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Meshfree methods for shear-deformable beams and plates based on mixed weak forms

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Overview

- Reissner-Mindlin problem.
- Kirchhoff limit and numerical locking.
- Some solutions?
- Mixed variational formulation with meshfree basis functions.
- An example with the Timoshenko beam problem.
- The Reissner-Mindlin problem:
  - Local patch projection method.
  - Stabilisation with the method of the augmented Lagrangian.
- Summary.

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The problem.

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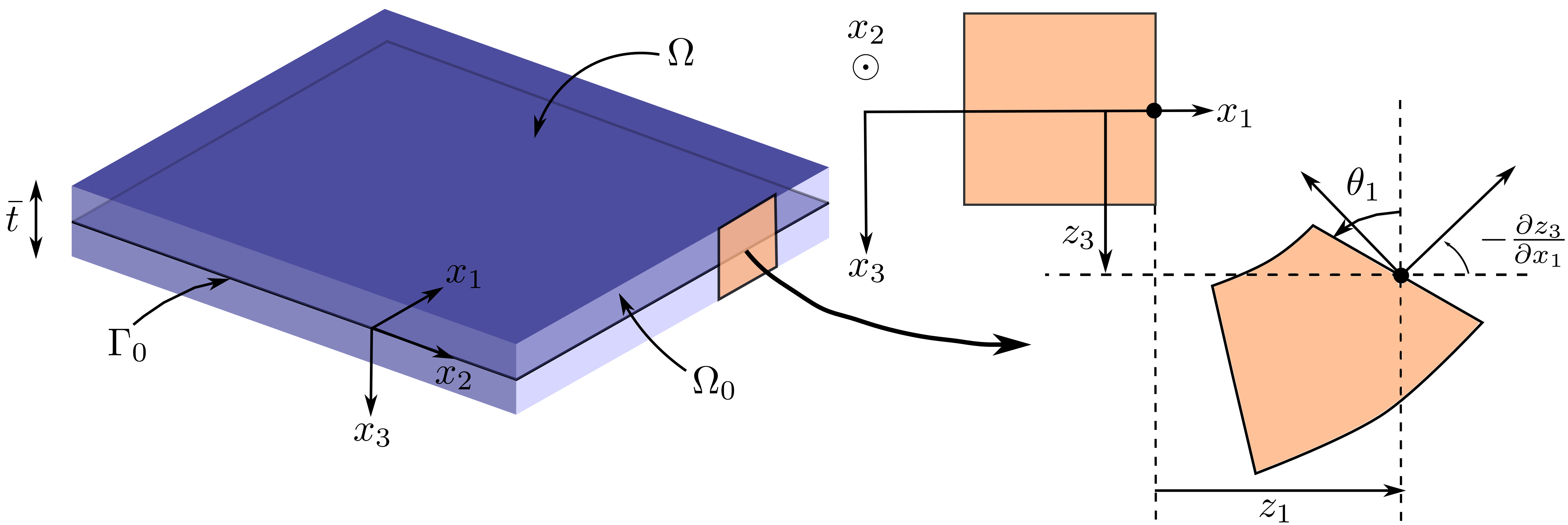
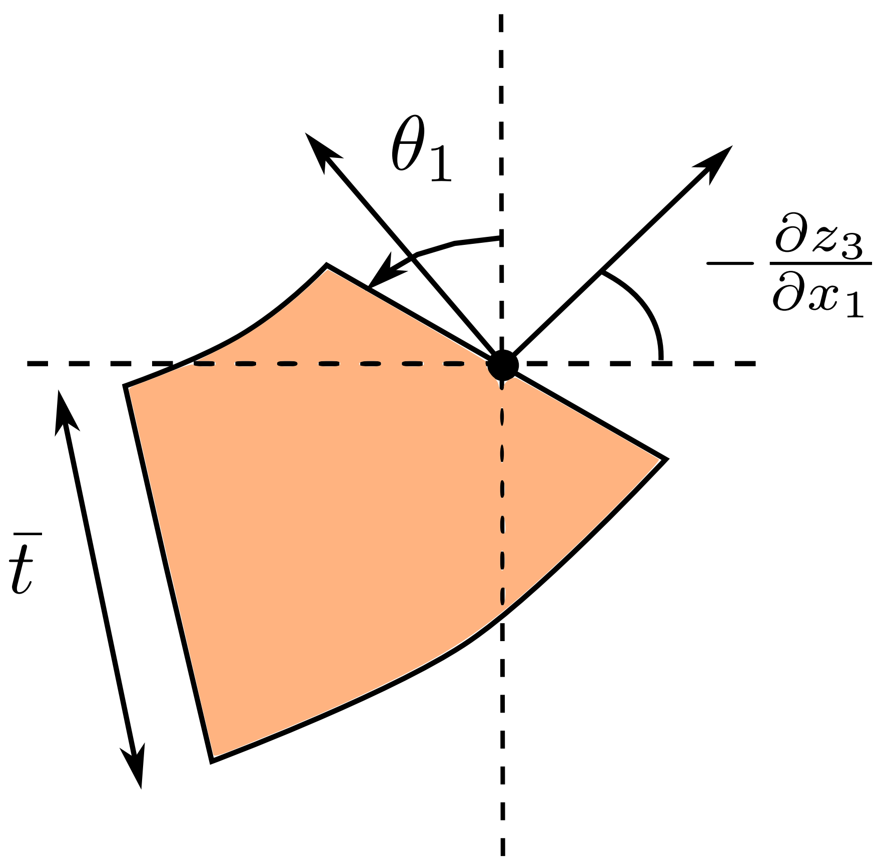
Reissner-Mindlin plate problem

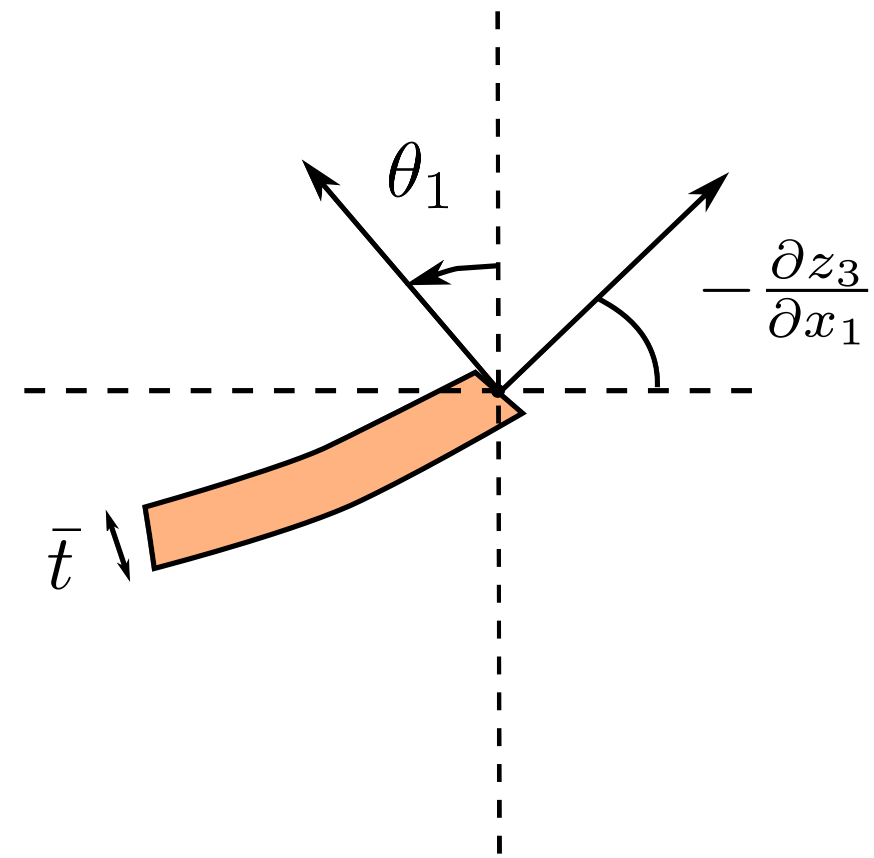


The Kirchhoff Limit



Reissner-Mindlin

$$\bar{t} \to 0$$



Kirchhoff

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Problems with small parameters crop up nearly everywhere!

incompressible elasticity,
incompressible fluid flow,
plates and shells,
Cosserat elasticity…

The issue:
numerical locking.

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The equations

Find $(z_{3h}, \boldsymbol{\theta}_h) \in (\mathcal{V}_{3h} \times \mathcal{R}_h)$ such that for all $(y_3, \boldsymbol{\eta}) \in (\mathcal{V}_{3h} \times \mathcal{R}_h)$:

$$\int_{\Omega_0} L\epsilon(\boldsymbol{\theta}_h) : \epsilon(\boldsymbol{\eta}) \; d\Omega + \lambda \bar{t}^{-2} \int_{\Omega_0} (\nabla z_3 - \boldsymbol{\theta}_h) \cdot (\nabla y_3 - \boldsymbol{\eta}) \; d\Omega = \int_{\Omega_0} g y_3 \; d\Omega$$

or:

$$a_b(\boldsymbol{\theta}_h; \eta) + \lambda \bar{t}^{-2} a_s(\boldsymbol{\theta}_h, z_3; \boldsymbol{\eta}, y_3) = f(y_3)$$

