R}^n$ is called *H-immersible*. Any G-structure on a manifold M which is *locally* equivalent to a H-immersible structure is also called H-immersible.

The following Theorem is due to Rosly and Schwarz.

Theorem 4.2 ([17])**.** *Let M be a m-dimensional manifold endowed with a G-structure $\pi : P \to M$ and $H \subset \mathrm{GL}_n(\mathbb{R})$ a Lie group which is of finite type and satisfies*

$$G = \{g \in \mathrm{GL}_m(\mathbb{R}) \,|\, \exists\, h \in H \text{ with } h = \begin{pmatrix} g & * \\ 0 & * \end{pmatrix}\} \ .$$

Then the G-structure $\pi : P \to M$ is H-immersible around $x_o \in M$ if and only if there exists

$$\omega^{\rm ind} = \sum_{p \geq -1} \omega^p \in \Omega^1_{loc}(M, \mathfrak{h}^\infty)\,, \quad \omega^p \in \Omega^1_{loc}(M, \mathfrak{h}^p)\,,$$

defined around x_o and satisfying

i) $\omega^{-1} : TM \to \mathbb{R}^n$ takes values into $\mathbb{R}^m$ and

$$\omega^{-1} = \sigma^* \theta$$

for some local section $\sigma : M \to P$ of the G-structure $\pi : P \to M$, whose soldering form has been denoted by $\theta : TP \to \mathbb{R}^m$;

ii) the Maurer-Cartan equation

$$d\omega^{ind} + \frac{1}{2}[\omega^{ind}, \omega^{ind}] = 0 \ . \tag{4.2}$$

Proof. Let $\tilde{H}$ be the simply connected Lie group whose Lie algebra is $\mathfrak{h}^\infty$ and consider the homogeneous manifold $N = \tilde{H}/\tilde{K}$, where $\tilde{K}$ is the analytic closed subgroup of $\tilde{H}$ corresponding to the following subalgebra:

$$\tilde{\mathfrak{k}} = \mathfrak{h}^0 + \mathfrak{h}^1 + \mathfrak{h}^2 + \cdots \ .$$

The Theorem follows by checking that the $\tilde{K}/K$-structure $p : \tilde{H}/K \to \tilde{H}/\tilde{K}$ associated to the natural *abelian* complement $\mathfrak{h}^{-1} = \mathbb{R}^n$ to $\tilde{\mathfrak{k}}$ in $\tilde{\mathfrak{h}}$ (see the discussion at the beginning of §3) is integrable around $o \in N$, by noticing that $\tilde{\mathfrak{k}}/\mathfrak{k} \simeq \mathfrak{h}$ as Lie algebras, and by applying Theorem 3.1. □

The following observation is also due to Rosly and Schwarz [17]. Let M be a m-dimensional manifold endowed with a G-structure $\pi : P \to M$. Let us fix a (local) frame

$$\epsilon : \mathbb{R}^m \to TM \ , \quad \epsilon_x \in P|_x$$

and denote the inverse coframe by $\omega^{-1} : TM \to \mathbb{R}^m$. Since the maximal transitive prolongation of a Lie group H admits a natural $\mathbb{Z}$-gradation (4.1), the Maurer-Cartan equation (4.2) boils down to a family of equations parameterized by $k \in \mathbb{N}$:

$$P^{k-1} := d\omega^{k-1} + \frac{1}{2} \sum_{0 \leq r \leq k-1} [\omega^r, \omega^{k-1-r}] = -[\omega^{-1}, \omega^k] \, . \tag{4.3}$$

This fact suggests a recursive procedure to construct solutions of (4.2). More precisely, assume a set $\{\omega^i\}_{i<k}$ which is a solution of the equations $\{P^{i-1} = -[\omega^{-1}, \omega^i]\}_{i<k}$ has been determined; (4.3) is then an equation in the unknown $\omega^k \in \Omega^1_{loc}(M, \mathfrak{h}^k)$, with

$$P^{k-1} = P^{k-1}(\omega^{-1}, \dots, \omega^{k-1}) \in \Omega^2_{loc}(M, \mathfrak{h}^{k-1}) \ .$$

We conclude §4 by showing how the well-known Gauss-Codazzi-Ricci equations underlying the embedding problem of a Riemannian manifold (M, g) into the flat space $\mathbb{R}^n$ fit into the scheme of immersible G-structures. Similar equations are then obtained for the analogous problem up to conformal changes of metric. The obtained results are basic knowledge in Mathematics, except perhaps the conformal case.

4.1. **The Riemannian case.** Let $p : Q \to N$ be the standard $O_n(\mathbb{R})$-structure

$$Q = \{(v_1, \dots, v_n) \text{ frame of } T_xN \,|\, g'(v_i, v_j) = \delta_{ij} \text{ for all } 1 \leq i, j \leq n\}$$

associated to a n-dimensional Riemannian manifold (N, g'). Of course, any submanifold $\imath : M \to N$ is regular and the $O_m(\mathbb{R})$-structure $\pi : P^{\text{ind}} \to M$ induced by the ambient geometry (N, Q) is the standard $O_m(\mathbb{R})$-structure associated to the pull-back metric $\imath^* g'$.
We consider m-dimensional Riemannian manifolds (M, g) and study the existence of local isometric immersions into the flat space $(\mathbb{R}^n, (\cdot, \cdot))$, for a fixed codimension $p := n - m \geq 0$.

