Direct image-analysis methods for surgical simulation and mixed meshfree methods

Jack S. Hale
Marie Curie Postdoctoral Fellow
Research Unit in Engineering Science
Université du Luxembourg

Collaborators:

Stephane P. A. Bordas, Universite du Luxembourg/Cardiff University
Pierre Kerfriden, Cardiff University
Juan J. Rodenas Garcia, Universidad Politecnica de Valencia

Pedro M. Baiz, Imperial College London
Alejandro Ortiz-Bernardin, Universidad de Chile
Christian J. Cyron, Yale University

Université Pierre-et-Marie-Curie
Institut Jean le Rond d'Alembert
26th May 1400

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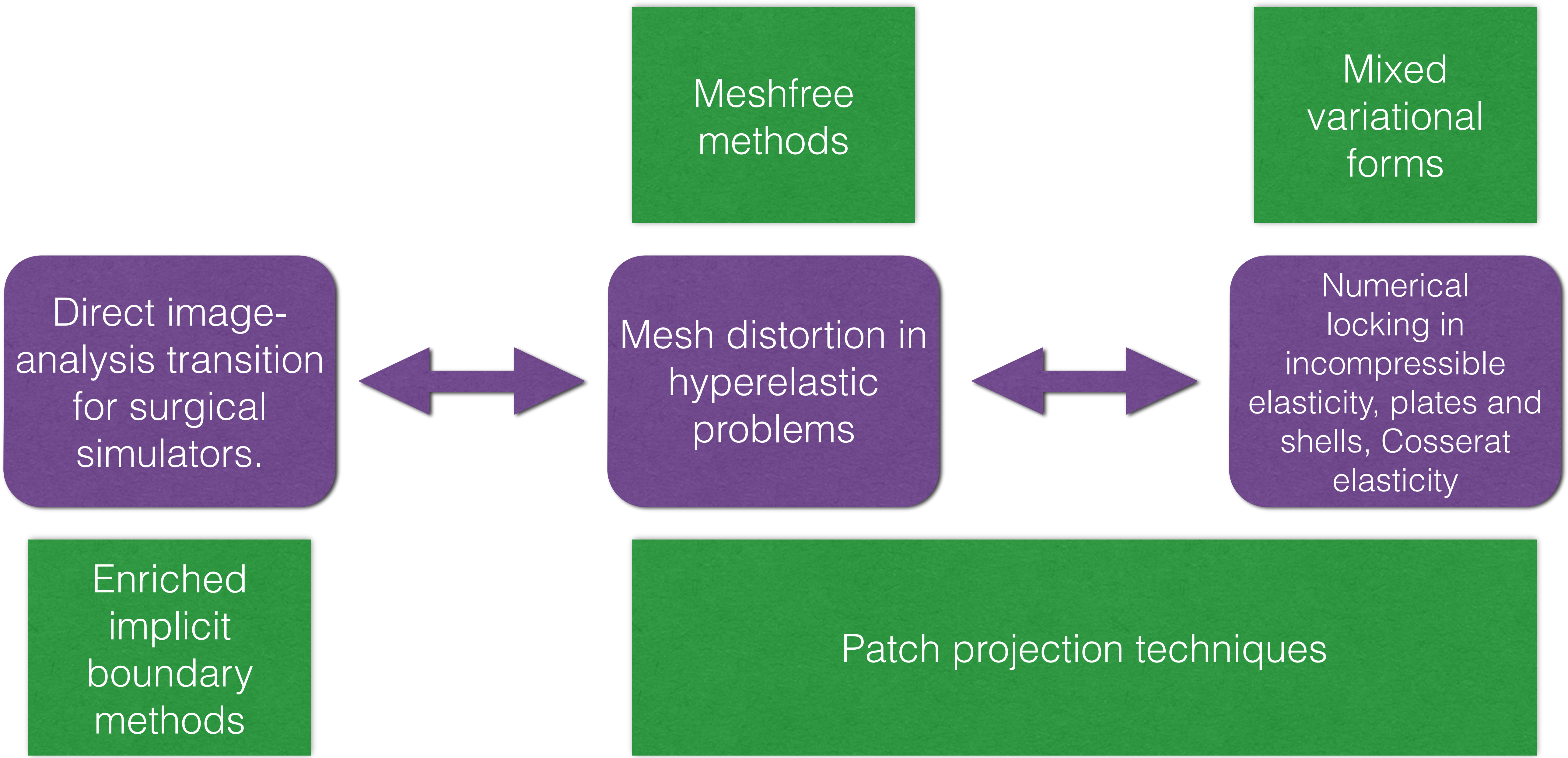
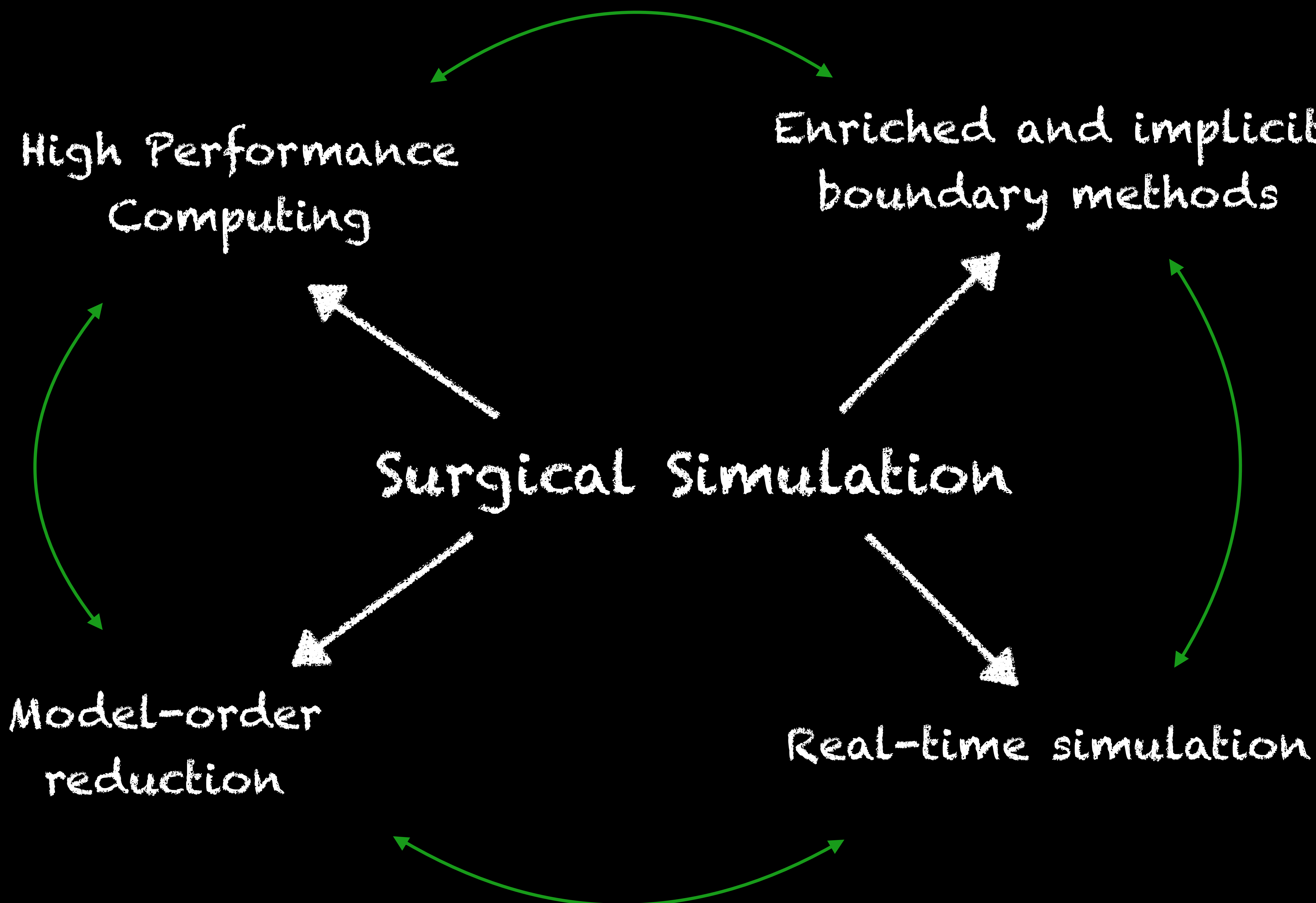
An Overview

