On an axiomatization of the quasi-arithmetic mean values without the symmetry axiom

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Abstract

Kolmogoroff and Nagumo proved that the quasi-arithmetic means correspond exactly to the decomposable sequences of continuous, symmetric, strictly increasing in each variable and reflexive functions. We replace decomposability and symmetry in this characterization by a generalization of the decomposability.

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1 Introduction

A considerable amount of literature about the concept of mean (or average) and the properties of several means (such as the median, the arithmetic mean, the geometric mean, the power mean, the harmonic mean, etc.) has been already produced in the 19th century and has often treated the significance and the interpretation of these specific aggregation functions.

Cauchy [5] considered in 1821 the mean of n independent variables $x_1, \ldots, x_n$ as a function $M(x_1, \ldots, x_n)$ which should be internal to the set of x_i values:

$$\min\{x_1, \ldots, x_n\} \leq M(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\}. \tag{1}$$

The concept of mean as an *average* is usually ascribed to Chisini [6], who gives in 1929 the following definition (p. 108):

> Let $y = g(x_1, \ldots, x_n)$ be a function of n independent variables $x_1, \ldots, x_n$ representing homogeneous quantities. A mean of $x_1, \ldots, x_n$ with respect to the function g is a number M such that, if each of $x_1, \ldots, x_n$ is replaced by M, the function value is unchanged, that is,
>
> $$g(M, \ldots, M) = g(x_1, \ldots, x_n).$$

When g is considered as the sum, the product, the sum of squares, the sum of inverses, the sum of exponentials, or is proportional to $[(\sum_i x_i^2)/(\sum_i x_i)]^{1/2}$ as for the duration of oscillations of a composed pendulum of n elements of same weights, the solution of Chisini's equation corresponds respectively to the arithmetic mean, the geometric mean, the quadratic mean, the harmonic mean, the exponential mean and the antiharmonic mean, which is defined as

$$M(x_1,\ldots,x_n) = \Big(\sum_i x_i^2\Big)/\Big(\sum_i x_i\Big).$$

Unfortunately, as noted by de Finetti [7, p. 378] in 1931, Chisini's definition is so general that it does not even imply that the "mean" (provided there exists a real and unique solution to the above equation) fulfils the Cauchy's internality property. The following quote from Ricci [13, p. 39] could be considered as another possible criticism to Chisini's view:

> ... when all values become equal, the mean equals any of them too. The inverse proposition is not true. If a function of several variables takes their common value when all variables coincide, this is not sufficient evidence for calling it a mean. For example, the function
>
> $$g(x_1,x_2,\ldots,x_n) = x_n + (x_n - x_1) + (x_n - x_2) + \cdots + (x_n - x_{n-1})$$
>
> equals x_n when $x_1 = \cdots = x_n$, but it is even greater than x_n as long as x_n is greater than every other variable.

In 1930, Kolmogoroff [9] and Nagumo [12] considered that the mean should be more than just a Cauchy mean or an average in the sense of Chisini. They defined a *mean value* to be an infinite sequence of continuous, symmetric and strictly increasing (in each variable) real functions

$$M_1(x_1) = x_1, M_2(x_1,x_2),\ldots,M_n(x_1,\ldots,x_n),\ldots$$

satisfying the *reflexive* law: $M_n(x,\ldots,x) = x$ for all n and all x, and a certain kind of associative law:

$$M_k(x_1,\ldots,x_k) = x \;\;\Rightarrow\;\; M_n(x_1,\ldots,x_k,x_{k+1},\ldots,x_n) = M_n(x,\ldots,x,x_{k+1},\ldots,x_n) \qquad (2)$$

for every natural integer $k \leq n$. They proved, independently of each other, that these conditions are necessary and sufficient for the quasi-arithmeticity of the mean, that is, for the existence of a continuous strictly monotonic function f such that M_n may be written in the form

$$M_n(x_1,\ldots,x_n) = f^{-1}\Big[\frac{1}{n}\sum_{i=1}^{n} f(x_i)\Big] \qquad (3)$$

for all $n \in \mathbb{N}_0$ ($\mathbb{N}_0$ denotes the set of strictly positive integers).