By identifying each $T_xM \simeq \mathbb{R}^m$ by means of a local orthonormal coframe

$$\omega^{-1}|_x : T_xM \to \mathbb{R}^m \, , \tag{4.4}$$

equation (4.3) with $k = 0$ is equivalent to

$$[v, \xi_x(w)] + [\xi_x(v), w] = -d\omega^{-1}|_x(v, w) \tag{4.5}$$

for any $v, w \in \mathbb{R}^m$ in the unknown $\xi_x \in \mathbb{R}^{m*} \otimes \mathfrak{so}_n(\mathbb{R})$. The decomposition

$$\mathfrak{so}_n(\mathbb{R}) = \{ \begin{pmatrix} A & -B^T \\ B & D \end{pmatrix} \mid A \in \mathfrak{so}_m(\mathbb{R}) \, , \; B \in \mathrm{Mat}(p, m; \mathbb{R}) \, , \; D \in \mathfrak{so}_p(\mathbb{R}) \} \tag{4.6}$$

determines an identification $\xi_x \simeq \varpi_x + \Pi_x + \psi_x$, for unique linear maps

$$\varpi_x \in \mathbb{R}^{m*} \otimes \mathfrak{so}_m(\mathbb{R}) \, , \; \Pi_x \in \mathbb{R}^{m*} \otimes \mathrm{Mat}(p, m; \mathbb{R}) \, , \; \psi_x \in \mathbb{R}^{m*} \otimes \mathfrak{so}_p(\mathbb{R}) \, , \tag{4.7}$$

and (4.5) decomposes into the following two equations

$$-\varpi_x(w)(v) + \varpi_x(v)(w) = -d\omega^{-1}|_x(v, w) \, , \quad -\Pi_x(w)(v) + \Pi_x(v)(w) = 0 \, .$$

It is well-known that the first equation has always a unique solution ϖ_x; the first order H-immersibility conditions implies also $\Pi_x \in S^2\mathbb{R}^{m*} \otimes \mathbb{R}^p$ and (4.5) admits always a smooth solution

$$\omega^0 \in \Omega^1_{loc}(M, \mathfrak{so}_n(\mathbb{R})) \, . \tag{4.8}$$

We denote the forms (4.7) associated to a solution (4.8) still by

$$\varpi \in \Omega^1_{loc}(M, \mathfrak{so}_m(\mathbb{R})) \, , \;\; \Pi \in \Omega^1_{loc}(M, \mathrm{Mat}(p, m; \mathbb{R})) \, , \;\; \psi \in \Omega^1_{loc}(M, \mathfrak{so}_p(\mathbb{R})) \, ;$$

they determine the connection form of the Levi-Civita covariant derivative ∇^{LC} on M and a symmetric tensor field $\alpha \in \Gamma_{loc}(S^2TM^* \otimes E)$ taking values in a (locally defined) vector bundle $E \simeq M \times \mathbb{R}^p \to M$ which is endowed with a connection $\nabla^\perp$ compatible with the fiber metric $g_E \simeq (\cdot, \cdot)|_{\mathbb{R}^p}$.

As the first prolongation $\mathfrak{so}_n(\mathbb{R})^1 = 0$, equation (4.3) with $k = 1$ is equivalent to

$$0 = d\omega^0(v, w) + [\omega^0(v), \omega^0(w)]$$

and it boils down to the vanishing of the following local two-forms:

$$\mathbf{G}(v, w) := d\varpi(v, w) + [\varpi(v), \varpi(w)] - \Pi(v)^T \cdot \Pi(w) + \Pi(w)^T \cdot \Pi(v) = 0 \, ,$$
$$\mathbf{C}(v, w) := d\Pi(v, w) + \Pi(v) \cdot \varpi(w) - \Pi(w) \cdot \varpi(v) + \psi(v) \cdot \Pi(w) - \psi(w) \cdot \Pi(v) = 0 \, ,$$
$$\mathbf{R}(v, w) := d\psi(v, w) + [\psi(v), \psi(w)] - \Pi(v) \cdot \Pi(w)^T + \Pi(w) \cdot \Pi(v)^T = 0 \, .$$

These second order H-immersibility conditions for the existence of an isometric immersion into $\mathbb{R}^n$ are the usual Gauss-Codazzi-Ricci equations for vector fields $X, Y, Z, W \in \mathfrak{X}(M)$ and sections $\mu, \nu \in \Gamma(E)$, i.e.

