k-intolerant capacities and Choquet integrals

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Abstract

We define an aggregation function to be (at most) k-intolerant if it is bounded from above by its kth lowest input value. Applying this definition to the discrete Choquet integral and its underlying capacity, we introduce the concept of k-intolerant capacities which, when varying k from 1 to n, cover all the possible capacities on n objects. Just as the concepts of k-additive capacities and p-symmetric capacities have been previously introduced essentially to overcome the problem of computational complexity of capacities, k-intolerant capacities are proposed here for the same purpose but also for dealing with intolerant or tolerant behaviors of aggregation. We also introduce axiomatically indices to appraise the extent to which a given capacity is k-intolerant and we apply them on a particular recruiting problem.

Keywords: multi-criteria analysis, interacting criteria; capacities; Choquet integral.

1 Introduction

In a previous work [11] the author investigated the intolerant behavior of the discrete Choquet integral when used to aggregate interacting criteria. Roughly speaking, the Choquet integral $\mathcal{C}_v$, or equivalently its associated capacity v, has an intolerant behavior if its output (aggregated) value is often close to the lowest of its input values. More precisely, regard the domain $[0,1]^n$ of $\mathcal{C}_v$ as a probability space, with uniform distribution, and consider the mathematical expectation of $\mathcal{C}_v$, which expresses the typical position of $\mathcal{C}_v$ within the unit interval. A low expectation then means that the Choquet integral is rather intolerant and behaves nearly like the minimum on average. Similarly, a high expectation means that the Choquet integral is rather tolerant and behaves nearly like the maximum on average. Note that such an analysis is meaningless when criteria are independent since, in that case, the Choquet integral boils down to a weighted arithmetic mean whose expectation is always one half (neither tolerant nor intolerant.)

In this paper we pursue this idea by defining k-intolerant Choquet integrals.[1] The case $k=1$ corresponds to the unique most intolerant Choquet integral, namely the minimum.

[1]Equivalently, we define k-intolerant capacities since there is a one-to-one correspondence between n-variable Choquet integrals and capacities defined on n objects.

The case $k = 2$ corresponds to the subclass of Choquet integrals that are bounded from above by their second lowest input values. Those Choquet integrals are more or less intolerant but not as much as the minimum. As an example, the following 3-variable Choquet integral

$$\mathcal{C}_v(x_1, x_2, x_3) = \frac{1}{2}\min(x_1, x_2) + \frac{1}{2}\min(x_1, x_3)$$

is clearly 2-intolerant, while being different from the minimum.

More generally, denoting by $x_{(1)}, \ldots, x_{(n)}$ the order statistics resulting from reordering $x_1, \ldots, x_n$ in the nondecreasing order, we say that an n-variable Choquet integral $\mathcal{C}_v$, or equivalently its underlying capacity v, is at most k-intolerant if

$$\mathcal{C}_v(x) \leq x_{(k)} \qquad (x \in [0,1]^n) \tag{1}$$

and it is exactly k-intolerant if, in addition, there is $x^* \in [0,1]^n$ such that $\mathcal{C}_v(x^*) > x^*_{(k-1)}$, with convention that $x_{(0)} := 0$.

Interestingly, condition (1) clearly implies that the output value of $\mathcal{C}_v$ is zero whenever at least k input values are zeros. We will see in Section 3 that the converse holds true as well.

At first glance, defining k-intolerant aggregation functions may appear as a pure mathematical exercise without any real application behind. In fact, in many real-life decision problems, experts or decision-makers are or must be intolerant. This is often the case when, in a given selection problem, we search for most qualified candidates among a wide population of potential alternatives. It is then sensible to reject every candidate which fails at least k criteria.

Example 1.1. Consider a (simplified) problem of selecting candidates applying for a university permanent position and suppose that the evaluation procedure is handled by appointed expert-consultants on the basis of the following academic selection criteria:

1. Scientific value of curriculum vitae,
2. Teaching effectiveness,
3. Ability to supervise staff and work in a team environment,
4. Ability to communicate easily in English,
5. Work experience in the industry,
6. Recommendations by faculty and other individuals.

Assume also that one of the rules of the evaluation procedure states that the complete failure of any two of these criteria results in automatic rejection of the applicant. This quite reasonable rule forces the Choquet integral, when used for the aggregation procedure, to be 2-intolerant, thus restricting the class of possible Choquet integrals for such a selection problem.

On the other hand, there are real-life situations where it is recommended to be tolerant, especially if the criteria are hard to meet simultaneously and if the potential alternatives are not numerous. To deal with such situations, we introduce k-tolerant aggregation functions and we will say that an n-variable Choquet integral $\mathcal{C}_v$, or equivalently its underlying capacity v, is at most k-tolerant if

$$\mathcal{C}_v(x) \geq x_{(n-k+1)} \qquad (x \in [0,1]^n).$$

In that case, the output value of $\mathcal{C}_v$ is one whenever at least k input values are ones.

Example 1.2. Consider a family who consults a Real Estate agent to buy a house. The parents propose the following house buying criteria:

1. Close to a school,
2. With parks for their children to play in,
3. With safe neighborhood for children to grow up in,
4. At least 100 meters from the closest major road,
5. At a fair distance from the nearest shopping mall,
6. Within reasonable distance of the airport.

Feeling that it is likely unrealistic to satisfy all six criteria simultaneously, the parents are ready to accept a house that would fully succeed any five over the six criteria. If a 6-variable Choquet integral is used in this selection problem, it must be 5-tolerant.

Considering k-intolerant and k-tolerant capacities can also be viewed as a way to make real applications easier to model from a computational viewpoint. Those "simplified" capacities indeed require less parameters than classical capacities (actually $O(n^{k-1})$ parameters instead of $O(2^n)$; see Section 3). Moreover, when varying k from 1 to n, we clearly recover all the possible capacities on n objects.

Notice however that this idea of partitioning capacities into subclasses is not new. Grabisch [4] proposed the k-additive capacities, which gradually cover all the possible capacities starting from additive capacities ($k = 1$). Later, Miranda et al. [16] introduced the p-symmetric capacities, also covering the possible capacities but starting from symmetric capacities ($p = 1$). Note also that other approaches to overcome the exponential complexity of capacities have also been previously proposed in the literature: Sugeno λ-measures [20], $\perp$-decomposable measures (see e.g. [7]), hierarchically decomposable measures [21], distorted probabilities (see e.g. [17]) to name a few.