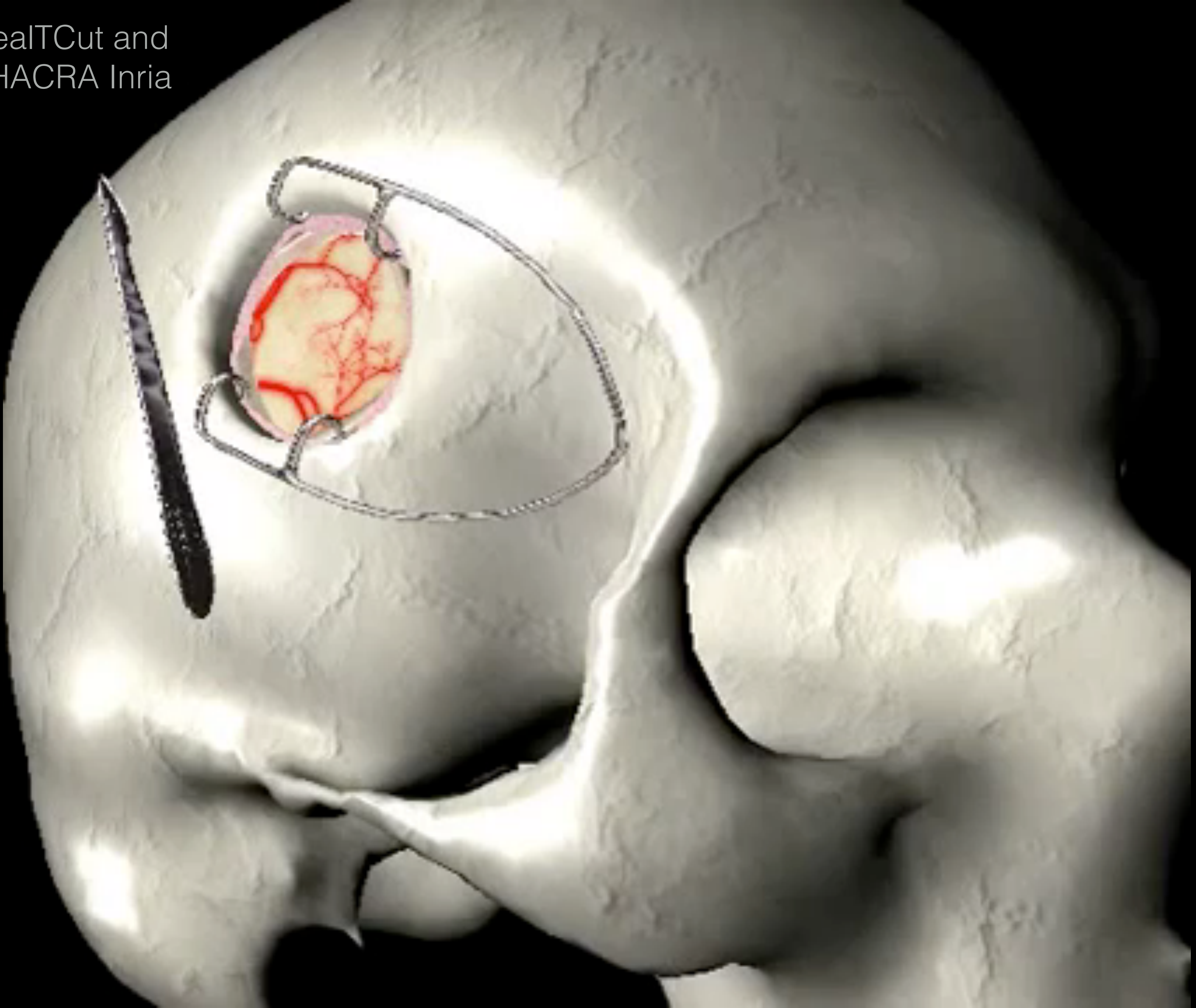
The problems and my research

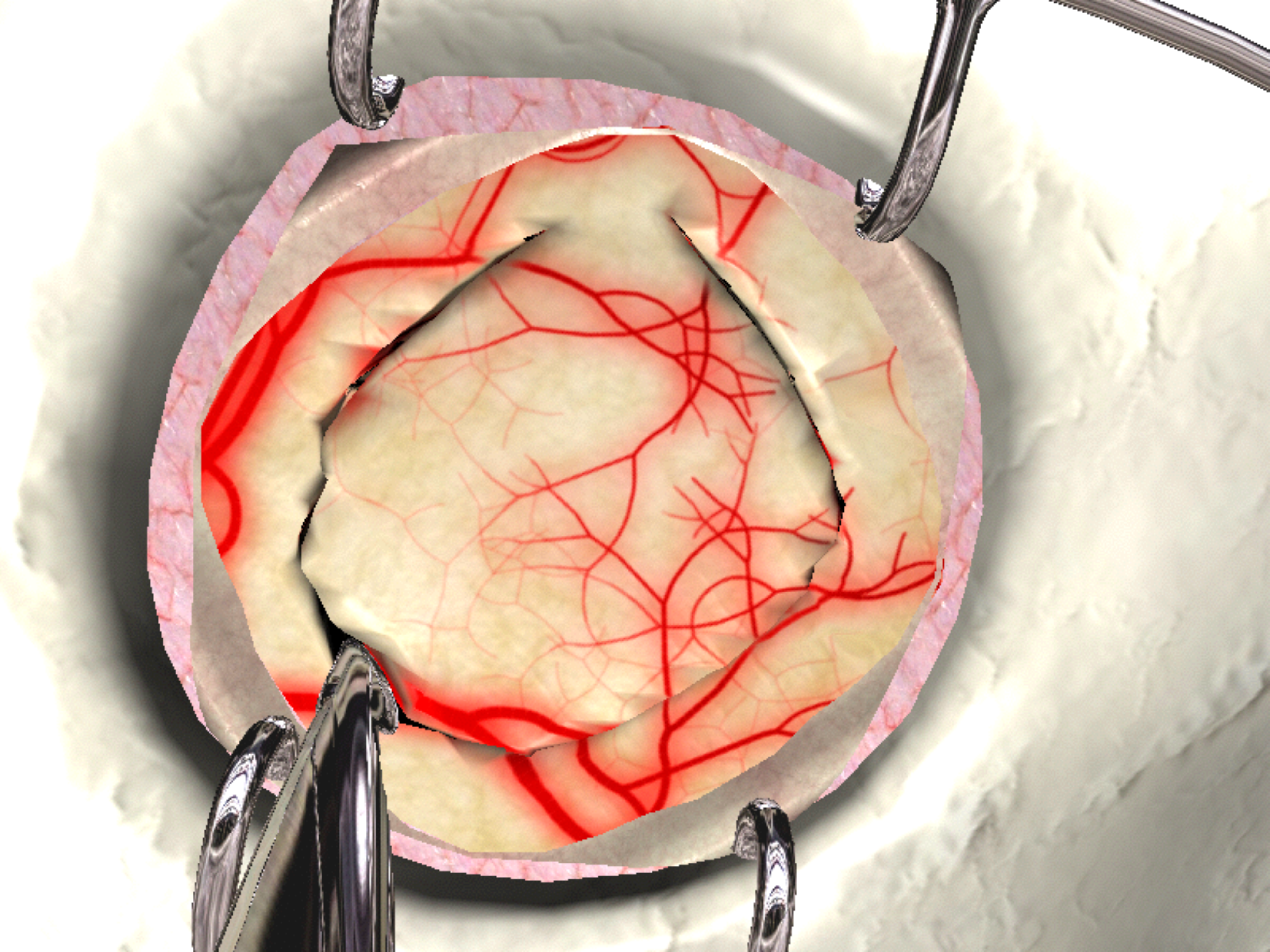


Project 1: Direct image to mesh analysis for surgical simulation

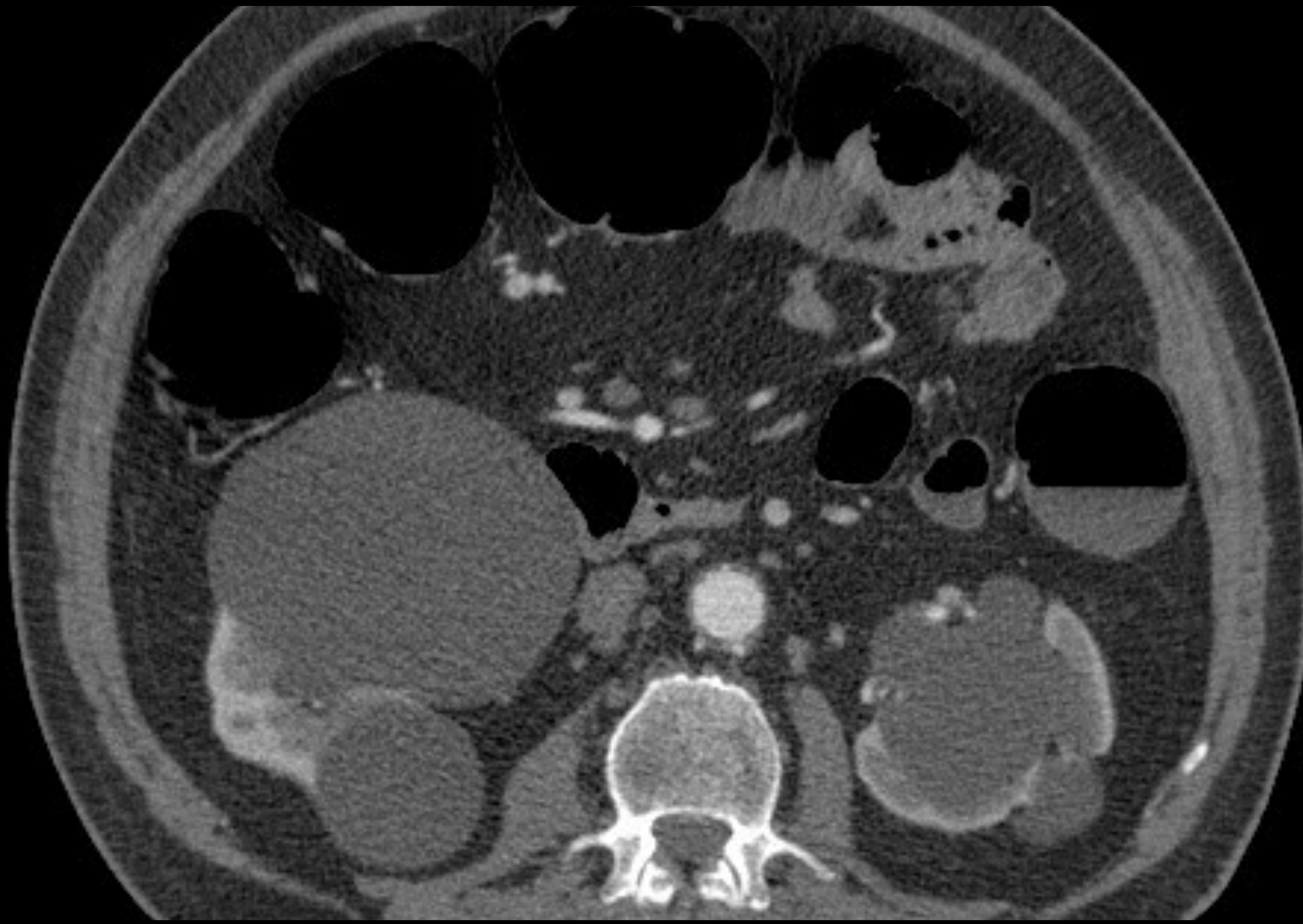

High Performance Computing
Enriched and implicit boundary methods
Surgical Simulation
Model-order reduction
Real-time simulation

