Preassociativity for aggregation functions

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Associative functions

Let X be a nonempty set

$G\colon X^2 \to X$ is *associative* if

$$G(G(a,b),c) \ = \ G(a,G(b,c))$$

Examples: $G(a,b) = a+b$ on $X = \mathbb{R}$
$G(a,b) = a \wedge b$ on $X = L$ (lattice)

Associative functions

$$G(G(a,b),c) \;=\; G(a,G(b,c))$$

Extension to *n*-ary functions

$$\begin{aligned} G_3(a,b,c) &:= G(G(a,b),c) \;=\; G(a,G(b,c)) \\ G_4(a,b,c,d) &:= G(G_3(a,b,c),d) \;=\; G(a,G(b,c),d) \;=\; \cdots \\ &\text{etc.} \end{aligned}$$

$G : \bigcup_{n\geq 2} X^n \to X : \mathbf{x} \in X^n \mapsto G_n(\mathbf{x})$

Associative functions with indefinite arity

Let

$$X^* \;=\; \bigcup_{n\in\mathbb{N}} X^n$$

$F\colon X^* \to X$ is *associative* if

$$\begin{aligned} & F(x_1,\ldots,x_p,\quad y_1,\ldots,y_q,\quad z_1,\ldots,z_r) \\ =\; & F(x_1,\ldots,x_p,F(y_1,\ldots,y_q),z_1,\ldots,z_r) \end{aligned}$$

Example: $F(x_1,\ldots,x_n) = x_1 + \cdots + x_n$ on $X = \mathbb{R}$

$F(x_1,\ldots,x_n) = x_1 \wedge \cdots \wedge x_n$ on $X = L$ (lattice)

Notation

We regard n-tuples $\mathbf{x}$ in X^n as *n-strings* over X

0-string: ε
1-strings: $x, y, z, \ldots$
n-strings: $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$
$|\mathbf{x}| \ = \$ length of $\mathbf{x}$

X^* is endowed with concatenation (X^* is the free monoid generated by X)

For $F : X^* \to Y$ we set

$$F_n := F|_{X^n}.$$

We assume

$$F(\mathbf{x}) \ = \ \varepsilon \quad \iff \quad \mathbf{x} \ = \ \varepsilon$$

Associative functions with indefinite arity

$F\colon X^* \to X$ is *associative* if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

F_1 may differ from the identity map!

Proposition

Let $F\colon X^* \to X$ and $G\colon X^* \to X$ be two associative functions such that $F_1 = G_1$ and $F_2 = G_2$. Then $F = G$.

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Proposition

A binary function $G\colon X^2 \to X$ is associative if and only if there exists an associative function $F\colon X^* \to X$ such that $F_2 = G$.

Preassociative functions

Let Y be a nonempty set

Definition. We say that $F\colon X^* \to Y$ is *preassociative* if

$$F(\mathbf{y}) \;=\; F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{xyz}) \;=\; F(\mathbf{xy}'\mathbf{z})$$

Examples: $F_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2 \quad (X = Y = \mathbb{R})$

$F_n(\mathbf{x}) = |\mathbf{x}| \quad (X \text{ arbitrary},\; Y = \mathbb{N})$

Associative functions are preassociative

$$\boxed{F(\mathbf{y}) \ = \ F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{xyz}) \ = \ F(\mathbf{xy}'\mathbf{z})}$$

Fact

If $F\colon X^* \to X$ is associative, then it is preassociative

Proof. Suppose $F(\mathbf{y}) = F(\mathbf{y}')$
Then $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}) = F(\mathbf{x}F(\mathbf{y}')\mathbf{z}) = F(\mathbf{xy}'\mathbf{z})$ □

Construction of preassociative functions

$$F(\mathbf{y}) \;=\; F(\mathbf{y}') \quad\Rightarrow\quad F(\mathbf{x}\mathbf{y}\mathbf{z}) \;=\; F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

Proposition (right composition)

If $F\colon X^* \to Y$ is preassociative, then so is the function

$$x_1 \cdots x_n \;\mapsto F_n(g(x_1)\cdots g(x_n))$$

for every function $g\colon X \to X$

Example: $F_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2 \quad (X = Y = \mathbb{R})$

