# Preassociativity for aggregation functions

MARICHAL Jean-Luc and TEHEUX Bruno

*Mathematics Research Unit - University of Luxembourg*
*6, rue Richard Coudenhove-Kalergi, 1359 Luxembourg*
*Luxembourg*
*E-mail:* jean-luc.marichal@uni.lu
*E-mail:* bruno.teheux@uni.lu

Let $X$ be an nonempty set and let $F : X^* \to X$ be a function with varying arity where $X^* = \bigcup_{n\geq 0} X^n$. For every $n \geq 0$ we denote by $F_n$ the function defined as $F_n = F|_{X^n}$ (we convey that $F(\epsilon) = \epsilon$ where $\epsilon$ is the 0-tuple). For elements $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ of $X^*$ we denote by $F(\mathbf{xy})$ the element $F(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and similarly for more than two tuples. The function $F$ is *associative* if it satisfies

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}), \qquad \mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*.$$

Associative functions have been considered by different authors (see [1, 2, 3, 4, 5, 6]). An example of associative function is given by the function $F : \mathbb{R}^* \to \mathbb{R}$ defined by $F(\mathbf{x}) = \sum_{i=1}^n x_i$ for every $n > 0$ and every $\mathbf{x} \in X^n$.

In this talk, we investigate a generalization of associativity called *preassociativity* (see [7]). If $X$ and $Y$ are two non-empty sets, a function $F : X^* \to Y$ is *preassociative* if it satisfies

$$F(\mathbf{y}) = F(\mathbf{y}') \implies F(\mathbf{xyz}) = F(\mathbf{xy}'\mathbf{z}), \qquad \mathbf{x}, \mathbf{y}, \mathbf{y}', \mathbf{z} \in X^*.$$

Preassociativity allows us to consider functions with varying arity $F : X^* \to Y$ with distinct domain $X$ and co-domain $Y$. Any associative function is easily seen to be preassociative. Actually, associative functions can be characterized among preassociative ones as stated below.

**Proposition 1** *A function $F : X^* \to X$ is associative if and only if it is preassociative and $F_1 \circ F = F$.*

If $f : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing, the function $F : \mathbb{R}^* \to \mathbb{R}$ defined by $F(\mathbf{x}) = f(\sum_{i=1}^n x_i)$ for every $n > 0$ and every $\mathbf{x} \in X^n$ is an example of preassociative function which is not associative.

Under some conditions stated in the two next results, new preassociative functions can be constructed from existing ones by function composition.

**Proposition 2 (Right composition)** *If $F : X^* \to Y$ is preassociative then, for every function $g : X \to X$, the function $H : X^* \to Y$ defined by $H(\mathbf{x}) = F\big(g(x_1), \ldots, g(x_n)\big)$ for every $x \in X^n$ and every $n > 0$ is preassociative.*

For instance, the squared distance function $F\colon \mathbb{R}^* \to \mathbb{R}$ defined by $F(\mathbf{x}) = \sum_{i=1}^n x_i^2$ for every $n > 0$ and every $\mathbf{x} \in X^n$ is preassociative.

**Proposition 3 (Left composition)** *Let $F : X^* \to Y$ be a preassociative function and let $g\colon Y \to Y$ be a function. If $g|_{\mathrm{ran}(F)}$ is constant or one-to-one, then the function $H : X^* \to Y$ defined by $H(\mathbf{x}) = g(F(\mathbf{x}))$ for every $\mathbf{x} \in X^*$ is preassociative.*

For instance, the function $F:\mathbb{R}^*\to\mathbb{R}$ defined by $F(\mathbf{x})=\exp(\sum_{i=1}^n x_i)$ for every $n>0$ and every $\mathbf{x}\in X^n$ is preassociative.

Among the proper subclasses of the class of preassociative functions $F:X^*\to X$ there is a class that contains properly the class of associative functions and that shares with the latter very interesting properties. It is the class of the preassociative functions $F:X^*\to X$ that satisfy $\mathrm{ran}(F)=\mathrm{ran}(F_1)$. The following factorization result allows us to generalize to this specific class some of the properties that are known for associative functions.

**Theorem 4** *Let $F:X^*\to Y$ be a function. The following assertions are equivalent.*

*(i)* *$F$ is preassociative and* $\mathrm{ran}(F)=\mathrm{ran}(F_1)$.

*(ii)* *There exists an associative function $H:X^*\to X$ and a one-to-one function $f:\mathrm{ran}(H)\to Y$ such that $F=f\circ H$.*

In the talk, we will also illustrate how this result can be used to give necessary and sufficient conditions for a preassociative function $F:X^*\to Y$ that satisfies $F_1\circ F=F$ to be characterized by $F_1$ and $F_2$ (a property that holds for any associative function). We will also show with concrete examples (Aczélian semigroups, t-norms, Ling's class) how to derive axiomatizations of classes of preassociative functions from certain existing axiomatizations of classes of associative functions.

## References


[1] Calvo, Tomasa and Mayor, Gaspar and Mesiar, Radko: *Aggregation Operators: Properties, Classes and Construction Methods.* In: Aggregation Operators: Properties, Classes and Construction Methods. Studies in Fuzziness and Soft Computing **97**, Physica-Verlag HD, Heidelberg, Germany, 2002.

[2] Couceiro, Miguel and Marichal, Jean-Luc: *Associative Polynomial Functions over Bounded Distributive Lattices.* Order **28** (2011) 1–8.

[3] Couceiro, Miguel and Marichal, Jean-Luc: *Aczélian $n$-ary semigroups.* Semigroup Forum **85**, 2012.

[4] Grabisch, Michel, Marichal, Jean-Luc, Mesiar, Radko, and Pap, Endre: *Aggregation functions.* Encyclopedia of Mathematics and its Applications **127**. Cambridge University Press, Cambridge, 2009.

[5] Klement, Erich Peter, Mesiar, Radko and Pap, Endre: *Triangular norms.* In: Trends in Logic - Studia Logica Library **8**, Kluwer Academic, Dordrecht, 2000.

[6] Marichal, Jean-Luc, *Aggregation operators for multicriteria decision aid.*, PhD thesis, Department of Mathematics, University of Liège, Liège, Belgium, 1998.

[7] Marichal, Jean-Luc and Teheux, Bruno: *Associative and preassociative functions*, arXiv:1309.7303.