Preassociative Aggregation Functions

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Abstract. We investigate the associativity property for varying-arity aggregation functions and introduce the more general property of preassociativity, a natural extension of associativity. We discuss this new property and describe certain classes of preassociative functions.

Keywords: associativity · preassociativity · aggregation · axiomatizations.

1 Introduction

Let X, Y be nonempty sets (e.g., nontrivial real intervals) and let $F\colon X^* \to Y$ be a varying-arity function, where $X^* = \cup_{n\geqslant 0} X^n$. The *n-th component* F_n of F is the restriction of F to X^n, i.e., $F_n = F|_{X^n}$. We convey that $X^0 = \{\varepsilon\}$ and that $F_0(\varepsilon) = \varepsilon$, where ε denotes the 0-tuple. For tuples $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ in X^*, the notation $F(\mathbf{x}, \mathbf{y})$ stands for $F(x_1, \ldots, x_n, y_1, \ldots, y_m)$, and similarly for more than two tuples. The *length* $|\mathbf{x}|$ of a tuple $\mathbf{x} \in X^*$ is a nonnegative integer defined in the usual way: we have $|\mathbf{x}| = n$ if and only if $\mathbf{x} \in X^n$.

In this note we are first interested in the associativity property for varying-arity functions. Actually, there are different equivalent definitions of this property (see, e.g., [6, 7, 11, 13, 15]). Here we use the one introduced in [15, p. 24].

Definition 1 ([15]). *A function $F\colon X^* \to X$ is said to be* associative *if for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*$ we have $F(\mathbf{x}, \mathbf{y}, \mathbf{z}) = F(\mathbf{x}, F(\mathbf{y}), \mathbf{z})$.*

As an example, the real-valued function $F\colon \mathbb{R}^* \to \mathbb{R}$ defined by $F_n(\mathbf{x}) = \sum_{i=1}^n x_i$ is associative.

Associative varying-arity functions are closely related to associative binary functions $G\colon X^2 \to X$, which are defined as the solutions of the functional equation

$$G(G(x, y), z) \;=\; G(x, G(y, z)), \qquad x, y, z \in X.$$

In fact, we show (Corollary 6) that a binary function $G\colon X^2 \to X$ is associative if and only if there exists an associative function $F\colon X^* \to X$ such that $G = F_2$.

Based on a recent investigation of associativity (see [7, 8]), we show that an associative function $F\colon X^* \to X$ is completely determined by its first two components F_1 and F_2. We also provide necessary and sufficient conditions on the components F_1 and F_2 for a function $F\colon X^* \to X$ to be associative (Theorem 7). These results are gathered in Section 3.

The main aim of this note is to introduce and investigate the following generalization of associativity, called *preassociativity*.

Definition 2. *We say that a function $F\colon X^* \to Y$ is* preassociative *if for every $\mathbf{x}, \mathbf{y}, \mathbf{y}', \mathbf{z} \in X^*$ we have*

$$F(\mathbf{y}) \;=\; F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{x}, \mathbf{y}, \mathbf{z}) \;=\; F(\mathbf{x}, \mathbf{y}', \mathbf{z}).$$

For instance, any real-valued function $F\colon \mathbb{R}^* \to \mathbb{R}$ defined as $F_n(\mathbf{x}) = f(\sum_{i=1}^{n} x_i)$ for every $n \in \mathbb{N}$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function, is preassociative.

It is immediate to see that any associative function $F\colon X^* \to X$ necessarily satisfies the equation $F_1 \circ F = F$ (take $\mathbf{x} = \varepsilon$ and $\mathbf{z} = \varepsilon$ in Definition 1). Actually, we show (Proposition 8) that a function $F\colon X^* \to X$ is associative if and only if it is preassociative and satisfies $F_1 \circ F = F$.

It is noteworthy that, contrary to associativity, preassociativity does not involve any composition of functions and hence allows us to consider a codomain Y that may differ from the domain X. For instance, the length function $F\colon X^* \to \mathbb{R}$ defined as $F(\mathbf{x}) = |\mathbf{x}|$ is preassociative.