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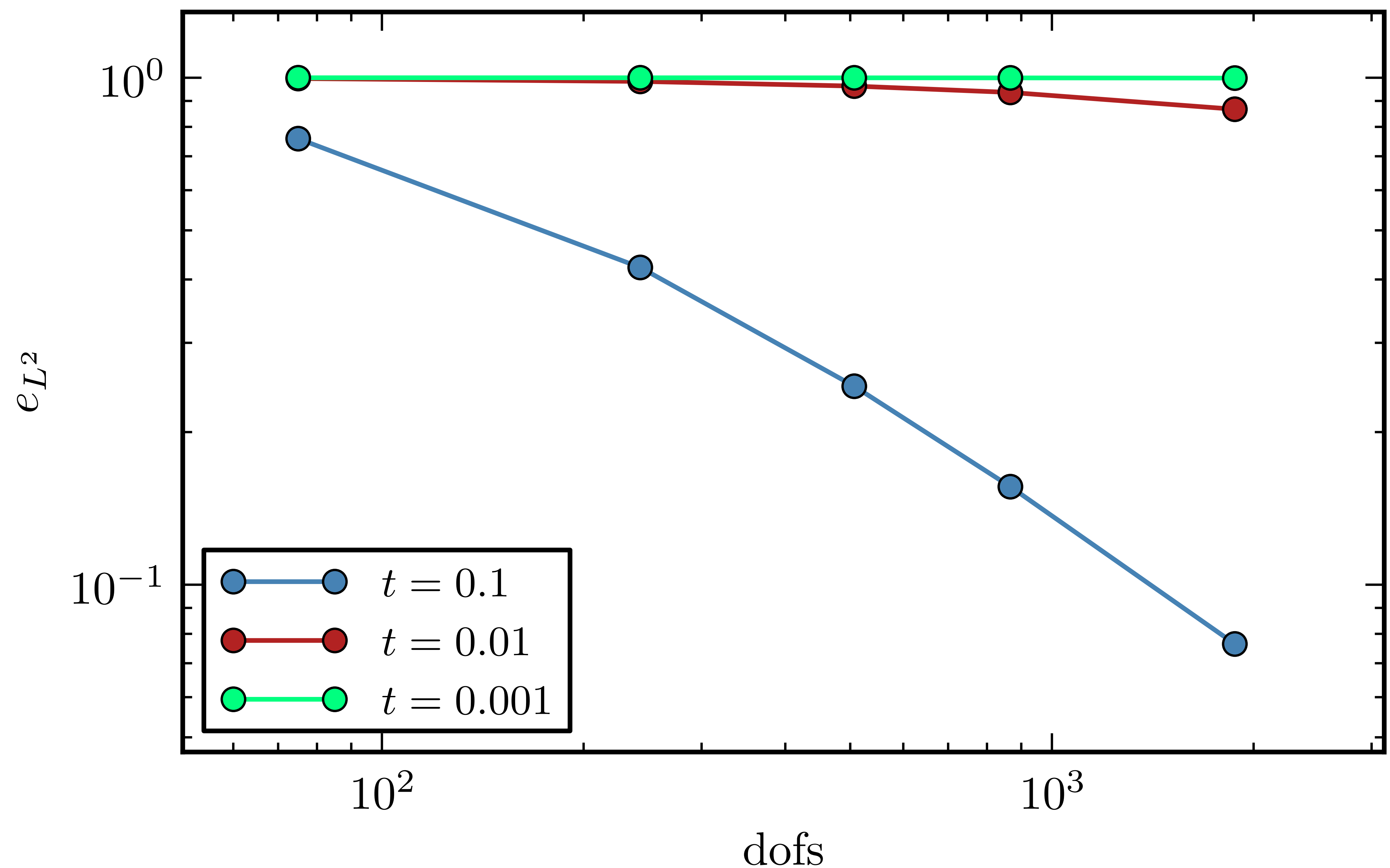
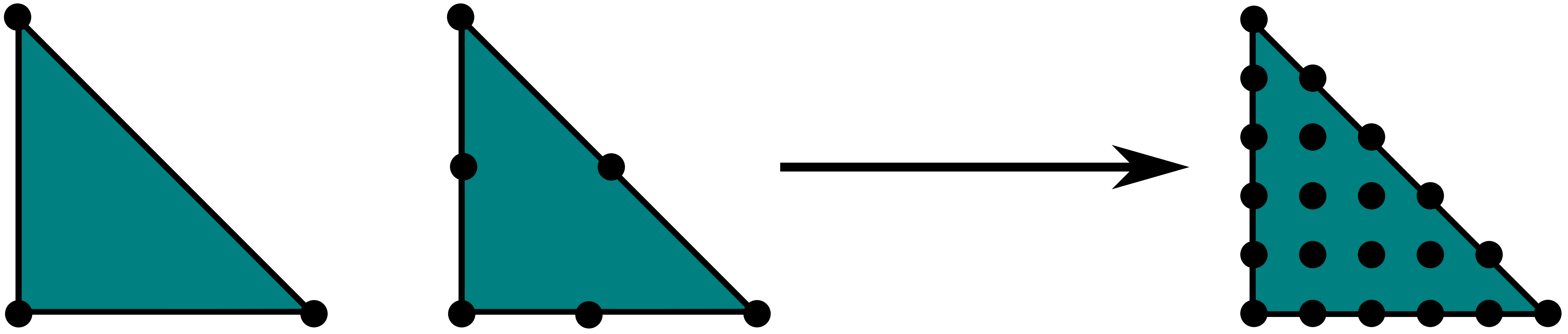
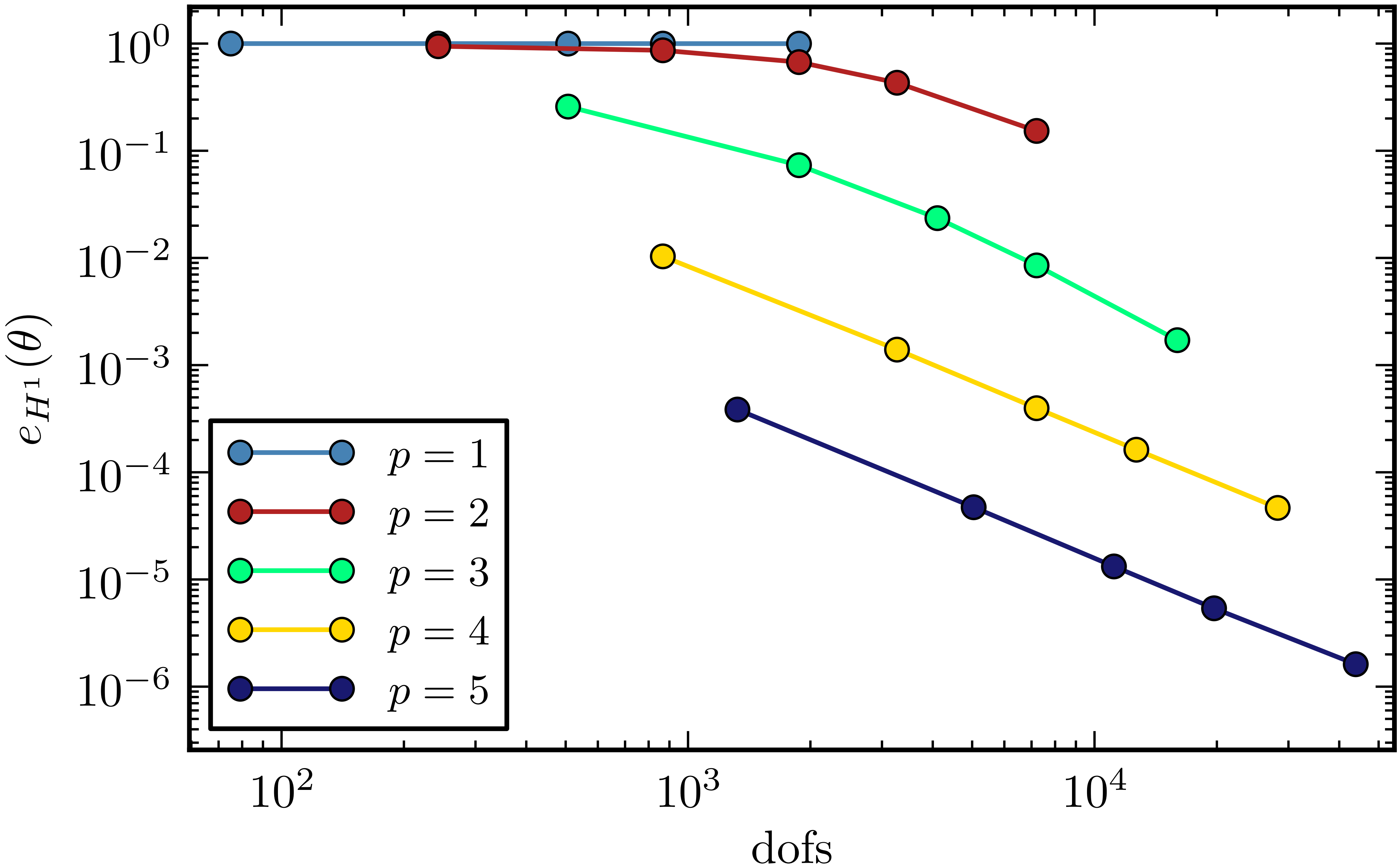
Locking



p-refinement



p-refinement



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Locking

Inability of the basis functions to satisfy the constraint imposed whilst still having optimal approximation properties.

$$||u - u_h|| \leq C(1/\bar{t})h^p||u''||$$

Conclusion: We can never fully eliminate locking with these approaches.

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Mixed weak form

Find $(z_{3h}, \boldsymbol{\theta}_h, \boldsymbol{\gamma}_h) \in (\mathcal{V}_{3h}, \mathcal{R}_h, \mathcal{S}_h)$
such that for all $(y_{3h}, \boldsymbol{\eta}, \boldsymbol{\psi}) \in (\mathcal{V}_{3h}, \mathcal{R}_h, \mathcal{S}_h)$:

$$a_b(\boldsymbol{\theta}_h; \boldsymbol{\eta}) + (\boldsymbol{\gamma}_h; \nabla y_3 - \boldsymbol{\eta})_{L^2} = f(y_3)$$

$$(\nabla z_{3h} - \boldsymbol{\theta}_h; \boldsymbol{\psi})_{L^2} - \frac{\bar{t}^2}{\lambda}(\boldsymbol{\gamma}_h; \boldsymbol{\psi})_{L^2} = 0$$

So the problem is solved?

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Stability

Brezzi (Braess): The classical saddle point problem ($\bar{t} = 0$) is stable, if and only if, the following conditions hold:

1. ($\mathcal{Z}$-Ellipticity of a) There exists a constant $\alpha \geq 0$ such that:

$$a(v, v) \geq \alpha \|v\|_{\mathcal{X}}^2 \quad \forall v \in \mathcal{Z}$$

where $\mathcal{Z}$ is the kernel of the bilinear form b:

$$\mathcal{Z} := \{v \in \mathcal{X} \mid b(v, q) = 0 \quad \forall q \in \mathcal{M}\}$$

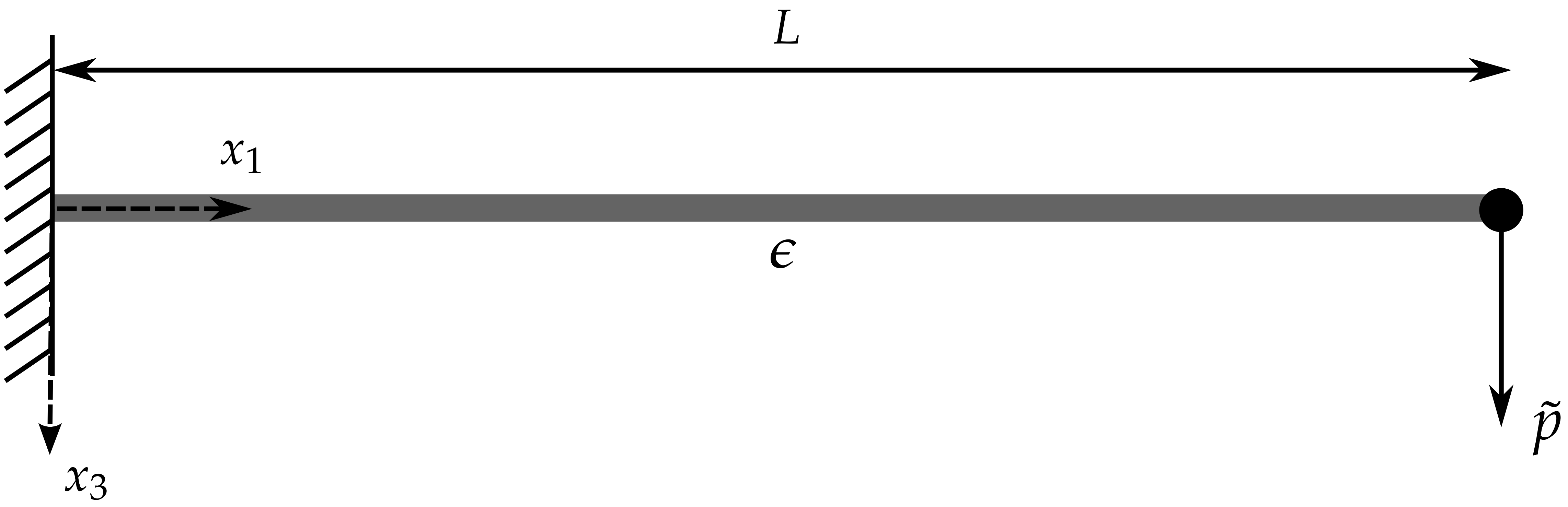
2. (inf-sup condition on b) The bilinear form b satisfies an inf-sup condition:

$$\inf_{q \in \mathcal{M}} \sup_{v \in \mathcal{X}} \frac{b(v, q)}{\|v\|_{\mathcal{X}} \|q\|_{\mathcal{M}}} = \beta > 0$$

