Quasi-Lovász extensions on bounded chains

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Motivation

Let X and Y be "structured" sets.

Aggregation function: A mapping $g\colon X^n \to Y$ that

- merges elements of X into an element of Y
- "preserves" X's structure in Y's.

Traditionally: X and Y are ordered and g is "order-preserving" (nondecreasing)

Well-known examples:

- **Choquet integral:** Numerical aggregation over $\mathbb{R}$
- **Sugeno integral:** Ordinal aggregation over chains C

Problem: Mixed aggregation **e.g.:** $g\colon C^n \to \mathbb{R}$

Motivation: decision under uncertainty

Decomposable model:

- Alternatives over C^n (**e.g.** mental states over the year $n = 12$)
- Evaluation of each state by a local utility function $\varphi \colon C \to \mathbb{R}$
- Overall evaluation by a global utility function $U \colon C^n \to \mathbb{R}$:

$$U(x_1, \ldots, x_n) := \mathrm{C}(\varphi(x_1), \ldots, \varphi(x_n))$$

where $\mathrm{C} \colon \mathbb{R}^n \to \mathbb{R}$ is a Choquet integral

Problems: Given $g \colon C^n \to \mathbb{R}$

1. Decide if g is Choquet decomposable
2. Determine all possible Choquet decompositions

Outline

I Preliminaries: order simplexes

II Choquet integrals as Lovász extensions

III Generalization: quasi-Lovász extensions

IV Axiomatizations of quasi-Lovász extensions

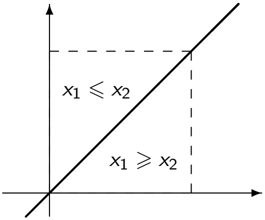
V Decompositions of quasi-Lovász extensions

Order simplexes

Let σ be a permutation on $[n] = \{1, \ldots, n\}$ $\quad (\sigma \in S_n)$:

- $\mathbb{R}^n_\sigma \; := \; \left\{ \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \; : \; x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)} \right\}$
- $\{0,1\}^n_\sigma \; := \; \mathbb{R}^n_\sigma \cap \{0,1\}^n$

Example : $n = 2$ $\quad (2! = 2$ permutations $\quad \Rightarrow \quad$ 2 simplexes!)



In general: $\mathbb{R}^n$ has $n!$ simplexes and each contains exactly $n+1$ points of $\{0,1\}^n$

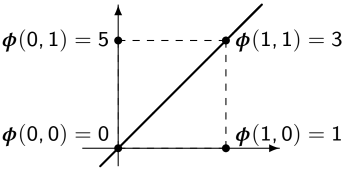
Let $\boldsymbol{\phi}\colon \{0,1\}^n \to \mathbb{R}$ be a (*pseudo-Boolean*) function **s.t.** $\boldsymbol{\phi}(\mathbf{0}) = 0$.

Definition (Lovász, 1983)

The **Lovász extension** of $\boldsymbol{\phi}\colon \{0,1\}^n \to \mathbb{R}$ is the function $f_{\boldsymbol{\phi}}\colon \mathbb{R}^n \to \mathbb{R}$ whose restriction to each $\mathbb{R}^n_\sigma$ is the unique linear function that interpolates $\boldsymbol{\phi}$ at the $n+1$ points of $\{0,1\}^n_\sigma$

In particular: $f_{\boldsymbol{\phi}}|_{\{0,1\}^n} = \boldsymbol{\phi}$

Example :



$$x_1 \geqslant x_2 \quad \Rightarrow \quad f_\phi(x_1, x_2) = x_1 + 2x_2$$

$$x_1 \leqslant x_2 \quad \Rightarrow \quad f_\phi(x_1, x_2) = -2x_1 + 5x_2$$

On $\mathbb{R}^2$:

$$f_\phi(x_1, x_2) = x_1 + 5x_2 - 3\min(x_1, x_2)$$

In general: f_ϕ can always be written in the form

$$f_\phi(\mathbf{x}) = \sum_{S\subseteq[n]} a_\phi(S) \min_{i\in S} x_i \qquad (\mathbf{x}\in\mathbb{R}^n)$$

where the coefficients $a_\phi(S)$ are given by the **Möbius transform** of ϕ

Consequence: f_ϕ is always piecewise linear and continuous!

...and on each order simplex $\mathbb{R}^n_\sigma$?

Choquet integral as a Lovász extension

On each order simplex $\mathbb{R}^n_\sigma$:

$$f_\phi(\mathbf{x}) = f_\phi(\mathbf{0}) + \sum_{i\in[n]} x_{\sigma(i)}\big(f_\phi(\mathbf{1}_{\{\sigma(i),\ldots,\sigma(n)\}}) - f_\phi(\mathbf{1}_{\{\sigma(i+1),\ldots,\sigma(n)\}})\big) \qquad (\mathbf{x}\in\mathbb{R}^n_\sigma)$$

Definition

A **Choquet integral** is a nondecreasing Lovász extension (vanishing at 0).