It is also noteworthy that, in a given multi-criteria sorting or ordering procedure, when the capacity must be learnt from a set of examples, it is sometimes interesting or even recommended to restrict the admissible capacities to k-intolerant capacities,[2] starting from $k = 1$ and incrementing this value until a solution is found. This makes it possible to simplify the aggregation model as much as possible while keeping an interpretation of the solution. In Section 7 we reconsider Example 1.1 on the basis of such a supervised learning method.

The outline of the paper is as follows. In Section 2 we introduce and formalize the concepts of k-intolerance and k-tolerance for arbitrary aggregation functions. In Section 3 we apply these concepts to the Choquet integral, thus introducing the k-intolerant and

[2] or k-tolerant, or k-additive, etc., according to the feeling of the decision maker.

k-tolerant capacities. In Sections 4 and 5 we investigate some behavioral indices when used with those particular capacities. The indices we focus on are: the Shapley importance index, the entropy, and the veto and favor indices. In Section 6 we axiomatically introduce new indices measuring the extent to which the Choquet integral is at most k-intolerant or k-tolerant. Finally, Section 7 is devoted to a real application based on Example 1.1.

2 Basic definitions

Let $F : [0,1]^n \to [0,1]$ be an aggregation function. By considering the cube $[0,1]^n$ as a probability space with uniform distribution, we can compute the mathematical expectation of F, that is,

$$E(F) := \int_{[0,1]^n} F(x)\,\mathrm{d}x. \tag{2}$$

This value gives the average position of F within the interval $[0,1]$.

When F is *internal* (i.e., $\min \leq F \leq \max$) then it is convenient to rescale $E(F)$ within the interval $[E(\min), E(\max)]$. This leads to the following normalized and mutually complementary values [1, 11]:

$$\mathrm{andness}(F) := \frac{E(\max) - E(F)}{E(\max) - E(\min)} \tag{3}$$

$$\mathrm{orness}(F) := \frac{E(F) - E(\min)}{E(\max) - E(\min)} \tag{4}$$

Thus defined, the degree of *andness* (resp. *orness*) of F represents the degree or intensity (between 0 and 1) to which the average value of F is close to that of "min" (resp. "max"). In some sense, it also reflects the extent to which F behaves like the minimum (resp. the maximum) on average.

Define the kth order statistic function $\mathrm{OS}_k : [0,1]^n \to [0,1]$ as

$$\mathrm{OS}_k(x) := x_{(k)} \qquad (x \in [0,1]^n),$$

where $x_{(k)}$ is the kth lowest coordinate of x. It can be proved [11] that

$$E(\mathrm{OS}_k) = \frac{k}{n+1} \qquad (k \in \{1, \ldots, n\})$$

and hence the set $\{E(\mathrm{OS}_k) \mid k = 1, \ldots, n\}$ partitions the unit interval $[0,1]$ into $n+1$ equal-length subintervals.

Now, as mentioned in the introduction, when a function $F : [0,1]^n \to [0,1]$ is used to aggregate decision criteria, it is clear that the lower $E(F)$, the more F has an intolerant behavior. This suggests the following definition.

Definition 2.1. Let $k \in \{1, \ldots, n\}$. An aggregation function $F : [0,1]^n \to [0,1]$ is *at most k-intolerant* if $F \leq \mathrm{OS}_k$. It is *k-intolerant* if, in addition, $F \nleq \mathrm{OS}_{k-1}$, where $\mathrm{OS}_0 := 0$ by convention.

It follows immediately from this definition that, for any k-intolerant function F, we have $E(F) \leq E(\mathrm{OS}_k)$ and, if F is internal, we have $\mathrm{andness}(F) \geq \mathrm{andness}(\mathrm{OS}_k)$ and $\mathrm{orness}(F) \leq \mathrm{orness}(\mathrm{OS}_k)$.

Example 2.1. The product $F(x) = \prod_i x_i$, defined on $[0,1]^n$, is 1-intolerant and we have $E(F) = 1/2^n$.

By duality, we can also introduce k-tolerant functions as follows.

Definition 2.2. Let $k \in \{1,\ldots,n\}$. An aggregation function $F : [0,1]^n \to [0,1]$ is *at most k-tolerant* if $F \geq \mathrm{OS}_{n-k+1}$. It is *$k$-tolerant* if, in addition, $F \ngeq \mathrm{OS}_{n-k+2}$, where $\mathrm{OS}_{n+1} := 1$ by convention.

It is immediate to see that when a function $F : [0,1]^n \to [0,1]$ is k-intolerant, its *dual* $F^* : [0,1]^n \to [0,1]$, defined by

$$F^*(x_1,\ldots,x_n) := 1 - F(1-x_1,\ldots,1-x_n) \qquad (x \in [0,1]^n) \tag{5}$$

is k-tolerant and vice versa.

In the next section we investigate the particular case where F is the Choquet integral and we define the concepts of k-intolerant and k-tolerant capacities.

3 Case of Choquet integrals and capacities

The use of the Choquet integral has been proposed by many authors as an adequate substitute to the weighted arithmetic mean to aggregate interacting criteria; see e.g. [2, 9]. In the weighted arithmetic mean model, each criterion is given a weight representing the importance of this criterion in the decision. In the Choquet integral model, where criteria can be dependent, a capacity is used to define a weight on each combination of criteria, thus making it possible to model the interaction existing among criteria.

Let us first recall the formal definitions of these concepts. Throughout, we will use the notation $N := \{1,\ldots,n\}$ for the set of criteria.

Definition 3.1. A *capacity* on N is a set function $v : 2^N \to [0,1]$, that is nondecreasing with respect to set inclusion and such that $v(\varnothing) = 0$ and $v(N) = 1$.

Definition 3.2. Let v be a capacity on N.[3] The *Choquet integral* of $x : N \to \mathbb{R}$ with respect to v is defined by

$$\mathcal{C}_v(x) := \sum_{i=1}^{n} x_{(i)}\,[v(A_{(i)}) - v(A_{(i+1)})], \tag{6}$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)} \leq \ldots \leq x_{(n)}$. Furthermore $A_{(i)} := \{(i),\ldots,(n)\}$ and $A_{(n+1)} := \varnothing$.