The *quasi-arithmetic means* (3) comprise most of the algebraic means of common use, and allow one to specify f in relation to operational conditioning, see Table 1. Some means however do not belong to this family: de Finetti [7, p. 380] observed that the antiharmonic mean is not increasing in each variable and that the median is not associative in the sense of (2).

The above properties defining a mean value seem to be natural enough[1]. For instance, one can readily see that, for increasing means, the reflexivity property is equivalent to Cauchy's internality (1), and both are accepted by all statisticians as requisites for means.

[1]Note that Fodor and Marichal [8] generalized the Kolmogoroff-Nagumo's theorem above by relaxing the condition that the means be strictly increasing, requiring only that they be increasing. The family obtained, which has a rather intricate structure, naturally includes the "min" and "max" operations.

$f(x)$	$M_n(x_1,\dots,x_n)$	name
x	$\frac{1}{n}\sum x_i$	arithmetic
x^2	$\sqrt{\frac{1}{n}\sum x_i^2}$	quadratic
$\log x$	$\sqrt[n]{\prod x_i}$	geometric
x^{-1}	$\frac{1}{\frac{1}{n}\sum \frac{1}{x_i}}$	harmonic
x^α $(\alpha \in \mathbb{R}_0)$	$\left(\frac{1}{n}\sum x_i^\alpha\right)^{\frac{1}{\alpha}}$	power
$e^{\alpha\, x}$ $(\alpha \in \mathbb{R}_0)$	$\frac{1}{\alpha}\ln\left[\frac{1}{n}\sum e^{\alpha\, x_i}\right]$	exponential

Table 1: Examples of quasi-arithmetic means

Associativity of means (2) has been introduced first in 1926 by Bemporad [4, p. 87] in a characterization of the arithmetic mean. Under reflexivity, this condition seems more natural, for it becomes equivalent to

$$\begin{gathered} M_k(x_1,\dots,x_k) = M_k(x'_1,\dots,x'_k) \\ \Downarrow \\ M_n(x_1,\dots,x_k,x_{k+1},\dots,x_n) = M_n(x'_1,\dots,x'_k,x_{k+1},\dots,x_n) \end{gathered}$$

which says that the mean does not change when altering some values without modifying their partial mean. More recently, Marichal and Roubens [11] proposed to call this property "decomposability" in order not to confuse it with the classical associativity property.

Observe however that this concept has been defined for symmetric means. When symmetry is not assumed, it is necessary to rewrite the decomposability property in such a way that the first variables are not privileged. Marichal *et al.* [10] proposed the following general form, called "strong decomposability":

$$M_k(x_{i_1},\dots,x_{i_k}) = x \quad \Rightarrow \quad M_n\Big(\sum_{i\in K} x_i\, e_i + \sum_{i\notin K} x_i\, e_i\Big) = M_n\Big(\sum_{i\in K} x\, e_i + \sum_{i\notin K} x_i\, e_i\Big)$$

for every subset $K = \{i_1,\dots,i_k\} \subseteq \{1,\dots,n\}$ with $i_1 < \cdots < i_k$ (e_i represents the vector of $\{0,1\}^n$ in which only the i-th component is 1). Of course, under symmetry, decomposability and strong decomposability are equivalent.

The aim of this paper is to show that symmetry is not necessary in the Kolmogoroff-Nagumo's characterization, provided that decomposability is replaced by strong decomposability[2]. Thus we show that any strongly decomposable sequence $(M_n)_{n\in\mathbb{N}_0}$ of continuous, strictly increasing and reflexive functions is a quasi-arithmetic mean value (3).

