On Order Invariant Synthesizing Functions

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Abstract

This paper gives a description of the class of continuous functions that are comparison meaningful in the sense of measurement theory. When idempotency is assumed, this class reduces to the Boolean max-min functions (lattice polynomials). In that case, continuity can be replaced by increasing monotonicity, provided the range of variables is open. The particular cases of order statistics and projection functions are also studied.

Keywords: Aggregation function; Ordinal scale; Comparison meaningfulness; Ordinal stability.

1 Introduction

Consider a set of real numbers defining an ordinal scale, i.e., a scale where only order matters, and not numbers. For example, a scale of evaluation of a scientific paper by a referee such as

1=Poor, 2=Below Average, 3=Average,
4=Very Good, 5=Excellent

is a (finite) ordinal scale, despite the coding by numbers 1 to 5. These numbers are actually meaningless since any other numbers that preserve order could have been used. For instance,

-6.5=Poor, -1.2=Below Average, 8.7=Average,
205.6=Very Good, 750=Excellent.

Thus, the numbers that are assigned to that scale are defined up to a continuous and strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$. For a general discussion of ordinal scales and for definitions of other scale types; see for instance Roberts [19] and Luce et al. [8].

Now, let $x_1, \ldots, x_n$ be real numbers given according to an ordinal scale. It is clear that any aggregation of these numbers cannot be made by means of usual arithmetic operations, unless these operations involve only order. For example, computing the arithmetic mean is forbidden, but the median or any order statistic is permitted. More precisely, the aggregated value can be calculated only by a synthesizing function $M : \mathbb{R}^n \to \mathbb{R}$ satisfying the following condition:

$$\begin{gathered} M(x_1, \ldots, x_n) \left\{ \begin{smallmatrix} < \\ = \end{smallmatrix} \right\} M(x'_1, \ldots, x'_n) \\ \Downarrow \\ M(\phi(x_1), \ldots, \phi(x_n)) \left\{ \begin{smallmatrix} < \\ = \end{smallmatrix} \right\} M(\phi(x'_1), \ldots, \phi(x'_n)) \end{gathered}$$

for any $x, x' \in \mathbb{R}^n$ and any continuous and strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$. Such an order invariant function is said to be *comparison meaningful (from an ordinal scale)*; see Orlov [14].

A typical example of comparison meaningful function is given by the *Boolean max-min functions* [9, 11], also called *lattice polynomials* [4, 16, 17]. These functions are of the form

$$M(x) = \bigvee_{T \in \mathcal{T}} \bigwedge_{i \in T} x_i \qquad (x \in \mathbb{R}^n),$$

where $\mathcal{T}$ is a non-empty family of non-empty subsets of $\{1, \ldots, n\}$. Moreover, symbols $\vee$ and $\wedge$ denote maximum and minimum, respectively.

Marichal and Mathonet [12] described the family of continuous and comparison meaningful functions $M : [a, b]^n \to \mathbb{R}$, where $[a, b]$ is an arbitrary bounded closed interval of the real line. They have also showed that, when idempotency is assumed (that is, $M(x, \ldots, x) = x$), these functions are exactly the Boolean max-min functions. In this case, increasing monotonicity can be substituted to continuity without change.

In this paper we generalize these results by removing two restrictions. Firstly, we assume that M is defined in E^n, where E is any real interval, possibly unbounded. Secondly, we assume that the set of functions ϕ (which lead from a scale value to an equivalent one) is reduced to increasing bijections, thus weakening the comparison meaningfulness property. This latter relaxation is essential, as the admissible transformations characterizing an ordinal scale actually should be only bijections preserving order (automorphisms); see [8].

The organization of the paper is the following. In Section 2 we present the Boolean max-min functions as well as some of their properties. In Section 3 we present the aggregation properties that will be used and we point out some connections between them. Section 4 is devoted to the main results mentioned above. Section 5 deals with the cases of order statistics and projection functions.

Note that similar studies have been done for ratio scales and interval scales by Aczél and Roberts [2] and Aczél, Roberts, and Rosenbaum [3].

2 Boolean max-min functions

In this section we investigate the Boolean max-min functions, which play a central role in this paper. The word 'Boolean' refers to the fact that these functions are generated by $\{0, 1\}$-valued set functions. They are also order invariant extensions in $\mathbb{R}^n$ of non-constant and increasing (monotone) Boolean functions.

To simplify the notation, we set $N := \{1, \ldots, n\}$.

Definition 2.1 *For any non-constant set function* $c: 2^N \to \{0,1\}$ *such that* $c(\emptyset) = 0$*, the Boolean max-min function* $\mathrm{B}_c^{\vee\wedge}: \mathbb{R}^n \to \mathbb{R}$ *associated to* c *is defined by*

$$\mathrm{B}_c^{\vee\wedge}(x) := \bigvee_{\substack{T \subseteq N \\ c(T)=1}} \bigwedge_{i \in T} x_i \qquad (x \in \mathbb{R}^n).$$

It can be proved [4, Chap. 2, Sect. 5] that any expression constructed from the real variables $x_1, \ldots, x_n$ and the symbols $\wedge$, $\vee$ (and, of course, parentheses) is a Boolean max-min function. This shows that the concept of Boolean max-min function is very natural despite its rather strange definition.