A Question of Balance

$$\mathcal{X}_h \qquad \qquad \mathcal{M}_h$$

Example: Timoshenko Beam



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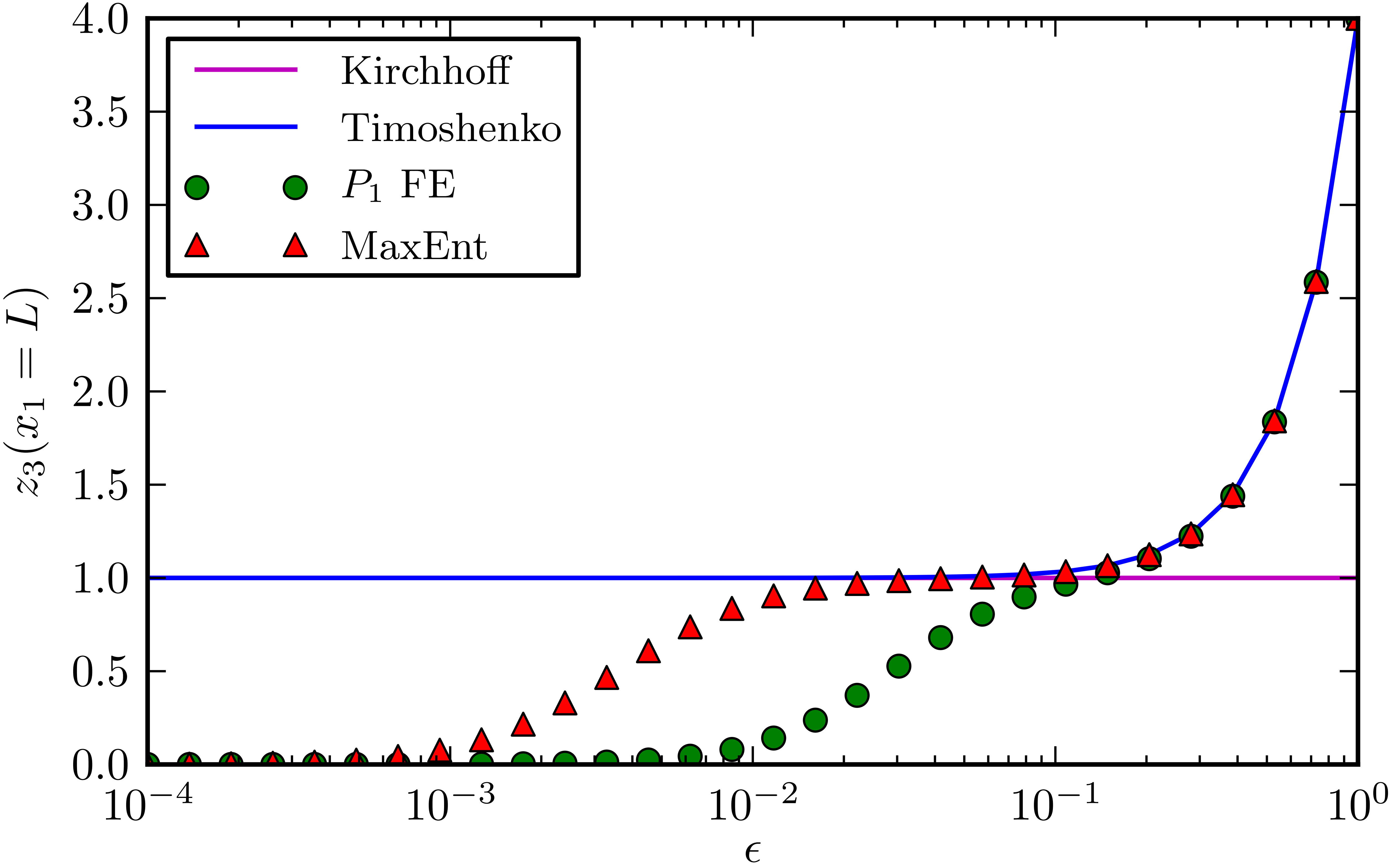

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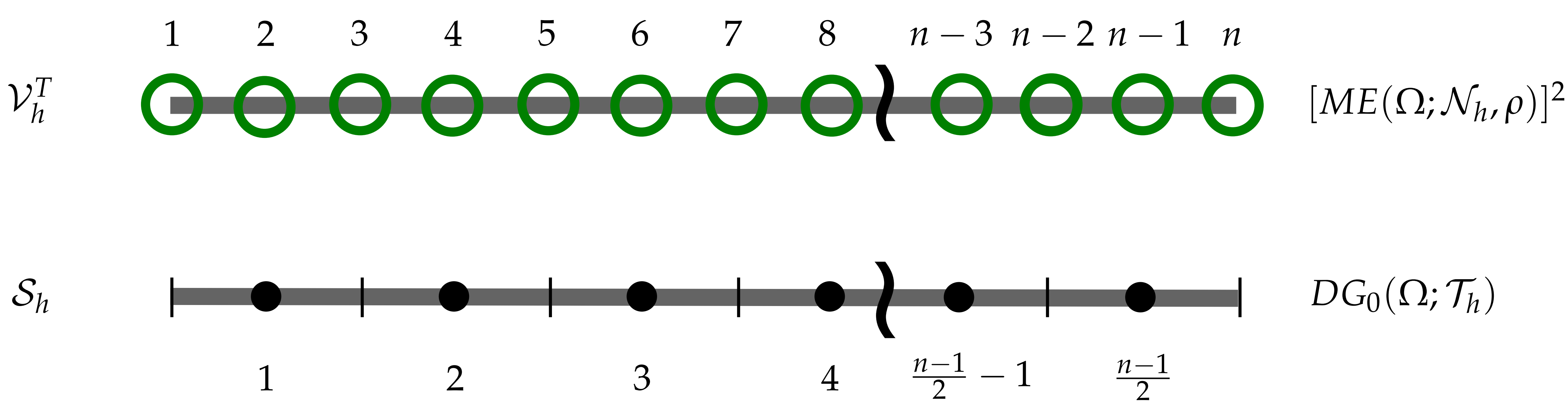
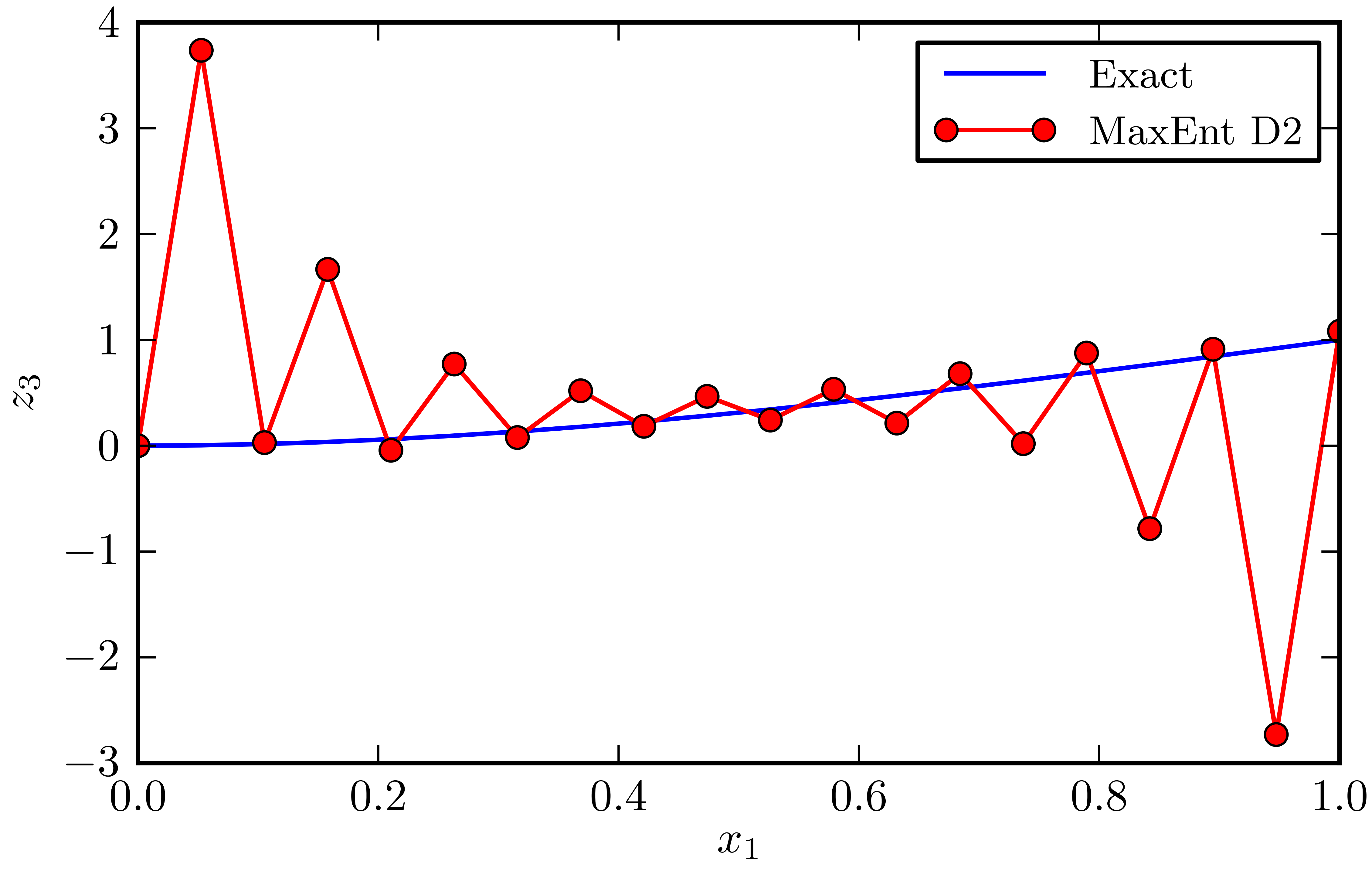
An unstable discretisation