$$\boxed{\begin{aligned} &0 = g(R_{XY}Z, W) - g_E(\alpha(Y, Z), \alpha(X, W)) + g_E(\alpha(X, Z), \alpha(Y, W)) \\ &0 = (\nabla_X \alpha)(Y, Z) - (\nabla_Y \alpha)(X, Z) \\ &0 = g_E(R_{XY}\mu, \nu) + \sum_i g_E(\alpha(X, E_i), \mu) g_E(\alpha(Y, E_i), \nu) + \\ &\quad - \sum_i g_E(\alpha(X, E_i), \nu) g_E(\alpha(Y, E_i), \mu) \end{aligned}}$$

Here $\nabla := \nabla^{LC} + \nabla^\perp$ denotes the covariant derivative on the vector bundle $TM \oplus E \to M$, compatible with the fiber metric $g \oplus g_E$, determined by the connection form $\varpi \oplus \psi$; R is the curvature of ∇ and $\{E_i\}$ any local orthonormal frame of M.

As $\mathfrak{so}_n(\mathbb{R})^1 = 0$, equations (4.3) are trivially satisfied when $k \geq 2$.

The above discussion together with Theorem 4.2 furnishes yet another proof of the well-known fact that any tuple $(E, g_E, \alpha, \nabla^\perp)$ satisfying the Gauss-Codazzi-Ricci equations determines a (local) isometric immersion $M \subset \mathbb{R}^n$ (see e.g. [15, pag. 47] or [23, pag. 49]).
Vice-versa, to any isometric immersion into flat space $\mathbb{R}^n$, one can always associate a tuple $(E, g_E, \alpha, \nabla^\perp)$ satisfying the Gauss-Codazzi-Ricci equations: It is given by the second fundamental form α, the normal bundle $E = NM$ together with the induced metric $g_E := (\cdot, \cdot)|_E$, and finally the projection $\nabla^\perp : TM \otimes NM \to NM$ along NM of the Levi-Civita connection of $\mathbb{R}^n$.

4.2. **The conformal case.** We retain the notation of §4.1. We want to determine the H-immersibility conditions for a m-dimensional Riemannian manifold (M, g) to admit a *conformal* immersion into the flat space $\mathbb{R}^n$, for a fixed $n \geq 3$. Considering the standard $CO(\mathbb{R})$-structures associated to (M, g) and $\mathbb{R}^n$, we are led to study equations (4.3) in the case $\mathfrak{h} = \mathfrak{co}_n(\mathbb{R})$. Recall that the maximal transitive prolongation $\mathfrak{co}_n(\mathbb{R})^\infty$ is finite-dimensional [10, 22] and isomorphic to

$$\mathfrak{so}(n+1, 1; \mathbb{R}) \simeq \mathbb{R}^n + \mathfrak{co}_n(\mathbb{R}) + \mathbb{R}^{n*} = \mathbb{R}^n + \mathfrak{co}_n(\mathbb{R}) + \mathfrak{co}(n, \mathbb{R})^1 \ ,$$

the identification $\mathbb{R}^{n*} = \mathfrak{co}_n(\mathbb{R})^1$ being given by

$$u^* \mapsto \{u \mapsto u \otimes u^* - \phi(u^* \otimes u) + u^*(u)\,\mathrm{Id}\} \ ,$$

where $\phi : \mathbb{R}^{n*} \otimes \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^{n*}$ denotes the duality isomorphism induced by the standard scalar product of $\mathbb{R}^n$.

From now on, we will identify each $T_xM \simeq \mathbb{R}^m$ by means of a local *orthonormal* coframe (4.4). For any $\xi_x \in \mathbb{R}^{m*} \otimes \mathfrak{co}_n(\mathbb{R})$, the decompositions (4.6) and $\mathfrak{co}_n(\mathbb{R}) = \mathfrak{so}_n(\mathbb{R}) \oplus \mathbb{R}\,\mathrm{Id}$ induce an identification $\xi_x \simeq \varpi_x + \Pi_x + \psi_x + \lambda_x$, where

$$\begin{aligned} &\varpi_x \in \mathbb{R}^{m*} \otimes \mathfrak{so}_m(\mathbb{R}) \,,\ \Pi_x \in \mathbb{R}^{m*} \otimes \mathrm{Mat}(p, m; \mathbb{R}) \\ &\psi_x \in \mathbb{R}^{m*} \otimes \mathfrak{so}_p(\mathbb{R}) \,,\ \lambda_x \in \mathbb{R}^{m*} \,, \end{aligned}$$

and equation (4.3) with $k = 0$ decomposes into

$$\begin{aligned} -\varpi_x(w)(v) + \varpi_x(v)(w) - \lambda_x(w)(v) + \lambda_x(v)(w) &= -d\omega^{-1}|_x(v, w) \ , \\ -\Pi_x(w)(v) + \Pi_x(v)(w) &= 0 \ . \end{aligned}$$