Now, we can readily see that any Boolean max-min function $\mathrm{B}_c^{\vee\wedge}$ fulfills the following property:

$$\mathrm{B}_c^{\vee\wedge}(x) \in \{x_1, \ldots, x_n\} \qquad (x \in \mathbb{R}^n).$$

Actually, we can point out a stronger property. Let Π denote the set of all permutations on N, and let us introduce the following sets which cover $\mathbb{R}^n$:

$$\mathcal{O}_\pi := \{x \in \mathbb{R}^n \,|\, x_{\pi(1)} \leq \cdots \leq x_{\pi(n)}\} \qquad (\pi \in \Pi).$$

Clearly, any Boolean max-min function $M = \mathrm{B}_c^{\vee\wedge}$ fulfills the following property:

$$\forall\, \pi \in \Pi,\ \exists\, k \in N \text{ such that } M(x) = x_k \ \ \forall x \in \mathcal{O}_\pi.$$

More precisely, we have the following result:

Proposition 2.1 *For any* $\pi \in \Pi$*, we have*

$$\mathrm{B}_c^{\vee\wedge}(x) = x_{\pi(j)} \qquad (x \in \mathcal{O}_\pi),$$

with

$$j = \bigvee_{\substack{T \subseteq N \\ c(T)=1}} \bigwedge_{\pi(i) \in T} i.$$

Proof. We simply have

$$\mathrm{B}_c^{\vee\wedge}(x) = \bigvee_{\substack{T \subseteq N \\ c(T)=1}} \bigwedge_{i \in T} x_i = \bigvee_{\substack{T \subseteq N \\ c(T)=1}} \bigwedge_{\pi(i) \in T} x_{\pi(i)},$$

which leads to the result. ∎

The set function c that defines $\mathrm{B}_c^{\vee\wedge}$ is not unique. For example, we have

$$x_1 \vee (x_1 \wedge x_2) = x_1 \qquad (x_1, x_2 \in \mathbb{R}).$$

It can be shown, however, that there is a unique increasing (monotone) set function c that defines $\mathrm{B}_c^{\vee\wedge}$, which is given by

$$c(T) = \mathrm{B}_c^{\vee\wedge}(e_T) \qquad (T \subseteq N),$$

where, for any $T \subseteq N$, e_T is the characteristic vector of T in $\{0,1\}^n$. In the appendix we present all the possible set functions that define the same Boolean max-min function.

Using classical distributivity of $\vee$ and $\wedge$, we can see that any Boolean max-min function can also be put in the form:

$$\bigwedge_{\substack{T\subseteq N\\ d(T)=0}} \bigvee_{i\in T} x_i,$$

with an appropriate set function $d : 2^N \to \{0,1\}$. In the appendix we give the conversion formulas between the set functions c and d as well as some other representations of $\mathrm{B}_c^{\vee\wedge}$.

Let $f : \{0,1\}^n \to \{0,1\}$ be a non-constant increasing Boolean function. Then the function $\mathrm{B}_c^{\vee\wedge}$, defined with $c(T) = f(e_T)$ $(T \subseteq N)$, is an extension in $\mathbb{R}^n$ of f since $f(e_T) = c(T) = \mathrm{B}_c^{\vee\wedge}(e_T)$ for all $T \subseteq N$. Consequently, any Boolean max-min function is an order invariant extension in $\mathbb{R}^n$ of a non-constant and increasing Boolean function. Moreover, we have the following axiomatic characterization; see [10, Lemmas 4.1 and 4.2].

Proposition 2.2 *The function $M : \mathbb{R}^n \to \mathbb{R}$ satisfies the following properties:*

- *increasing monotonicity (in each argument),*
- *$M(e_T) \in \{0,1\}$ for all $T \subseteq N$,*
- *interval invariance property:*

$$M(r\,x_1 + s, \ldots, r\,x_n + s) = r\,M(x_1, \ldots, x_n) + s,$$

  for all $x \in \mathbb{R}^n$, all $r > 0$, and all $s \in \mathbb{R}$,

if and only if there exists a set function c such that $M = \mathrm{B}_c^{\vee\wedge}$.

Consider now the case of symmetric Boolean max-min functions. For this purpose we recall the concept of *order statistic* (cf. van der Waerden [20, Sect. 17]).

Definition 2.2 *For any $k \in N$, the order statistic function $\mathrm{OS}_k : \mathbb{R}^n \to \mathbb{R}$ associated to the kth argument is defined by*

$$\mathrm{OS}_k(x) = x_{(k)} \qquad (x \in \mathbb{R}^n),$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)} \leq \cdots \leq x_{(n)}$.

By Proposition 2.1, we immediately see that any symmetric Boolean max-min function is an order statistic. Conversely, any order statistic is a symmetric Boolean max-min function (see the appendix).

Note that when n is odd, $n = 2k-1$, the particular order statistic $x_{(k)}$ is the well-known *median* function:

$$\mathrm{median}(x_1, \ldots, x_{2k-1}) = x_{(k)}.$$

Another particular case of Boolean max-min functions is given by the projection functions.

Definition 2.3 *For any $k \in N$, the projection function $\mathrm{P}_k : \mathbb{R}^n \to \mathbb{R}$ associated to the kth argument is defined by*

$$\mathrm{P}_k(x) = x_k \qquad (x \in \mathbb{R}^n).$$

The projection function P_k consists in projecting $x \in \mathbb{R}^n$ onto the kth axis. As a particular synthesizing function, it corresponds to a dictatorial aggregation.

3 Aggregation properties

Let E be any real interval, bounded or not. Its interior is denoted E°. The automorphism group of E, that is the group of all increasing bijections $\phi : E \to E$ is denoted by $\Phi(E)$ and the set of all continuous and strictly increasing functions $\phi : E \to E$ by $\Phi'(E)$. We clearly have $\Phi(E) \subset \Phi'(E)$. For the sake of simplicity, we also denote the vector $(\phi(x_1), \ldots, \phi(x_n))$ by $\phi(x)$.

In this section we present some aggregation properties that we will use to characterize the set of Boolean max-min functions. The main one is the comparison meaningfulness property, introduced by Orlov [14]. Let us recall its definition.

Definition 3.1 *A function $M : E^n \to \mathbb{R}$ is Φ-comparison meaningful (Φ-**CM**) if, for any $\phi \in \Phi(E)$ and any $x, x' \in E^n$, we have*

$$M(x) \left\{ \begin{smallmatrix} < \\ = \end{smallmatrix} \right\} M(x') \quad \Rightarrow \quad M(\phi(x)) \left\{ \begin{smallmatrix} < \\ = \end{smallmatrix} \right\} M(\phi(x')).$$

A stronger requirement is the Φ-ordinal stability, proposed by Marichal and Roubens [13].

