Associative string functions

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Associative functions

Let X be a nonempty set

$F\colon X^2 \to X$ is *associative* if

$$F(F(x,y),z) \;=\; F(x,F(y,z))$$

Example: $F(x,y) = x + y$ on $X = \mathbb{R}$

Associative functions

Extension to a function with an indefinite arity

$F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$

$$F(x_1,\ldots,x_n) \;=\; F(F(x_1,\ldots,x_{n-1}),x_n) \qquad n\geqslant 3$$

Example

$$\begin{aligned} F(x_1,x_2) &= x_1+x_2 \\ F(x_1,x_2,x_3) &= F(F(x_1,x_2),x_3) = x_1+x_2+x_3 \\ \text{etc.} \end{aligned}$$

Notation

We regard n-tuples $\mathbf{x}$ in X^n as *n-strings* over X

0-string: ε
1-strings: $x, y, z, \ldots$
n-strings: $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$

$$X^* \ = \ \bigcup_{n \in \mathbb{N}} X^n$$

X^* is endowed with concatenation

Example: $\mathbf{x} \in X^n$, $y \in X$, $\mathbf{z} \in X^m \quad \Rightarrow \quad \mathbf{x}y\mathbf{z} \in X^{n+1+m}$

$$|\mathbf{x}| \ = \ \text{length of } \mathbf{x}$$

Functions of indefinite arities

$$F\colon X^* \to X$$

n-ary part of F :

$$F_n\colon X^n \to X$$

$$F_n \;=\; F|_{X^n}$$

We assume that

$$F(\mathbf{x}) \;=\; \varepsilon \quad\Longleftrightarrow\quad \mathbf{x} \;=\; \varepsilon$$

Associative functions with indefinite arity

Binary associativity

$$F(F(xy)z) \;=\; F(x\,F(yz))$$

Induction formula

$$F(\mathbf{x}y) \;=\; F(F(\mathbf{x})y) \qquad |\mathbf{x}| \geqslant 2$$

Proposition

Binary associativity $+$ induction formula

$$\Updownarrow$$

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad |\mathbf{y}| \geqslant 2,\; |\mathbf{xz}| \geqslant 1$$

Associative functions with indefinite arity

$F\colon X^* \to X$ is *associative* if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

Theorem

$F\colon X^* \to X$ is associative if and only if

(i) Binary associativity + induction formula

(ii) $F_1 \circ F_1 = F_1$

(iii) $F_1 \circ F_2 = F_2$

(iv) $F_2(xy) = F_2(F_1(x)\,y) = F_2(x\,F_1(y))$

Observation: $F_1 = \mathrm{id}$ can always be considered

Associative functions with indefinite arity

Example

$$F\colon \mathbb{R}^* \to \mathbb{R}$$

$$F_n(x_1 \cdots x_n) \;=\; \sqrt{|x_1|^2 + \cdots + |x_n|^2} \qquad n \geqslant 2$$

$$F_1(x) \;=\; x$$

$$F_1(x) \;=\; |x|$$

Preassociative functions

Let Y be a nonempty set

Definition. We say that $F\colon X^* \to Y$ is *preassociative* if

$$F(\mathbf{y}) \;=\; F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) \;=\; F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

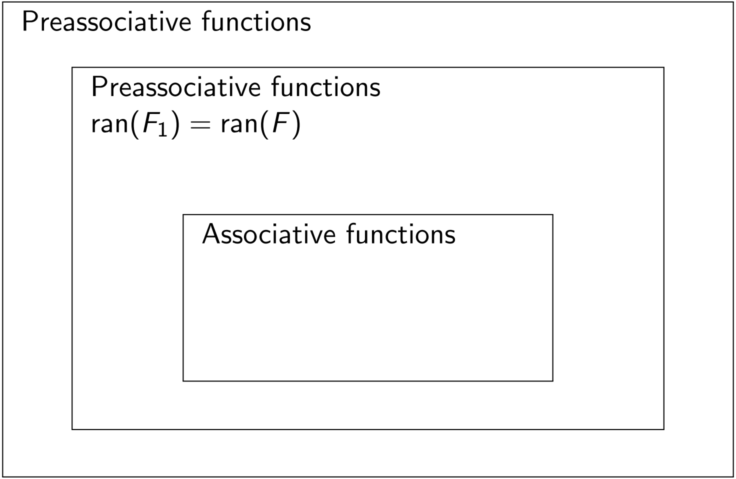
Examples: $F_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2 \quad (X = Y = \mathbb{R})$

$F_n(\mathbf{x}) = |\mathbf{x}| \quad (X \text{ arbitrary},\ Y = \mathbb{N})$

Proposition

$F\colon X^* \to X$ is associative $\iff$ F is preassociative and $F(F(\mathbf{x})) = F(\mathbf{x})$

Preassociative functions



Preassociative functions

Theorem

Let $F\colon X^* \to Y$. The following assertions are equivalent:

(i) F is preassociative and satisfies $\mathrm{ran}(F_1) = \mathrm{ran}(F)$

(ii) F can be factorized into

$$F = f \circ H$$

where $H\colon X^* \to X$ is associative
$f\colon \mathrm{ran}(H) \to Y$ is one-to-one