In particular, as in §4.1, it always exists a unique smooth solution of the form (4.8), that is we restrict to the case $\lambda = 0$ in the decomposition

$$\omega^0 \simeq \varpi + \Pi + \psi + \lambda \ .$$

Equation (4.3) with $k = 1$ is given by

$$-[v, \eta_x(w)] - [\eta_x(v), w] = d\omega^0|_x(v, w) + [\omega^0|_x(v), \omega^0|_x(w)] \tag{4.9}$$

and it is an equation in the unknown

$$\eta_x \in \mathbb{R}^{m*} \otimes \mathfrak{co}_n(\mathbb{R})^1 \ . \tag{4.10}$$

To proceed further, we recall that $\mathfrak{co}_n(\mathbb{R})^1 \simeq \mathbb{R}^{n*} = \mathbb{R}^{m*} \oplus \mathbb{R}^{p*}$ so that any element (4.10) can be decomposed as

$$\eta_x(v) = \vec{u}^*_x(v) + u^{\perp *}_x(v) \in \mathbb{R}^{m*} + \mathbb{R}^{p*} \ .$$

With this in mind, equation (4.9) split into the following four equations:

$$\mathbf{G}(v,w) = -w \otimes \vec{u}^*(v) + \phi(\vec{u}^*(v) \otimes w) + v \otimes \vec{u}^*(w) - \phi(\vec{u}^*(w) \otimes v) \in \mathfrak{so}_m(\mathbb{R})$$

$$\mathbf{C}(v,w) = \phi(u^{\perp *}(v) \otimes w) - \phi(u^{\perp *}(w) \otimes v) \in \mathrm{Mat}(p,m;\mathbb{R})$$

$$\mathbf{R}(v,w) = 0 \in \mathfrak{so}_p(\mathbb{R})$$

$$\vec{u}^*(v)(w) - \vec{u}^*(w)(v) = 0 \in \mathbb{R}\,\mathrm{Id} \ .$$

A smooth solution $\omega^1 \in \Omega^1_{loc}(M, \mathfrak{co}_n(\mathbb{R})^1)$ of (4.9) is thus identifiable with a pair (B, D) formed by a symmetric tensor field $B \in \Gamma_{loc}(S^2T^*M)$ and a $D \in \Gamma_{loc}(T^*M \otimes E)$, satisfying the following conformal version of the usual Gauss-Codazzi-Ricci equations for $X, Y, Z, W \in \mathfrak{X}(M)$ and $\mu, \nu \in \Gamma(E)$:

$$\boxed{\begin{aligned}
&g(R_{XY}Z, W) - g_E(\alpha(Y,Z), \alpha(X,W)) + g_E(\alpha(X,Z), \alpha(Y,W)) = \\
&\quad = -g \owedge B(X,Y,Z,W) \\
&(\nabla_X \alpha)(Y,Z) - (\nabla_Y \alpha)(X,Z) = D(X)g(Y,Z) - D(Y)g(X,Z) \\
&0 = g_E(R_{XY}\mu, \nu) + \sum_i g_E(\alpha(X,E_i),\mu) g_E(\alpha(Y,E_i),\nu) + \\
&\qquad - \sum_i g_E(\alpha(X,E_i),\nu) g_E(\alpha(Y,E_i),\mu)
\end{aligned}}$$

Here E, α, g_E, $\nabla = \nabla^{LC} + \nabla^\perp$, R and $\{E_i\}$ have to be interpreted as in §4.1 while the symbol $g \owedge B$ denotes the Kulkarni-Nomizu product of the symmetric tensors g and B of type $(0,2)$ (see [2, pag.47] for the definition of the Kulkarni-Nomizu product).

As $\mathfrak{co}_n(\mathbb{R})^2 = 0$ but $\mathfrak{co}_n(\mathbb{R})^1 \neq 0$, equations (4.3) for $k = 2$ are not trivially satisfied. They are given by

$$d\omega^1(v,w) + [\omega^0(v), \omega^1(w)] + [\omega^1(v), \omega^0(w)] = 0$$

and furnish the following third order H-immersibility conditions

$$\boxed{\begin{aligned}
&(\nabla_X B)(Y,Z) + g_E(D(X), \alpha(Y,Z)) && \text{symmetric in } X \text{ and } Y \\
&(\nabla_X D)(Y,\mu) + \sum_i g_E(\alpha(X,E_i),\mu) B(Y,E_i) && \text{symmetric in } X \text{ and } Y
\end{aligned}}$$

where $X, Y, Z \in \mathfrak{X}(M)$, $\mu \in \Gamma(E)$ and, in the second equation, D has to be interpreted as a section of $T^*M \otimes E^*$ using the duality associated to g_E.

The above discussion together with Theorem 4.2 proves that it is possible to associate a local conformal immersion of M into flat space $\mathbb{R}^n$ to any

tuple $(E, g_E, \alpha, \nabla^\perp, B, D)$ which satisfies the above mentioned conformal Gauss-Codazzi-Ricci equations and third order conditions.