In this note we mainly focus on those preassociative functions $F\colon X^* \to Y$ for which F_1 and F have the same range. (When $Y = X$, the latter condition is an immediate consequence of the condition $F_1 \circ F = F$ and hence those preassociative functions include the associative ones). Similarly to associative functions, we show that those functions are completely determined by their first two components (Proposition 12) and we provide necessary and sufficient conditions on the components F_1 and F_2 for a function $F\colon X^* \to Y$ to be preassociative and have the same range as F_1 (Theorem 15). We also give a description of these functions as compositions of the form $F = f \circ H$, where $H\colon X^* \to X$ is associative and $f\colon \mathrm{ran}(H) \to Y$ is one-to-one (Theorem 13). This is done in Section 4. Finally, in Section 5 we focus on some noteworthy axiomatized classes of associative functions and show how they can be extended to classes of preassociative functions.

The terminology used throughout this paper is the following. We denote by $\mathbb{N}$ the set $\{1, 2, 3, \ldots\}$ of strictly positive integers. The domain and range of any function f are denoted by $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$, respectively. The identity function is the function $\mathrm{id}\colon X \to X$ defined by $\mathrm{id}(x) = x$.

The proofs of our results have intentionally been omitted due to space limitation but will be available in an extended version of this note.

2 Preliminaries

Recall that a function $F\colon X^n \to X$ is said to be *idempotent* (see, e.g., [11]) if $F(x, \ldots, x) = x$ for every $x \in X$. A function $F\colon X^* \to X$ is said to be *idempotent* if F_n is idempotent for every $n \in \mathbb{N}$.

We now introduce the following definitions. We say that $F\colon X^* \to X$ is *unarily idempotent* if $F_1(x) = x$ for every $x \in X$, i.e., $F_1 = \mathrm{id}$. We say that $F\colon X^* \to X$ is *unarily range-idempotent* if $F(x) = x$ for every $x \in \mathrm{ran}(F)$, or equivalently, $F_1 \circ F = F$. We say that $F\colon X^* \to Y$ is *unarily quasi-range-idempotent* if $\mathrm{ran}(F_1) = \mathrm{ran}(F)$. Since this property is a consequence of the condition $F_1 \circ F = F$, we see that unarily range-idempotent functions are necessarily unarily quasi-range-idempotent.

We now show that any unarily quasi-range-idempotent function $F\colon X^* \to Y$ can always be factorized as $F = F_1 \circ H$, where $H\colon X^* \to X$ is a unarily range-idempotent function. First recall that a function g is a *quasi-inverse* [17, Sect. 2.1] of a function f if

$$f \circ g|_{\mathrm{ran}(f)} = \mathrm{id}|_{\mathrm{ran}(f)},$$
$$\mathrm{ran}(g|_{\mathrm{ran}(f)}) = \mathrm{ran}(g).$$

For any function f, denote by $Q(f)$ the set of its quasi-inverses. This set is nonempty whenever we assume the Axiom of Choice (AC), which is actually just another form of the statement "every function has a quasi-inverse."

Proposition 3. *Assume AC and let $F\colon X^* \to Y$ be a unarily quasi-range-idempotent function. For any $g \in Q(F_1)$, the function $H\colon X^* \to X$ defined as $H = g \circ F$ is a unarily range-idempotent solution of the equation $F = F_1 \circ H$.*

3 Associative Functions

As observed in [15, p. 25] (see also [5, p. 15] and [11, p. 33]), associative functions $F\colon X^* \to X$ are completely determined by their unary and binary components. Indeed, by associativity we have

$$F_n(x_1,\ldots,x_n) \;=\; F_2(F_{n-1}(x_1,\ldots,x_{n-1}),x_n), \qquad n \geqslant 3, \tag{1}$$

or equivalently,

$$F_n(x_1,\ldots,x_n) \;=\; F_2(F_2(\ldots F_2(F_2(x_1,x_2),x_3)\ldots),x_n), \qquad n \geqslant 3. \tag{2}$$

We state this immediate result as follows.

Proposition 4. *Let $F\colon X^* \to X$ and $G\colon X^* \to X$ be two associative functions such that $F_1 = G_1$ and $F_2 = G_2$. Then $F = G$.*

A natural and important question now arises: Find necessary and sufficient conditions on the components F_1 and F_2 for a function $F\colon X^* \to X$ to be associative. To answer this question we first yield the following characterization of associative functions.