Definition 3.2 *A function $M : E^n \to E$ is Φ-ordinally stable (Φ-**OS**) if, for any $\phi \in \Phi(E)$ and any $x \in E^n$, we have*

$$M(\phi(x)) = \phi(M(x)).$$

Replacing Φ by Φ' in the two definitions above, we define the Φ'-comparison meaningfulness (Φ'-**CM**) and the Φ'-ordinal stability (Φ'-**OS**).

The following proposition was proved by Ovchinnikov [15, Theorem 4.1] in the particular case of means; see also [6, 13, 16].

Proposition 3.1 *Let the function $M : E^n \to E$ fulfill Φ-**OS**. Then*

$$M(x) \in \{x_1, \ldots, x_n\} \cup \{\inf E, \sup E\} \qquad (x \in E^n).$$

*Furthermore, if E is open or if M fulfills Φ'-**OS** then*

$$M(x) \in \{x_1, \ldots, x_n\} \qquad (x \in E^n).$$

Proof. Consider $x = (x_1, \ldots, x_n) \in E^n$ reordered as $x_{(1)} \leq \ldots \leq x_{(n)}$ and set $x_0 := M(x)$. Suppose the result is false. We then have three exclusive cases:

- If $x_{(i)} < x_0 < x_{(i+1)}$ for one $i \in \{1, \ldots, n-1\}$ then there are elements $u, v \in E$ and a function $\phi \in \Phi(E)$ such that $x_{(i)} < u < x_0 < v < x_{(i+1)}$, $\phi(t) = t$ in $E \setminus [x_{(i)}, x_{(i+1)}]$, and $\phi(u) = v$. This implies $\phi(x_0) > x_0$, which is impossible because

$$\phi(x_0) = \phi(M(x)) = M(\phi(x)) = M(x) = x_0.$$

- If $x_0 < x_{(1)}$ then there are $v \in E$ and a function $\phi \in \Phi(E)$ such that $x_0 < v < x_{(1)}$, $\phi(t) = t$ for all $t \geq x_{(1)}$, and $\phi(x_0) = v$. This implies $\phi(x_0) > x_0$, a contradiction.

- The case $x_{(n)} < x_0$ can be dealt with as the previous one.

The second part of the statement is then immediate. ■

In the second part of Proposition 3.1, the assumption that E is open is necessary when not considering Φ'-**OS**. Indeed, if $a := \inf E \in E$ for example, then any $\phi \in \Phi(E)$ is such that $\phi(a) = a$ and thus the constant function $M = a$ fulfills Φ-**OS**.

When E is open, a complete description of the class of functions $M : E^n \to E$ fulfilling Φ-**OS** was given by Ovchinnikov [16, Theorem 5.1] as follows. Let R be a total preorder on N. A *cell* $\mathcal{O}_R$ in E^n is defined by

$$\mathcal{O}_R := \{x \in E^n \,|\, x_i < x_j \;\Leftrightarrow\; i\,P\,j \;\text{ and }\; x_i = x_j \;\Leftrightarrow\; i\,I\,j\},$$

where P and I are the asymmetric and symmetric parts of R, respectively. The set of all cells forms a partition of E^n. The result is then the following.

Proposition 3.2 *Assume that E is open. The function $M : E^n \to E$ fulfills Φ-**OS** if and only if for each total preorder R on N, there exists $k \in N$ such that $M(x) = x_k$ for all $x \in \mathcal{O}_R$.*

The most often encountered synthesizing functions in the literature on aggregation are means or averaging functions, such as the weighted arithmetic means. Cauchy [5] considered in 1821 the mean of n independent variables $x_1, \ldots, x_n$ as a function $M(x_1, \ldots, x_n)$ which should be internal to the set of x_i values.

Definition 3.3 *A function $M : E^n \to \mathbb{R}$ is internal (**Int**) if*

$$\min x_i \leq M(x_1, \ldots, x_n) \leq \max x_n \qquad (x \in E^n).$$

Such means satisfy trivially the property of *idempotency*, i.e., if all x_i are identical, $M(x)$ restitutes the common value.

Definition 3.4 *A function $M : E^n \to \mathbb{R}$ is idempotent (**Id**) if*

$$M(x, \ldots, x) = x \qquad (x \in E).$$

The characterizations we will present in the next section are mainly devoted to idempotent functions. We shall also use two other aggregation properties: continuity (**Co**) and increasing monotonicity in each argument (**In**).

The **Id** property seems natural enough, even when values to be aggregated are defined on an ordinal scale. Besides, one can readily see that, for functions fulfilling **In**, it is equivalent to Cauchy's internality **Int**, and both are accepted by all statisticians as requisites for means and typical values.

The following result, adapted from Lemma 2.2 in [15], shows that ordinal stability and comparison meaningfulness are closely related properties.

Proposition 3.3 *Consider a function $M : E^n \to E$.*

i) *If M fulfills* **Id** *and* Φ-**CM** *then it fulfills* Φ-**OS**.
ii) *If M fulfills* Φ-**OS** *then it fulfills* Φ-**CM**.
iii) *If E is open then M fulfills* **Id** *and* Φ-**CM** *if and only if it fulfills* Φ-**OS**.
iv) *M fulfills* **Id** *and* Φ'-**CM** *if and only if it fulfills* Φ'-**OS**.

Proof. Let us prove *i*), *ii*) and *iii*). The proof is identical for *iv*).

Let $M : E^n \to E$ fulfill **Id** and Φ-**CM**. Let $x \in E^n$ and set $x_0 := M(x)$. By **Id**, we have $M(x) = M(x_0, \ldots, x_0)$ and thus, for any $\phi \in \Phi(E)$,

$$M(\phi(x)) \stackrel{\mathbf{CM}}{=} M(\phi(x_0), \ldots, \phi(x_0)) \stackrel{\mathbf{Id}}{=} \phi(x_0) = \phi(M(x))$$

and M fulfills Φ-**OS**. Conversely, it is clear that if M fulfills Φ-**OS**, it also fulfills Φ-**CM**. Moreover, if E is open then, by Proposition 3.1, M fulfills **Id**. ■

4 Main results

In the present section we give the axiomatic characterizations stated in the introduction. Consider first the case where E is a bounded closed interval $[a, b]$. Marichal and Mathonet [12, Theorem 4.1] proved the following result.