2 The result

We first show that, for any strongly decomposable sequence $(M_n)_{n\in\mathbb{N}_0}$ of reflexive functions, the two-place function M_2 fulfils the *bisymmetry functional equation*[3]

$$M_2(M_2(x_1,x_2),M_2(x_3,x_4)) = M_2(M_2(x_1,x_3),M_2(x_2,x_4)). \tag{4}$$

To do this, we need some intermediate results. Let us first introduce the following practical notation: for all $k\in\mathbb{N}_0$, we define $k\odot x := x,\ldots,x$ (k times). For instance, we have

$$M_5(2\odot x, 3\odot y) = M_5(x,x,y,y,y).$$

Now, we present a technical lemma which is adapted from Nagumo [12, §1]. It will be very useful as we continue.

Lemma 2.1 *Consider a strongly decomposable sequence $(M_n)_{n\in\mathbb{N}_0}$ of reflexive functions. Then we have, for all $k,n\in\mathbb{N}_0$ with $n\geq 2$,*

$$M_{k.n}(k\odot x_1,\ldots,k\odot x_n) = M_n(x_1,\ldots,x_n), \tag{5}$$
$$M_{k.n}(x_{11},\ldots,x_{1k};\ldots;x_{n1},\ldots,x_{nk}) = M_n(M_k(x_{11},\ldots,x_{1k});\ldots;M_k(x_{n1},\ldots,x_{nk})), \tag{6}$$
$$M_n(x_1,\ldots,x_n) = M_n(x'_n,\ldots,x'_1),\quad \textit{where } x'_j = M_{n-1}(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n). \tag{7}$$

Proof. Let us fix $k,n\in\mathbb{N}_0$ with $n\geq 2$. We then have

$$\begin{aligned} M_{k.n}(k\odot x_1,\ldots,k\odot x_n) &= M_{k.n}((k.n)\odot M_n(x_1,\ldots,x_n)) \quad \text{(strong decomposability)}\\ &= M_n(x_1,\ldots,x_n) \quad \text{(reflexivity)} \end{aligned}$$

which proves (5). Next, we have

$$\begin{aligned} & M_{k.n}(x_{11},\ldots,x_{1k};\ldots;x_{n1},\ldots,x_{nk})\\ =\ & M_{k.n}(k\odot M_k(x_{11},\ldots,x_{1k});\ldots;k\odot M_k(x_{n1},\ldots,x_{nk})) \quad \text{(strong decomposability)}\\ =\ & M_n(M_k(x_{11},\ldots,x_{1k}),\ldots,M_k(x_{n1},\ldots,x_{nk})) \quad \text{(by (5))} \end{aligned}$$

which proves (6). Finally, by (5), we have

$$M_n(x_1,\ldots,x_n) = M_{n(n-1)}((n-1)\odot x_1,\ldots,(n-1)\odot x_n)$$

[2] One could think that strong decomposability implies symmetry. Nevertheless, the sequence of non-symmetric means defined for all $n\in\mathbb{N}_0$ by $M_n(x_1,\ldots,x_n)=x_1$ is strongly decomposable.

[3] The bisymmetry property (also called mediality) is very easy to handle and has been investigated from the algebraic point of view by using it mostly in structures without the property of associativity — in a certain respect, it has been used as a substitute for associativity and also for symmetry. For a list of references see [2, §6.4] (see also [3, Chap. 17]).

and by using strong decomposability with subset $K_j = \{j, n+j, 2n+j, \ldots, (n-2)n+j\}$ for $j = 1, \ldots, n$, we obtain

$$M_{n(n-1)}((n-1) \odot x_1, \ldots, (n-1) \odot x_n) = M_{n(n-1)}(x'_n, \ldots, x'_1; \ldots ; x'_n, \ldots, x'_1).$$

Therefore, we have

$$\begin{aligned} M_n(x_1, \ldots, x_n) &= M_{n(n-1)}(x'_n, \ldots, x'_1; \ldots ; x'_n, \ldots, x'_1) \\ &= M_{n-1}((n-1) \odot M_n(x'_n, \ldots, x'_1)) \quad \text{(by (6))} \\ &= M_n(x'_n, \ldots, x'_1) \quad \text{(reflexivity)} \end{aligned}$$

which proves (7). ∎

We now have at hand all the necessary tools to establish that M_2 is a solution of the bisymmetry equation.