RealTCut and
SHACRA Inria



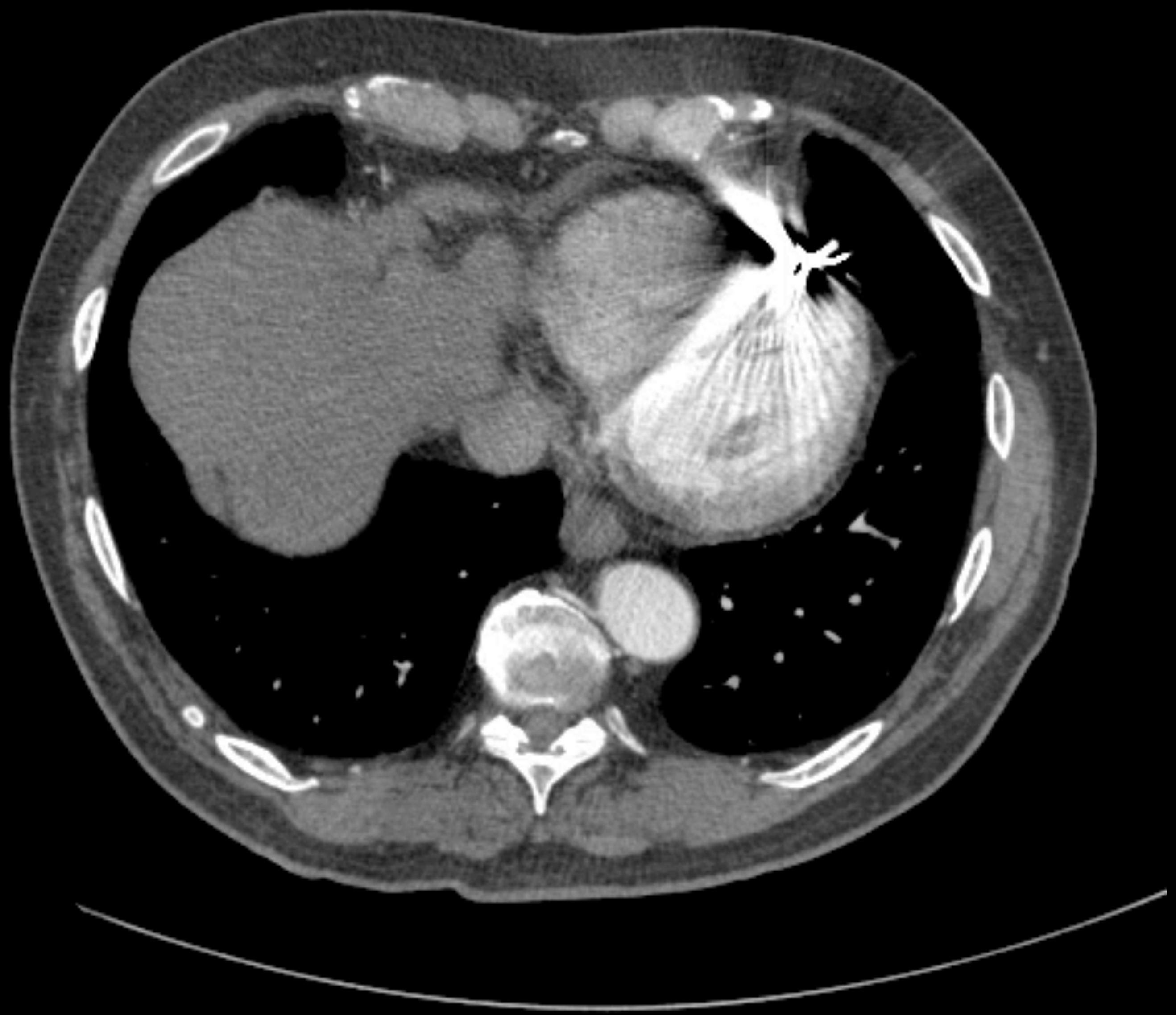


How can we move from an image…



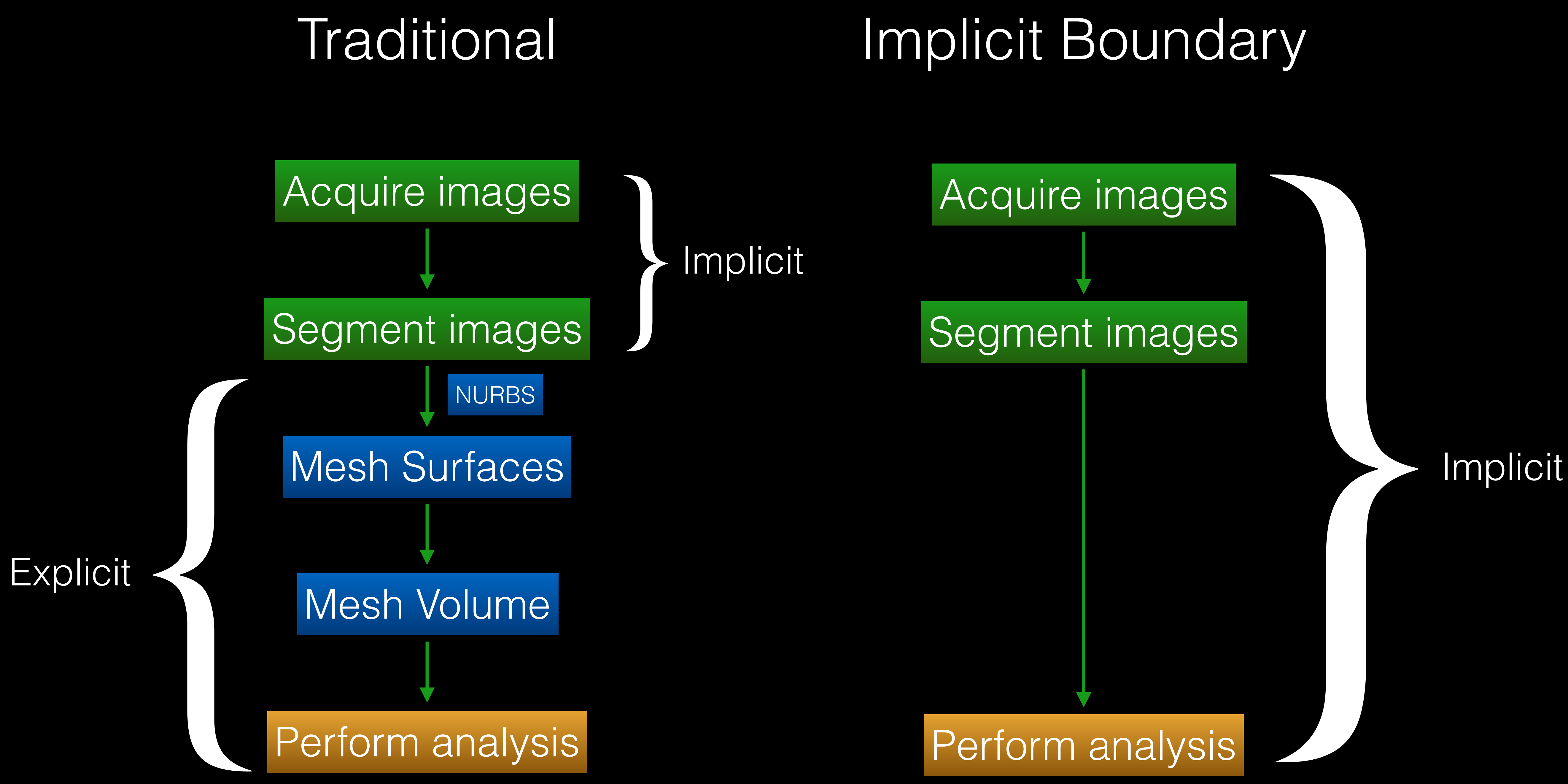
COLONIX, OSIRIX

…or perhaps a series of images…



Source: COLONIX, OSIRIX

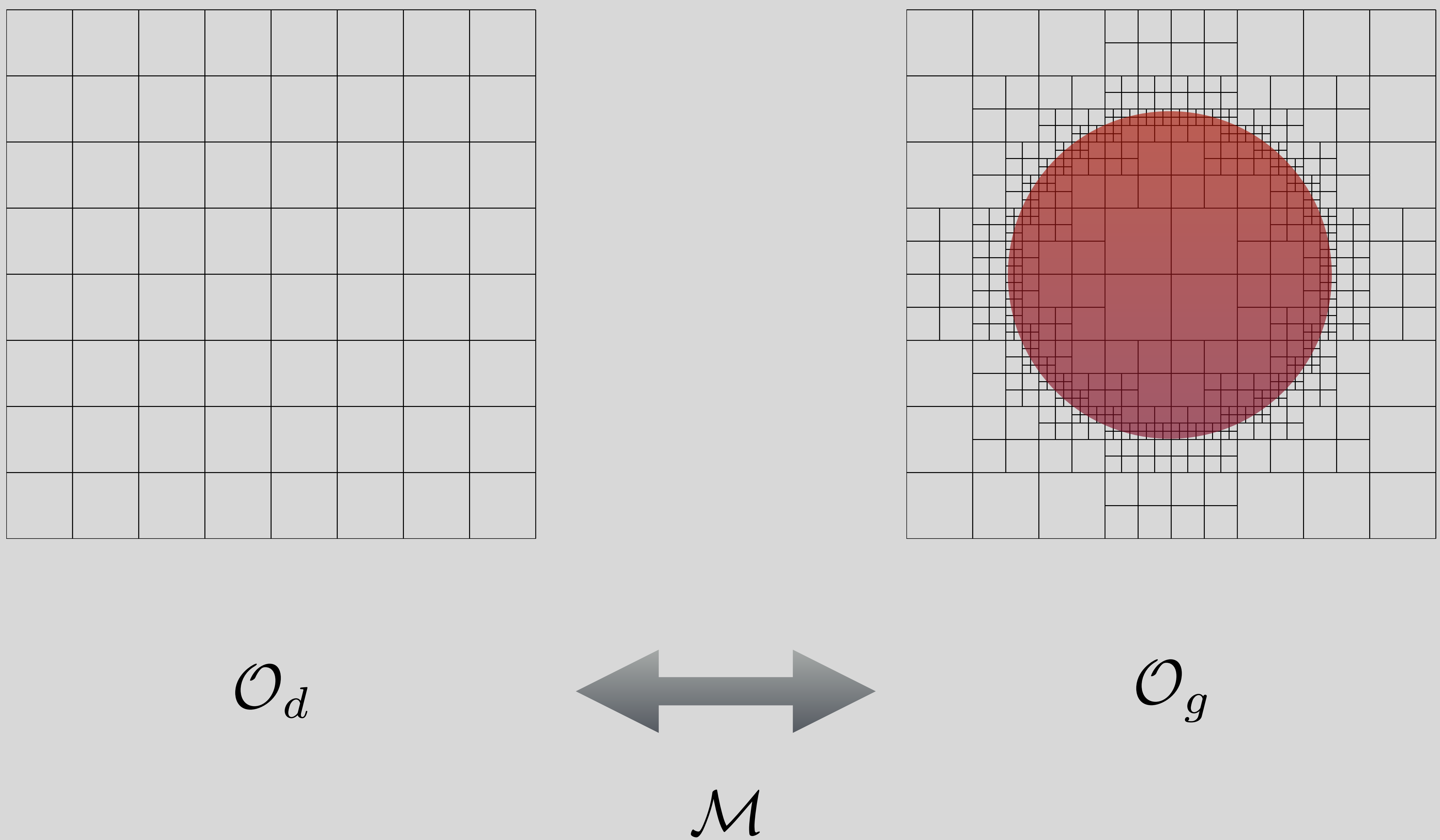
Pipelines to analysis



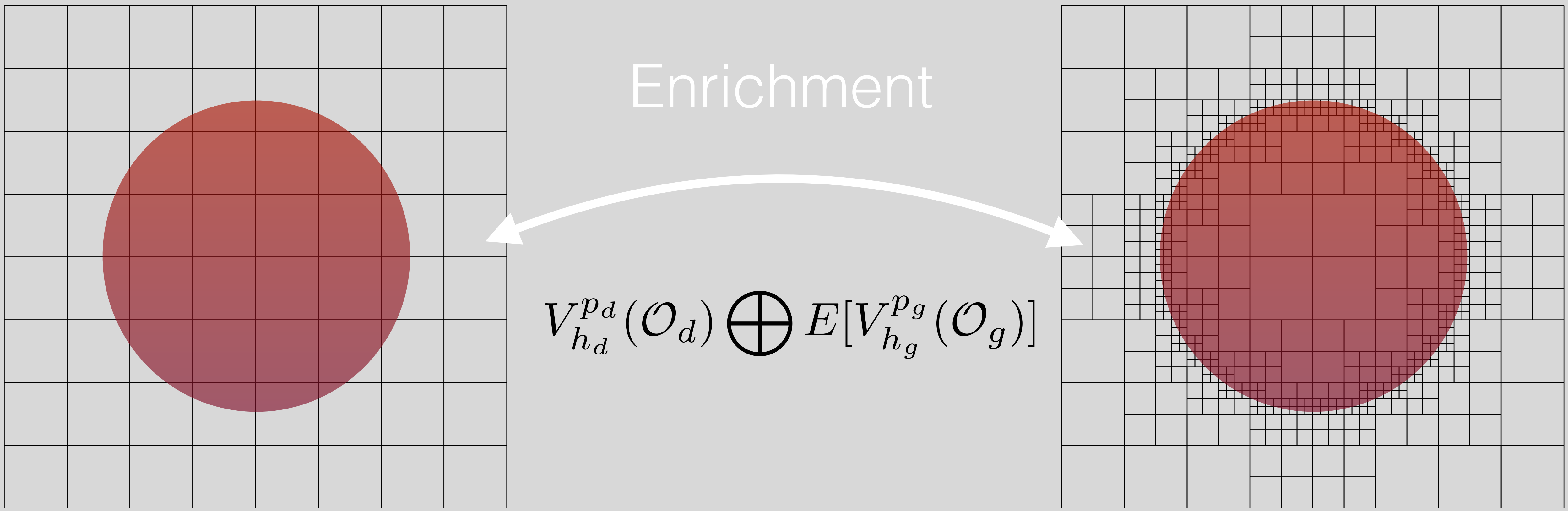
Nested Octree

Discretisation

Geometry



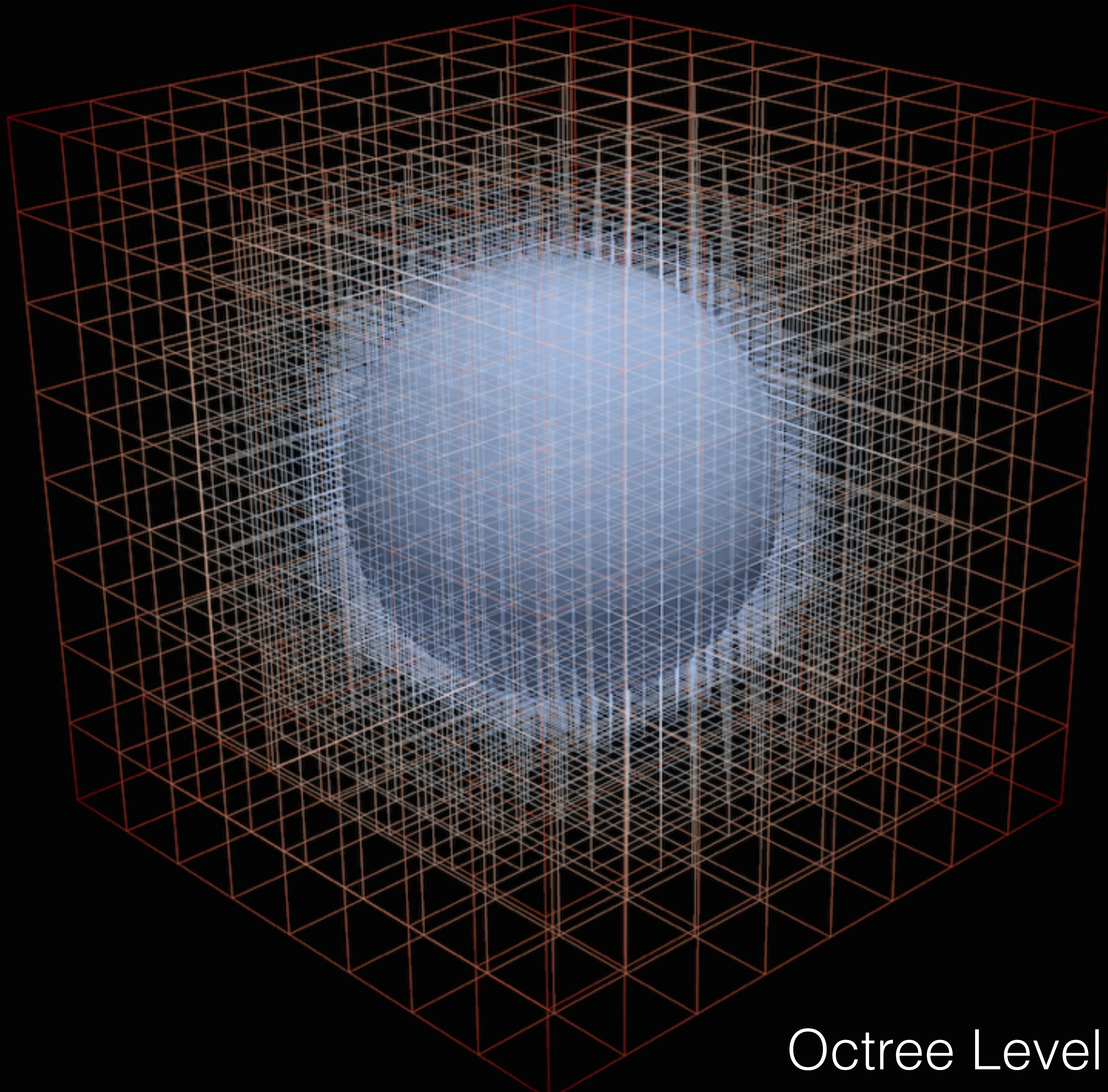
How to transfer geometric information back to the discretisation?



$$V_{h_d}^{p_d}(\mathcal{O}_d) \bigoplus E[V_{h_g}^{p_g}(\mathcal{O}_g)]$$

$V_{h_d}^{p_d}(\mathcal{O}_d)$ $\quad$ $p_d > p_g$ $\quad$ $V_{h_g}^{p_g}(\mathcal{O}_g)$

$h_d > h_g$ $\quad$ $p_g = 1$



Octree Level 5/Level 3

6a2e86c

Surface

Features

- 2D and 3D problems on the same code-path.
- parallel hybrid MPI/TBB assembly and solution.
- fast and robust computational geometry using CGAL.