Exact
MaxEnt D2
4
3
2
1
0
−1
−2
−3
z_3
0.0
0.2
0.4
0.6
0.8
1.0
x_1

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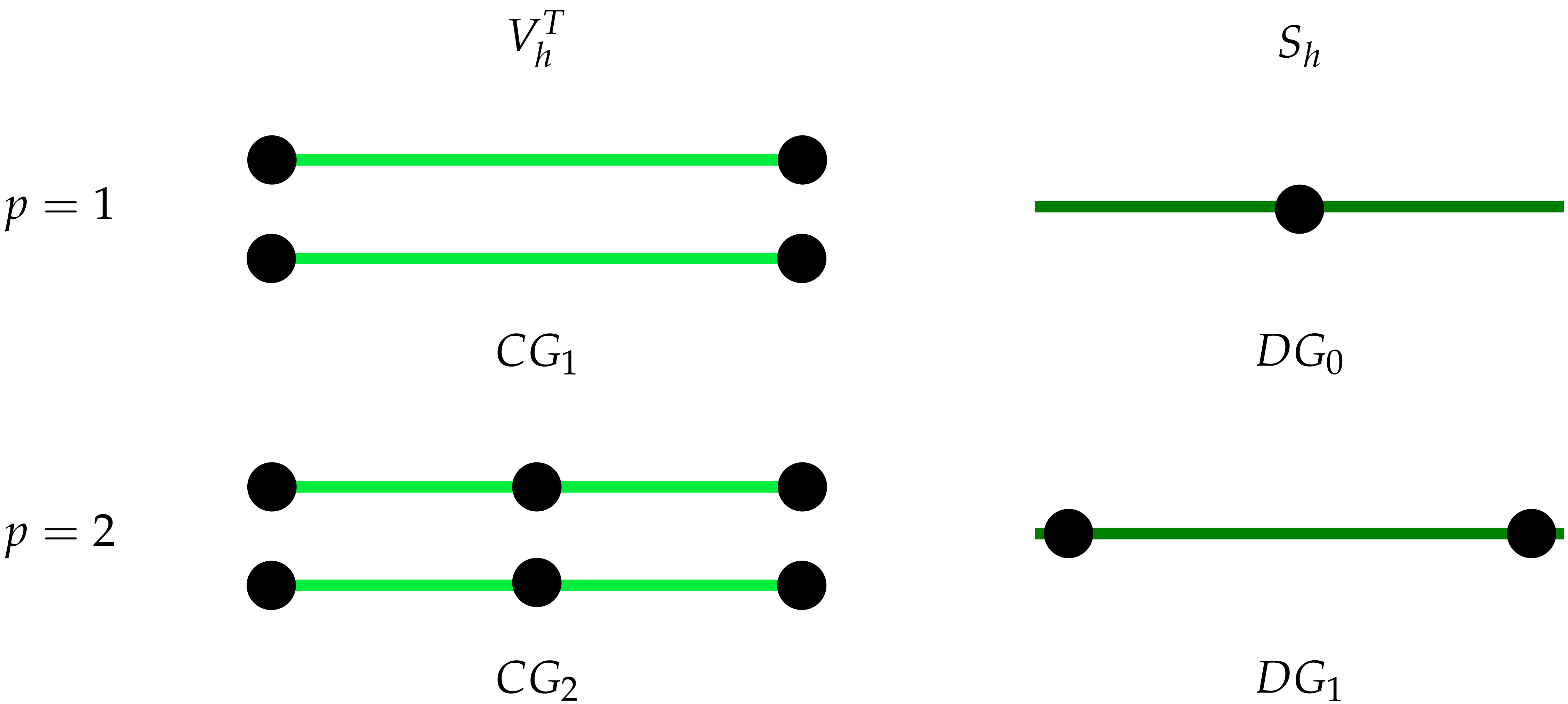
What about FEM?



Chapelle and Bathe

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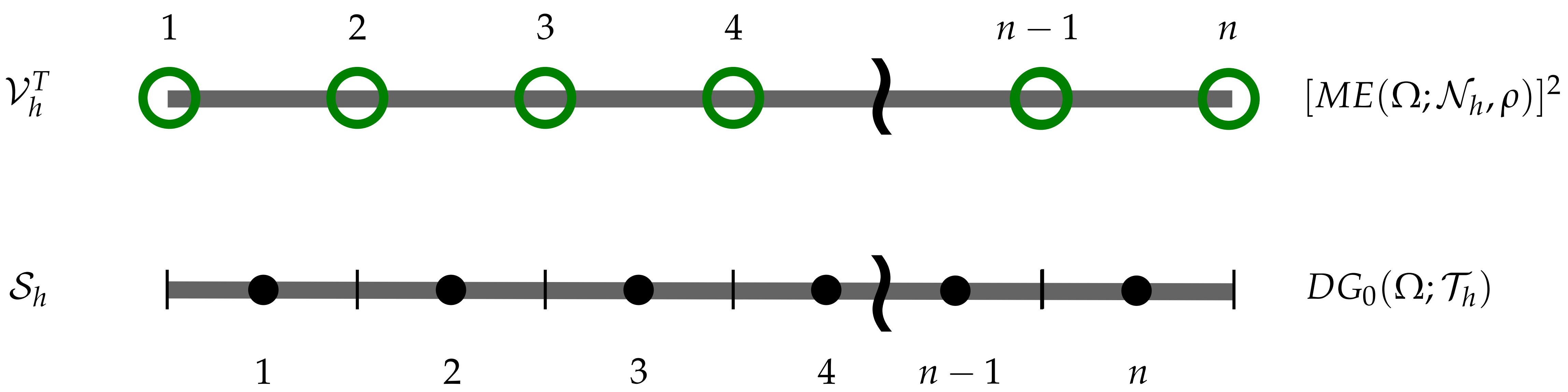
A stable discretisation