Theorem 4.1 *The function $M : [a, b]^n \to \mathbb{R}$ fulfills* **Co** *and* Φ'-**CM** *if and only if*

- *either M is constant,*
- *or there exists a set function c and a continuous and strictly monotonic function $g : [a, b] \to \mathbb{R}$ such that $M = g \circ \mathrm{B}_c^{\vee\wedge}$.*

We intend to generalize this theorem by assuming that M is defined on any real interval E and also by replacing Φ' by Φ. We then have the following characterization.

Theorem 4.2 *The function $M : E^n \to \mathbb{R}$ fulfills* **Co** *and* Φ-**CM** *if and only if*

- *either M is constant,*
- *or there exists a set function c and a continuous and strictly monotonic function $g : E \to \mathbb{R}$ such that $M = g \circ \mathrm{B}_c^{\vee\wedge}$.*

Proof. (Sufficiency) Trivial.
(Necessity) Let $(a_m)_{m\in\mathbb{N}}$ and $(b_m)_{m\in\mathbb{N}}$ be two arbitrary sequences of E° such that

$$\lim_{m\to\infty} a_m = \inf E, \qquad \lim_{m\to\infty} b_m = \sup E,$$

and

$$a_{m+1} < a_m < b_m < b_{m+1} \qquad (m \in \mathbb{N}).$$

For any fixed $m \in \mathbb{N}$, the restriction of M to $[a_m, b_m]^n$ fulfills **Co** and Φ'-**CM**. By Theorem 4.1, either there exists a constant C_m such that $M = C_m$ in $[a_m, b_m]^n$, or there exists an increasing set function c_m and a continuous and strictly monotonic function $g_m : [a_m, b_m] \to \mathbb{R}$ such that $M = g_m \circ \mathrm{B}_{c_m}^{\vee\wedge}$ in $[a_m, b_m]^n$. Similarly, either there exists a constant C_{m+1} such that $M = C_{m+1}$ in $[a_{m+1}, b_{m+1}]^n$, or there exists an increasing set function c_{m+1} and a continuous and strictly monotonic function $g_{m+1} : [a_{m+1}, b_{m+1}] \to \mathbb{R}$ such that $M = g_{m+1} \circ \mathrm{B}_{c_{m+1}}^{\vee\wedge}$ in $[a_{m+1}, b_{m+1}]^n$.

Clearly, M is constant in $[a_m, b_m]^n$ if and only if M is constant in $[a_{m+1}, b_{m+1}]^n$. Hence, either we have

$$M = C_m = C_{m+1} \ \text{ on } \ [a_m, b_m]^n,$$

or

$$M = g_m \circ \mathrm{B}_{c_m}^{\vee\wedge} = g_{m+1} \circ \mathrm{B}_{c_{m+1}}^{\vee\wedge} \ \text{ on } \ [a_m, b_m]^n.$$

However, for any $t \in [a_m, b_m]$, we have

$$g_m(t) = (g_m \circ \mathrm{B}_{c_m}^{\vee\wedge})(t, \ldots, t) = (g_{m+1} \circ \mathrm{B}_{c_{m+1}}^{\vee\wedge})(t, \ldots, t) = g_{m+1}(t),$$

and hence $g_m = g_{m+1}$ in $[a_m, b_m]$. Moreover, defining $\psi_m : [0,1] \to [a_m, b_m]$ by

$$\psi_m(t) = a_m + t\,(b_m - a_m) \qquad (t \in [0,1]),$$

we have, for any $T \subseteq N$,

$$\begin{aligned} \psi_m(c_m(T)) &= \psi_m\Big[\mathrm{B}_{c_m}^{\vee\wedge}(e_T)\Big] = \mathrm{B}_{c_m}^{\vee\wedge}(\psi_m(e_T)) \\ &= \mathrm{B}_{c_{m+1}}^{\vee\wedge}(\psi_m(e_T)) = \psi_m\Big[\mathrm{B}_{c_{m+1}}^{\vee\wedge}(e_T)\Big] = \psi_m(c_{m+1}(T)), \end{aligned}$$

and hence, $c_m = c_{m+1}$.

Consequently, either M is constant in $(E^\circ)^n$, or there exists a set function c and a continuous and strictly monotonic function $g : E^\circ \to \mathbb{R}$ such that $M = g \circ \mathrm{B}_c^{\vee\wedge}$. By **Co**, this result still holds in E^n. ■

The following two characterizations follow immediately from Theorem 4.2.

Corollary 4.1 *The function $M : E^n \to \mathbb{R}$ fulfills* **Co**, **Id**, *and* Φ-**CM** *if and only if there exists a set function c such that $M = \mathrm{B}_c^{\vee\wedge}$.*

Proof. Trivial. ■

Note that the result in Corollary 4.1 was stated and proved first in social choice theory by Yanovskaya [21, Theorem 1] when $E = \mathbb{R}$.

Corollary 4.2 *Assume that E is open. Then the function $M : E^n \to E$ fulfills* **Co** *and* Φ-**OS** *if and only if there exists a set function c such that $M = \mathrm{B}_c^{\vee\wedge}$.*

Proof. The proof follows immediately from Proposition 3.3 and Corollary 4.1. ■

The result in Corollary 4.2 was stated and proved by Ovchinnikov [16, Theorem 5.3] in the more general setting where the range of variables is a doubly homogeneous linear order (i.e., a linear order X fulfilling the following property: for any $x_1, x_2, y_1, y_2 \in X$, with $x_1 < x_2$ and $y_1 < y_2$, there is an automorphism $\phi : X \to X$ such that $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$).

Note also that the extension of this result to the (infinite) case of functional operators can be found in [18].

We have already observed in the remark regarding Proposition 3.1 that, when E is not open, there exist functions $M : E^n \to E$ fulfilling **Co** and Φ-**OS** other than $\mathrm{B}_c^{\vee\wedge}$. The complete description of that family is given in the following result.