  - automatic Delaunay tessellation of integration subdomains.
- completely separate representation of discretisation and geometry via nested octree data structures.
  - constructs to represent soft and hard segmentations of image data.
  - implicit representation via level-sets, inside-outside functions.
- independent hpe-type adaptivity on discretisation and geometry.
  - fast refinement and coarsening operations.

Outlook

- We are developing a cartesian grid implicit boundary/enriched finite element method toolkit within deal.ii.
- By uncoupling discretisation and geometry we hope to produce a method particularly suited to image-based analysis.
- Many challenges ahead, particularly with imposition of Dirichlet boundary conditions on hyperelastic soft-tissue model.

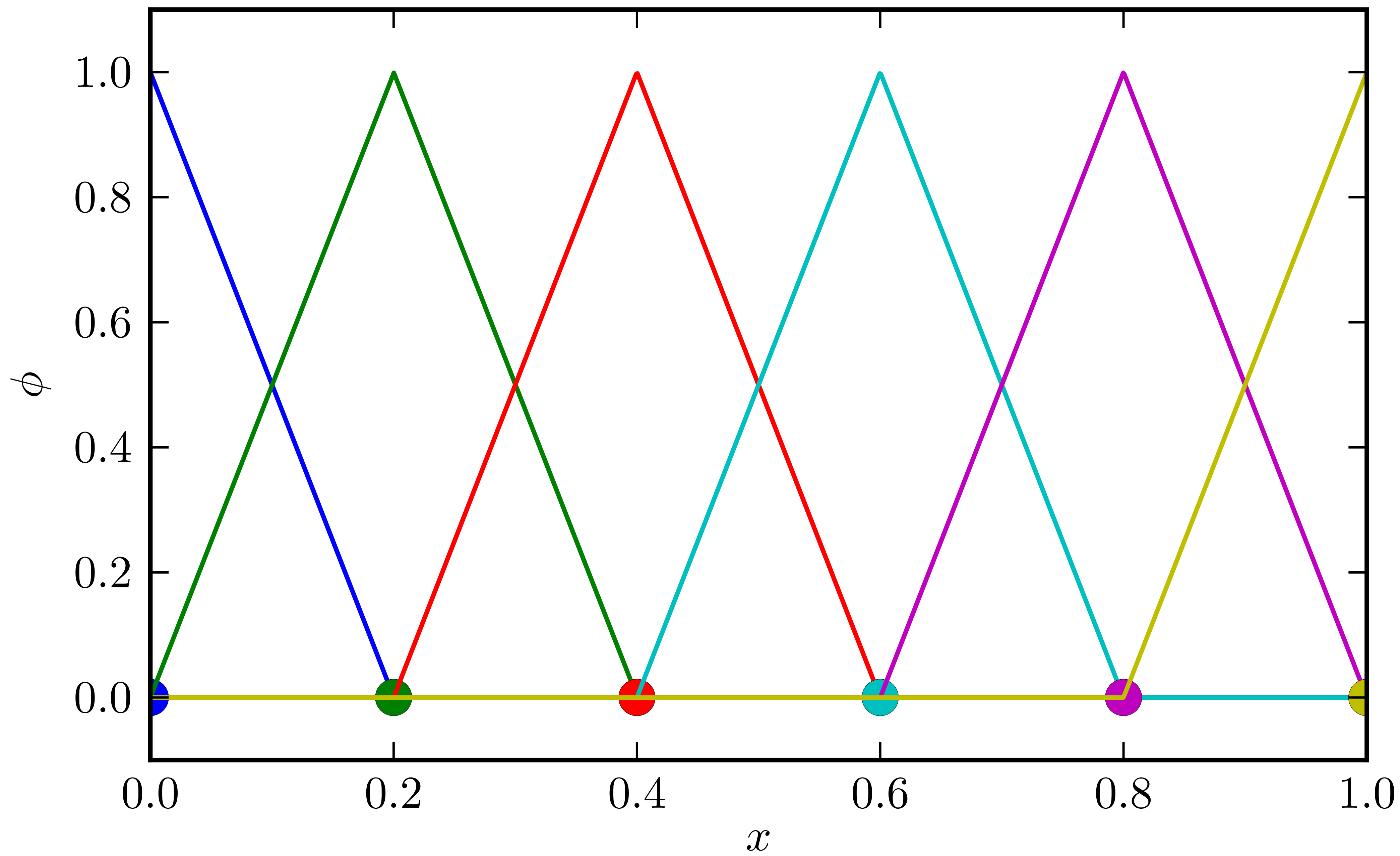
Project 2: Mesh-based and Meshfree Partitions of Unity