Proposition 2.1 *For any strongly decomposable sequence $(M_n)_{n \in \mathbb{N}_0}$ of reflexive functions, the two-place function M_2 fulfils the bisymmetry functional equation (4).*

Proof. We have successively,

$$\begin{aligned} & M_2(M_2(x_1, x_2), M_2(x_3, x_4)) \\ = \;& M_4(x_1, x_2, x_3, x_4) \quad \text{(by (6))} \\ = \;& M_4(M_2(x_1, x_3), M_2(x_2, x_4), M_2(x_1, x_3), M_2(x_2, x_4)) \quad \text{(strong decomposability)} \\ = \;& M_2(M_2(M_2(x_1, x_3), M_2(x_2, x_4)), M_2(M_2(x_1, x_3), M_2(x_2, x_4))) \quad \text{(by (6))} \\ = \;& M_2(M_2(x_1, x_3), M_2(x_2, x_4)) \quad \text{(reflexivity)} \end{aligned}$$

which proves the result. ∎

Now, consider a strongly decomposable sequence $(M_n)_{n \in \mathbb{N}_0}$ of continuous, strictly increasing and reflexive functions. Since M_2 fulfils the bisymmetry equation, it must have a particular form. Actually, it has been proved by Aczél [1] (see also [2, §6.4] and [3, Chap. 17]) that the general continuous, strictly increasing, reflexive real solution of the bisymmetry equation (4) is given by the *quasi-linear mean*. The statement of this result is formulated as follows.

Theorem 2.1 *Let I be any real interval, finite or infinite. A two-place function $M : I^2 \to \mathbb{R}$ is continuous, strictly increasing in each variable, reflexive and fulfils the bisymmetric equation (4) if and only if there exists a continuous strictly monotonic function $f : I \to \mathbb{R}$ and a real number $\theta \in]0, 1[$ such that*

$$M(x_1, x_2) = f^{-1}\Big[\theta\, f(x_1) + (1-\theta)\, f(x_2)\Big], \quad \forall x_1, x_2 \in I. \tag{8}$$

According to this result, M_2 is of the form (8). By using strong decomposability, we will show later that the number θ occuring in this form must be $1/2$, so that M_2 is symmetric. Next, we will show that every function M_n is also symmetric. Before going on, consider two lemmas.

Lemma 2.2 *If A corresponds to the matrix*

$$A = \begin{pmatrix} \theta & \theta & 0 \\ 1-\theta & 0 & \theta \\ 0 & 1-\theta & 1-\theta \end{pmatrix}, \qquad \theta \in]0,1[,$$

then

$$\lim_{i\to+\infty} A^i = \frac{1}{D}\begin{pmatrix} \theta^2 & \theta^2 & \theta^2 \\ \theta(1-\theta) & \theta(1-\theta) & \theta(1-\theta) \\ (1-\theta)^2 & (1-\theta)^2 & (1-\theta)^2 \end{pmatrix}$$

with $D = \theta^2 + \theta(1-\theta) + (1-\theta)^2$.

Proof. The eigenvalues of A correspond to the solutions of $\det(A - zI) = 0$ or

$$(z-1)[\theta(1-\theta) - z^2] = 0.$$

Three distinct eigenvalues are obtained: $z_1 = 1, z_2 = \sqrt{\theta(1-\theta)}, z_3 = -\sqrt{\theta(1-\theta)}$ and A can be diagonalized:

$$\Delta = S^{-1}AS = \mathrm{diag}\Big(1, \sqrt{\theta(1-\theta)}, -\sqrt{\theta(1-\theta)}\,\Big).$$