Vice-versa, to any conformal immersion $\imath : M \to \mathbb{R}^n$ with $\imath^*(\cdot,\cdot) = e^{2f} g$, one associate a tuple $(E, g_E, \alpha, \nabla^\perp, B, D)$ as follows. The tensor field $\alpha \in \Gamma(S^2 T^*M \otimes E) = \Gamma(S^2 T^*M \otimes NM)$ is the second fundamental form of the isometric immersion $\imath : (M, e^{2f} g) \to (\mathbb{R}^n, (\cdot,\cdot))$ while

$$\begin{aligned} g_E &:= e^{-2f} (\cdot,\cdot)|_E \,, \\ \nabla^\perp_X \mu &:= \boldsymbol{\nabla}_X \mu - df(X)\mu \,, \\ B &:= Hf - df \odot df + \frac{1}{2} g(df, df) g \,, \\ D &:= \alpha(\cdot, \nabla f) \,, \end{aligned}$$

where $\boldsymbol{\nabla}$ is the projection along NM of the Levi-Civita connection of $\mathbb{R}^n$, Hf and ∇f denote the Hessian and the gradient of f w.r.t. g respectively. Using [2, Thm. 1.159], one can check that such a tuple satisfies the conformal Gauss-Codazzi-Ricci equations and that the fields B and D do indeed satisfy also the third order conditions. The quite long but straightforward calculations are omitted.

Specializing to the $p = 0$ case furnishes yet another proof of the well-known fact that a Riemannian manifold (M, g) of dimension $m \geq 3$ is locally conformally flat if and only if the Weyl-Schouten tensor (if $m = 3$) or the Weyl tensor (if $m = 4$) vanishes (see Example 5.3 in §5). On the other hand the case $p > 0$, although a basic result, is not available in the Literature at the best of the author's knowledge.

Last Sec. §5 is devoted to the introduction of some cohomology groups which are the target spaces of the obstructions to costructing the equivalence mentioned in Theorem 4.2.

5. Restricted Spencer cohomology groups

Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ and consider the decomposition (2.4) for some fixed $m \leq n$. Rosly and Schwarz remarked in [17] that, whenever a set of solutions $\{\omega^i\}_{i<k}$ of equations $\{P^{i-1} = -[\omega^{-1}, \omega^i]\}_{i<k}$ has been determined, equation (4.3) in the unknown $\omega^k \in \Omega^1_{loc}(M, \mathfrak{h}^k)$ is linear algebraic at any point $x \in M$. This says that solutions of (4.3) exist if and only if P^{k-1} satisfies some algebraic conditions.

The main purpose of this Section is to define and study the basic properties of some cohomological groups which are the natural spaces where these algebraic obstructions live.

We decided to call these groups the *restricted Spencer cohomology groups*. They are a necessary ingredient to study problems which are more complicated than those treated in the examples of Section §4. For example, we will

use them in [19] to study the necessary and sufficient conditions under which a CR quaternionic manifold is immersible into the quaternionic projective space $\mathbb{H}\mathrm{P}^n$ with its canonical quaternionic structure.

Definition 5.1. The *restricted Spencer cohomology groups* $H_R^{p,q}(\mathfrak{h})$ (of the Lie algebra $\mathfrak{h}$ w.r.t. the fixed subspace $\mathbb{R}^m$ of $\mathbb{R}^n$) are the cohomology groups associated to the differential complex

$$\cdots \to C_R^{p+1,q-1}(\mathfrak{h}) \to C_R^{p,q}(\mathfrak{h}) \to C_R^{p-1,q+1}(\mathfrak{h}) \to \cdots ,$$

where

i) the space of (p,q)-cochains is $C_R^{p,q}(\mathfrak{h}) = \mathfrak{h}^{p-1} \otimes \Lambda^q \mathbb{R}^{m\,*}$, $\forall p, q \geq 0$,
ii) the coboundary map $\partial^{p,q} : C_R^{p,q}(\mathfrak{h}) \to C_R^{p-1,q+1}(\mathfrak{h})$ is given by

$$\partial^{p,q}(c)(x_1, \ldots, x_{q+1}) := \sum_{i=1}^{q+1} (-1)^i [c(x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_{q+1}), x_i]$$

for any $c \in C_R^{p,q}(\mathfrak{h})$ and $x_1, \ldots, x_{q+1} \in \mathbb{R}^m$.

The main result of this section is the following.

Proposition 5.2. *Let M be a m-dimensional manifold endowed with a G-structure $\pi : P \to M$ with soldering form $\theta : TP \to \mathbb{R}^m$, and $H \subset \mathrm{GL}_n(\mathbb{R})$ a Lie group which satisfies*

$$G = \{ g \in \mathrm{GL}_m(\mathbb{R}) \,|\, \exists\, h \in H \text{ with } h = \begin{pmatrix} g & * \\ 0 & * \end{pmatrix} \} \ .$$