Partition of Unity

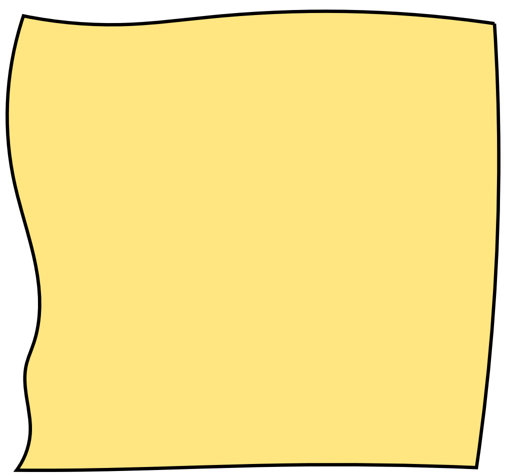
$$u_h(x) = \sum_{i=1}^{N} \phi_i u_i$$

$$\sum_{i=1}^{N} \phi_i = 1$$

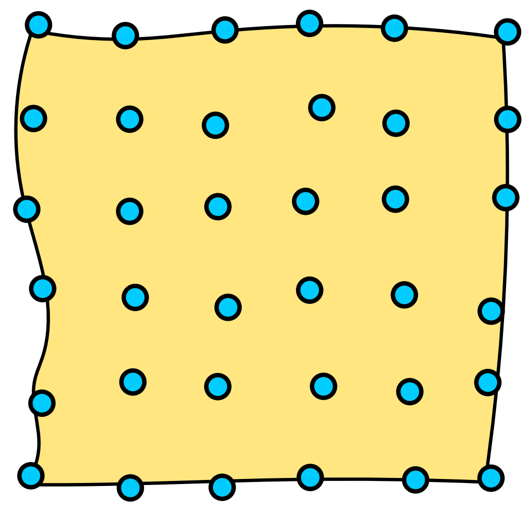
Finite elements



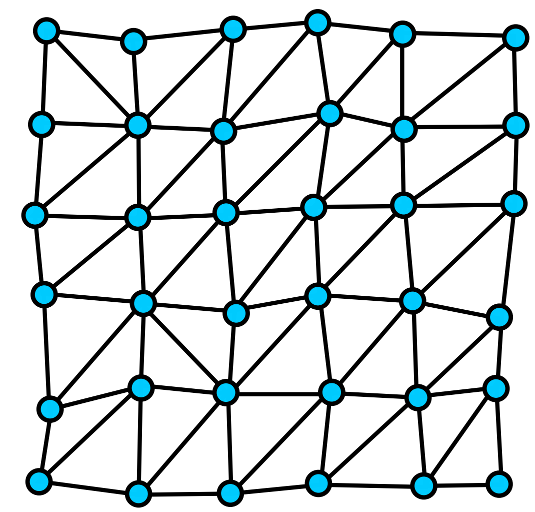
Mesh-based Partition of Unity



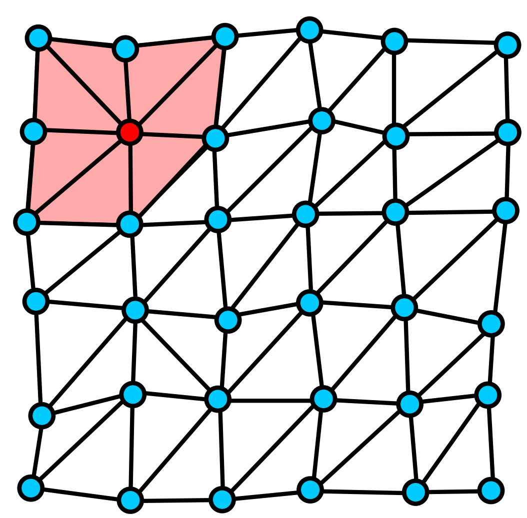
(a) Problem domain Ω described in CAD program



(b) Problem domain seeded with nodes.

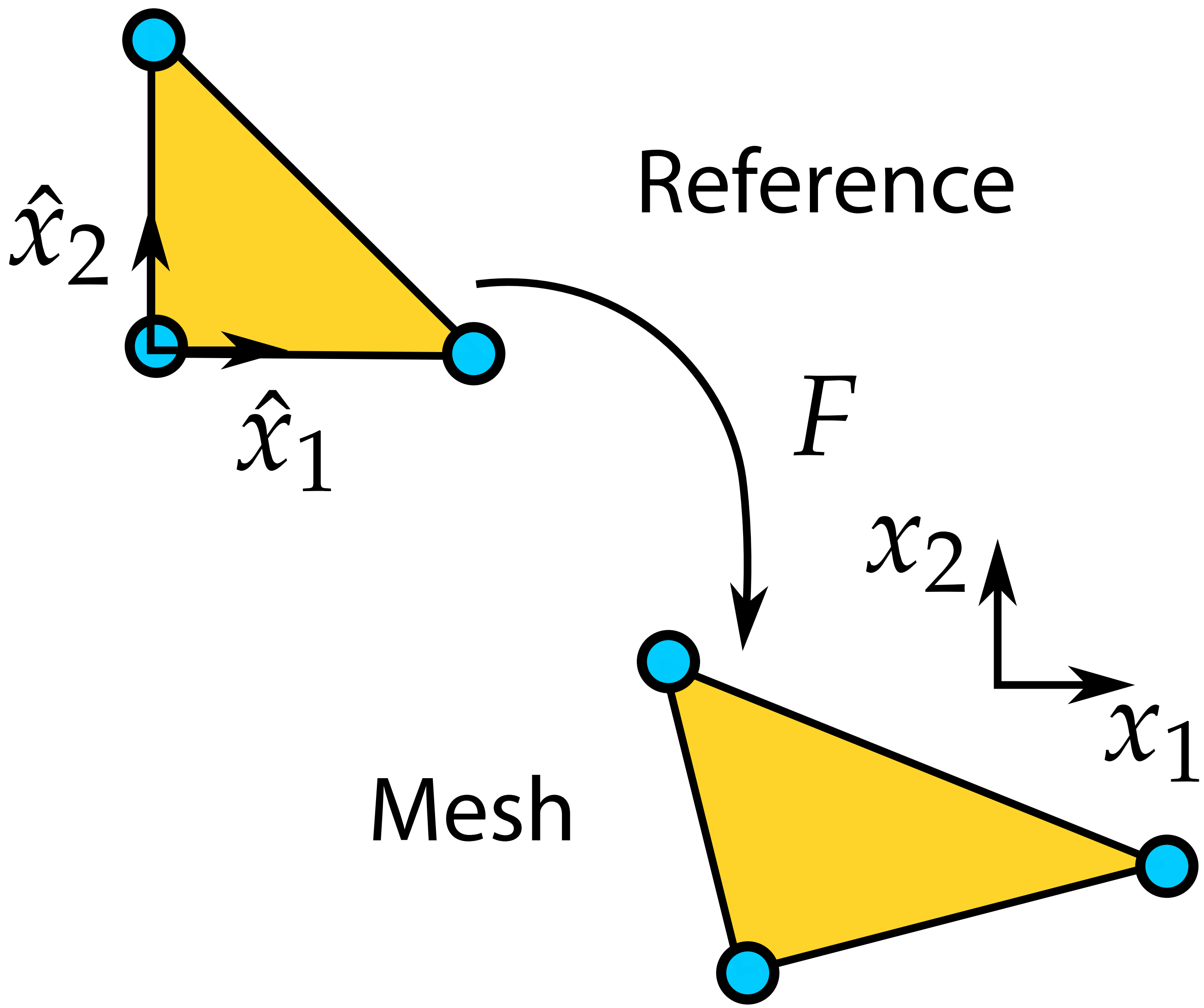


(c) Algorithm meshes the nodes.

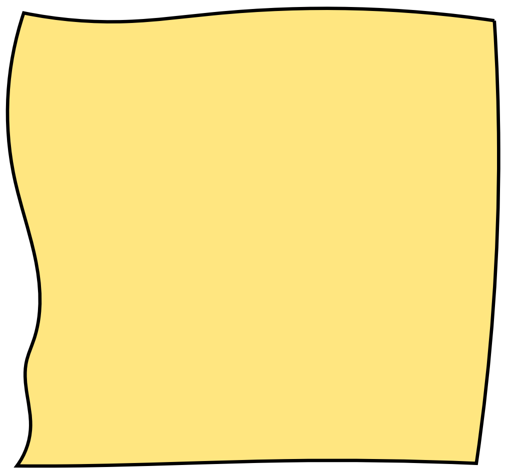


(d) The support and connectivity of each basis function is directly linked to the underlying mesh.

Finite Element Method



Meshfree-based Partition of Unity



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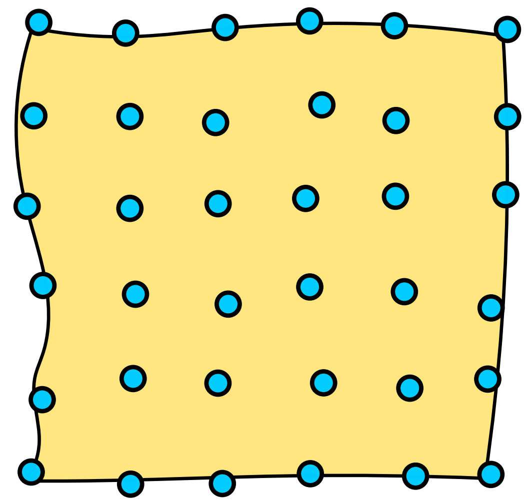

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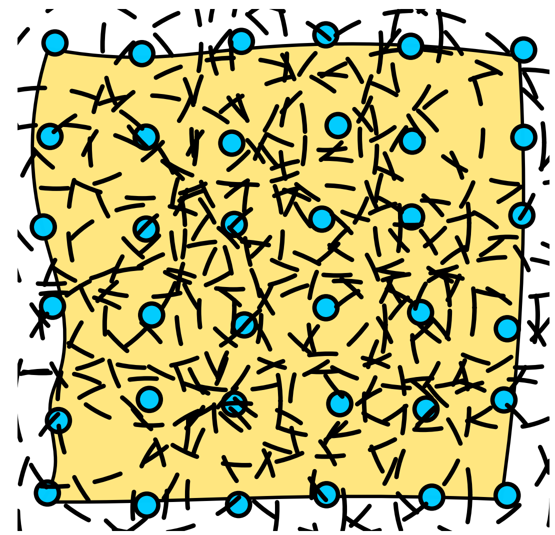
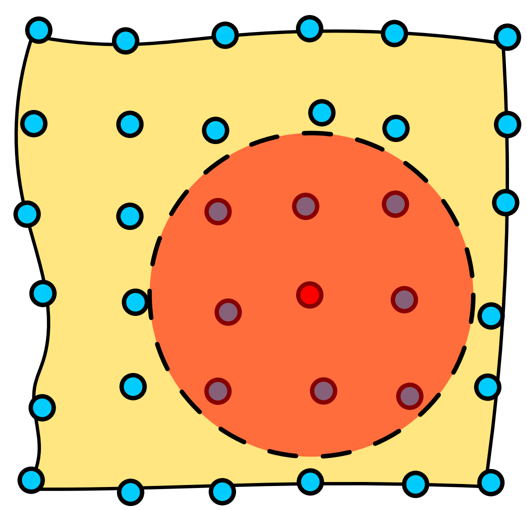


(c) Every node is given a support domain.