Corollary 4.3 *The function $M : E^n \to E$ fulfills* **Co** *and* Φ-**OS** *if and only if*

- *either $M = \inf E$ (unless $\inf E \notin E$),*
- *or $M = \sup E$ (unless $\sup E \notin E$),*
- *or there exists a set function c such that $M = \mathrm{B}_c^{\vee\wedge}$.*

Proof. (Sufficiency) Trivial.
(Necessity) By Proposition 3.3, M fulfills **Φ-CM**. Moreover, by Theorem 4.2, we have two exclusive cases:

- M is constant. By Proposition 3.1, we have $M = \inf E$ (unless $\inf E \notin E$) or $M = \sup E$ (unless $\sup E \notin E$).

- There exists a set function c' and a continuous and strictly monotonic function $g : E \to \mathbb{R}$ such that $M = g \circ \mathrm{B}_{c'}^{\vee\wedge}$. By Proposition 3.1, the restriction of M to $(E^\circ)^n$ ranges in E° and fulfills the assumptions of Corollary 4.2. Hence, there exists a set function c such that $M = \mathrm{B}_c^{\vee\wedge}$ in $(E^\circ)^n$ and even in E^n since M fulfills **Co**.

■

It follows from Corollary 4.3 that the functions $M : E^n \to E$ that fulfill **Co**, **Id**, and **Φ-OS** are exactly the Boolean max-min functions.

Now, let us turn to the case of increasing functions. Marichal and Mathonet [12, Theorem 3.1] proved the following result.

Theorem 4.3 *The function $M : [a, b]^n \to \mathbb{R}$ fulfills* **In**, **Id**, *and* **Φ'-CM** *if and only if there exists a set function c such that $M = \mathrm{B}_c^{\vee\wedge}$.*

As already mentioned in [12], Φ' cannot be replaced by Φ in Theorem 4.3. Indeed, since $\Phi([a, b])$ is the set of all continuous and strictly increasing functions $\phi : [a, b] \to [a, b]$ with boundary conditions $\phi(a) = a$ and $\phi(b) = b$, we immediately see that the function $M^* : [a, b]^n \to \mathbb{R}$, defined by

$$M^*(x) = \begin{cases} b, & \text{if } \max_i x_i = b, \\ \min_i x_i, & \text{else,} \end{cases} \tag{1}$$

fulfills **In**, **Id**, and **Φ-CM**, but is not a Boolean max-min function in $[a, b]^n$.

This example shows that if E is not open, the set of Boolean max-min functions in E^n cannot be characterized by the properties **In**, **Id**, and **Φ-CM**. The following result shows that this characterization holds when E is open.

Theorem 4.4 *Assume that E is open. Then the function $M : E^n \to \mathbb{R}$ fulfills* **In**, **Id**, *and* **Φ-CM** *if and only if there exists a set function c such that $M = \mathrm{B}_c^{\vee\wedge}$.*

Proof. The proof is similar to that of Theorem 4.2. Note however that the absence of **Co** forces E to be open. ■

Theorem 4.4 shows that the discontinuities as in (1) occur only on the border of E^n. Thus, any function $M : E^n \to \mathbb{R}$ fulfilling **In**, **Id**, and **Φ-CM** is a Boolean max-min function in $(E^\circ)^n$.

Corollary 4.4 *Assume that E is open. Then the function $M : E^n \to E$ fulfills* **In** *and* **Φ-OS** *if and only if there exists a set function c such that $M = \mathrm{B}_c^{\vee\wedge}$.*

Proof. The proof follows from Proposition 3.3 and Theorem 4.4. ■

5 Order statistics and projection functions

We now intend to characterize the order statistics and the projection functions, which are particular Boolean max-min functions.

Since the order statistics are exactly the symmetric Boolean max-min functions, we immediately have the following three characterizations. The notation **Sy** stands for the symmetry property.

Corollary 5.1 *The function $M : E^n \to \mathbb{R}$ fulfills* **Sy**, **Co**, *and* Φ-**CM** *if and only if*

- *either M is constant,*
- *or there exists $k \in N$ and a continuous and strictly monotonic function $g : E \to \mathbb{R}$ such that $M = g \circ \mathrm{OS}_k$.*

Corollary 5.2 *The function $M : E^n \to \mathbb{R}$ fulfills* **Sy**, **Co**, **Id**, *and* Φ-**CM** *if and only if there exists $k \in N$ such that $M = \mathrm{OS}_k$.*

Corollary 5.3 *Assume that E is open. Then the function $M : E^n \to \mathbb{R}$ fulfills* **Sy**, **In**, **Id**, *and* Φ-**CM** *if and only if there exists $k \in N$ such that $M = \mathrm{OS}_k$.*

Note that the second characterization, when **Int** replaces **Id**, was proved first by Orlov [14] in $\mathbb{R}^n$, then by Marichal and Roubens [13, Theorem 1] in E^n, and finally by Ovchinnikov [15, Theorem 4.3] in the more general framework of ordered sets. The two other characterizations were previously unknown.

Now, let us characterize the median function, which is a particular order statistic. For that purpose we introduce the following property.

Definition 5.1 *Let $\psi : E \to E$ be a decreasing bijection. A function $M : E^n \to \mathbb{R}$ is stable with respect to a ψ-reversal of the scale (ψ-**SR**) if for any $x, x' \in E^n$, we have*

$$M(x) = M(x') \quad \Rightarrow \quad M(\psi(x)) = M(\psi(x')),$$

where the notation $\psi(x)$ means $(\psi(x_1), \ldots, \psi(x_n))$.

We then have the following lemma.

Lemma 5.1 *Assume that n is odd. For any decreasing bijection $\psi : E \to E$, the function $M : E^n \to \mathbb{R}$ is an order statistic fulfilling ψ-**SR** if and only if $M = \mathrm{median}$.*

Proof. (Sufficiency) Trivial.
(Necessity) As in the proof of Proposition 3.3, we have $M(\psi(x)) = \psi(M(x))$ for all $x \in E^n$.