Given a local section $\sigma : M \to P$ of $\pi : P \to M$, use the coframe

$$\omega^{-1} = \sigma^* \theta \in \Omega^1_{loc}(M, \mathbb{R}^m) \tag{5.1}$$

to identify local q-forms on M with values into $\mathfrak{h}^{p-1}$ to local functions on M with values in $C_R^{p,q}(\mathfrak{h})$. Then

i) *Equation* (4.3) *with $k = 0$ has a solution $\omega^0 \in \Omega^1_{loc}(M, \mathfrak{h}^0)$ if and only if*

$$[P^{-1}|_x] = 0 \in H_R^{0,2}(\mathfrak{h})$$

for all $x \in M$.

ii) *Let $\omega^0 \in \Omega^1_{loc}(M, \mathfrak{h}^0), \ldots, \omega^{p-1} \in \Omega^1_{loc}(M, \mathfrak{h}^{p-1})$ be a sequence of forms which satisfy equations* (4.3) *for all $0 \leq k < p$. Then equation* (4.3) *with $k = p$ has a solution $\omega^p \in \Omega^1_{loc}(M, \mathfrak{h}^p)$ if and only if*

$$[P^{p-1}|_x] = 0 \in H_R^{p,2}(\mathfrak{h})$$

for all $x \in M$.

Proof. Equation (4.3) with $k = 0$ is equivalent to the existence of an element $\xi_x \in C_R^{1,1}(\mathfrak{h})$ such that

$$[v, \xi_x(w)] + [\xi_x(v), w] = -P^{-1}|_x(v, w)$$

for any $v, w \in \mathbb{R}^m$ and $x \in M$. Equivalently $\partial^{1,1}(\xi_x) = -P^{-1}|_x \in C_R^{0,2}(\mathfrak{h})$ at any $x \in M$, which proves the necessary implication. Vice-versa, consider a vector space direct sum

$$C_R^{1,1}(\mathfrak{h}) = \operatorname{Ker} \partial^{1,1} \oplus \mathtt{C}^{1,1} \ ,$$

and decompose each $\xi_x = \xi'_x + \xi''_x$ accordingly. As $\partial^{1,1} : C_R^{1,1}(\mathfrak{h}) \to C_R^{0,2}(\mathfrak{h})$ induces an isomorphism between $\mathtt{C}^{1,1}$ and $\operatorname{Im} \partial^{1,1}$, one gets that

$$x \to \omega^0|_x := \xi''_x = -(\partial^{1,1}|_{\mathtt{C}^{1,1}})^{-1}(P^{-1}|_x)$$

is the required smooth solution of (4.3) with $k = 0$. In this case, by Exercise 5.20 of Chapter I of [21] one has that

$$\begin{aligned} 0 = d[\omega^{-1}, \omega^0] &= [d\omega^{-1}, \omega^0] - [\omega^{-1}, d\omega^0] = -[[\omega^{-1}, \omega^0], \omega^0] - [\omega^{-1}, d\omega^0] \\ &= [[\omega^0, \omega^0], \omega^{-1}] + [[\omega^0, \omega^{-1}], \omega^0] - [\omega^{-1}, d\omega^0] \\ = [[\omega^0, \omega^0], \omega^{-1}] - \frac{1}{2}[[\omega^0, \omega^0], \omega^{-1}]] - [\omega^{-1}, d\omega^0] &= -[\omega^{-1}, P^0] = 0 \, , \end{aligned}$$

that is $P^0|_x \in C_R^{1,2}(\mathfrak{h})$ is a cocycle.
Equation (4.3) with $k = 1$ is equivalent to the existence of a $\eta_x \in C_R^{2,1}(\mathfrak{h})$ such that

$$[v, \eta_x(w)] + [\eta_x(v), w] = -P^0|_x(v, w)$$

i.e. $\partial^{2,1}(\eta_x) = -P^0|_x$ at any $x \in M$. One proceed as above to construct a smooth solution $\omega^1 \in \Omega^1_{loc}(M, \mathfrak{h}^1)$.
The next cases $k \geq 2$ are proved by a straightforward induction argument which we omit. □

At the best of the author's knowledge, the groups $H_R^{p,q}(\mathfrak{h})$ introduced in Definition 5.1 have never been considered in the Literature. They are related to the cohomology groups $H^{p,2}(\mathfrak{h}|_{\mathbb{R}^m})$ considered in [17] as follows.

Denote by $\mathfrak{h}|_{\mathbb{R}^m} \subset \operatorname{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ the space of maps made up of restrictions to $\mathbb{R}^m$ of elements of $\mathfrak{h}$ and by $C^{p,q}(\mathfrak{h}|_{\mathbb{R}^m})$ the associated Spencer cochains [7, 22]. There exists a natural restriction morphism of differential complexes $C_R^{p,q}(\mathfrak{h}) \to C^{p,q}(\mathfrak{h}|_{\mathbb{R}^m})$ which induces a map at the level of cohomologies

$$\mathcal{RS} : H_R^{p,q}(\mathfrak{h}) \to H^{p,q}(\mathfrak{h}|_{\mathbb{R}^m}) \ . \tag{5.2}$$