Construction of preassociative functions

$$F(\mathbf{y}) \;=\; F(\mathbf{y}') \quad\Rightarrow\quad F(\mathbf{x}\mathbf{y}\mathbf{z}) \;=\; F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

Proposition (left composition)

If $F\colon X^* \to Y$ is preassociative, then so is

$$g \circ F \;:\; \mathbf{x} \mapsto g(F(\mathbf{x}))$$

for every function $g\colon Y \to Y$ such that $g|_{\mathrm{ran}(F)}$ is one-to-one

Example: $F_n(\mathbf{x}) = \exp(x_1^2 + \cdots + x_n^2) \quad (X = Y = \mathbb{R})$

Construction of preassociative functions

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Example: $F_n(\mathbf{x}) = \exp(x_1^2 + \cdots + x_n^2) \quad (X = Y = \mathbb{R})$

Question: Given a preassociative F, which are the g such that $g \circ F$ is preassociative?

Being preassociative is a strong property

$$F(\mathbf{y}) \;=\; F(\mathbf{y}') \quad\Rightarrow\quad F(\mathbf{x}\mathbf{y}\mathbf{z}) \;=\; F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

Proposition

Assume $F\colon X^* \to Y$ is preassociative. If F_n is constant, then so is F_{n+1}.

Proof. If $F_n(\mathbf{y}) = F_n(\mathbf{y}')$ for all $\mathbf{y}, \mathbf{y}' \in X^n$, then $F_{n+1}(x\mathbf{y}) = F_{n+1}(x\mathbf{y}')$ and hence F_{n+1} depends only on its first argument... □

Associative $\iff$ Preassociative with 'constrained' F_1

$$F(\mathbf{y}) \;=\; F(\mathbf{y}') \quad\Rightarrow\quad F(\mathbf{x}\mathbf{y}\mathbf{z}) \;=\; F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

Proposition

$F\colon X^* \to X$ is associative if and only if it is preassociative and

$$F_1(F(\mathbf{x})) = F(\mathbf{x})$$

We relax the constraint on F_1

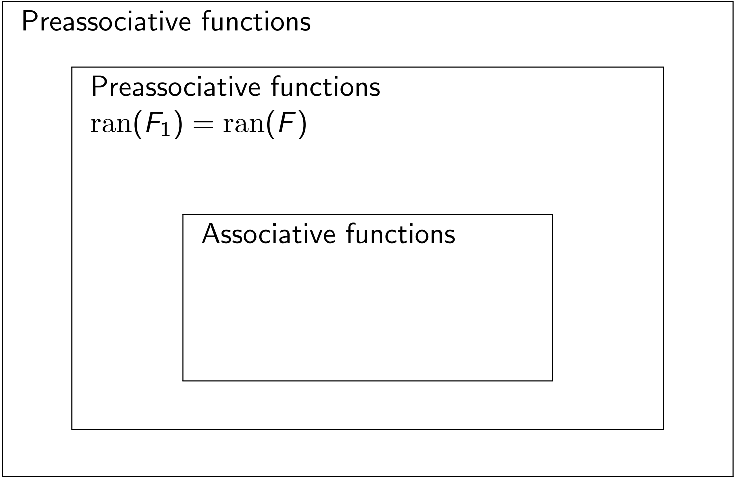
Relaxation of $F_1(F(\mathbf{x})) = F(\mathbf{x})$:

$$\boxed{\mathrm{ran}(F_1) \;=\; \mathrm{ran}(F)}$$

where

$$\begin{aligned}\mathrm{ran}(F_1) &= \{F_1(x) : x \in X\}\\ \mathrm{ran}(F) &= \{F(\mathbf{x}) : \mathbf{x} \in X^*\}\end{aligned}$$

Nested classes of preassociative functions



If $\mathrm{ran}(F_1) = \mathrm{ran}(F)$

Proposition

Let $F\colon X^* \to Y$ and $G\colon X^* \to Y$ be two preassociative functions such that $\mathrm{ran}(F_1) = \mathrm{ran}(F)$, $\mathrm{ran}(G_1) = \mathrm{ran}(G_1)$, $F_1 = G_1$ and $F_2 = G_2$. Then $F = G$.