In this section we apply the ideas of k-intolerance and k-tolerance (cf. Definitions 2.1 and 2.2) to the Choquet integral. Since this integral is internal, it can be seen as a function from $[0,1]^n$ to $[0,1]$. Its expectation is then given by (see [11, §3])

$$E(\mathcal{C}_v) = \frac{1}{n+1}\sum_{t=0}^{n}\frac{1}{\binom{n}{t}}\sum_{\substack{T\subseteq N\\ |T|=t}} v(T). \tag{7}$$

[3] Actually, in the definition of the Choquet integral, we can relax the condition $v(N) = 1$.

It is useful here to remember the following identity (see e.g. [9])

$$v(T) = \mathcal{C}_v(\mathbf{1}_T) \qquad (T \subseteq N), \tag{8}$$

where $\mathbf{1}_T$ denotes the characteristic vector of subset T in $\{0,1\}^n$. Recall also that the Choquet integral is stable for the positive linear transformations (see e.g. [9]), that is, it fulfills the following functional equation

$$\mathcal{C}_v(rx_1 + s, \ldots, rx_n + s) = r\mathcal{C}_v(x_1, \ldots, x_n) + s$$

for all $x \in \mathbb{R}^n$, $r > 0$, $s \in \mathbb{R}$.

Let us denote by $\mathcal{F}_N$ the set of all capacities on N. The following proposition, inspired from [11, §4], gives equivalent conditions for a Choquet integral to be at most k-intolerant.

Proposition 3.1. *Let $k \in \{1, \ldots, n\}$ and $v \in \mathcal{F}_N$. Then the following assertions are equivalent:*

i) $\mathcal{C}_v(x) \leq x_{(k)}$ $\forall x \in [0,1]^n$,

ii) $\mathcal{C}_v(x) \leq x_{(k)}$ $\forall x \in \{0,1\}^n$,

iii) $v(T) = 0$ $\forall T \subseteq N$ *such that* $|T| \leq n - k$,

iv) $\mathcal{C}_v(x) = 0$ $\forall x \in [0,1]^n$ *such that* $x_{(k)} = 0$,

v) $\mathcal{C}_v(x)$ *is independent of* $x_{(k+1)}, \ldots, x_{(n)}$,

vi) $\forall \lambda \in [0,1]$, $\forall x \in [0,1]^n$, *we have* $x_{(k)} \leq \lambda \ \Rightarrow\ \mathcal{C}_v(x) \leq \lambda$,

vii) $\forall \lambda \in [0,1]$, $\forall K \subseteq N$ *with* $|K| = k$, *we have* $\mathcal{C}_v(\mathbf{1}_{N\setminus K} + \lambda\,\mathbf{1}_K) = \lambda$,

viii) $\exists \lambda \in [0,1)$ *such that* $\forall x \in [0,1]^n$ *we have* $x_{(k)} \leq \lambda \ \Rightarrow\ \mathcal{C}_v(x) \leq \lambda$,

ix) $\exists \lambda \in [0,1)$ *such that,* $\forall K \subseteq N$ *with* $|K| = k$, *we have* $\mathcal{C}_v(\mathbf{1}_{N\setminus K} + \lambda\,\mathbf{1}_K) = \lambda$.

Proof. We shall prove the equivalence by establishing the chain of implications:

$$i) \Rightarrow ii) \Rightarrow iii) \Rightarrow v) \Rightarrow vii) \Rightarrow i) \Rightarrow vi) \Rightarrow viii) \Rightarrow ix) \Rightarrow iv) \Rightarrow iii)$$

which provides a rather short proof.

i) $\Rightarrow$ *ii)* Trivial.

ii) $\Rightarrow$ *iii)* Immediate from Eq. (8).

iii) $\Rightarrow$ *v)* For all $i = k+1, \ldots, n$, the coefficient of $x_{(i)}$ in Eq. (6) is zero.

v) $\Rightarrow$ *vii)* For all $\lambda \in [0,1]$ and all $K \subseteq N$, with $|K| = k$, we have

$$\mathcal{C}_v(\mathbf{1}_{N\setminus K} + \lambda\,\mathbf{1}_K) = \mathcal{C}_v(\lambda\,\mathbf{1}_{N\setminus K} + \lambda\,\mathbf{1}_K) = \lambda.$$

vii) $\Rightarrow$ *i)* Let $x \in [0,1]^n$ and choose $K(x) \subseteq \{i \in N \mid x_i \leq x_{(k)}\}$ with $|K(x)| = k$. By increasing monotonicity of $\mathcal{C}_v$, we have

$$\mathcal{C}_v(x) \leq \mathcal{C}_v(\mathbf{1}_{N\setminus K(x)} + x_{(k)}\,\mathbf{1}_{K(x)}) = x_{(k)}.$$

i) $\Rightarrow$ *vi)* We merely have $\mathcal{C}_v(x) \leq x_{(k)} \leq \lambda$.

vi) $\Rightarrow$ *viii)* $\Rightarrow$ *ix)* Trivial.

ix) $\Rightarrow$ *iv)* For any $K \subseteq N$ with $|K| = k$, we have

$$\lambda = \mathcal{C}_v(\mathbf{1}_{N\setminus K} + \lambda\,\mathbf{1}_K) = \lambda + (1-\lambda)\,\mathcal{C}_v(\mathbf{1}_{N\setminus K})$$

and hence $\mathcal{C}_v(\mathbf{1}_{N\setminus K}) = 0$.

iv) $\Rightarrow$ *iii)* Immediate from Eq. (8). $\square$

As we can see, some assertions of Proposition 3.1 are natural and can be interpreted easily. Some others are more surprising and show that the Choquet integral may have unexpected behaviors.

First, assertion (*iii*) enables us to define k-intolerant capacities as follows.

Definition 3.3. Let $k \in \{1, \ldots, n\}$. A capacity $v \in \mathcal{F}_N$ is *k-intolerant* if $v(T) = 0$ for all $T \subseteq N$ such that $|T| \leq n - k$ and there is $T^* \subseteq N$, with $|T^*| = n - k + 1$, such that $v(T^*) \neq 0$.

Assertion (*iv*) says that the output value of the Choquet integral is zero whenever at least k input values are zeros. This is actually a straightforward consequence of k-intolerance.

Assertion (*v*) is more surprising. It says that the output value of the Choquet integral does not take into account the values of $x_{(k+1)}, \ldots, x_{(n)}$.[4] Back to Example 1.1, only the two lowest scores are taken into account to provide a overall evaluation, regardless of the other scores.