Let $k \in N$. By ψ-**SR**, we have

$$\psi(x_{(k)}) = \psi(\mathrm{OS}_k(x)) = \mathrm{OS}_k(\psi(x)) = \psi(x_{(n-k+1)}),$$

for all $x \in E^n$. Consequently, we have $\mathrm{OS}_k = \mathrm{OS}_{n-k+1}$ in E^n, and hence $n = 2k - 1$. ∎

Combining Lemma 5.1 with Corollaries 5.1 to 5.3, we have immediately the following results.

Corollary 5.4 *Assume that n is odd. There exists a decreasing bijection $\psi : E \to E$ such that the function $M : E^n \to \mathbb{R}$ fulfills* **Sy**, **Co**, Φ-**CM**, *and* ψ-**SR** *if and only if*

- *either M is constant,*
- *or there exists a continuous and strictly monotonic function $g : E \to \mathbb{R}$ such that $M = g \circ \mathrm{median}$.*

Corollary 5.5 *Assume that n is odd. There exists a decreasing bijection $\psi : E \to E$ such that the function $M : E^n \to \mathbb{R}$ fulfills* **Sy**, **Co**, **Id**, Φ-**CM**, *and* ψ-**SR** *if and only if $M = \mathrm{median}$.*

Corollary 5.6 *Assume that E is open and that n is odd. There exists a decreasing bijection $\psi : E \to E$ such that the function $M : E^n \to \mathbb{R}$ fulfills* **Sy**, **In**, **Id**, Φ-**CM**, *and* ψ-**SR** *if and only if $M = \mathrm{median}$.*

The two latter results are to be compared with the following characterization in $\mathbb{R}^n$ of the arithmetic mean; see Aczél [1, Sect. 5.3.1].

Proposition 5.1 *The function $M : \mathbb{R}^n \to \mathbb{R}$ fulfills* **Sy**, **Id**, *and the following additivity property:*

$$M(x_1 + x'_1, \dots, x_n + x'_n) = M(x_1, \dots, x_n) + M(x'_1, \dots, x'_n) \qquad (x, x' \in \mathbb{R}^n),$$

if and only if $M(x) = (\sum_i x_i)/n$ for all $x \in \mathbb{R}^n$.

Now, let us turn to the case of projection functions. As we can easily see, the projection functions fulfill the following property [2, 7].

Definition 5.2 *A function $M : E^n \to \mathbb{R}$ is Φ-comparison meaningful from independent ordinal scales (Φ-**CMI**) if, for any $\phi_1, \dots, \phi_n \in \Phi(E)$ and any $x, x' \in E^n$, we have*

$$M(x) \left\{ \begin{smallmatrix} < \\ = \end{smallmatrix} \right\} M(x') \quad \Rightarrow \quad M(\phi(x)) \left\{ \begin{smallmatrix} < \\ = \end{smallmatrix} \right\} M(\phi(x')),$$

where the notation $\phi(x)$ means $(\phi_1(x_1), \dots, \phi_n(x_n))$.

We shall prove that the projection functions are exactly those Boolean max-min functions which fulfill Φ-**CMI**. On this issue, Kim [7, Corollary 1.2] proved the following result.

Proposition 5.2 *The function $M : \mathbb{R}^n \to \mathbb{R}$ fulfills* **Co** *and* Φ-**CMI** *if and only if*

- *either M is constant,*
- *or there exists $k \in N$ and a continuous and strictly monotonic function $g : \mathbb{R} \to \mathbb{R}$ such that $M = g \circ \mathrm{P}_k$.*

From this result, we can deduce the following lemma, which will enable us to characterize easily the projection functions.

Lemma 5.2 *The function $M : E^n \to \mathbb{R}$ is a Boolean max-min function fulfilling* Φ-**CMI** *if and only if there exists $k \in N$ such that $M = \mathrm{P}_k$.*

Proof. (Sufficiency) Trivial.
(Necessity) Suppose that the Boolean max-min function $\mathrm{B}_c^{\vee\wedge}$ fulfills **Φ-CMI** in E^n. If $E=\mathbb{R}$ then, by Proposition 5.2, $\mathrm{B}_c^{\vee\wedge}$ is trivially a projection function.

Assume that E is an arbitrary open real interval. Let $x,x'\in\mathbb{R}^n$, let $\phi_1,\dots,\phi_n\in\Phi(\mathbb{R})$, and let $\psi:E\to\mathbb{R}$ be any increasing bijection. Setting $y_i:=\psi^{-1}(x_i)$, $y'_i:=\psi^{-1}(x'_i)$, and $\psi_i:=\psi^{-1}\circ\phi_i\circ\psi$ for all $i\in N$, we have

$$\begin{aligned}
& \mathrm{B}_c^{\vee\wedge}(x)\left\{\begin{smallmatrix}<\\=\end{smallmatrix}\right\}\mathrm{B}_c^{\vee\wedge}(x')\\
\Rightarrow\ & \mathrm{B}_c^{\vee\wedge}(y)\left\{\begin{smallmatrix}<\\=\end{smallmatrix}\right\}\mathrm{B}_c^{\vee\wedge}(y')\\
\Rightarrow\ & \mathrm{B}_c^{\vee\wedge}(\psi_1(y_1),\dots,\psi_n(y_n))\left\{\begin{smallmatrix}<\\=\end{smallmatrix}\right\}\mathrm{B}_c^{\vee\wedge}(\psi_1(y'_1),\dots,\psi_n(y'_n))\\
\Rightarrow\ & \mathrm{B}_c^{\vee\wedge}(\phi_1(x_1),\dots,\phi_n(x_n))\left\{\begin{smallmatrix}<\\=\end{smallmatrix}\right\}\mathrm{B}_c^{\vee\wedge}(\phi_1(x'_1),\dots,\phi_n(x'_n)).
\end{aligned}$$

Hence, $\mathrm{B}_c^{\vee\wedge}$ fulfills **Φ-CMI** in $\mathbb{R}^n$ and is thus a projection function.