Let C be a chain with minimum 0 and maximum 1.

Definition

A **quasi-Lovász extension** is a function $U\colon C^n \to \mathbb{R}$ decomposable as

$$U(x_1, \ldots, x_n) := f(\varphi(x_1), \ldots, \varphi(x_n)) \quad \text{where}$$

1. $f\colon \mathbb{R}^n \to \mathbb{R}$ is a Lovász extension $(f = f_\phi$ for some $\phi)$
2. $\varphi\colon C \to \mathbb{R}$ is a nondecreasing function verifying $\varphi(0) = 0$.

In DMU: φ is a local utility function and f a Choquet integral.

Problem: Axiomatize the class of quasi-Lovász extensions

Comonotonic modularity

As before: For $\sigma \in S_n$, let $C^n_\sigma := \left\{(x_1, \ldots, x_n) \in C^n \;:\; x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\right\}$

Definition

$g\colon C^n \to \mathbb{R}$ is **comonotonically modular** if for all $\mathbf{x}, \mathbf{x}' \in C^n_\sigma$, for some $\sigma \in S_n$,

$$g(\mathbf{x}) + g(\mathbf{x}') = g(\mathbf{x} \vee \mathbf{x}') + g(\mathbf{x} \wedge \mathbf{x}')$$

NB: (quasi-)Lovász and "(quasi-)lattice polynomials" are comonotonically modular!

Recall: lattice polynomials generalize Sugeno integrals...

NB2: U is comonotonically modular **iff** so is $U_0 = U - U(\mathbf{0})$

Description of comonotonically modular functions

Theorem: For any $U\colon C^n \to \mathbb{R}$ T.F.A.E.:

1. U is comonotonically modular

2. There is $g\colon C^n \to \mathbb{R}$ **s.t.** $\forall \sigma \in S_n$ and $\forall \mathbf{x} \in C^n_\sigma$,

$$U_0(\mathbf{x}) = \sum_{i\in[n]} \big(g(x_{\sigma(i)} \wedge \mathbf{1}_{\{\sigma(i),\ldots,\sigma(n)\}}) - g(x_{\sigma(i)} \wedge \mathbf{1}_{\{\sigma(i+1),\ldots,\sigma(n)\}})\big)$$

3. U is **comonotonically separable:** $\forall \sigma \in S_n$, $\exists g_i^\sigma : C \to \mathbb{R}$, $i \in [n]$, **s.t.**

$$U(\mathbf{x}) = \sum_{i=1}^{n} g_i^\sigma(x_{\sigma(i)}) \qquad \text{for } \mathbf{x} \in C^n_\sigma.$$

NB: (quasi-)Lovász and (quasi-)lattice polynomials are comonotonically separable!

Axiomatization of quasi-Lovász extensions

Definition

$g\colon C^n \to \mathbb{R}$ is **weakly homogeneous** if there is an order-preserving $\varphi\colon C \to \mathbb{R}$ **s.t.**

$$\varphi(0) = 0 \quad \text{and} \quad g(x \wedge \mathbf{1}_A) = \varphi(x)g(\mathbf{1}_A) \quad \text{for every} \quad x \in C \ \text{and} \ A \subseteq [n]$$

Theorem: For any nonconstant $U\colon C^n \to \mathbb{R}$ T.F.A.E.:

1. U is a quasi-Lovász with $U_0(\mathbf{1}_A) \neq 0$ for some $A \subseteq [n]$.
2. U is comonotonically modular and U_0 is weakly homogeneous
3. There is an order-preserving $\varphi_U\colon C \to \mathbb{R}$ **s.t.**

$$\varphi_U(0) = 0 \quad \text{and} \quad \varphi_U(1) = 1 \ \text{and} \quad U = f_{U|_{\{0,1\}^n}} \circ \varphi_U$$

Moreover: We can choose $\varphi_U(x) = \frac{U_0(x \wedge \mathbf{1}_A)}{U_0(\mathbf{1}_A)}$.

Question: Is the previous decomposition unique?

Theorem: For a quasi-Lovász extension $U = f \circ \varphi$ T.F.A.E.:

1. U is nonconstant
2. There is $A \subseteq [n]$ **s.t.** $U_0(\mathbf{1}_A) \neq 0$
3. There is $a > 0$ **s.t.** $\varphi = a\,\varphi_U$ and $f_0 = f - f(\mathbf{0}) = \frac{1}{a}(f_{U|_{\{0,1\}^n}})_0$.

Thank you for your attention!