Assertion (*viii*) is also of interest. By imposing that $\mathcal{C}_v(x) \leq \lambda$ whenever $x_{(k)} \leq \lambda$ for a given threshold $\lambda \in [0, 1)$, we necessarily force $\mathcal{C}_v$ to be at most k-intolerant. For instance, consider the problem of evaluating students with respect to different courses and suppose that it is decided that if the lowest k marks obtained by a student are less than 18/20 then his/her overall mark must be less than 18/20. In this case, the Choquet integral utilized is at most k-intolerant.

Yet more surprising is the following phenomenon. Suppose that whenever a student gets $x = 11/20$ for any k courses and $y = 12/20$ everywhere else it is decided that the overall mark is $x = 11/20$. Then the Choquet integral is at most k-intolerant. Indeed, for any $K \subseteq N$, with $|K| = k$, we simply have, since $y > x$,

$$x = \mathcal{C}_v(y\,\mathbf{1}_{N\setminus K} + x\,\mathbf{1}_K) = x + (y - x)\,\mathcal{C}_v(\mathbf{1}_{N\setminus K}),$$

which implies $\mathcal{C}_v(\mathbf{1}_{N\setminus K}) = 0$, retrieving condition (*iv*). Thus, increasing the marks on courses $N \setminus K$ has no effect on the overall evaluation.

Proposition 3.1 can be easily rewritten for k-tolerance by considering the dual $\mathcal{C}_v^*$ of the Choquet integral $\mathcal{C}_v$ as defined in Eq. (5). On this issue, Grabisch et al. [6, §4] showed that the dual $\mathcal{C}_v^*$ of $\mathcal{C}_v$ is the Choquet integral $\mathcal{C}_{v^*}$ defined from the *dual capacity* v^*, which is constructed from v by

$$v^*(T) = 1 - v(N \setminus T) \qquad (T \subseteq N).$$

We then have

$$\mathcal{C}_v \geq \mathrm{OS}_{n-k+1} \quad \Leftrightarrow \quad \mathcal{C}_{v^*} \leq \mathrm{OS}_k.$$

Proposition 3.2. *Let $k \in \{1, \ldots, n\}$ and $v \in \mathcal{F}_N$. Then the following assertions are equivalent:*

[4] More formally, we say that $F(x)$ is independent of $x_{(k+1)}, \ldots, x_{(n)}$ if it remains unchanged when $x_{(k+1)}, \ldots, x_{(n)}$ are replaced with any other values $\geq x_{(k)}$.

i) $\mathcal{C}_v(x) \geq x_{(n-k+1)}$ $\forall x \in [0,1]^n$,
ii) $\mathcal{C}_v(x) \geq x_{(n-k+1)}$ $\forall x \in \{0,1\}^n$,
iii) $v(T) = 1$ $\forall T \subseteq N$ *such that* $|T| \geq k$,
iv) $\mathcal{C}_v(x) = 1$ $\forall x \in [0,1]^n$ *such that* $x_{(n-k+1)} = 1$,
v) $\mathcal{C}_v(x)$ *is independent of* $x_{(1)}, \ldots, x_{(n-k)}$,
vi) $\forall \lambda \in [0,1]$, $\forall x \in [0,1]^n$, *we have* $x_{(n-k+1)} \geq \lambda \Rightarrow \mathcal{C}_v(x) \geq \lambda$,
vii) $\forall \lambda \in [0,1]$, $\forall K \subseteq N$ *with* $|K| = k$, *we have* $\mathcal{C}_v(\lambda\,\mathbf{1}_K) = \lambda$,
viii) $\exists \lambda \in (0,1]$ *such that* $\forall x \in [0,1]^n$ *we have* $x_{(n-k+1)} \geq \lambda \Rightarrow \mathcal{C}_v(x) \geq \lambda$,
ix) $\exists \lambda \in (0,1]$ *such that,* $\forall K \subseteq N$ *with* $|K| = k$, *we have* $\mathcal{C}_v(\lambda\,\mathbf{1}_K) = \lambda$.

Here again, some assertions are of interest. First, assertion (*iii*) enables us to define k-tolerant capacities as follows.

Definition 3.4. Let $k \in \{1, \ldots, n\}$. A capacity $v \in \mathcal{F}_N$ is *k-tolerant* if $v(T) = 1$ for all $T \subseteq N$ such that $|T| \geq k$ and there is $T^* \subseteq N$, with $|T^*| = k - 1$, such that $v(T^*) \neq 1$.

Assertion (*iv*) says that the output value of the Choquet integral is one whenever at least k input values are ones.

Assertion (*v*) says that the output value of the Choquet integral does not take into account the values of $x_{(1)}, \ldots, x_{(n-k)}$. As an application, consider students who are evaluated according to n homework assignments and assume that the evaluation procedure states that the two lowest homework scores of each student are dropped, which implies that each student can miss two homework assignments without affecting his/her final grade. If a n-variable Choquet integral is used to aggregate the homework scores, it should not take $x_{(1)}$ and $x_{(2)}$ into consideration and hence it is at most $(n-2)$-tolerant.

4 k-intolerant capacities and Shapley indices

In this section we intend to measure the effects of k-intolerance and k-tolerance on two particular behavioral[5] indices: the Shapley importance index and the entropy. Further indices, namely veto and favor indices, will be discussed in the next section.

The overall importance of each criteria can be measured through the concept of Shapley importance index, which was originally introduced in cooperative game theory as a power index [19]. Formally, the *Shapley importance index* of criterion $j \in N$ with respect to $v \in \mathcal{F}_N$ is defined as

$$\phi(v,j) := \sum_{T \subseteq N \setminus \{j\}} \frac{(n - |T| - 1)!\,|T|!}{n!}\,[v(T \cup \{j\}) - v(T)].$$

It is clear that this index can be interpreted as a weighted average value of the marginal contribution $v(T \cup \{j\}) - v(T)$ of element j alone in all combinations $T \subseteq N \setminus \{j\}$. To make this clearer, it is informative to rewrite the index as follows

$$\phi(v,j) = \frac{1}{n} \sum_{t=0}^{n-1} \frac{1}{\binom{n-1}{t}} \sum_{\substack{T \subseteq N \setminus \{j\} \\ |T| = t}} [v(T \cup \{j\}) - v(T)]. \tag{9}$$

[5]Here 'behavioral' refers to the behavior of the decision-maker or, equivalently, that of the Choquet integral used to aggregate criteria.

Thus, the average value of $v(T \cup \{j\}) - v(T)$ is computed first over all the subsets of the same size t and then over all the possible sizes. In particular, we see that $0 \leq \phi(v, j) \leq 1$ for all $v \in \mathcal{F}_N$.