Assume now that E is an arbitrary interval, bounded or not. For any open interval $\Omega\subseteq E$, the function $\mathrm{B}_c^{\vee\wedge}$ fulfills **Φ-CMI** in Ω^n and is thus a projection function. We then conclude as in the proof of Theorem 4.2. ■

Combining Lemma 5.2 with the results obtained in Section 4, we deduce immediately the following three characterizations.

Corollary 5.7 *The function $M:E^n\to\mathbb{R}$ fulfills* **Co** *and* **Φ-CMI** *if and only if*
- *either M is constant,*
- *or there exists $k\in N$ and a continuous and strictly monotonic function $g:E\to\mathbb{R}$ such that $M=g\circ \mathrm{P}_k$.*

Corollary 5.8 *The function $M:E^n\to\mathbb{R}$ fulfills* **Co**, **Id**, *and* **Φ-CMI** *if and only if there exists $k\in N$ such that $M=\mathrm{P}_k$.*

Corollary 5.9 *Assume that E is open. Then the function $M:E^n\to\mathbb{R}$ fulfills* **In**, **Id**, *and* **Φ-CMI** *if and only if there exists $k\in N$ such that $M=\mathrm{P}_k$.*

A Representations of Boolean max-min functions

The results we present here can be easily extracted from [11], where the discrete Sugeno integral (i.e., a weighted generalization of Boolean max-min functions) is investigated.

The set function c that defines the Boolean max-min function $\mathrm{B}_c^{\vee\wedge}$ is not uniquely determined. The following result gives all the possible set functions that define the same Boolean max-min function.

Observe first that we have

$$\mathrm{B}_c^{\vee\wedge}(e_T)=\bigvee_{K\subseteq T}c(K)\qquad(T\subseteq N).$$

Proposition A.1 *Let c and c' be set functions defining $\mathrm{B}_c^{\vee\wedge}$ and $\mathrm{B}_{c'}^{\vee\wedge}$, respectively. Then the following three assertions are equivalent:*

i) $\mathrm{B}_{c'}^{\vee\wedge}=\mathrm{B}_c^{\vee\wedge}$

ii) $\mathrm{B}_{c'}^{\vee\wedge}(e_T) = \mathrm{B}_c^{\vee\wedge}(e_T), \quad T \subseteq N$

iii) for any $T \subseteq N$, $T \neq \emptyset$, *we have*

$$\begin{cases} c'(T) = c(T), & \text{if } c(T) > \bigvee\limits_{K \subsetneq T} c(K), \\ 0 \leq c'(T) \leq \bigvee\limits_{K \subseteq T} c(K), & \text{if } c(T) \leq \bigvee\limits_{K \subsetneq T} c(K). \end{cases}$$

Let c be any set function defining $\mathrm{B}_c^{\vee\wedge}$ and let $T \subseteq N$, $T \neq \emptyset$. If $c(T) = 1$ and $c(K) = 0$ for all $K \subsetneq T$ then, according to the previous result, $c(T)$ cannot be modified without altering $\mathrm{B}_c^{\vee\wedge}$. In the other case, it can be replaced by 0.

If all the $c(T)$'s are taken as small as possible then we say that $\mathrm{B}_c^{\vee\wedge}$ is put in its *canonical* form. By contrast, if c is such that

$$c(T) = \bigvee_{K \subseteq T} c(K) \qquad (T \subseteq N),$$

then the $c(T)$'s are taken as large as possible and we say that $\mathrm{B}_c^{\vee\wedge}$ is put in its *complete* form. Actually, $\mathrm{B}_c^{\vee\wedge}$ is put in its complete form if and only if c is increasing. We then have the following immediate result.

Proposition A.2 *We can determine the complete form of any function* $\mathrm{B}_c^{\vee\wedge}$ *by taking* $c(T) = \mathrm{B}_c^{\vee\wedge}(e_T)$ *for all* $T \subseteq N$. *We then get its canonical form by considering successively the* T*'s in the decreasing cardinality order and setting* $c(T) = 0$ *whenever there exists* $i \in T$ *such that* $c(T \setminus \{i\}) = 1$.

Thus, the canonical form of $\mathrm{B}_c^{\vee\wedge}$ is obtained by the following algorithm:

- For all $T \subseteq N$, repeat: $c(T) \leftarrow \mathrm{B}_c^{\vee\wedge}(e_T)$
- For $t = n, \ldots, 2, 1$, repeat:
  - For all $T \subseteq N$ with $|T| = t$, repeat:
    - If $\exists\, i \in T$ such that $c(T \setminus \{i\}) = 1$ then $c(T) \leftarrow 0$.

Now, by exchanging the position of $\vee$ and $\wedge$ in Definition 2.1, we can define the Boolean min-max functions as follows.

Definition A.1 *For any non-constant set function* $d : 2^N \to \{0,1\}$ *such that* $d(\emptyset) = 1$, *the Boolean min-max function* $\mathrm{B}_d^{\wedge\vee} : \mathbb{R}^n \to \mathbb{R}$ *associated to* d *is defined by*

$$\mathrm{B}_d^{\wedge\vee}(x) := \bigwedge_{\substack{T \subseteq N \\ d(T)=0}} \bigvee_{i \in T} x_i \qquad (x \in \mathbb{R}^n).$$

Observe that we have

$$\mathrm{B}_d^{\wedge\vee}(e_T) = \bigwedge_{K \subseteq N \setminus T} d(K) \qquad (T \subseteq N).$$

Moreover, we have a result similar to Proposition A.1.

