Preassociative aggregation functions

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Associative functions

$G\colon X^2 \to X$ is *associative* if

$$G(G(a,b),c) \;=\; G(a,G(b,c))$$

Examples: $G(a,b) = a + b$ on $X = \mathbb{R}$
$G(a,b) = a \wedge b$ on $X = L$ (lattice)

From associative functions to functions with indefinite arity

$$G(G(a,b),c) \;=\; G(a,G(b,c))$$

Extension to *n*-ary functions

$$\begin{aligned}
G_3(a,b,c) &:= G(G(a,b),c) \\
G_4(a,b,c,d) &:= G(G_3(a,b,c),d)\cdots \\
G_{n+1}(x_1,\ldots,x_{n+1}) &:= G(G_n(x_1,\ldots,x_n),x_{n+1})
\end{aligned}$$

By induction we construct

$$G^e : \bigcup_{n\geq 2} X^n \to X : \mathbf{x} \in X^n \mapsto G_n(\mathbf{x})$$

Associative functions with indefinite arity

Fact

If $G : X^2 \to X$ is associative and $p + q + r \geq 2$ then

$$\begin{aligned} & G^e(x_1, \ldots, x_p, \quad y_1, \ldots, y_q, \quad z_1, \ldots, z_r) \\ = & \; G^e(x_1, \ldots, x_p, G^e(y_1, \ldots, y_q), z_1, \ldots, z_r) \end{aligned}$$

Definition.

$F\colon \bigcup_{n\geq 0} X^n \to X$ is *associative* if for every $p + q + r \geq 0$

$$\begin{aligned} & F(x_1, \ldots, x_p, \quad y_1, \ldots, y_q, \quad z_1, \ldots, z_r) \\ = & \; F(x_1, \ldots, x_p, F(y_1, \ldots, y_q), z_1, \ldots, z_r) \end{aligned}$$

We use more comfortable notations

We regard n-tuples $\mathbf{x}$ in X^n as *n-strings* over X

1-strings: $x, y, z, \ldots$
n-strings: $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$

0-string: ε

$|\mathbf{x}| \;=\;$ length of $\mathbf{x}$

For $F : X^* \to Y$ we set

$$F_n := F|_{X^n}.$$

We convey

$$F(\mathbf{x}) \;=\; F(\varepsilon) \quad \Longleftrightarrow \quad \mathbf{x} \;=\; \varepsilon$$

$$F(\mathbf{x}) \;=\; \varepsilon \qquad \text{if } Y = X.$$

Associative functions with indefinite arity

$F\colon X^* \to X$ is *associative* if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\, \mathbf{xyz} \in X^*$$

F_1 may differ from the identity map!

Proposition

Let $F\colon X^* \to X$ and $G\colon X^* \to X$ be two associative functions such that $F_1 = G_1$ and $F_2 = G_2$. Then $F = G$.

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Preassociative functions

Definition. We say that $F\colon X^* \to Y$ is *preassociative* if

$$F(\mathbf{y}) \ = \ F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{xyz}) \ = \ F(\mathbf{xy}'\mathbf{z})$$

Examples: $F_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2$
$F_n(\mathbf{x}) = |\mathbf{x}|$

Associative functions are preassociative

$$F(\mathbf{y}) \;=\; F(\mathbf{y}') \quad\Rightarrow\quad F(\mathbf{xyz}) \;=\; F(\mathbf{xy'z})$$

Fact

If $F\colon X^* \to X$ is associative, then it is preassociative

Proof. Suppose $F(\mathbf{y}) = F(\mathbf{y}')$
Then $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}) = F(\mathbf{x}F(\mathbf{y}')\mathbf{z}) = F(\mathbf{xy'z})$ □

Construction of preassociative functions

$$F(\mathbf{y}) = F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

Proposition (**right** composition)

If $F\colon X^* \to Y$ is preassociative, then so is the function

$$x_1 \cdots x_n \mapsto F_n(g(x_1) \cdots g(x_n))$$

for every function $g\colon X \to X$

Example: $F_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2 \quad (X = Y = \mathbb{R})$

Construction of preassociative functions

$$F(\mathbf{y}) \; = \; F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{xyz}) \; = \; F(\mathbf{xy}'\mathbf{z})$$

Proposition (left composition)

If $F\colon X^* \to Y$ is preassociative, then so is

$$g \circ F \;\; : \;\; \mathbf{x} \mapsto g(F(\mathbf{x}))$$

for every function $g\colon Y \to Y$ such that $g|_{\mathrm{ran}(F)}$ is one-to-one

Example: $F_n(\mathbf{x}) = \exp(x_1^2 + \cdots + x_n^2) \quad (X = Y = \mathbb{R})$

Construction of preassociative functions

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Example: $F_n(\mathbf{x}) = \exp(x_1^2 + \cdots + x_n^2) \quad (X = Y = \mathbb{R})$

Question: Given a preassociative F, which are the g such that $g \circ F$ is preassociative?