(d) The support and connectivity of each basis function is a natural consequence of the node positions and support domains.

Meshfree partition of unity is constructed in *global* space

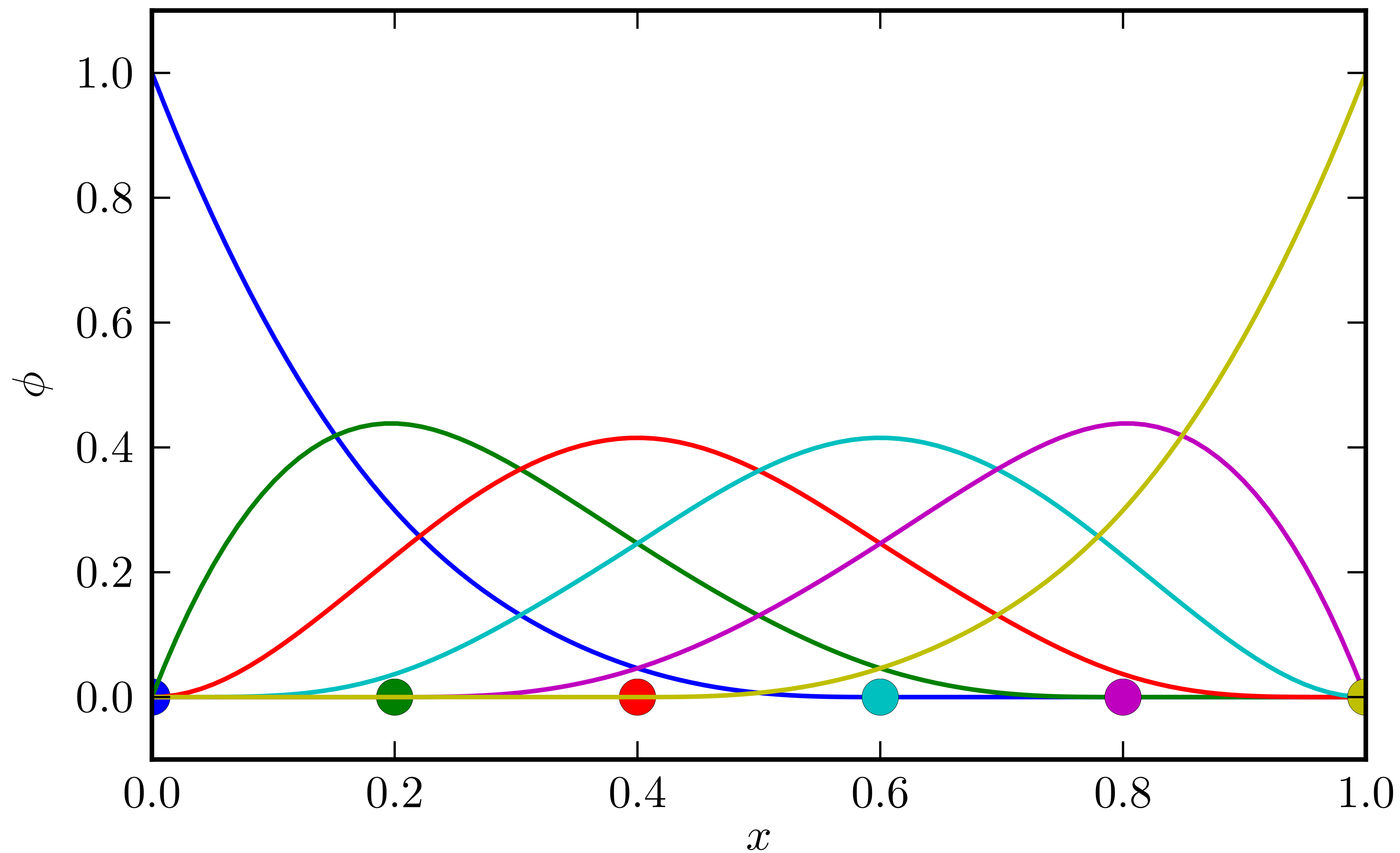
Good:

- improved approximation properties.
  - high-order continuity.
  - less dofs for given accuracy.
- **less sensitive to poor quality node distributions.**
- eases mesh generation.

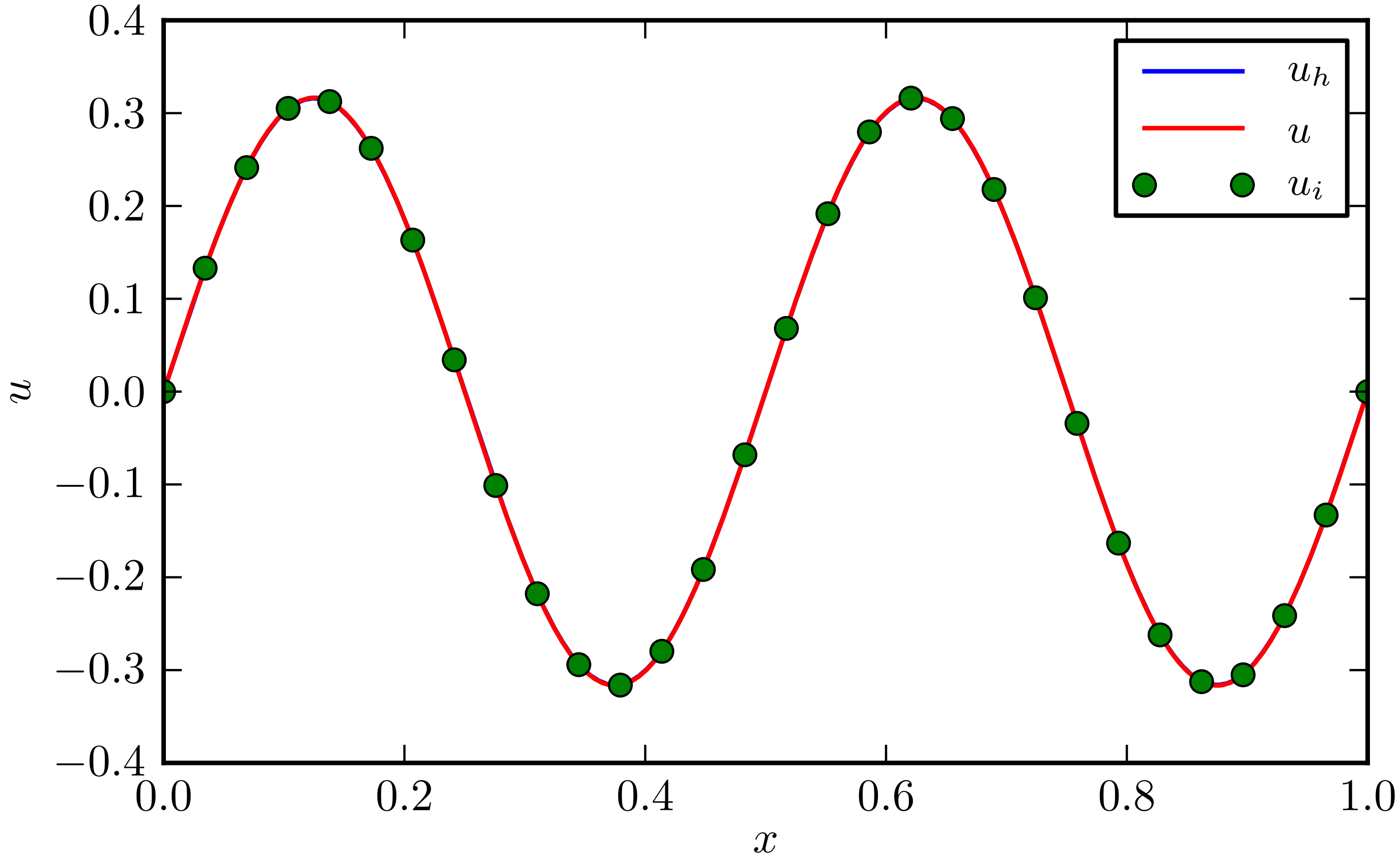
Not so good:

- expensive.
- extra flexibility also means extra complexity:
  - **integration.**
  - adaptivity/error measures.
  - mathematical proof.
  - **stability.**
  - **boundary conditions.**