As $P^{-1}|_x$ takes always values into the subspace $\mathbb{R}^m \otimes \Lambda^2 \mathbb{R}^{m\,*}$ of $C_R^{0,2}(\mathfrak{h})$, one can show that

$$[P^{-1}|_x] = 0 \in H_R^{0,2}(\mathfrak{h}) \qquad \Longleftrightarrow \qquad \mathcal{RS}[P^{-1}|_x] = 0 \ .$$

In general a similar property does not hold for the classes $[P^{p-1}|_x]$ if $p \geq 1$ and (5.2) is not injective. For example, a closer look at §4.1 reveals that

$$\begin{aligned} &H^{1,2}(\mathfrak{so}_n(\mathbb{R})|_{\mathbb{R}^m}) = H^{1,2}(\mathfrak{so}_m(\mathbb{R}) + \operatorname{Hom}(\mathbb{R}^m, \mathbb{R}^p)) \simeq H^{1,2}(\mathfrak{so}_m(\mathbb{R})) \, , \\ &H_R^{1,2}(\mathfrak{so}_n(\mathbb{R})) \simeq H^{1,2}(\mathfrak{so}_m(\mathbb{R})) + \mathbb{R}^p \otimes R^{2,1} + \mathfrak{so}_p(\mathbb{R}) \otimes \Lambda^2 \mathbb{R}^{m*} \, , \end{aligned}$$

where $R^{2,1}$ is the $\mathfrak{gl}_m(\mathbb{R})$-irreducible submodule of $\mathbb{R}^{m*} \otimes \Lambda^2 \mathbb{R}^{m*}$ different from $\Lambda^3 \mathbb{R}^{m*}$, the second order obstructions to immersibility being given by $\mathbf{G}|_x \in H^{1,2}(\mathfrak{so}_m(\mathbb{R}))$, $\mathbf{C}|_x \in \mathbb{R}^p \otimes R^{2,1}$ and $\mathbf{R}|_x \in \mathfrak{so}_p(\mathbb{R}) \otimes \Lambda^2 \mathbb{R}^{m*}$.

We conclude the paper with two examples.

Example 5.3. We obtained that a Riemannian manifold (M, g) of dimension $n \geq 3$ is locally conformally flat if and only if there exists a $B \in S^2T^*M$ such that

$$\boxed{\begin{aligned} &g(R_{XY}Z, W) = -g \owedge B(X, Y, Z, W) \\ &(\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z) \end{aligned}}$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$. In this case the codimension $p = 0$ and the restricted Spencer groups coincide with $H^{p,2}(\mathfrak{co}_n(\mathbb{R}))$; it is known that the only non-trivial ones are $H^{1,2}(\mathfrak{co}_n(\mathbb{R}))$ if $n \geq 4$ and $H^{2,2}(\mathfrak{co}_3(\mathbb{R}))$ [14].

In case $n \geq 4$, the first condition is equivalent to the vanishing of the Weyl tensor (see Theorem 1.114 [2]), the second one being automatically satisfied. In case $n = 3$, the first condition is automatically satisfied while the second one is equivalent to the vanishing of the Weyl-Schouten tensor (see Section 16.4 [2]).

Example 5.4. Let (N, J) be an almost complex manifold of real dimension $\dim N = 2n$ and fix a natural number $1 \leq r \leq n$. We denote by $p : Q \to N$ the H-structure determined by the frames $(v_1, \dots, v_{2n})$ of T_xN satisfying

$$v_{r+i} = Jv_i \quad \forall\, 1 \leq i \leq r \qquad \text{and} \qquad v_{r+n+j} = Jv_{2r+j} \quad \forall\, 1 \leq j \leq n - r \ .$$

The Lie group H is isomorphic to $GL_n(\mathbb{C})$ and it is possible to decompose its Lie algebra $\mathfrak{h}$ in terms of $\mathfrak{gl}_r(\mathbb{C})$-modules as follows:

$$\mathfrak{h} \simeq \mathfrak{gl}_r(\mathbb{C}) + \mathbb{R}^{2r} \otimes \mathbb{R}^{n-r*} + T' + \mathbb{R}^{n-r} \otimes \mathbb{R}^{2r*} + T'' \ , \tag{5.3}$$

where T' and T'' are two copies of the trivial $\mathfrak{gl}_r(\mathbb{C})$-module $\mathbb{R}^{n-r} \otimes \mathbb{R}^{n-r*}$ (T' is a Lie subalgebra while $[T'', T''] \subset T'$).

The structure groups of the bundles (2.2) and (2.3) associated to a regular submanifold $M \subset N$ of dimension $\dim M = n + r$ are isomorphic and correspond, respectively, to the analytic subgroups $H' \subset H$ and $G \subset \mathrm{GL}_{n+r}(\mathbb{R})$ associated to the Lie subalgebra $\mathfrak{gl}_r(\mathbb{C}) + \mathbb{R}^{2r} \otimes \mathbb{R}^{n-r*} + T'$ inside (5.3).
It follows that M is endowed with a distribution $\mathcal{D} \subset TM$ of $\operatorname{rk}\mathcal{D} = 2r$ together with a field of partial almost complex structures $J : \mathcal{D} \to \mathcal{D}$: We are dealing with $n + r$-dimensional almost CR manifolds $(M, \mathcal{D}, J)$ of CR dimension r and CR codimension $n - r$.