Factorizing preassociative functions with associative ones

Theorem (Factorization)

Let $F\colon X^* \to Y$. The following assertions are equivalent:

(i) F is preassociative and satisfies $\mathrm{ran}(F_1) = \mathrm{ran}(F)$

(ii) F can be factorized into

$$F \ = \ f \circ H$$

where $H\colon X^* \to X$ is associative and $f\colon \mathrm{ran}(H) \to Y$ is one-to-one.

The factorization theorem can be used to obtain axiomatizations of functions classes

A three step technique:

(Binary) Start with a class associative functions $F : X^2 \to X$,

(Source) Axiomatize all their associative extensions $F : X^* \to X$,

(Target) Use factorization theorem to weaken this axiomatization to capture preassociativity.

Aczélian semigroups

Theorem (Aczél 1949)

$H\colon \mathbb{R}^2 \to \mathbb{R}$ is

- continuous
- one-to-one in each argument
- associative

if and only if

$$H(xy) \;=\; \varphi^{-1}(\varphi(x)+\varphi(y))$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous and strictly monotone

Source class:

$$H_n(\mathbf{x}) \;=\; \varphi^{-1}(\varphi(x_1)+\cdots+\varphi(x_n))$$

Preassociative functions from Aczélian semigroups

Target Theorem

Let $F\colon \mathbb{R}^* \to \mathbb{R}$. The following assertions are equivalent:

(i) F is preassociative and satisfies $\mathrm{ran}(F_1) = \mathrm{ran}(F)$, F_1 and F_2 are continuous and one-to-one in each argument

(ii) we have

$$F_n(\mathbf{x}) \;=\; \psi\big(\varphi(x_1) + \cdots + \varphi(x_n)\big)$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $\psi\colon \mathbb{R} \to \mathbb{R}$ are continuous and strictly monotone

Preassocitive functions from T-norms

A *T-norm* is a function $T\colon [0,1]^2 \to [0,1]$ which is nondecreasing in each argument, symmetric, associative, and such that $T(1x) = x$.

Target Theorem

Let $F\colon [0,1]^* \to \mathbb{R}$ be such that F_1 is strictly increasing. The following assertions are equivalent:

(i) F is preassociative and $\mathrm{ran}(F_1) = \mathrm{ran}(F)$, F_2 is symmetric, nondecreasing, and $F_2(1x) = F_1(x)$

(ii) we have

$$F = f \circ T$$

where $f\colon [0,1] \to \mathbb{R}$ is strictly increasing and $T\colon [0,1]^* \to [0,1]$ is a triangular norm

Associative range-idempotent functions

Source Theorem

Let $H\colon \mathbb{R}^* \to \mathbb{R}$ be a function. The following assertions are equivalent:

(i) H is associative and satisfies $H(H(x)H(x)) = H(x)$, and H_1 and H_2 are symmetric, continuous, and nondecreasing

(ii) there exist $a, b, c \in \mathbb{R}$, $a \leqslant c \leqslant b$, such that

$$H_n(\mathbf{x}) \;=\; \mathrm{med}\Big(a, \mathrm{med}\Big(\bigwedge_{i=1}^{n} x_i, c, \bigvee_{i=1}^{n} x_i\Big), b\Big)$$

Preassociative functions from range-idempotent ones

Theorem

Let $F\colon \mathbb{R}^* \to \mathbb{R}$ be a function and let $[a, b]$ be a closed interval. The following assertions are equivalent:

(i) F is preassociative and satisfies $\mathrm{ran}(F_1) = \mathrm{ran}(F)$, there exists a continuous and strictly increasing function $f\colon [a, b] \to \mathbb{R}$ such that $F_1(x) = (f \circ \mathrm{med})(a, x, b)$, F_2 is continuous, nondecreasing and satisfies $F_2(xx) = F_1(x)$

(ii) there exist $c \in [a, b]$ such that

$$F_n(\mathbf{x}) \;=\; (f \circ \mathrm{med})\Big(a, \mathrm{med}\Big(\bigwedge_{i=1}^{n} x_i, c, \bigvee_{i=1}^{n} x_i\Big), b\Big)$$

Open problems

(1) Find new axiomatizations of classes of preassociative functions from existing axiomatizations of classes of associative functions

(2) Find interpretations of preassociativity in fuzzy logic, artificial intelligence,...

(3) Given a preassociative F, which are the g such that $g \circ F$ is preassociative?