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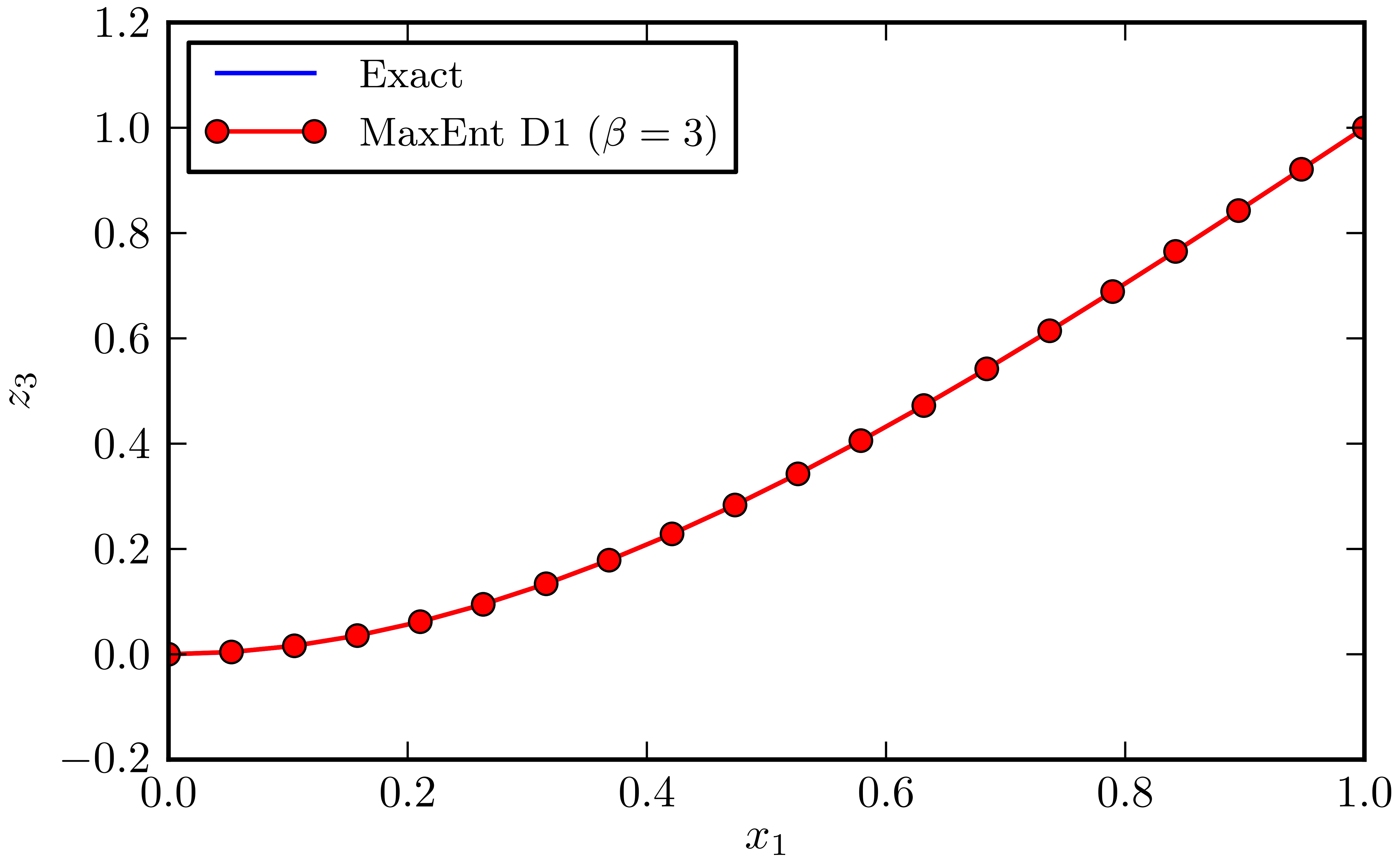
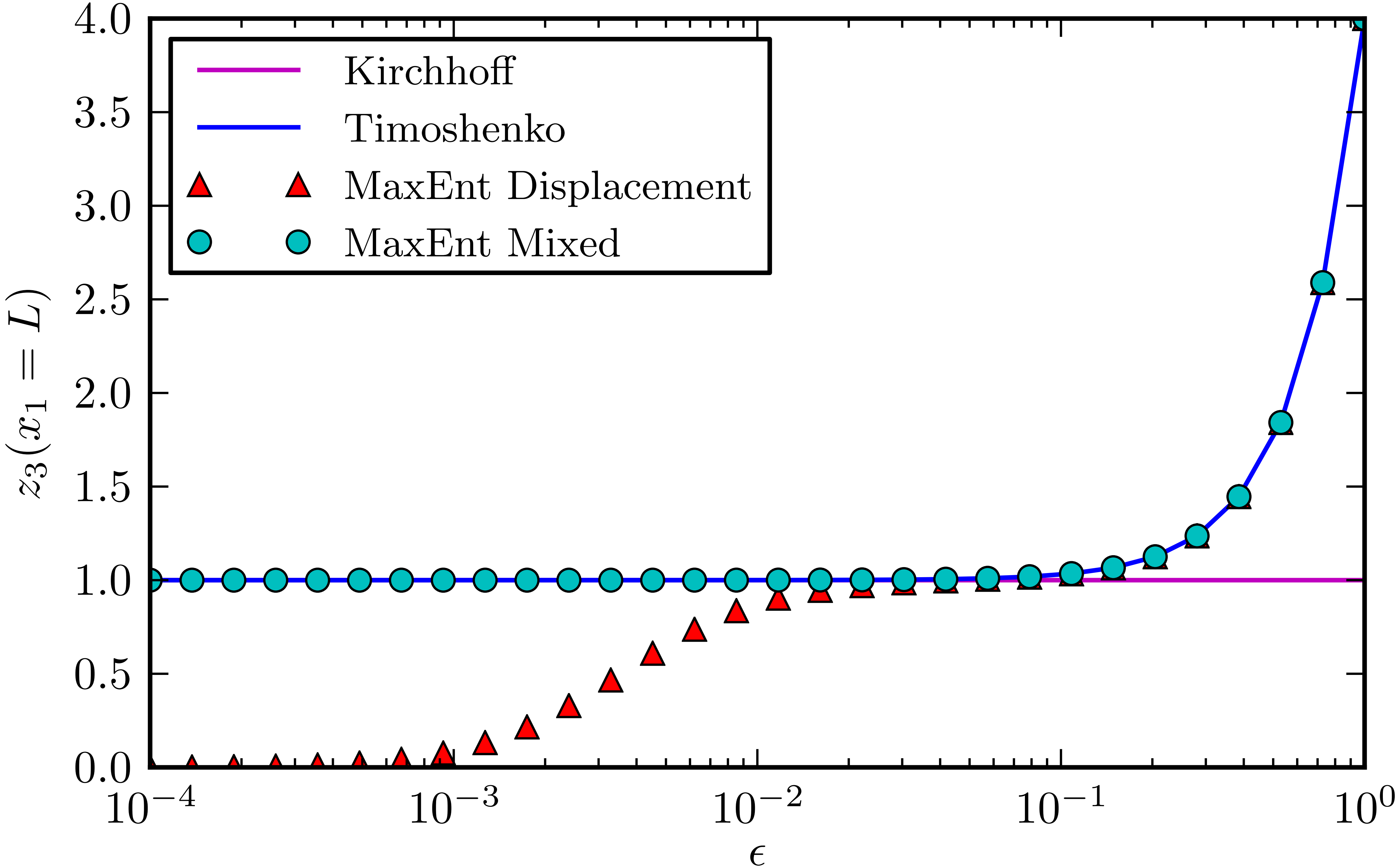
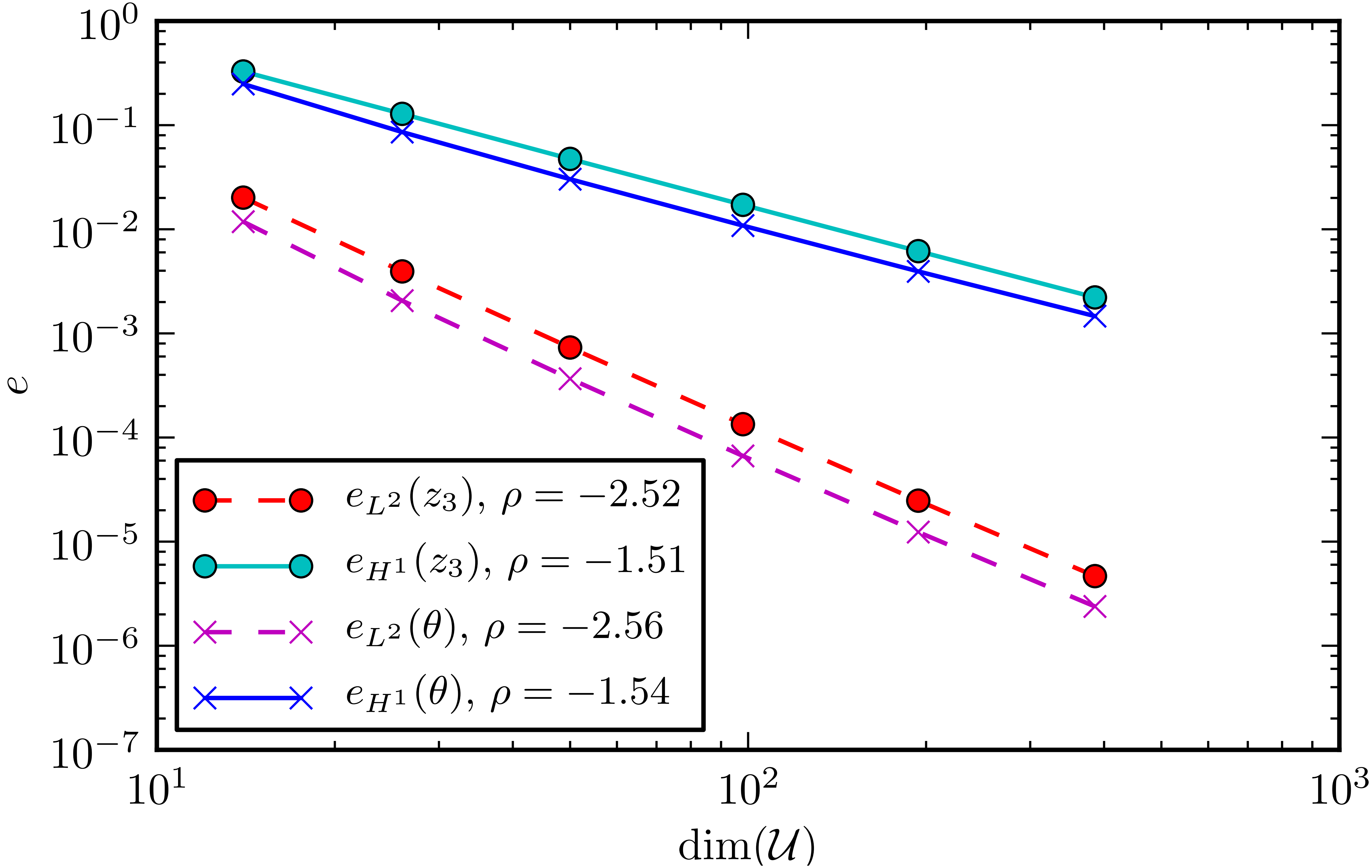




Kirchhoff
Timoshenko
MaxEnt Displacement
MaxEnt Mixed
$z_3(x_1 = L)$
ϵ
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}


10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
e
10^1
10^2
10^3
$\dim(\mathcal{U})$
$e_{L^2}(z_3),\ \rho = -2.52$
$e_{H^1}(z_3),\ \rho = -1.51$
$e_{L^2}(\theta),\ \rho = -2.56$
$e_{H^1}(\theta),\ \rho = -1.54$

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Sketch proof

1. *Arroyo and Ortiz:* In the local limit $\beta \to 0$ Maximum-Entropy basis functions converge to the finite element method CG_1.

2. *Chapelle and Bathe:* The CG_p/DG_{p-1} finite element satisfies the kernel coercivity and inf-sup condition.

Implies the proposed meshfree method is stable.

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Back to the Reissner-Mindlin problem.

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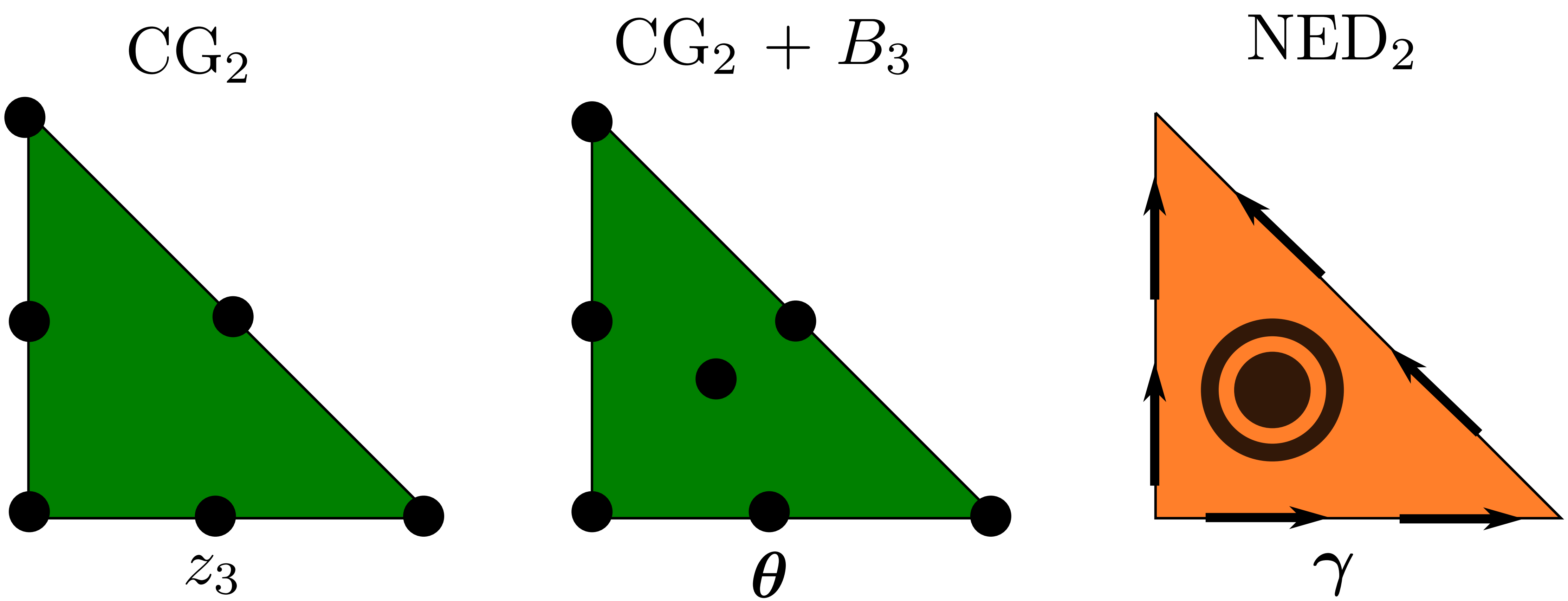
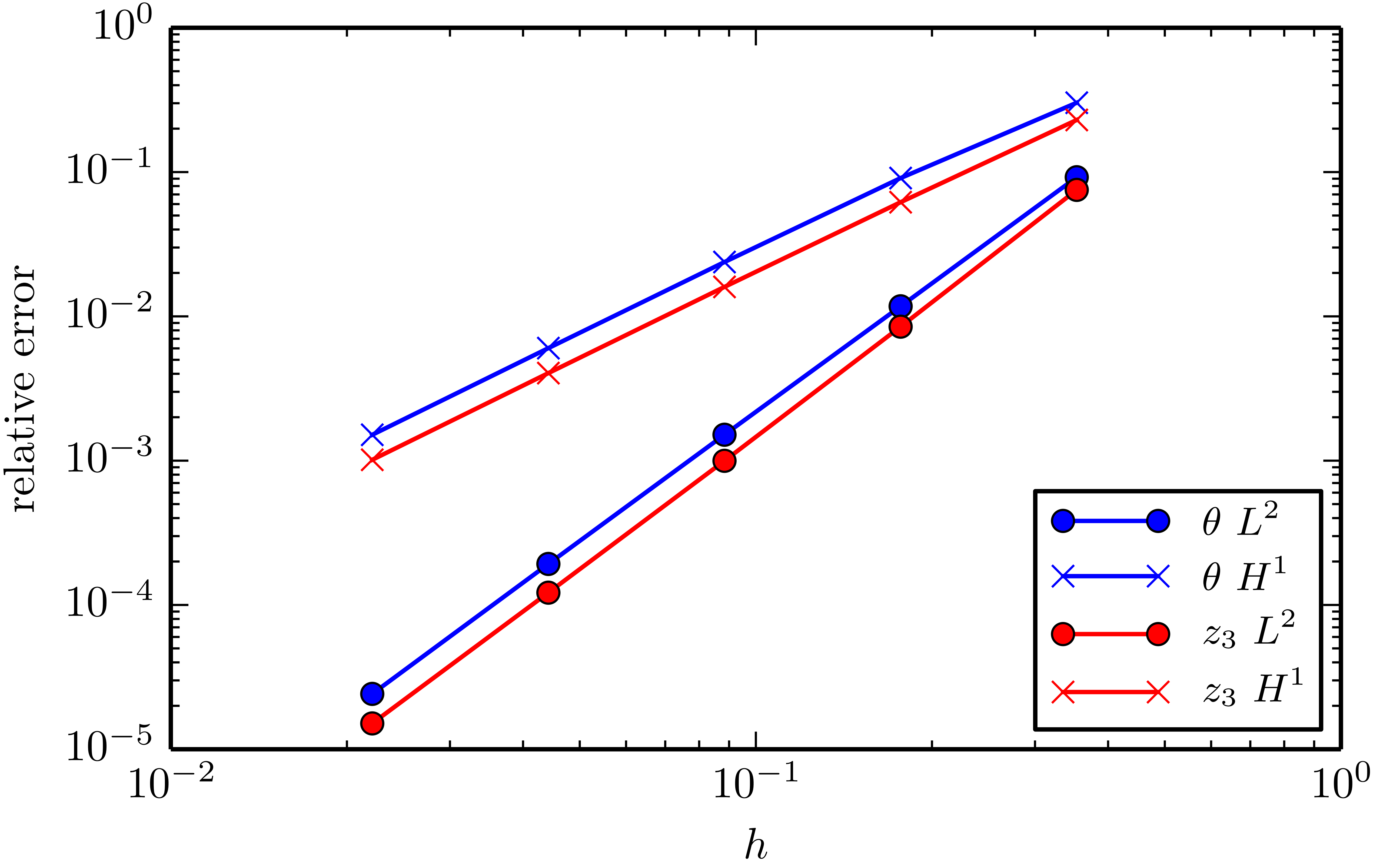
What about FEM?