Theorem 5. *A function $F\colon X^* \to X$ is associative if and only if*

(i) $F_1 \circ F_1 = F_1$ *and* $F_1 \circ F_2 = F_2$,

(ii) $F_2(x_1,x_2) = F_2(F_1(x_1),x_2) = F_2(x_1,F_1(x_2))$,
(iii) F_2 *is associative, and*
(iv) *condition (1) or (2) holds.*

Corollary 6. *A binary function $F\colon X^2 \to X$ is associative if and only if there exists an associative function $G\colon X^* \to X$ such that $F = G_2$.*

Theorem 5 enables us to answer the question raised above. We state the result in the following theorem.

Theorem 7. *Let $F_1\colon X \to X$ and $F_2\colon X^2 \to X$ be two functions. Then there exists an associative function $G\colon X^* \to X$ such that $G_1 = F_1$ and $G_2 = F_2$ if and only if conditions (i)–(iii) of Theorem 5 hold. Such a function G is then uniquely determined by $G_n(x_1,\ldots,x_n) = G_2(G_{n-1}(x_1,\ldots,x_{n-1}),x_n)$ for $n \geqslant 3$.*

Thus, two functions $F_1\colon X \to X$ and $F_2\colon X^2 \to X$ are the unary and binary components of an associative function $F\colon X^* \to X$ if and only if these functions satisfy conditions (i)–(iii) of Theorem 5. In the case when only a binary function F_2 is given, any unary function F_1 satisfying conditions (i) and (ii) can be considered, for instance the identity function. Note that it may happen that the identity function is the sole possibility for F_1, for instance when we consider the binary function $F_2\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $F_2(x_1,x_2) = x_1+x_2$. However, there are examples where F_1 may differ from the identity function. For instance, for any real number $p \geqslant 1$, the p-norm $F\colon \mathbb{R}^* \to \mathbb{R}$ defined by $F_n(\mathbf{x}) = (\sum_{i=1}^n |x_i|^p)^{1/p}$ is associative but not unarily idempotent (here $|x|$ denotes the absolute value of x). Of course an associative function F that is not unarily idempotent can be made unarily idempotent simply by setting $F_1 = \mathrm{id}$. By Theorem 5 the resulting function is still associative.

4 Preassociative Functions

In this section we investigate the preassociativity property (see Definition 2) and describe certain classes of preassociative functions.

As mentioned in the introduction, any associative function $F\colon X^* \to X$ is preassociative. More precisely, we have the following result.

Proposition 8. *A function $F\colon X^* \to X$ is associative if and only if it is preassociative and unarily range-idempotent (i.e., $F_1 \circ F = F$).*

Remark 9. The function $F\colon \mathbb{R}^* \to \mathbb{R}$ defined as $F_n(\mathbf{x}) = 2\sum_{i=1}^n x_i$ is an instance of preassociative function which is not associative.

Let us now see how new preassociative functions can be generated from given preassociative functions by left and right compositions with unary maps.

Proposition 10 (Right composition). *If $F\colon X^* \to Y$ is preassociative then, for every function $g\colon X \to X$, the function $H\colon X^* \to Y$, defined as $H_n = F_n \circ (g,\ldots,g)$ for every $n \in \mathbb{N}$, is preassociative. For instance, the squared distance function $F\colon \mathbb{R}^* \to \mathbb{R}$ defined as $F_n(\mathbf{x}) = \sum_{i=1}^n x_i^2$ is preassociative.*

Proposition 11 (Left composition). *Let $F\colon X^* \to Y$ be a preassociative function and let $g\colon Y \to Y$ be a function. If $g|_{\mathrm{ran}(F)}$ is constant or one-to-one, then the function $H\colon X^* \to Y$ defined as $H = g \circ F$ is preassociative. For instance, the function $F\colon \mathbb{R}^* \to \mathbb{R}$ defined as $F_n(\mathbf{x}) = \exp(\sum_{i=1}^{n} x_i)$ is preassociative.*

We now focus on those preassociative functions which are unarily quasi-range-idempotent, that is, such that $\mathrm{ran}(F_1) = \mathrm{ran}(F)$. As we will now show, this special class of functions has interesting properties. First of all, just as for associative functions, preassociative and unarily quasi-range-idempotent functions are completely determined by their unary and binary components.

Proposition 12. *Let $F\colon X^* \to Y$ and $G\colon X^* \to Y$ be preassociative and unarily quasi-range-idempotent functions such that $F_1 = G_1$ and $F_2 = G_2$, then $F = G$.*

We now give a description of the preassociative and unarily quasi-range-idempotent functions as compositions of associative functions with unary maps.

