Reasoning About Coalitional Effectivity in Modal Extension of Łukasiewicz Logic

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Łukasiewicz logic

Replace the $\{0,1\}$-valued judgements with the **quantitative assessment of propositions** ϕ in the scale $Ł_k = \left\{0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1\right\}$, $k \in \mathbb{N}$. The logical connectives are:

negation	$\neg x$	$1-x$
strong disjunction	$x \oplus y$	$\min\{x+y, 1\}$
strong conjunction	$x \odot y$	$\max\{x+y-1, 0\}$
implication	$x \to y$	$\min\{1-x+y, 1\}$
lattice disjunction	$x \vee y$	$\max\{x, y\}$
lattice conjunction	$x \wedge y$	$\min\{x, y\}$

Axiomatization, completeness, complexity, game semantics!

Many-valued effectivity functions

$N = \{1, \dots, n\}$ player set

S finite set of states

Σ_i finite strategy space of player $i \in N$

$o \colon \prod_{i \in N} \Sigma_i \to S$ outcome function

Definition Let $G = (N, \{\Sigma_i \mid i \in N\}, S, o)$ be a game form. The $Ł_k$**-valued effectivity function of** G is the map $E_G \colon \mathcal{P}N \times Ł_k^S \to Ł_k$ defined by

$$E_G(C, f) = \max_{\sigma_C} \min_{\sigma_{\overline{C}}} f(o(\sigma_C \sigma_{\overline{C}})), \qquad C \in \mathcal{P}N, f \in Ł_k^S.$$

Example

$u_i \colon \prod_{j \in N} \Sigma_j \to Ł_k$ **utility function** of player $i \in N$

$o(\sigma_1, \dots, \sigma_n) = (u_i(\sigma_1, \dots, \sigma_n))_{i \in N} \in Ł_k^N$

$S = Ł_k^N$ states

$G = (N, \{\Sigma_i \mid i \in N\}, Ł_k^N, o)$

How to assess the utility of coalitions $C \subseteq N$?

Idea Use an aggregation function $f_C \colon Ł_k^N \to Ł_k$. For example:

- $f_C^m(x_1, \dots, x_n) = \min\{x_i \mid i \in C\}$
- $f_C^a(x_1, \dots, x_n) = \frac{1}{n} \sum_{i \in N} x_i$

Interpretation: $E_G(C, f_C)$ is the minimum guaranteed utility that the coalition C is able to ensure provided that the cooperative outcomes are evaluated by f_C

Playable $Ł_k$-valued effectivity functions

An $Ł_k$**-valued effectivity function** is a mapping $E \colon \mathcal{P}N \times Ł_k^S \to Ł_k$.

1. **outcome monotonic**: $E(C, f) \geq E(C, g)$ for every $C \in \mathcal{P}N$ and $f, g \in Ł_k^S$, $f \geq g$
2. idempotizable: $E(C, f \oplus f) = E(C, f) \oplus E(C, f)$, $E(C, f \odot f) = E(C, f) \odot E(C, f)$ for every $C \in \mathcal{P}N$ and every $f \in Ł_k^S$
3. N**-maximal**: $\neg E(\varnothing, \neg f) \leq E(N, f)$ for every $f \in Ł_k^S$
4. **superadditive**: $E(C, f) \wedge E(D, g) \leq E(C \cup D, f \wedge g)$ for every $C, D \in \mathcal{P}N$ with $C \cap D = \varnothing$ and every $f, g \in Ł_k^S$
5. **regular**: $E(C, f) \leq \neg E(\overline{C}, \neg f)$ for every $C \in \mathcal{P}N$ and $f \in Ł_k^S$
6. **coalition monotonic**: $E(C, f) \leq E(D, f)$ for every $C \subseteq D \in \mathcal{P}N$ and every $f \in Ł_k^S$

Definition We say that $E \colon \mathcal{P}N \times Ł_k^S \to Ł_k$ is **playable** if $E(C, 1) = 1$, $E(C, 0) = 0$, and the conditions 1.-4. are satisfied.

Theorem An $Ł_k$-valued effectivity function E is playable iff there is a game form G such that $E = E_G$.

Language of many-valued coalitional logic

Let $\mathcal{L}$ be the language $\{\to, \neg, 1\} \cup \{[C] \mid C \in \mathcal{P}N\}$.

The set $\mathsf{Form}_\mathcal{L}$ of formulas is defined inductively from the countably infinite set of propositional variables Prop by the rules:

$$\phi ::= 1 \mid p \mid \phi \to \phi \mid \neg\phi \mid [C]\phi \qquad p \in \mathsf{Prop},\ C \in \mathcal{P}N$$

The formulas are essentially the same as the formulas in Pauly's Coalitional Logic, while the intended semantics is completely different: every $\phi \in \mathsf{Form}_\mathcal{L}$ is evaluated in the non-boolean chain $Ł_k$.

Axioms of $Ł_k$-valued coalitional logic

- The axioms of Łukasiewicz $Ł_k$-valued logic
- The axioms:

  $\neg[C]0$

  $[C]1$

  $\neg[\varnothing]\neg p \to [N]p$

  $([C]p \wedge [D]q) \to [C \cup D](p \wedge q)$

  $[C](p \odot p) \leftrightarrow [C]p \odot [C]p$

  $[C](p \oplus p) \leftrightarrow [C]p \oplus [C]p$

  for every $C, D \in \mathcal{P}N$ with $C \cap D = \varnothing$
- Closed under: MP, Equivalence, Uniform Substitution, Monotonicity

A many-valued modal logic with a neighborhood semantics *à la* Coalition Logic.

Definition We define the $Ł_k$**-valued coalitional logic** $\mathbf{C}_k$ to be the intersection of all the $Ł_k$-valued logics satisfying the above axioms.

Generalized neighborhood semantics

An $Ł_k$**-valued coalitional frame** is a pair $\mathfrak{F} = (S, E)$, where $E : S \to (\mathcal{P}N \times Ł_k^S \to Ł_k)$ assigns a playable effectivity function $E(s)$ to every state $s \in S$.

Definition An $Ł_k$**-valued coalitional model** is a pair $\mathcal{M} = (\mathfrak{F}, \mathrm{Val})$, where $\mathfrak{F} = (S, E)$ is an $Ł_k$-valued coalitional frame and Val is a truth valuation $S \times \mathsf{Prop} \to Ł_k$.

We need to extend Val to modal formulas:

$$\mathrm{Val}(s, [C]\phi) = E(s)\big(C, \mathrm{Val}(-, \phi)\big), \qquad C \in \mathcal{P}N, s \in S, \phi \in \mathsf{Form}_\mathcal{L}$$

Completeness

We use the technique of the **canonical model** to prove the following:

Theorem Let $\phi \in \mathsf{Form}_\mathcal{L}$ and $\mathcal{M}$ be an $Ł_k$-valued coalitional model. Then

$$\mathbf{C}_k \vdash \phi \quad \text{iff} \quad \mathcal{M} \models \phi.$$

Equivalent presentation of idempotizability

Let $i \leq k$. The following are equivalent:

- the truth value of formula '*coalition C can enforce a state in which ϕ holds*' is $\geq \frac{i}{k}$
- coalition C can enforce a state in which the truth value of ϕ is $\geq \frac{i}{k}$

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