Now, if $v \in \mathcal{F}_N$ is at most k-intolerant then we necessarily have

$$\phi(v, j) \leq \frac{k}{n} \qquad (j \in N). \tag{10}$$

Indeed, considering Eq. (9) and noting that $v(T \cup \{j\}) - v(T) \leq 1$, we merely have

$$\phi(v, j) \leq \frac{1}{n} \sum_{t=n-k}^{n-1} 1 = \frac{k}{n}.$$

Eq. (10) shows that, for any k-intolerant capacity, the overall importance of any criterion is bounded above by k/n. In Example 1.1, no criterion has an overall importance exceeding $1/3$.

Notice also that, for a fixed $j \in N$, inequality (10) is tight for the fuzzy measure $v_k^j \in \mathcal{F}_N$ defined for all $T \subseteq N$ by

$$v_k^j(T) = \begin{cases} 1, & \text{if } T \ni j \text{ and } |T| \geq n - k + 1, \\ 0, & \text{else.} \end{cases}$$

Indeed, we have $\phi(v_k^j, j) = k/n$ and

$$\phi(v_k^j, i) = \frac{1}{n-1}\,(1 - \frac{k}{n}) \qquad (i \in N \setminus \{j\}).$$

By duality and since $\phi(v^*, j) = \phi(v, j)$, inequality (10) is also valid for at most k-tolerant capacities and, for a fixed $j \in N$, it is tight for $(v_k^j)^*$.

Another important behavioral index is the *entropy* of a capacity $v \in \mathcal{F}_N$, which was introduced by the author [8, 10, 13] to appraise the dispersion of the values of v. The entropy of a capacity $v \in \mathcal{F}_N$ is defined as

$$H(v) := \sum_{j=1}^{n} \frac{1}{n} \sum_{t=0}^{n-1} \frac{1}{\binom{n-1}{t}} \sum_{\substack{T \subseteq N \setminus \{j\} \\ |T| = t}} h[v(T \cup \{j\}) - v(T)],$$

where $h : [0, 1] \to \mathbb{R}$ is the function $h(x) = -x \ln x$, with range $[0, 1/e]$.

For any capacity $v \in \mathcal{F}_N$, we have $0 \leq H(v) \leq \ln n$, with $H(v) = \ln n$ if and only if v is additive and symmetric (i.e., $\mathcal{C}_v$ is the arithmetic mean). For any k-intolerant or k-tolerant capacity $v \in \mathcal{F}_N$, we clearly have

$$H(v) \leq \frac{k}{e} \tag{11}$$

which shows that those particular capacities are not very well dispersed, especially when k is low.[6]

[6]Note also that inequality (11) is not tight.

5 Links with veto and favor criteria

Definition of k-intolerant aggregation functions (cf. Definition 2.1) is actually inspired from the following concept of veto criterion, which was introduced in multi-criteria decision-making by Grabisch [3].

Let $F : [0,1]^n \to [0,1]$ be an arbitrary aggregation function. A criterion $j \in N$ is said to be a *veto* for F if

$$F(x) \leq x_j \qquad (x \in [0,1]^n).$$

Even though this definition resembles that of k-intolerance, it involves only one criterion. Clearly, the failure of this criterion necessarily entails a low overall score.

Similarly, a criterion $j \in N$ is a *favor* for F if

$$F(x) \geq x_j \qquad (x \in [0,1]^n).$$

Here the satisfaction of a favor criterion entails a high overall score.

When F is the Choquet integral, analog versions of Propositions 3.1 and 3.2 can be easily obtained (see also [11, §4]). Restricting ourselves to the main assertions, we obtain the following two results, whose interpretations are straightforward.

Proposition 5.1. *Let $j \in N$ and $v \in \mathcal{F}_N$. Then the following assertions are equivalent:*

i) $\mathcal{C}_v(x) \leq x_j \;\; \forall x \in [0,1]^n$,

ii) $v(T) = 0 \;\; \forall T \subseteq N$ *such that* $T \not\ni j$,

iii) $\mathcal{C}_v(x) = 0 \;\; \forall x \in [0,1]^n$ *such that* $x_j = 0$,

iv) $\mathcal{C}_v(x)$ *is independent of* x_i $(i \in N \setminus \{j\})$ *whenever* $x_i \geq x_j$,

v) $\exists \lambda \in [0,1)$ *such that* $\forall x \in [0,1]^n$ *we have* $x_j \leq \lambda \;\Rightarrow\; \mathcal{C}_v(x) \leq \lambda$.

Proposition 5.2. *Let $j \in N$ and $v \in \mathcal{F}_N$. Then the following assertions are equivalent:*

i) $\mathcal{C}_v(x) \geq x_j \;\; \forall x \in [0,1]^n$,

ii) $v(T) = 1 \;\; \forall T \subseteq N$ *such that* $T \ni j$,

iii) $\mathcal{C}_v(x) = 1 \;\; \forall x \in [0,1]^n$ *such that* $x_j = 1$,

iv) $\mathcal{C}_v(x)$ *is independent of* x_i $(i \in N \setminus \{j\})$ *whenever* $x_i \leq x_j$,

v) $\exists \lambda \in (0,1]$ *such that* $\forall x \in [0,1]^n$ *we have* $x_j \geq \lambda \;\Rightarrow\; \mathcal{C}_v(x) \geq \lambda$.

Since they present rather extreme behaviors, veto and favor criteria rarely occur in practical applications. It is then natural to wonder if one can define indices measuring the intensity (between 0 and 1) to which a given criterion $j \in N$ behaves like a veto or a favor for the Choquet integral $\mathcal{C}_v$.

Considering again $x \in [0,1]^n$ as a multi-dimensional random variable uniformly distributed, we could propose to define such indices as

$$\begin{aligned} \mathrm{veto}(\mathcal{C}_v, j) &:= \Pr[\mathcal{C}_v(x) \leq x_j \mid x \in [0,1]^n] \\ \mathrm{favor}(\mathcal{C}_v, j) &:= \Pr[\mathcal{C}_v(x) \geq x_j \mid x \in [0,1]^n] \end{aligned}$$

Unfortunately, as pointed out in [11, §4], these definitions lead to rather intricate formulas, which are not even continuous with respect to the capacity v.