Maximum-entropy basis functions

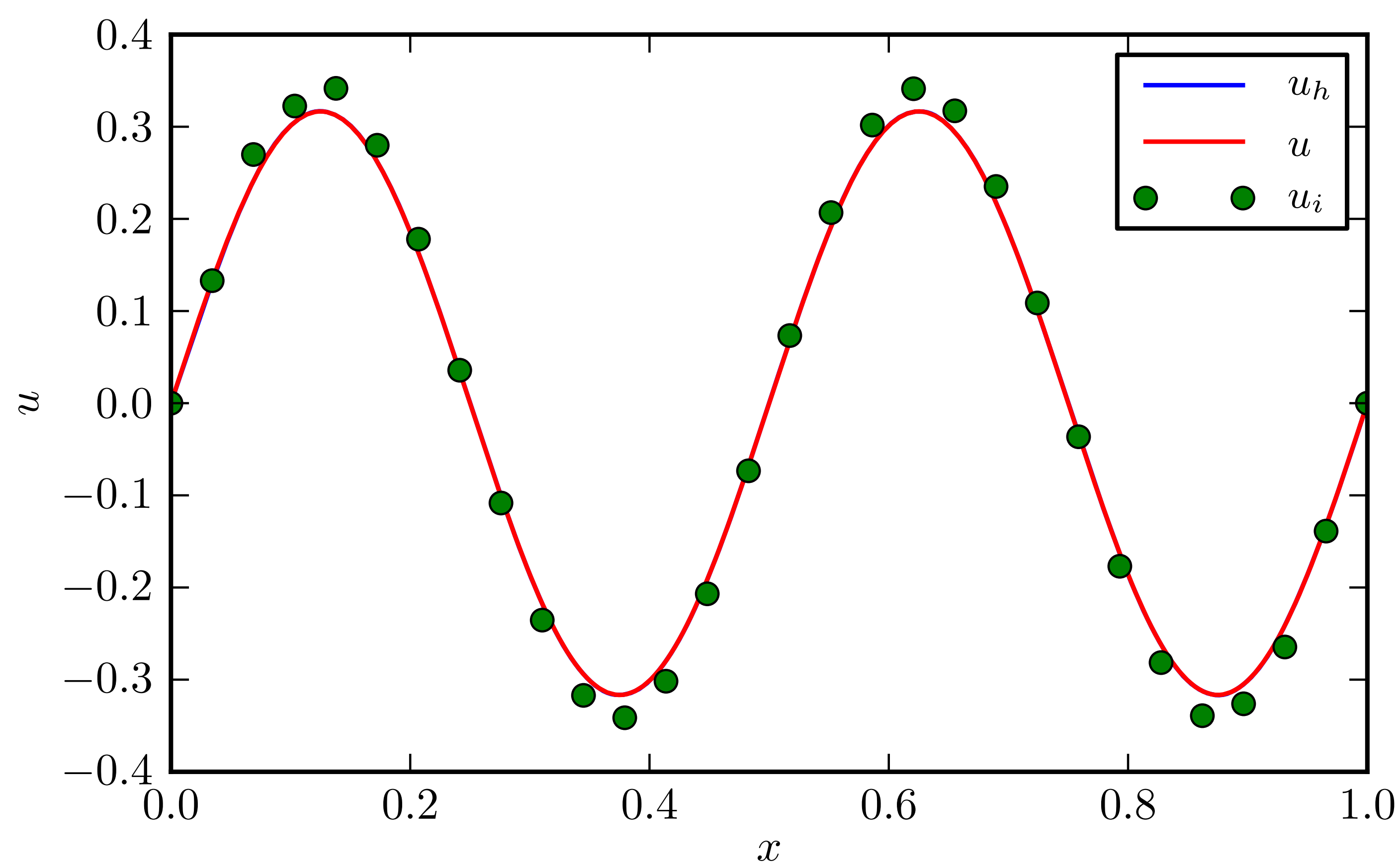


RBF

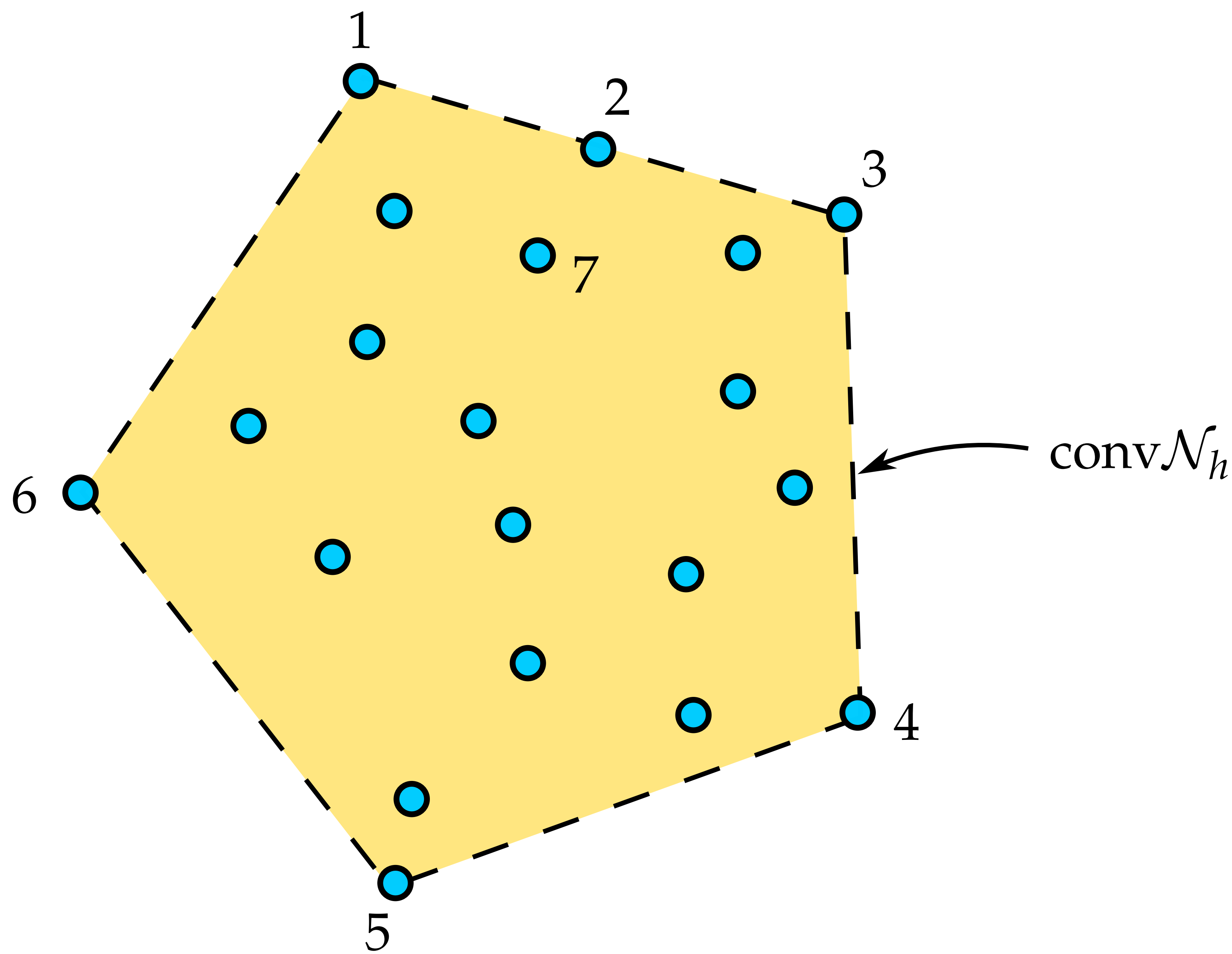


$$u_h(x) = \sum_{i=1}^{N} \phi_i u_i$$

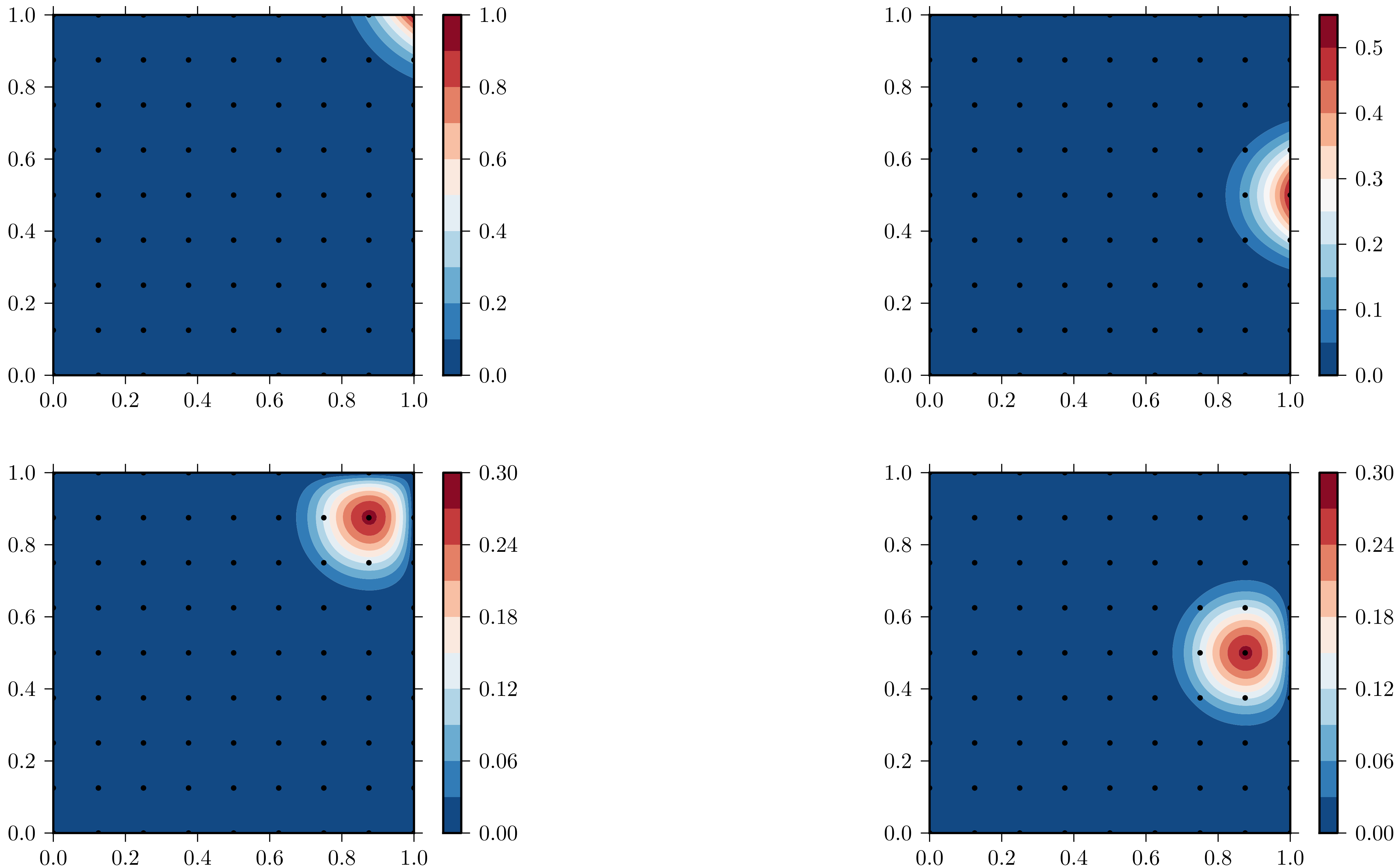
Maximum-Entropy



Maximum-entropy basis functions



Maximum-entropy basis functions



Maximum-entropy basis functions

All basis functions associated with interior nodes vanish on the convex hull.

$$u_h \in H_0^1(\Omega)$$

Vital for **easy** imposition of Dirichlet boundary conditions.

The problem.