We also have the following eigenvectors:

$$S_1 = \begin{pmatrix} s_{11} \\ s_{21} \\ s_{31} \end{pmatrix} = \begin{pmatrix} \theta^2 \\ \theta(1-\theta) \\ (1-\theta)^2 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} s_{12} \\ s_{22} \\ s_{32} \end{pmatrix} = \begin{pmatrix} -\sqrt{\theta} \\ \sqrt{\theta} - \sqrt{1-\theta} \\ \sqrt{1-\theta} \end{pmatrix},$$

$$S_3 = \begin{pmatrix} s_{13} \\ s_{23} \\ s_{33} \end{pmatrix} = \begin{pmatrix} \sqrt{\theta} \\ -\sqrt{\theta} - \sqrt{1-\theta} \\ \sqrt{1-\theta} \end{pmatrix}.$$

A can be expressed under the form: $A = S\Delta S^{-1}$ and

$$A^i = S\Delta^i S^{-1}, \quad \forall i \in \mathbb{N}_0.$$

Finally, setting $s'_{ij} := (S^{-1})_{ij}$, we have

$$\lim_{i\to+\infty} A^i = S\Big(\lim_{i\to+\infty} \Delta^i\Big)S^{-1} = (\, s'_{11}S_1 \quad s'_{12}S_1 \quad s'_{13}S_1 \,)\,.$$

Since $S^{-1}S = id$, we only have to determine $s'_{11}, s'_{12}, s'_{13}$ such that

$$(\, s'_{11} \quad s'_{12} \quad s'_{13} \,)\, S = (\, 1 \quad 0 \quad 0 \,)$$

and we can see that

$$s'_{11} = s'_{12} = s'_{13} = \frac{1}{D}.$$

∎

Lemma 2.3 *Consider a strongly decomposable sequence $(M_n)_{n\in\mathbb{N}_0}$ of functions. If M_2 is symmetric then, for all $n \in \mathbb{N}$, $n > 2$, M_n is also symmetric.*

Proof. Let us proceed by induction over $n \geq 2$. Assume that M_n is symmetric for a fixed $n \geq 2$. By strong decomposability, we have

$$\begin{aligned} M_{n+1}(x_1, \ldots, x_{n+1}) &= M_{n+1}(x_1, n \odot M_n(x_2, \ldots, x_{n+1})) \\ &= M_{n+1}(n \odot M_n(x_1, \ldots, x_n), x_{n+1}), \end{aligned}$$

and M_{n+1} is also symmetric. Hence the result. ∎

Now, we can turn to the main result. Before stating it, recall the Kolmogoroff-Nagumo's theorem.

Theorem 2.2 *Let I be any (finite or infinite) real interval, and $(M_n)_{n \in \mathbb{N}_0}$ be a decomposable sequence of continuous, symmetric, strictly increasing and reflexive functions $M_n : I^n \to \mathbb{R}$. Then and only then there exists a continuous strictly monotonic function $f : I \to \mathbb{R}$ such that, for all $n \in \mathbb{N}_0$,*

$$M_n(x_1, \ldots, x_n) = f^{-1}\Big[\frac{1}{n}\sum_{i=1}^{n} f(x_i)\Big], \quad \forall (x_1, \ldots, x_n) \in I^n.$$

As already announced, we show that the symmetry property is not necessary in Theorem 2.2 if we replace decomposability by strong decomposability. The statement is the following.

Theorem 2.3 *Let I be any (finite or infinite) real interval, and $(M_n)_{n \in \mathbb{N}_0}$ be a strongly decomposable sequence of continuous, strictly increasing and reflexive functions $M_n : I^n \to \mathbb{R}$. Then and only then there exists a continuous strictly monotonic function $f : I \to \mathbb{R}$ such that, for all $n \in \mathbb{N}_0$,*

$$M_n(x_1, \ldots, x_n) = f^{-1}\Big[\frac{1}{n}\sum_{i=1}^{n} f(x_i)\Big], \quad \forall (x_1, \ldots, x_n) \in I^n.$$

Proof. (Sufficiency) Trivial (see Theorem 2.2).