Alternative indices have been proposed axiomatically by the author [11, §4] as follows

$$\mathrm{veto}(\mathcal{C}_v, j) \;:=\; 1 - \frac{1}{n-1} \sum_{t=0}^{n-1} \frac{1}{\binom{n-1}{t}} \sum_{\substack{T \subseteq N \setminus \{j\} \\ |T| = t}} v(T)$$

$$\text{favor}(\mathcal{C}_v, j) \;:=\; \frac{1}{n-1}\sum_{t=0}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}} v(T\cup\{j\}) - \frac{1}{n-1}$$

Besides the advantage of being linear in terms of v, these indices can have a straightforward interpretation, as we will now show.

For any aggregation function $F : [0,1]^n \to [0,1]$, we define the *conditional expectation* of F given $x_j = \lambda \in \{0,1\}$ as

$$E(F \mid x_j = \lambda) := \int_{[0,1]^n} F(x \mid x_j = \lambda)\,\mathrm{d}x. \tag{12}$$

Now, by rewriting Eqs. (3) and (4) by means of the conditional expectation (12) instead of the classical expectation (2), we naturally define the *conditional* andness and orness degrees given $x_j = \lambda$. These new definitions then enable us to easily rewrite the veto and favor indices as in Eqs. (13) and (14) below, which shows that these indices somehow represent the intensity to which assertions (*iii*) in Propositions 5.1 and 5.2 are true.

Proposition 5.3. *For any $v \in \mathcal{F}_N$ and any $j \in N$, we have*

$$\text{veto}(\mathcal{C}_v, j) \;=\; \text{andness}(\mathcal{C}_v \mid x_j = 0), \tag{13}$$

$$\text{favor}(\mathcal{C}_v, j) \;=\; \text{orness}(\mathcal{C}_v \mid x_j = 1). \tag{14}$$

Proof. For any capacity $v \in \mathcal{F}_N$, the Choquet integral $\mathcal{C}_v$ can be written as (see e.g. [9])

$$\mathcal{C}_v(x) = \sum_{S\subseteq N} a(S)\,\min_{i\in S} x_i \qquad (x \in [0,1]^n),$$

where a is the Möbius transform (see e.g. [18]) of v, defined as

$$a(S) = \sum_{T\subseteq S} (-1)^{|S|-|T|} v(T) \qquad (S \subseteq N). \tag{15}$$

It follows that

$$\mathcal{C}_v(x \mid x_j = 0) = \sum_{S\subseteq N\setminus\{j\}} a(S)\,\min_{i\in S} x_i$$

and hence, denoting by v_{-j} the restriction of v to $N\setminus\{j\}$, we simply have

$$\mathcal{C}_v(x \mid x_j = 0) = \mathcal{C}_{v_{-j}}(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n).$$

Finally, by (7),

$$E(\mathcal{C}_v \mid x_j = 0) = \frac{1}{n}\sum_{t=0}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}} v(T),$$

which leads to Eq. (13).

The other equation follows by duality. Indeed, we have

$$\mathcal{C}_v(x \mid x_j = 1) = 1 - \mathcal{C}_{v^*}(\mathbf{1}_N - x \mid 1 - x_j = 0) = 1 - \mathcal{C}_{v^*}(y \mid y_j = 0)$$

where $y := \mathbf{1}_N - x$, and hence

$$E(\mathcal{C}_v \mid x_j = 1) \;=\; 1 - E(\mathcal{C}_{v^*} \mid x_j = 0) \;=\; \frac{1}{n}\sum_{t=0}^{n-1}\frac{1}{\binom{n-1}{t}}\sum_{\substack{T\subseteq N\setminus\{j\}\\|T|=t}} v(T\cup\{j\}).$$

□

Now, let us investigate the behavior of veto and favor indices when used with k-intolerant and k-tolerant capacities.

If $\mathcal{C}_v$ is k-intolerant then, from Eqs. (13) and (14), it follows immediately that, for any $j \in N$,

$$\mathrm{veto}(\mathcal{C}_v, j) \geq \mathrm{veto}(\mathrm{OS}_k, j) \quad \text{and} \quad \mathrm{favor}(\mathcal{C}_v, j) \leq \mathrm{favor}(\mathrm{OS}_k, j),$$

which shows that any criterion is more a veto for $\mathcal{C}_v$ than for OS_k and less a favor for $\mathcal{C}_v$ than for OS_k.

Similarly, if $\mathcal{C}_v$ is k-tolerant then

$$\mathrm{veto}(\mathcal{C}_v, j) \leq \mathrm{veto}(\mathrm{OS}_{n-k+1}, j) \quad \text{and} \quad \mathrm{favor}(\mathcal{C}_v, j) \geq \mathrm{favor}(\mathrm{OS}_{n-k+1}, j),$$

with similar interpretations.

Of course, these latter four inequalities are tight for $\mathcal{C}_v = \mathrm{OS}_k$ and $\mathcal{C}_v = \mathrm{OS}_{n-k+1}$, respectively.

6 Intolerance and tolerance indices

Exactly as for veto and favor phenomena, it is legitimate to wonder how we could define an index measuring the degree to which a given Choquet integral or its capacity is at most k-intolerant or at most k-tolerant. Again, we can think of the probabilities

$$\begin{aligned}
\mathrm{intol}_{\leq k}(\mathcal{C}_v) &:= \Pr[\mathcal{C}_v(x) \leq x_{(k)} \mid x \in [0,1]^n], \\
\mathrm{tol}_{\leq k}(\mathcal{C}_v) &:= \Pr[\mathcal{C}_v(x) \geq x_{(n-k+1)} \mid x \in [0,1]^n] = \mathrm{intol}_{\leq k}(\mathcal{C}_{v^*}),
\end{aligned}$$

which lead to nonlinear formulas.

Alternatively, we can proceed as in Proposition 5.3 and hence focus on assertions (iv) of Propositions 3.1 and 3.2.

First, define the following conditional expectations:

$$E(\mathcal{C}_v \mid x_{(k)} = 0) := \frac{1}{\binom{n}{k}} \sum_{\substack{K \subseteq N \\ |K|=k}} E(\mathcal{C}_v \mid x\,\mathbf{1}_K = 0\,\mathbf{1}_K), \tag{16}$$

$$E(\mathcal{C}_v \mid x_{(n-k+1)} = 1) := \frac{1}{\binom{n}{k}} \sum_{\substack{K \subseteq N \\ |K|=k}} E(\mathcal{C}_v \mid x\,\mathbf{1}_K = \mathbf{1}_K) = 1 - E(\mathcal{C}_{v^*} \mid x_{(k)} = 0), \tag{17}$$

where, for $\lambda \in \{0,1\}$,

$$E(\mathcal{C}_v \mid x\,\mathbf{1}_K = \lambda\,\mathbf{1}_K) := \int_{[0,1]^n} \mathcal{C}_v(x \mid x\,\mathbf{1}_K = \lambda\,\mathbf{1}_K)\,\mathrm{d}x.$$

These definitions are based on the idea that condition $x_{(k)} = 0$ (resp. $x_{(n-k+1)} = 1$) means that at least k coordinates of x are zeros (resp. ones).