MITC Family - *Bathe, Chapelle, Arnold, Fortin…*



Generalised Displacement Method



10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
relative error
10^{-2}
10^{-1}
10^{0}
h
θ L^2
θ H^1
z_3 L^2
z_3 H^1

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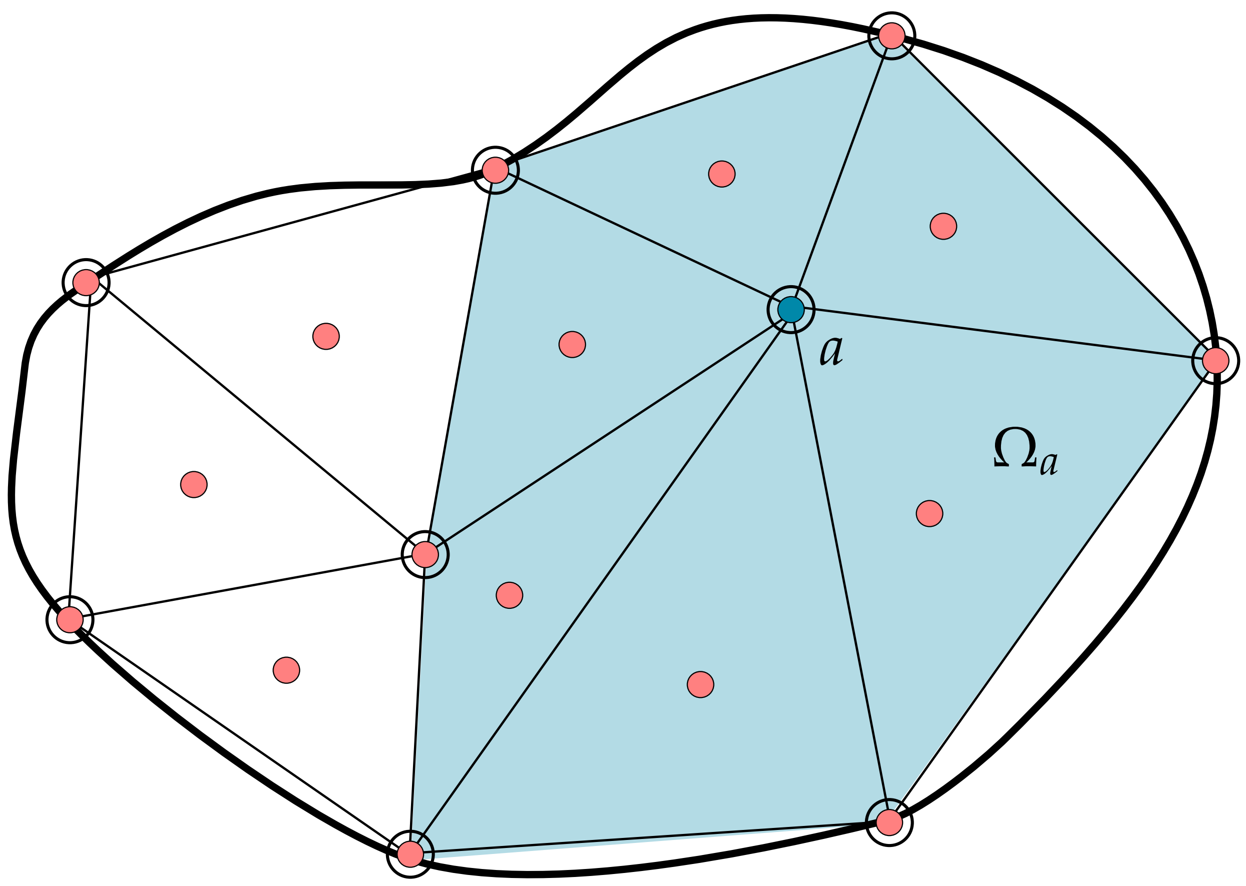
Local Patch Projection

For the nearly incompressible elasticity problem.

Ortiz, Puso, Sukumar.



$$p_a = -\lambda_e \sum_{b=1}^{N} \left\{ \frac{\int_{\Omega_a} N_{pa} \boldsymbol{m}^T \boldsymbol{B}_b \, d\Omega}{\int_{\Omega_a} N_{pa} \, d\Omega} \right\} \boldsymbol{u}_b$$

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Stability

Brezzi (Braess): The classical saddle point problem ($\bar{t} = 0$) is stable, if and only if, the following conditions hold:

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$$\inf_{q \in \mathcal{M}} \sup_{v \in \mathcal{X}} \frac{b(v, q)}{\|v\|_{\mathcal{X}} \|q\|_{\mathcal{M}}} = \beta > 0$$

A Question of Balance

$$\mathcal{X}_h \qquad\qquad \mathcal{M}_h$$

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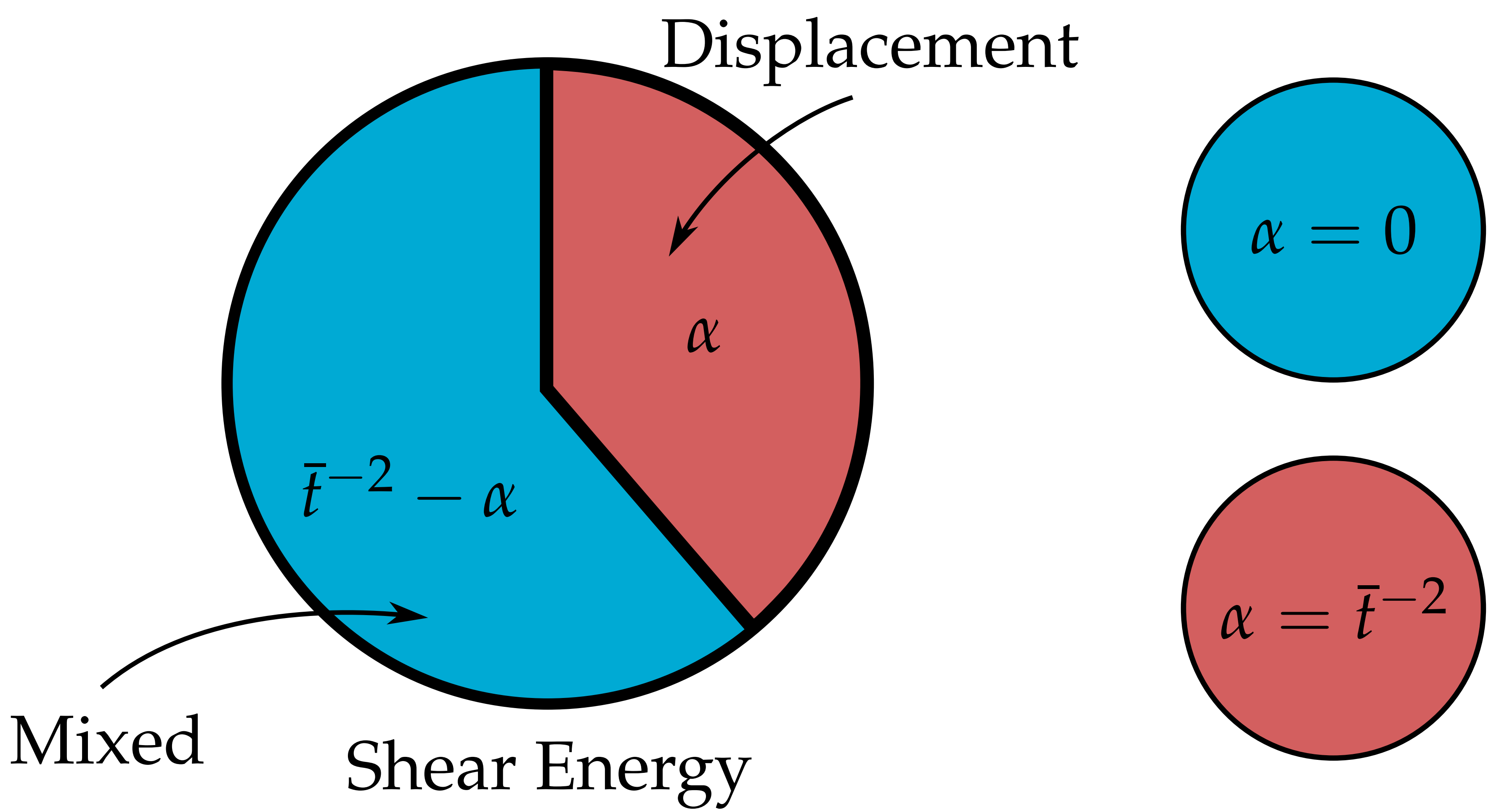
A solution

Make the Reissner-Mindlin problem look more like the incompressible elasticity problem.
Arnold, Lovadina, Chinosi.