Associative $\iff$ Preassociative with 'constrained' F_1

$$F(\mathbf{y}) = F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{xyz}) = F(\mathbf{xy'z})$$

Proposition

$F\colon X^* \to X$ is associative if and only if it is preassociative and

$$F_1(F(\mathbf{x})) = F(\mathbf{x}), \qquad \mathbf{x} \neq \varepsilon$$

We relax the constraint on F_1

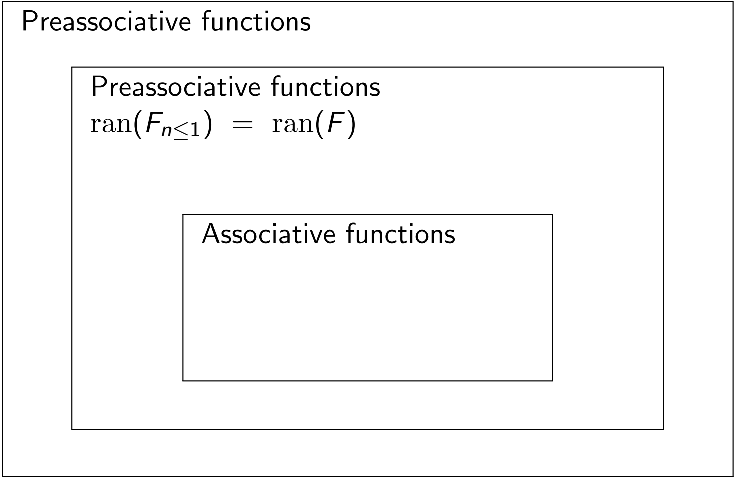
Relaxation of $F_1(F(\mathbf{x})) = F(\mathbf{x})$, $\mathbf{x} \neq \varepsilon$:

$$\boxed{\mathrm{ran}(F_{n\leq 1}) \;=\; \mathrm{ran}(F)}$$

where

$$\begin{aligned}
\mathrm{ran}(F_{n\leq 1}) &= \{F_1(x) : x \in X\} \cup \{F(\varepsilon)\} \\
\mathrm{ran}(F) &= \{F(\mathbf{x}) : \mathbf{x} \in X^*\}
\end{aligned}$$

Nested classes of preassociative functions



If $\operatorname{ran}(F_{n\leq 1}) = \operatorname{ran}(F)$

Proposition

Let $F\colon X^* \to Y$ and $G\colon X^* \to Y$ be two preassociative functions such that $\operatorname{ran}(F_{n\leq 1}) = \operatorname{ran}(F)$, $\operatorname{ran}(G_{n\leq 1}) = \operatorname{ran}(G)$, $F_0 = G_0$, $F_1 = G_1$ and $F_2 = G_2$. Then $F = G$.

Factorizing preassociative functions with associative ones

Theorem (Factorization)

Let $F\colon X^* \to Y$. The following assertions are equivalent:

(i) F is preassociative and satisfies $\mathrm{ran}(F_{n\leq 1}) = \mathrm{ran}(F)$

(ii) F can be factorized into

$$F = f \circ H$$

where $H\colon X^* \to X$ is associative and $f\colon \mathrm{ran}(H) \to Y$ is one-to-one.

Aczélian semigroups

Theorem (Aczél 1949)

$H\colon \mathbb{R}^2 \to \mathbb{R}$ is

- continuous
- one-to-one in each argument
- associative

if and only if

$$H(xy) \;=\; \varphi^{-1}(\varphi(x)+\varphi(y))$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous and strictly monotone

A class of associative functions:

$$H_n(\mathbf{x}) \;=\; \varphi^{-1}(\varphi(x_1)+\cdots+\varphi(x_n))$$

Preassociative functions from Aczélian semigroups

Theorem

Let $F\colon \mathbb{R}^* \to \mathbb{R}$. The following assertions are equivalent:

(i) F is preassociative and satisfies $\mathrm{ran}(F_{n\leq 1}) = \mathrm{ran}(F)$, F_1 and F_2 are continuous and one-to-one in each argument

(ii) we have

$$F_n(\mathbf{x}) = \psi\big(\varphi(x_1) + \cdots + \varphi(x_n)\big)$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $\psi\colon \mathbb{R} \to \mathbb{R}$ are continuous and strictly monotone

Open problems

(1) Find new axiomatizations of classes of preassociative functions from existing axiomatizations of classes of associative functions

(2) Find interpretations/applications of preassociativity in (fuzzy,modal) logic, artificial intelligence, machine learning, MCDM...

(3) Given a preassociative F, which are the g such that $g \circ F$ is preassociative?