Theorem 13. *Assume AC and let $F\colon X^* \to Y$ be a function. The following assertions are equivalent.*

(i) *F is preassociative and unarily quasi-range-idempotent.*
(ii) *There exists an associative function $H\colon X^* \to X$ and a one-to-one function $f\colon \mathrm{ran}(H) \to Y$ such that $F = f \circ H$. In this case we have $F = F_1 \circ H$, $f = F_1|_{\mathrm{ran}(H)}$, $f^{-1} \in Q(F_1)$, and we may choose $H = g \circ F$ for any $g \in Q(F_1)$.*

Remark 14. If condition (ii) of Theorem 13 holds, then by (1) we see that F can be computed recursively by

$$F_n(x_1, \ldots, x_n) \;=\; F_2((f^{-1} \circ F_{n-1})(x_1, \ldots, x_{n-1}), x_n), \qquad n \geqslant 3.$$

A similar observation was already made in a more particular setting for the so-called quasi-associative functions, see [18].

We now provide necessary and sufficient conditions on the unary and binary components for a function $F\colon X^* \to X$ to be preassociative and unarily quasi-range-idempotent. We have the following two results.

Theorem 15. *Assume AC. A function $F\colon X^* \to Y$ is preassociative and unarily quasi-range-idempotent if and only if $\mathrm{ran}(F_2) \subseteq \mathrm{ran}(F_1)$ and there exists $g \in Q(F_1)$ such that*

(i) $H_2(x_1, x_2) = H_2(H_1(x_1), x_2) = H_2(x_1, H_1(x_2))$,
(ii) *H_2 is associative, and*
(iii) *the following holds*

$$F_n(x_1, \ldots, x_n) \;=\; F_2((g \circ F_{n-1})(x_1, \ldots, x_{n-1}), x_n), \qquad n \geqslant 3,$$

or equivalently,

$$F_n(x_1, \ldots, x_n) \;=\; F_2(H_2(\ldots H_2(H_2(x_1, x_2), x_3) \ldots), x_n), \qquad n \geqslant 3,$$

where $H_1 = g \circ F_1$ *and* $H_2 = g \circ F_2$.

Theorem 16. *Assume AC and let* $F_1\colon X \to Y$ *and* $F_2\colon X^2 \to Y$ *be two functions. Then there exists a preassociative and unarily quasi-range-idempotent function* $G\colon X^* \to Y$ *such that* $G_1 = F_1$ *and* $G_2 = F_2$ *if and only if* $\mathrm{ran}(F_2) \subseteq \mathrm{ran}(F_1)$ *and there exists* $g \in Q(F_1)$ *such that conditions (i) and (ii) of Theorem 15 hold, where* $H_1 = g \circ F_1$ *and* $H_2 = g \circ F_2$*. Such a function* G *is then uniquely determined by* $G_n(x_1, \ldots, x_n) = G_2((g \circ G_{n-1})(x_1, \ldots, x_{n-1}), x_n)$ *for* $n \geqslant 3$.

5 Axiomatizations of some Function Classes

In this section we derive axiomatizations of classes of preassociative functions from certain existing axiomatizations of classes of associative functions. We restrict ourselves to a small number of classes. Further axiomatizations can be derived from known classes of associative functions.

5.1 Preassociative Functions Built from Aczélian Semigroups

Let us recall an axiomatization of the Aczélian semigroups due to Aczél [1] (see also [2, 8, 9]).

Proposition 17 ([1]). *Let* I *be a nontrivial real interval (i.e., nonempty and not a singleton). A function* $H\colon I^2 \to I$ *is continuous, one-to-one in each argument, and associative if and only if there exists a continuous and strictly monotonic function* $\varphi\colon I \to J$ *such that*

$$H(xy) = \varphi^{-1}\left(\varphi(x) + \varphi(y)\right),$$

where J *is a real interval of one of the forms* $]-\infty, b[$, $]-\infty, b]$, $]a, \infty[$, $[a, \infty[$ *or* $\mathbb{R} =]-\infty, \infty[$ $(b \leqslant 0 \leqslant a)$*. For such a function* H*, the interval* I *is necessarily open at least on one end. Moreover,* φ *can be chosen to be strictly increasing.*

Proposition 17 can be extended to preassociative functions as follows.