A solution

Problem 17 (Discrete stabilised mixed scaled Reissner-Mindlin problem). *Find the transverse deflection, rotations and transverse shear stresses* $(z_{3h}, \boldsymbol{\theta}_h, \boldsymbol{\gamma}_h) \in (\mathcal{V}_{3h}, \mathcal{R}_h, S_h)$ *such that for all* $(y_3, \boldsymbol{\eta}, \boldsymbol{\psi}) \in (\mathcal{V}_{3h}, \mathcal{R}_h, S_h)$:

$$a_b(\boldsymbol{\theta}_h, \boldsymbol{\eta}) + \lambda\alpha a_s(\boldsymbol{\theta}_h, z_{3h}; \boldsymbol{\eta}, y_3) + (\boldsymbol{\gamma}_h, \nabla y_3 - \boldsymbol{\eta})_{L^2(\Omega_0)} = g(y_3) \tag{6.10a}$$

$$(\nabla z_{3h} - \boldsymbol{\theta}_h, \boldsymbol{\psi})_{L^2(\Omega_0)} - \frac{\bar{t}^2}{\lambda(1-\alpha\bar{t}^2)}(\boldsymbol{\gamma}_h, \boldsymbol{\psi})_{L^2(\Omega_0)} = 0 \tag{6.10b}$$

$$a(v; v) := a_b(\boldsymbol{\theta}, \boldsymbol{\eta}) + \lambda\alpha a_s(\boldsymbol{\theta}, z_3; \boldsymbol{\eta}, y_3) \geq \delta \|v\|_{\mathcal{X}}^2 \quad \forall v \in \mathcal{X}$$

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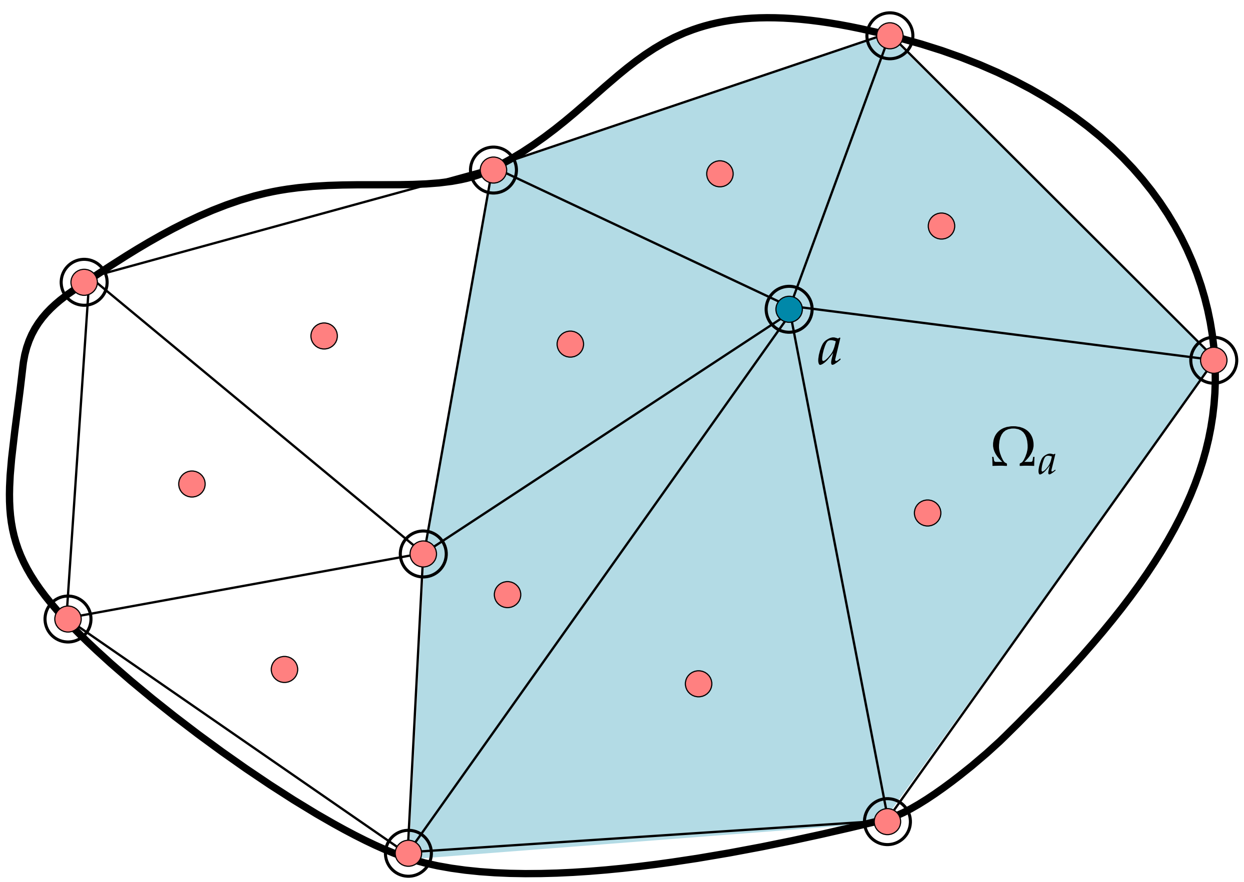
Final Formulation

$$a_b(\boldsymbol{\theta}_h, \boldsymbol{\eta}) + \lambda \alpha a_s(\boldsymbol{\theta}_h, z_{3h}; \boldsymbol{\eta}, y_3) + \frac{\lambda(1-\alpha\bar{t}^2)}{\bar{t}^2}(\Pi_h^p(\nabla z_{3h} - \boldsymbol{\theta}_h), \nabla y_3 - \boldsymbol{\eta})_{L^2(\Omega_0)} = g(y_3)$$



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$$\inf_{q \in \mathcal{M}} \sup_{v \in \mathcal{X}} \frac{b(v,q)}{\|v\|_{\mathcal{X}} \|q\|_{\mathcal{M}}} = \beta > 0$$

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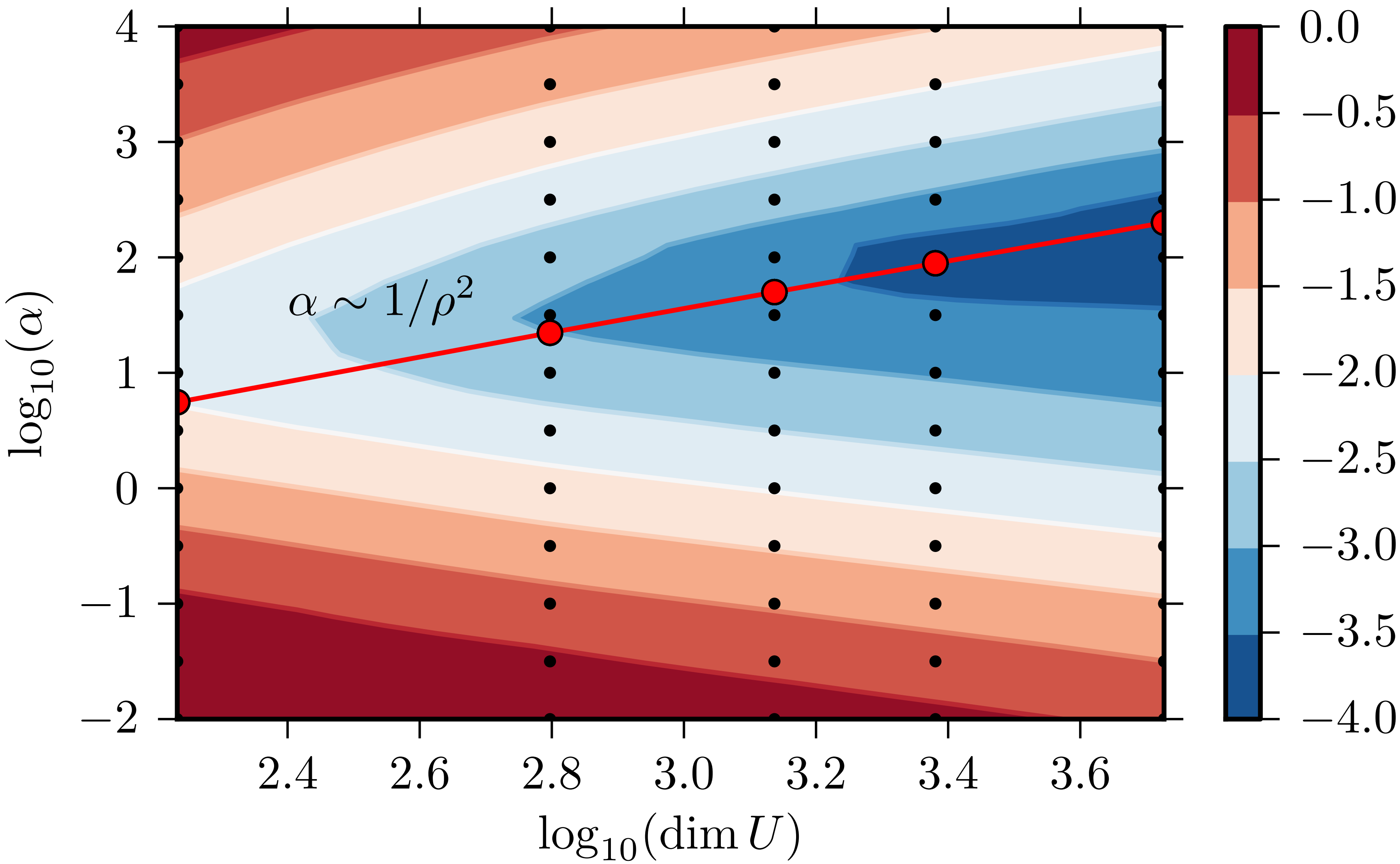
Choosing stability parameter