Proposition A.3 *Let* d *and* d' *be set functions defining* $\mathrm{B}_d^{\wedge\vee}$ *and* $\mathrm{B}_{d'}^{\wedge\vee}$, *respectively. Then the following three assertions are equivalent:*

i) $\mathrm{B}^{\wedge\vee}_{d'} = \mathrm{B}^{\wedge\vee}_{d}$

ii) $\mathrm{B}^{\wedge\vee}_{d'}(e_T) = \mathrm{B}^{\wedge\vee}_{d}(e_T), \quad T \subseteq N$

iii) for any $T \subseteq N$, $T \neq \emptyset$, *we have*

$$\begin{cases} d'(T) = d(T), & \text{if } d(T) < \bigwedge\limits_{K\subsetneq T} d(K), \\ \bigwedge\limits_{K\subseteq T} d(K) \leq d'(T) \leq 1, & \text{if } d(T) \geq \bigwedge\limits_{K\subsetneq T} d(K). \end{cases}$$

Let d be any set function defining $\mathrm{B}^{\wedge\vee}_{d}$ and let $T \subseteq N$, $T \neq \emptyset$. If $d(T) = 0$ and $d(K) = 1$ for all $K \subsetneq T$ then $d(T)$ cannot be modified without altering $\mathrm{B}^{\wedge\vee}_{d}$. In the other case, it can be replaced by 1.

If all the $d(T)$'s are taken as large as possible then we say that $\mathrm{B}^{\wedge\vee}_{d}$ is put in its *canonical* form. By contrast, if d is such that

$$d(T) = \bigwedge_{K\subseteq T} d(K) \qquad (T \subseteq N),$$

then the $d(T)$'s are taken as small as possible and we say that $\mathrm{B}^{\wedge\vee}_{d}$ is put in its *complete* form. Actually, $\mathrm{B}^{\wedge\vee}_{d}$ is put in its complete form if and only if d is decreasing.

Proposition A.4 *We can determine the complete form of any function* $\mathrm{B}^{\wedge\vee}_{d}$ *by taking* $d(T) = \mathrm{B}^{\wedge\vee}_{d}(e_{N\setminus T})$ *for all* $T \subseteq N$. *We then get its canonical form by considering successively the* T*'s in the decreasing cardinality order and setting* $d(T) = 1$ *whenever there exists* $i \in T$ *such that* $d(T \setminus \{i\}) = 0$.

Thus, the canonical form of $\mathrm{B}^{\wedge\vee}_{d}$ is obtained by the following algorithm:

- For all $T \subseteq N$, repeat: $d(T) \leftarrow \mathrm{B}^{\wedge\vee}_{d}(e_{N\setminus T})$
- For $t = n, \ldots, 2, 1$, repeat:
  - For all $T \subseteq N$ with $|T| = t$, repeat:
    - If $\exists\, i \in T$ such that $d(T \setminus \{i\}) = 0$ then $d(T) \leftarrow 1$.

As already mentioned in Section 2, any Boolean max-min function is also a Boolean min-max function. The following proposition gives the correspondence formulas between these two representations.

Proposition A.5 *Let* c *and* d *be set functions defining* $\mathrm{B}^{\vee\wedge}_{c}$ *and* $\mathrm{B}^{\wedge\vee}_{d}$, *respectively. Then we have*

$$\mathrm{B}^{\vee\wedge}_{c} = \mathrm{B}^{\wedge\vee}_{d} \quad \Leftrightarrow \quad \bigvee_{K\subseteq T} c(K) = \bigwedge_{K\subseteq N\setminus T} d(K) \qquad (T \subseteq N).$$

In particular, if c *and* d *are respectively increasing and decreasing then*

$$\mathrm{B}^{\vee\wedge}_{c} = \mathrm{B}^{\wedge\vee}_{d} \quad \Leftrightarrow \quad c(T) = d(N \setminus T) \qquad (T \subseteq N).$$

Therefore, if c is increasing then the Boolean max-min function $\mathrm{B}^{\vee\wedge}_{c}$ can be written in the following forms

$$\mathrm{B}^{\vee\wedge}_{c}(x) = \bigvee_{\substack{T\subseteq N \\ c(T)=1}} \bigwedge_{i\in T} x_i = \bigwedge_{\substack{T\subseteq N \\ c(N\setminus T)=0}} \bigvee_{i\in T} x_i.$$

By analogy to the theory of Boolean functions, the first form is called the disjunctive normal form (DNF) of $\mathrm{B}_c^{\vee\wedge}$ and the second one its conjunctive normal form (CNF).

Regarding the order statistics, the following equalities were proved by Ovchinnikov [15, Sect. 7]:

$$\mathrm{OS}_k(x) = \bigvee_{\substack{T\subseteq N\\|T|=n-k+1}} \bigwedge_{i\in T} x_i = \bigwedge_{\substack{T\subseteq N\\|T|=k}} \bigvee_{i\in T} x_i \qquad (k \in N).$$

Thus, the order statistic OS_k is a Boolean max-min function $\mathrm{B}_c^{\vee\wedge}$ whose canonical form is defined by $c(T) = 1$ if and only if $|T| = n - k + 1$, and the complete form by $c(T) = 1$ if and only if $|T| \geq n - k + 1$. It is also a Boolean min-max function $\mathrm{B}_d^{\wedge\vee}$ whose canonical form is defined by $d(T) = 0$ if and only if $|T| = k$, and the complete form by $d(T) = 0$ if and only if $|T| \geq k$.

Proposition A.6 *Let c be an increasing set function defining $\mathrm{B}_c^{\vee\wedge}$. Then the following assertions are equivalent:*

i) $c(T) = c(T')$ *whenever* $|T| = |T'|$.

ii) *There exists $k \in N$ such that* $\mathrm{B}_c^{\vee\wedge} = \mathrm{OS}_k$.

iii) $\mathrm{B}_c^{\vee\wedge}$ *is a symmetric function.*

Proof. *i)* $\Rightarrow$ *ii)* Since c is increasing, there exists $k \in N$ such that $c(T) = 1$ if and only if $|T| \geq n - k + 1$.

ii) $\Rightarrow$ *iii)* Trivial.

iii) $\Rightarrow$ *i)* Since c is increasing, we have $c(T) = \mathrm{B}_c^{\vee\wedge}(e_T) = \mathrm{B}_c^{\vee\wedge}(e_{T'}) = c(T')$. ∎

Now, it is clear that, for any $k \in N$, the projection function P_k is a Boolean max-min function $\mathrm{B}_c^{\vee\wedge}$ whose canonical form is defined by $c(T) = 1$ if and only if $T = \{k\}$, and the complete form by $c(T) = 1$ if and only if $T \ni k$.

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