We now recover the usual integrability conditions for $(M, \mathcal{D}, J)$ to be *locally embeddable*, i.e. locally induced by an immersion into $N = \mathbb{C}^n$ with its standard complex structure. The conditions are only formal as $\mathfrak{gl}_n(\mathbb{C})$ is of infinite type [10] and we can not apply Theorem 4.2.

First note that the cohomology groups $H^{p,2}_R(\mathfrak{gl}_n(\mathbb{C}))$ are zero if $p \geq 1$. In fact any restricted cocycle

$$\eta \in C^{p,2}_R(\mathfrak{gl}_n(\mathbb{C}))$$

can be extended to a unique cocycle $\tilde{\eta} \in C^{p,2}(\mathfrak{gl}_n(\mathbb{C}))$ by $\mathbb{C}$-linearity. The assertion is then an easy consequence of the fact $H^{p,2}(\mathfrak{gl}_n(\mathbb{C})) = 0$ for all $p \geq 1$ [7, 12].

We are led then to study the existence of a form $\omega^0 \in \Omega^1_{loc}(M, \mathfrak{gl}_n(\mathbb{C}))$ which solves the system

$$d\omega^{-1}|_x(v,w) + [v, \omega^0|_x(w)] + [\omega^0|_x(v), w] = 0 \tag{5.4}$$

where $v, w \in \mathbb{R}^{n+r}$ and we identified each tangent space $T_xM \simeq \mathbb{R}^{n+r}$ via an admissible coframe as usual. We now proceed in a slightly different way. Note that (5.4) admits a solution if and only if the torsion

$$T|_x \in \mathbb{R}^{n+r} \otimes \Lambda^2 \mathbb{R}^{n+r*}$$

of the covariant derivative ∇ associated to any connection 1-form

$$\varpi \in \Omega^1_{loc}(M, \mathfrak{g})$$

is restricted exact. We determine the space

$$\mathbb{R}^{n+r} \otimes \Lambda^2 \mathbb{R}^{n+r*} \cap \operatorname{Im} \partial^{1,1} \tag{5.5}$$

as follows. First note that $\operatorname{Im} \partial^{1,1} = \operatorname{Im} \partial\!\!\!/^{1,1}|_{\Lambda^2 \mathbb{R}^{n+r}}$ where

$$\partial\!\!\!/^{p,q} : C^{p,q}(\mathfrak{h}) \to C^{p-1,q+1}(\mathfrak{h}) \ , \quad C^{p,q}(\mathfrak{h}) = \mathfrak{h}^{p-1} \otimes \Lambda^q \mathbb{R}^{2n\,*}$$

denotes the Spencer operator between the usual Spencer cochains. Secondly it is not difficult to see that $\operatorname{Im} \partial\!\!\!/^{1,1} = \mathbb{R}^{2n} \otimes \Lambda^{(2,0)} \mathbb{R}^{2n*} + \mathbb{R}^{2n} \otimes \Lambda^{(1,1)} \mathbb{R}^{2n*} =$

$$= \{ T \in \mathbb{R}^{2n} \otimes \Lambda^2 \mathbb{R}^{2n*} \,|\, T(v,w) - T(Jv, Jw) = -JT(Jv, w) - JT(v, Jw) \} \ .$$

It follows that (5.5) equals the space of maps $T \in \mathbb{R}^{n+r} \otimes \Lambda^2 \mathbb{R}^{n+r*}$ which satisfy

$$T(v,w) - T(Jv,Jw) \in \mathbb{R}^{2r} \ , T(v,w) - T(Jv,Jw) = -JT(Jv,w) - JT(v,Jw) \ ,$$

for any $v, w \in \mathbb{R}^{2r}$. Hence $(M, \mathcal{D}, J)$ is formally locally embeddable in $\mathbb{C}^n$ if and only if there exists a connection ∇ with $\nabla \mathcal{D} \subset \mathcal{D}$, $\nabla J = 0$ and whose torsion satisfies

$$T(X,Y) - T(JX, JY) \in \mathcal{D} \, ,$$

$$T(X,Y) - T(JX, JY) = -JT(JX, Y) - JT(X, JY) \, ,$$

for any $X, Y \in \mathcal{D}$. Equivalently one gets

$$\boxed{\begin{array}{l} [X,Y] - [JX, JY] \in \mathcal{D} \\ [X,Y] - [JX, JY] = -J[JX, Y] - J[X, JY] \\ \text{for all } X, Y \in \mathcal{D} \, . \end{array}}$$

It is well-known [1, 3] that the above formal conditions are sufficient for local embeddability under the additional assumption of analyticity. It might be interesting to get an alternative proof of this fact by using the above discussion together with results on partial differential equations as presented in [6, 4, 16].

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