Axiomatizations of function classes

Theorem (Aczél 1949)

$H\colon \mathbb{R}^2 \to \mathbb{R}$ is

- continuous
- one-to-one in each argument
- associative

if and only if

$$H(xy) \;=\; \varphi^{-1}(\varphi(x)+\varphi(y))$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous and strictly monotone

$$H_n(x_1 \cdots x_n) \;=\; \varphi^{-1}(\varphi(x_1)+\cdots+\varphi(x_n))$$

Axiomatizations of function classes

Theorem

Let $F\colon \mathbb{R}^* \to \mathbb{R}$. The following assertions are equivalent:

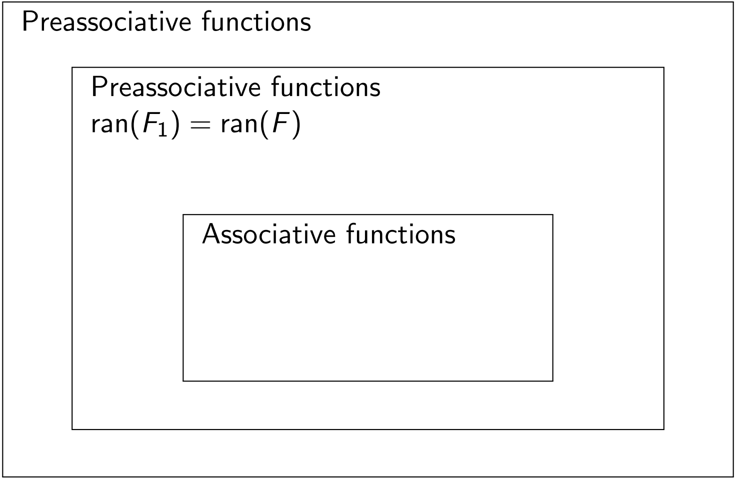
(i) F is preassociative and satisfies $\mathrm{ran}(F_1) = \mathrm{ran}(F)$, F_1 and F_2 are continuous and one-to-one in each argument

(ii) we have

$$F_n(x_1 \cdots x_n) \;=\; \psi\big(\varphi(x_1) + \cdots + \varphi(x_n)\big)$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $\psi\colon \mathbb{R} \to \mathbb{R}$ are continuous and strictly monotone

Preassociative functions



String functions

A *string function* if a function

$$F\colon X^* \to X^*$$

$F\colon X^* \to X^*$ is *associative* if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

Associative string functions

$F\colon X^* \to X^*$ is *associative* if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

Examples

- $F = \mathrm{id}$
- $F =$ sorting data in alphabetic order
- $F =$ transforming a string of letters into upper case
- $F =$ removing a given letter, say 'a'
- $F =$ removing all repeated occurrences of letters

$$F(\text{mathematics}) \;=\; \text{matheics}$$

Preassociative functions

Theorem

Let $F\colon X^* \to Y$. The following assertions are equivalent:

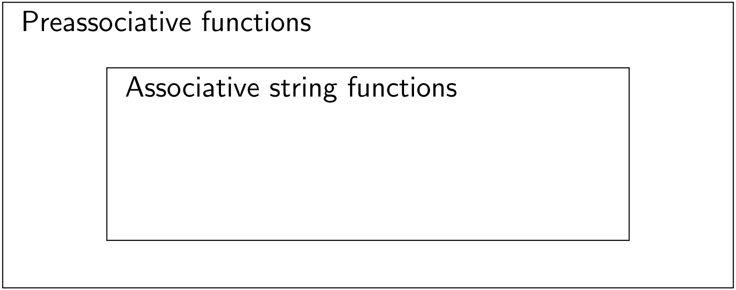
(i) F is preassociative

(ii) F can be factorized into

$$F \;=\; f \circ H$$

where $H\colon X^* \to X^*$ is associative
$f\colon \mathrm{ran}(H) \to Y$ is one-to-one

Preassociative functions



Open question:
Find characterizations of classes of associative string functions

Thank you for your attention !

A very short proof

Recall that $F\colon X^* \to X$ is

- *associative* if $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z})$
- *preassociative* if $F(\mathbf{y}) = F(\mathbf{y}')$ implies $F(\mathbf{xyz}) = F(\mathbf{xy}'\mathbf{z})$

Proposition

F is associative $\iff$ F is preassociative and $F(\mathbf{x}) = F(F(\mathbf{x}))$

($\Rightarrow$) If F is associative, then $F(\mathbf{y}) = F(F(\mathbf{y}))$ (just take $\mathbf{xz} = \varepsilon$)
Now if $F(\mathbf{y}) = F(\mathbf{y}')$, then

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}) = F(\mathbf{x}F(\mathbf{y}')\mathbf{z}) = F(\mathbf{xy}'\mathbf{z})$$

($\Leftarrow$) We have $F(\mathbf{y}) = F(F(\mathbf{y}))$ and therefore

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \square$$