Consistent with numerical experiments of *Lovadina*

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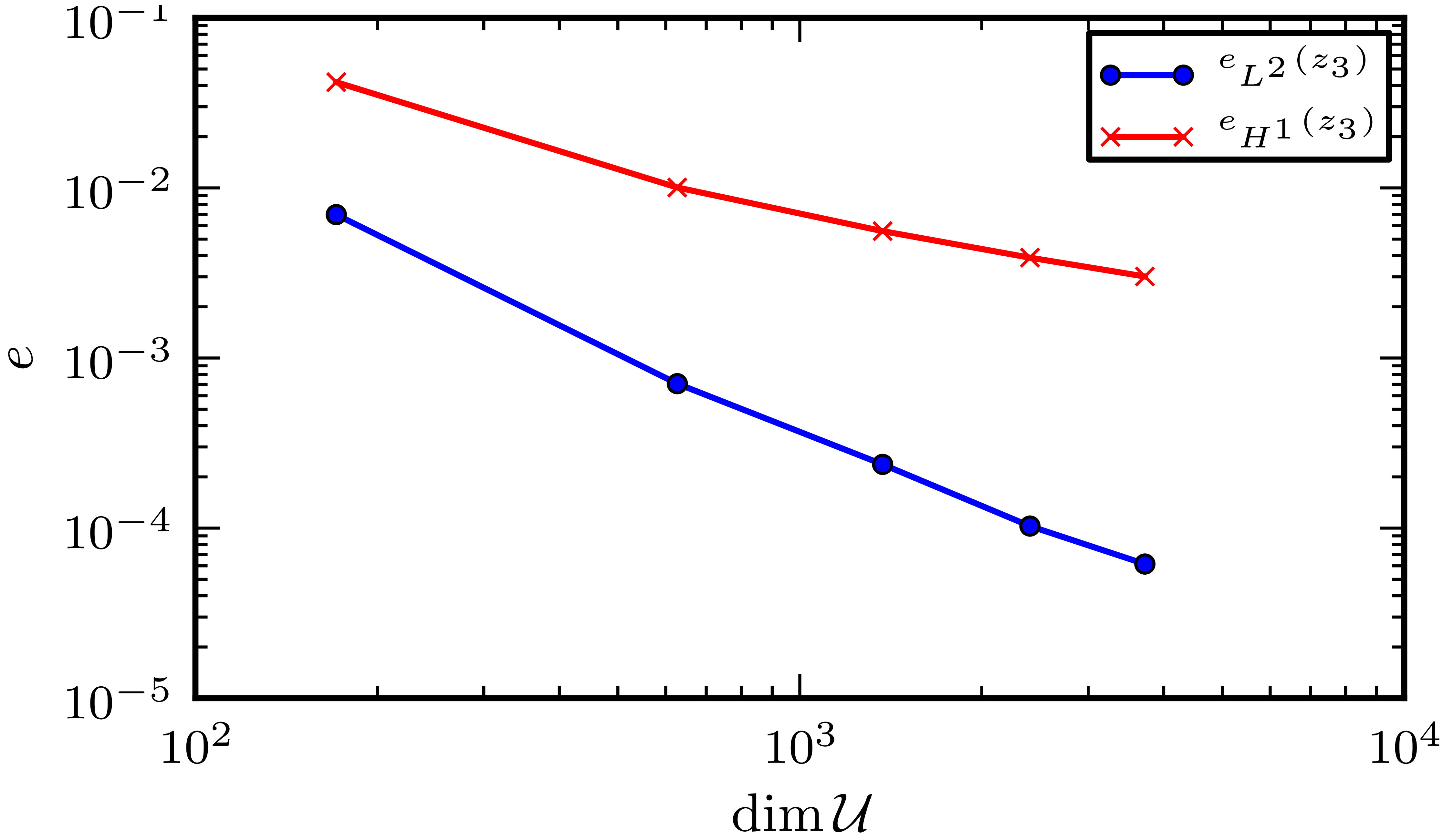
Convergence



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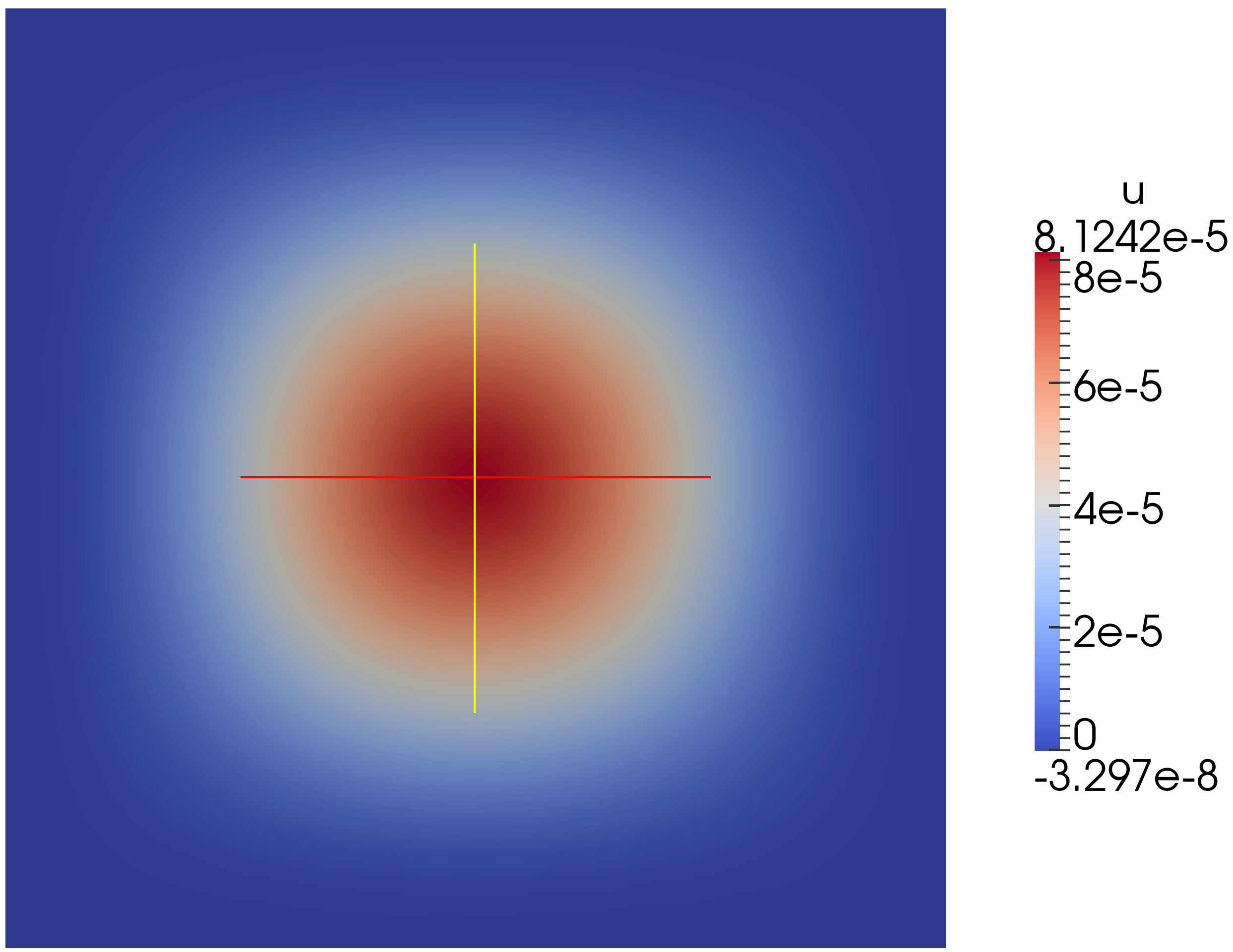

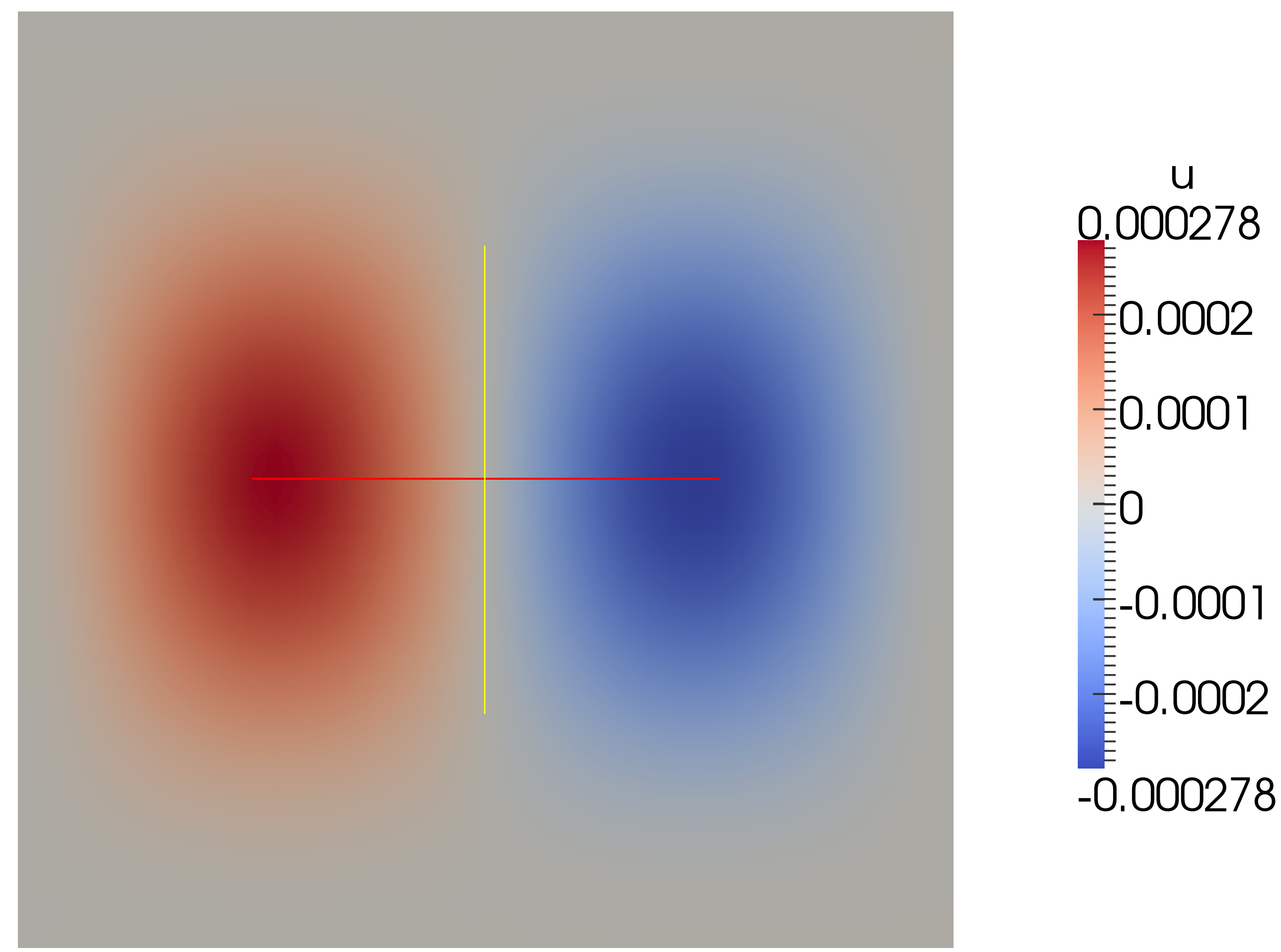

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Conclusions

- During my PhD I have worked to develop meshfree methods for the Reissner-Mindlin problem.
- The resulting method:
  - is based on a sound variational principle.
  - does not lock.
  - retains the mathematical properties of the original Reissner-Mindlin problem.
- More recently I have worked with Ortiz and Cyron to extend the definition of the patch projection operator to higher-order basis functions as well as meshfree basis functions for the Lagrange multiplier space. Additionally we have worked on solving outstanding issues related to numerical integration.