(Necessity) Consider a strongly decomposable sequence $(M_n)_{n \in \mathbb{N}_0}$ of continuous, strictly increasing and reflexive functions $M_n : I^n \to \mathbb{R}$. By Proposition 2.1, M_2 fulfils the bisymmetry functional equation (4). By Theorem 2.1, there exists a continuous strictly monotonic function $f : I \to \mathbb{R}$ and a real number $\theta \in]0, 1[$ such that

$$M_2(x_1, x_2) = f^{-1}\Big[\theta\, f(x_1) + (1-\theta)\, f(x_2)\Big], \quad \forall x_1, x_2 \in I.$$

Define $\Omega := f(I) = \{f(x) \,|\, x \in I\}$. The sequence $(F_n)_{n \in \mathbb{N}_0}$ defined by

$$F_n(z_1, \ldots, z_n) := f\Big[M_n(f^{-1}(z_1), \ldots, f^{-1}(z_n))\Big], \quad \forall (z_1, \ldots, z_n) \in \Omega^n, \forall n \in \mathbb{N}_0,$$

is also strongly decomposable and such that each F_n is continuous, strictly increasing and reflexive. Moreover, we have

$$F_2(z_1, z_2) = \theta\, z_1 + (1-\theta)\, z_2, \quad \forall z_1, z_2 \in \Omega. \tag{9}$$

Now, let us show that

$$F_3(z_1, z_2, z_3) = \frac{1}{D}\Big[\theta^2\, z_1 + \theta(1-\theta)\, z_2 + (1-\theta)^2\, z_3\Big], \quad \forall z_1, z_2, z_3 \in \Omega, \tag{10}$$

with $D = \theta^2 + \theta(1-\theta) + (1-\theta)^2$. We have successively

$$\begin{aligned} F_3(z_1, z_2, z_3) &= F_3(F_2(z_1, z_2), F_2(z_1, z_3), F_2(z_2, z_3)) \quad \text{(by (7))} \\ &= F_3(\theta\, z_1 + (1-\theta)\, z_2, \theta\, z_1 + (1-\theta)\, z_3, \theta\, z_2 + (1-\theta)\, z_3) \quad \text{(by (9))} \\ &= F_3((z_1, z_2, z_3)\, A) \end{aligned}$$

where A is the matrix defined in Lemma 2.2. By iteration, we obtain

$$\begin{aligned} F_3(z_1, z_2, z_3) &= F_3((z_1, z_2, z_3)\, A) = F_3((z_1, z_2, z_3)\, A^2) \\ &= F_3((z_1, z_2, z_3)\, A^i) \quad \forall i \in \mathbb{N}_0. \end{aligned}$$

We then have

$$\begin{aligned} F_3(z_1, z_2, z_3) &= \lim_{i\to+\infty} F_3((z_1, z_2, z_3)\, A^i) \quad \text{(constant numerical sequence)} \\ &= F_3((z_1, z_2, z_3) \lim_{i\to+\infty} A^i) \quad \text{(continuity)} \\ &= F_3\Big(3 \odot \frac{1}{D}\big[\theta^2\, z_1 + \theta(1-\theta)\, z_2 + (1-\theta)^2\, z_3\big]\Big) \quad \text{(Lemma 2.2)} \\ &= \frac{1}{D}\big[\theta^2\, z_1 + \theta(1-\theta)\, z_2 + (1-\theta)^2\, z_3\big] \quad \text{(reflexivity)} \end{aligned}$$

which proves (10).

Now we show that θ must be $1/2$. Strong decomposability implies

$$F_3(z_1, z_2, z_3) = F_3(F_2(z_1, z_3), z_2, F_2(z_1, z_3)).$$

By (9) and (10), this identity becomes

$$\theta(1-\theta)(1-2\,\theta)(z_3 - z_1) = 0,$$

that is $\theta = 1/2$.

Consequently, M_2 is symmetric. By Lemma 2.3, M_n is symmetric for all $n \in \mathbb{N}_0$. We then conclude by Theorem 2.2. ■

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