Next, by proceeding as in the proof of Proposition 5.3, we easily arrive at the following formulas

$$E(\mathcal{C}_v \mid x_{(k)} = 0) = \frac{1}{n-k+1} \sum_{t=0}^{n-k} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T|=t}} v(T), \tag{18}$$

$$E(\mathcal{C}_v \mid x_{(n-k+1)} = 1) = \frac{1}{n-k+1} \sum_{t=k}^{n} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T|=t}} v(T), \tag{19}$$

with particular cases (when $k = n$)

$$E(\mathcal{C}_v \mid x_{(n)} = 0) = 0 \quad \text{and} \quad E(\mathcal{C}_v \mid x_{(1)} = 1) = 1.$$

Finally, by rewriting Eqs. (3) and (4) by means of the conditional expectations (16) and (17), we obtain the following conditional andness and orness degrees (for $k \neq n$):

$$\begin{aligned} \mathrm{andness}(\mathcal{C}_v \mid x_{(k)} = 0) &= 1 - \frac{1}{n-k} \sum_{t=0}^{n-k} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T|=t}} v(T) && (20) \\ \mathrm{orness}(\mathcal{C}_v \mid x_{(n-k+1)} = 1) &= \frac{1}{n-k} \sum_{t=k}^{n} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T|=t}} v(T) - \frac{1}{n-k} && (21) \end{aligned}$$

Hence the following definition of intolerance and tolerance indices, constructed in the spirit of assertions (iv) of Propositions 3.1 and 3.2.

Definition 6.1. For any $k \in \{1, \dots, n-1\}$, we define the *k-intolerance* and *k-tolerance* indices, respectively, as

$$\begin{aligned} \mathrm{intol}_{\leq k}(\mathcal{C}_v) &:= \mathrm{andness}(\mathcal{C}_v \mid x_{(k)} = 0) \\ \mathrm{tol}_{\leq k}(\mathcal{C}_v) &:= \mathrm{orness}(\mathcal{C}_v \mid x_{(n-k+1)} = 1) = \mathrm{andness}(\mathcal{C}_{v^*} \mid x_{(k)} = 0). \end{aligned}$$

Thus defined, these indices have some interesting properties.

First, it is clear that $\mathcal{C}_v$ is at most k-intolerant (resp. at most k-tolerant) if and only if $\mathrm{intol}_{\leq k}(\mathcal{C}_v) = 1$ (resp. $\mathrm{tol}_{\leq k}(\mathcal{C}_v) = 1$). For example, for any $k \in \{1, \dots, n-1\}$ and any $l \in \{1, \dots, n\}$, we have

$$\mathrm{intol}_{\leq k}(\mathrm{OS}_l) = \mathrm{tol}_{\leq k}(\mathrm{OS}_{n-l+1}) = 1 - \frac{(l-k)^+}{n-k}$$

where $r^+ := \max(r, 0)$ means the positive part of $r \in \mathbb{R}$.

For a weighted arithmetic mean WAM_ω (Choquet integral generated from an additive capacity) with a weight vector ω, we simply have

$$\mathrm{intol}_{\leq k}(\mathrm{WAM}_\omega) = \mathrm{tol}_{\leq k}(\mathrm{WAM}_\omega) = \frac{n+k-1}{2n}$$

which is a linear increasing expression moving from $1/2$ to $1 - 1/n$ as k moves from 1 to $n-1$.

Note also that, according to our convention that $x_{(0)} = 0$ and $x_{(n+1)} = 1$, we can immediately extend Eqs. (18) and (19) to the case $k = 0$. This leads us to

$$E(\mathcal{C}_v \mid x_{(0)} = 0) = E(\mathcal{C}_v) = E(\mathcal{C}_v \mid x_{(n+1)} = 1)$$

and hence we can define

$$\mathrm{intol}_{\leq 0}(\mathcal{C}_v) := \mathrm{andness}(\mathcal{C}_v) \quad \text{and} \quad \mathrm{tol}_{\leq 0}(\mathcal{C}_v) := \mathrm{orness}(\mathcal{C}_v).$$

Surprisingly, Eqs. (20) and(21) show that we also have

$$\mathrm{intol}_{\leq 1}(\mathcal{C}_v) = \mathrm{andness}(\mathcal{C}_v) \quad \text{and} \quad \mathrm{tol}_{\leq 1}(\mathcal{C}_v) = \mathrm{orness}(\mathcal{C}_v).$$

The intolerance and tolerance indices proposed in Definition 6.1 have been defined in a constructive way. To fully justify their use, we need to propose an axiomatic characterization of them. The next result, inspired from [11, Theorem 4.1], deals with this issue.

For any capacity $v \in \mathcal{F}_N$ and any permutation π on N, πv will denote the capacity of $\mathcal{F}_N$ defined by $\pi v(\pi(S)) = v(S)$ for all $S \subseteq N$, where $\pi(S) = \{\pi(i) \mid i \in S\}$.

Theorem 6.1. *Let $k \in \{1, \dots, n-1\}$ and consider a family of real numbers $\{\psi_k(\mathcal{C}_v) \mid v \in \mathcal{F}_N\}$. These numbers*

- *are linear w.r.t. the capacity, that is, there exist real constants p_T^k $(T \subseteq N)$ such that*

$$\psi_k(\mathcal{C}_v) = \sum_{T \subseteq N} p_T^k \, v(T) \qquad (v \in \mathcal{F}_N)$$

- *fulfill the "symmetry" axiom, that is, for any permutation π on N, we have*

$$\psi_k(\mathcal{C}_v) = \psi_k(\mathcal{C}_{\pi v}) \qquad (v \in \mathcal{F}_N)$$

- *fulfill the "boundary" axiom, that is, for any $l \in \{1, \dots, n\}$, we have*

$$\psi_k(\mathrm{OS}_l) = 1 - \frac{(l-k)^+}{n-k} \qquad \left(\textit{resp. } \psi_k(\mathrm{OS}_{n-l+1}) = 1 - \frac{(l-k)^+}{n-k}\right)$$

if and only if $\psi_k(\mathcal{C}_v) = \mathrm{intol}_{\leq k}(\mathcal{C}_v)$ (resp. $\psi_k(\mathcal{C}_v) = \mathrm{tol}_{\leq k}(\mathcal{C}_v)$) for all $v \in \mathcal{F}_N$.