Theorem 18. *Let* I *be a nontrivial real interval (i.e., nonempty and not a singleton). A function* $F\colon I^* \to \mathbb{R}$ *is preassociative and unarily quasi-range-idempotent, and* F_1 *and* F_2 *are continuous and one-to-one in each argument if and only if there exist continuous and strictly monotonic functions* $\varphi\colon I \to J$ *and* $\psi\colon J \to \mathbb{R}$ *such that*

$$F_n(\mathbf{x}) \;=\; \psi(\varphi(x_1) + \cdots + \varphi(x_n)), \qquad n \in \mathbb{N},$$

where J *is a real interval of one of the forms* $]-\infty, b[$, $]-\infty, b]$, $]a, \infty[$, $[a, \infty[$ *or* $\mathbb{R} =]-\infty, \infty[$ $(b \leqslant 0 \leqslant a)$*. For such a function* F*, we have* $\psi = F_1 \circ \varphi^{-1}$ *and* I *is necessarily open at least on one end. Moreover,* φ *can be chosen to be strictly increasing.*

5.2 Preassociative Functions Built from t-norms and Related Functions

Recall that a *t-norm* (resp. *t-conorm*) is a function $H\colon [0,1]^2 \to [0,1]$ which is nondecreasing in each argument, symmetric, associative, and such that $H(1,x) = x$ (resp. $H(0,x) = x$) for every $x \in [0,1]$. Also, a *uninorm* is a function $H\colon [0,1]^2 \to [0,1]$ which is nondecreasing in each argument, symmetric, associative, and such that there exists $e \in]0,1[$ for which $H(e,x) = x$ for every $x \in [0,1]$. For general background see, e.g., [3, 10–13, 17].

T-norms can be extended to preassociative functions as follows.

Theorem 19. *Let $F\colon [0,1]^* \to \mathbb{R}$ be a function such that F_1 is strictly increasing (resp. strictly decreasing). Then F is preassociative and unarily quasi-range-idempotent, and F_2 is symmetric, nondecreasing (resp. nonincreasing) in each argument, and satisfies $F_2(1,x) = F_1(x)$ for every $x \in [0,1]$ if and only if there exists a strictly increasing (resp. strictly decreasing) function $f\colon [0,1] \to \mathbb{R}$ and a t-norm $H\colon [0,1]^* \to [0,1]$ such that $F = f \circ H$. In this case we have $f = F_1$.*

If we replace the condition "$F_2(1,x) = F_1(x)$" in Theorem 19 with "$F_2(0,x) = F_1(x)$" (resp. "$F_2(e,x) = F_1(x)$ for some $e \in]0,1[$"), then the result still holds provided that the t-norm is replaced with a t-conorm (resp. a uninorm).

5.3 Preassociative Functions Built from Ling's Axiomatizations

Recall an axiomatization due to Ling [14]; see also [4, 16].

Proposition 20 ([14]). *Let $[a,b]$ be a real closed interval, with $a < b$. A function $H\colon [a,b]^2 \to [a,b]$ is continuous, nondecreasing in each argument, associative, and such that $H(b,x) = x$ for all $x \in [a,b]$ and $H(x,x) < x$ for all $x \in]a,b[$, if and only if there exists a continuous and strictly decreasing function $\varphi\colon [a,b] \to [0,\infty[$, with $\varphi(b) = 0$, such that*

$$H(xy) \;=\; \varphi^{-1}(\min\{\varphi(x) + \varphi(y), \varphi(a)\}).$$

Proposition 20 can be extended to preassociative functions as follows.

Theorem 21. *Let $[a,b]$ be a real closed interval and let $F\colon [a,b]^* \to \mathbb{R}$ be a function such that F_1 is strictly increasing (resp. strictly decreasing). Then F is unarily quasi-range idempotent and preassociative, and F_2 is continuous and nondecreasing (resp. nonincreasing) in each argument, $F_2(b,x) = F_1(x)$ for every $x \in [a,b]$, $F_2(x,x) < F_1(x)$ (resp. $F_2(x,x) > F_1(x)$) for every $x \in]a,b[$ if and only if there exists a continuous and strictly decreasing function $\varphi\colon [a,b] \to [0,\infty[$, with $\varphi(b) = 0$, and a strictly decreasing (resp. strictly increasing) function $\psi\colon [0,\varphi(a)] \to \mathbb{R}$ such that*

$$F_n(\mathbf{x}) \;=\; \psi(\min\{\varphi(x_1) + \cdots + \varphi(x_n), \varphi(a)\}).$$

For such a function, we have $\psi = F_1 \circ \varphi^{-1}$.

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