Proof. (Sufficiency) Trivial.

(Necessity) The proof is constructed in the same spirit as that of Theorem 4.1 in [11]. The first two axioms imply that $\psi_k(\mathcal{C}_v)$ is necessarily of the form

$$\psi_k(\mathcal{C}_v) = \sum_{T \subseteq N} p_{|T|}^k \, v(T) \qquad (v \in \mathcal{F}_N).$$

Now, rewriting this latter equation for $\mathcal{C}_v = \mathrm{OS}_l$ (resp. $\mathcal{C}_v = \mathrm{OS}_{n-l+1}$) $(l = 1, \dots, n)$, for which we have

$$v(T) = \begin{cases} 1, & \text{if } |T| \geq n-l+1 \text{ (resp. } |T| \geq l), \\ 0, & \text{else,} \end{cases}$$

we arrive at the following $n \times n$ triangular linear system (with nonzero diagonal entries)

$$\sum_{\substack{t=n-l+1 \\ (\text{resp. } t=l)}}^{n} \binom{n}{t} p_t^k = 1 - \frac{(l-k)^+}{n-k} \qquad (l = 1, \dots, n).$$

As the solution of this system is unique, it must be given by $\mathrm{intol}_{\leq k}(\mathcal{C}_v)$ (resp. $\mathrm{tol}_{\leq k}(\mathcal{C}_v)$). $\square$

The axioms of Theorem 6.1 can be interpreted as follows. As for veto and favor indices, we ask the intolerance and tolerance indices to be linear with respect to the capacity. We also require these indices to be independent of the numbering of criteria. The third axiom is motivated by the following observation. For a fixed $k \in \{1, \dots, n-1\}$ the expression $\psi_k(\mathrm{OS}_l)$ (resp. $\psi_k(\mathrm{OS}_{n-l+1})$) must be

- one, whenever $1 \leq l \leq k$,
- zero, when $l = n$ (limit condition),
- a decreasing linear expression of l, when $k \leq l \leq n$.

7 Application

In this final section we investigate a practical use of Example 1.1. We assume that the Choquet integral is used to aggregate the selection criteria and we search for a feasible capacity on the basis of a set of six given prototypic applicants.

Suppose that each criterion is defined on the same 3-point ordinal scale

$$\text{Good} \succ \text{Medium} \succ \text{Bad}$$

and that the overall score of each applicant is rated on a 4-point ordinal scale

$$A \succ B_1 \succ B_2 \succ C.$$

Thus, this selection problem identifies with a classical multi-criteria sorting problem in which alternatives are sorted into four ordered classes.

The capacity v is learned from six prototypes whose profiles are defined as follows:

prot.	crit. 1	crit. 2	crit. 3	crit. 4	crit. 5	crit. 6	overall
p_1	B	G	G	G	G	G	C
p_2	G	G	G	B	G	G	C
p_3	G	B	G	G	G	G	B_2
p_4	G	G	G	G	B	G	B_1
p_5	G	G	B	G	G	M	B_1
p_6	G	G	B	G	G	G	A

Of course, in order to properly aggregate criteria, we have to consider profiles having numerical components. In the Tomaso method (see e.g. [12]), which is a recently introduced multi-criteria sorting method, it is proposed to use profiles whose components are numerical scores built on the basis of the evaluations of alternatives along the different criteria.

Without going into further details, we present the following feasible solution obtained by the Tomaso approach: $v(T) = 0$ for all $T \subseteq \{1, \ldots, 6\}$ except

$$\begin{aligned}
&v(\{1,2,4,5\}) = v(\{1,2,3,4,5\}) = v(\{1,3,4,5,6\}) = 1/3,\\
&v(\{1,2,3,4,6\}) = 2/3,\\
&v(\{1,2,4,5,6\}) = v(\{1,2,3,4,5,6\}) = 1.
\end{aligned}$$

Moreover, the overall scores A, B_1, B_2, and C correspond to the numerical values 1, 2/3, 1/3, and 0, respectively.

Clearly, this solution is 3-intolerant. Moreover, looking at the sequence

$$(\text{intol}_{\leq k}(\mathcal{C}_v))_{k=1,\ldots,5},$$

which is necessarily nondecreasing, we obtain $(0.92, 0.99, 1, 1, 1)$. Notice that the second value of this sequence is rather high, which suggests that a 2-intolerant solution might likely exist.

Actually, it turns out that, by restricting the set of feasible solutions to 2-intolerant capacities, TOMASO still obtains a solution, namely $v(T) = 0$ for all $T \subseteq \{1, \ldots, 6\}$ except

$$\begin{aligned} &v(\{1,3,4,5,6\}) = 1/4, \\ &v(\{1,2,3,4,6\}) = 1/2, \\ &v(\{1,2,4,5,6\}) = v(\{1,2,3,4,5,6\}) = 1. \end{aligned}$$

In this case the overall scores A, B_1, B_2, and C correspond to the numerical values 1, 1/2, 1/4, and 0, respectively, and the sequence $(\mathrm{intol}_{\leq k}(\mathcal{C}_v))_{k=1,\ldots,5}$ becomes $(0.94, 1, 1, 1, 1)$.

Clearly, no 1-intolerant solution exists.

8 Conclusion

In this paper, which can be considered as the sequel of [11], we have proposed the concepts of k-intolerant and k-tolerant Choquet integrals and capacities. Besides the obvious computational advantage of these concepts (comparable to that of k-additive and p-symmetric capacities), they can be easily interpreted in practical decision problems where the decision makers must be intolerant or tolerant (cf. Section 3).

We have also introduced axiomatically intolerance and tolerance indices which measure the degree to which the Choquet integral is k-intolerant and k-tolerant. These indices, when varying k from 1 to $n-1$, make it possible to identify and measure the intolerant or tolerant character of the decision maker.

As k-additive and p-symmetric capacities have been usefully extended to bi-capacities (see [5, 14, 15]), it could be interesting to extend k-intolerant and k-tolerant capacities to bi-capacities, too. This is a matter for further investigation.

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