Problems with small parameters crop up nearly everywhere!
Hyperelasticity
Incompressible fluid flow
Plates and Shells
Cosserat elasticity
and probably many more…

The problem:
numerical locking

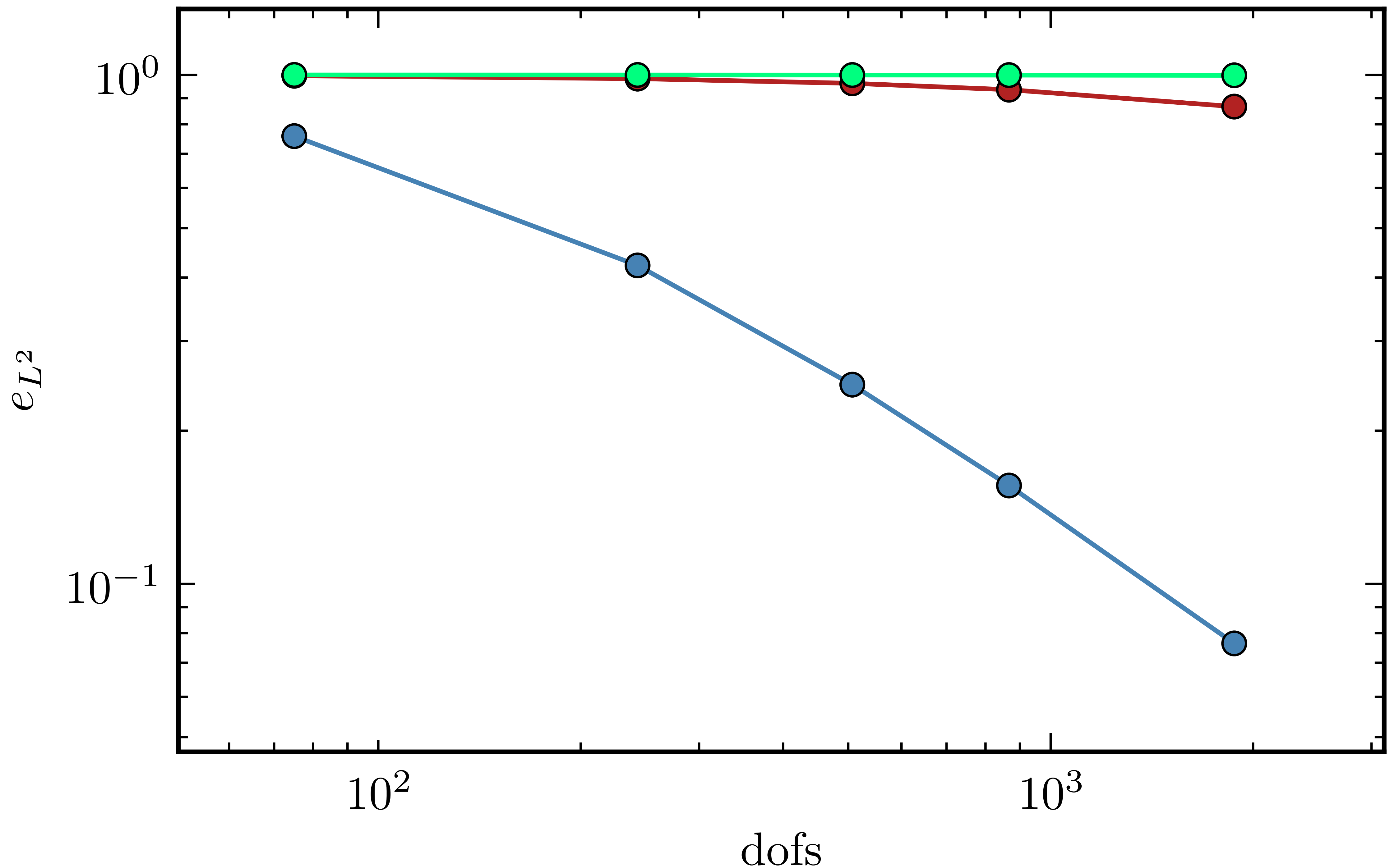
Nearly-incompressible elasticity

Find $u_h \in \mathcal{U}_h$ such that

$$\int_\Omega \mathbf{C}\epsilon(\mathbf{u}_h) : \epsilon(\mathbf{v}) \; d\Omega = \int_\Gamma \mathbf{f} \cdot \mathbf{v} \; d\Omega \quad \forall v \in H_0^1(\Omega)$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

Locking

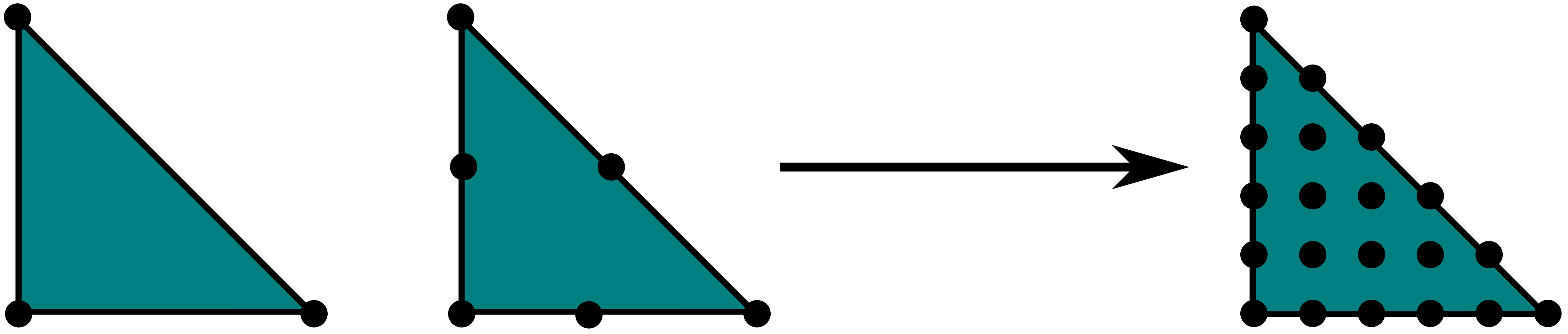


Locking

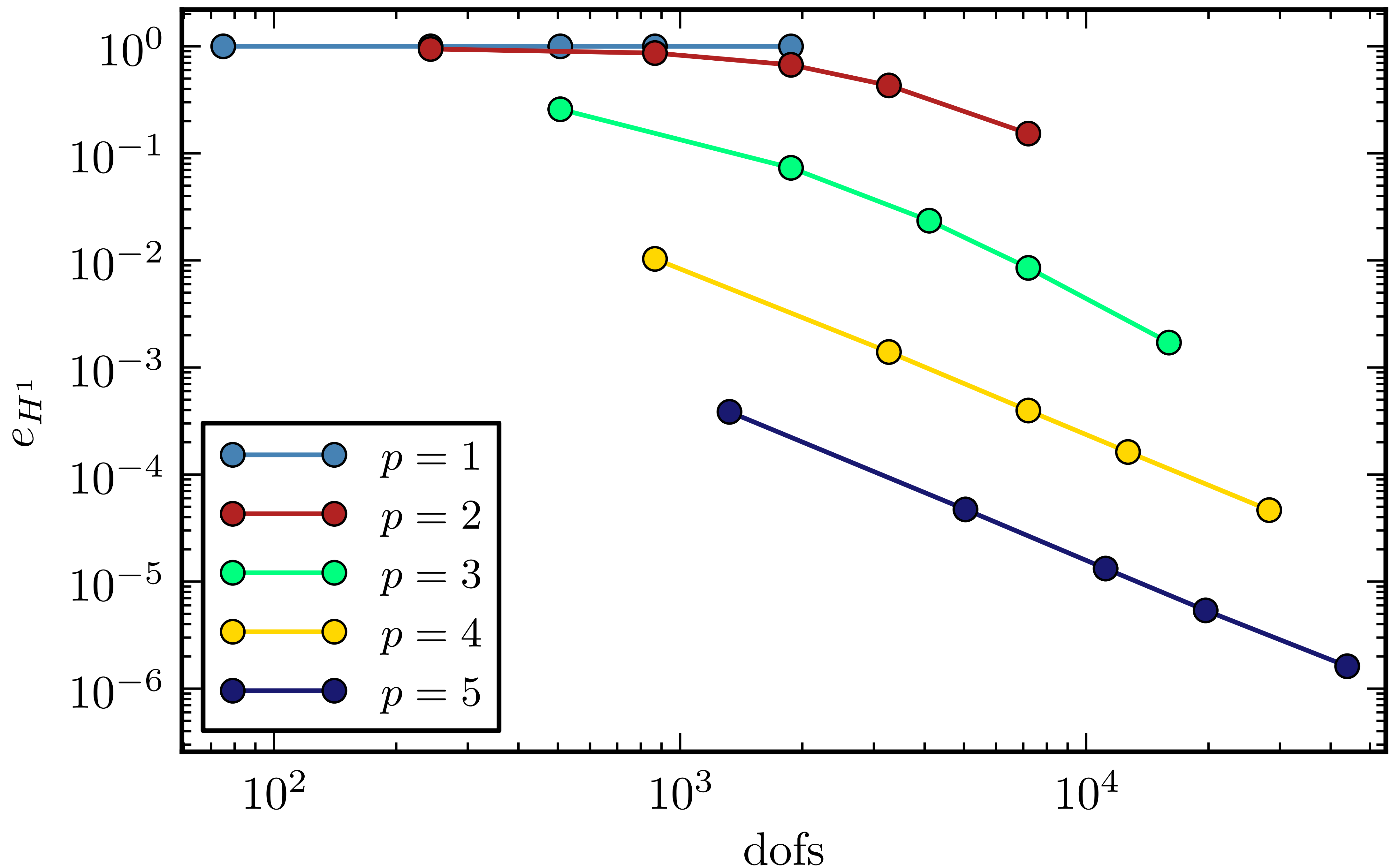
Inability of the basis functions to satisfy the constraint imposed whilst still having adequate approximation properties

$$||u - u_h|| \leq C(\lambda) h^p ||u''||$$

p-refinement



p-refinement



Mixed variational formulation

Find $\mathbf{u}_h \in \mathcal{P}_h$ and $p_h \in \mathcal{P}_h$ such that:

$$2\mu \int_\Omega \epsilon(\mathbf{u}_h) \cdot \epsilon(\mathbf{v}) \, d\Omega + \int_\Omega p_h \nabla \cdot \mathbf{v} \, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\Omega \qquad \forall \mathbf{v} \in \mathcal{U}_h$$

$$\int_\Omega \nabla \cdot \mathbf{u}_h \; q \, d\Omega - \frac{1}{\lambda} \int_\Omega p_h q \, d\Omega = 0 \qquad \forall q \in \mathcal{P}_h$$

Looking better

Great, so the problem is solved right?

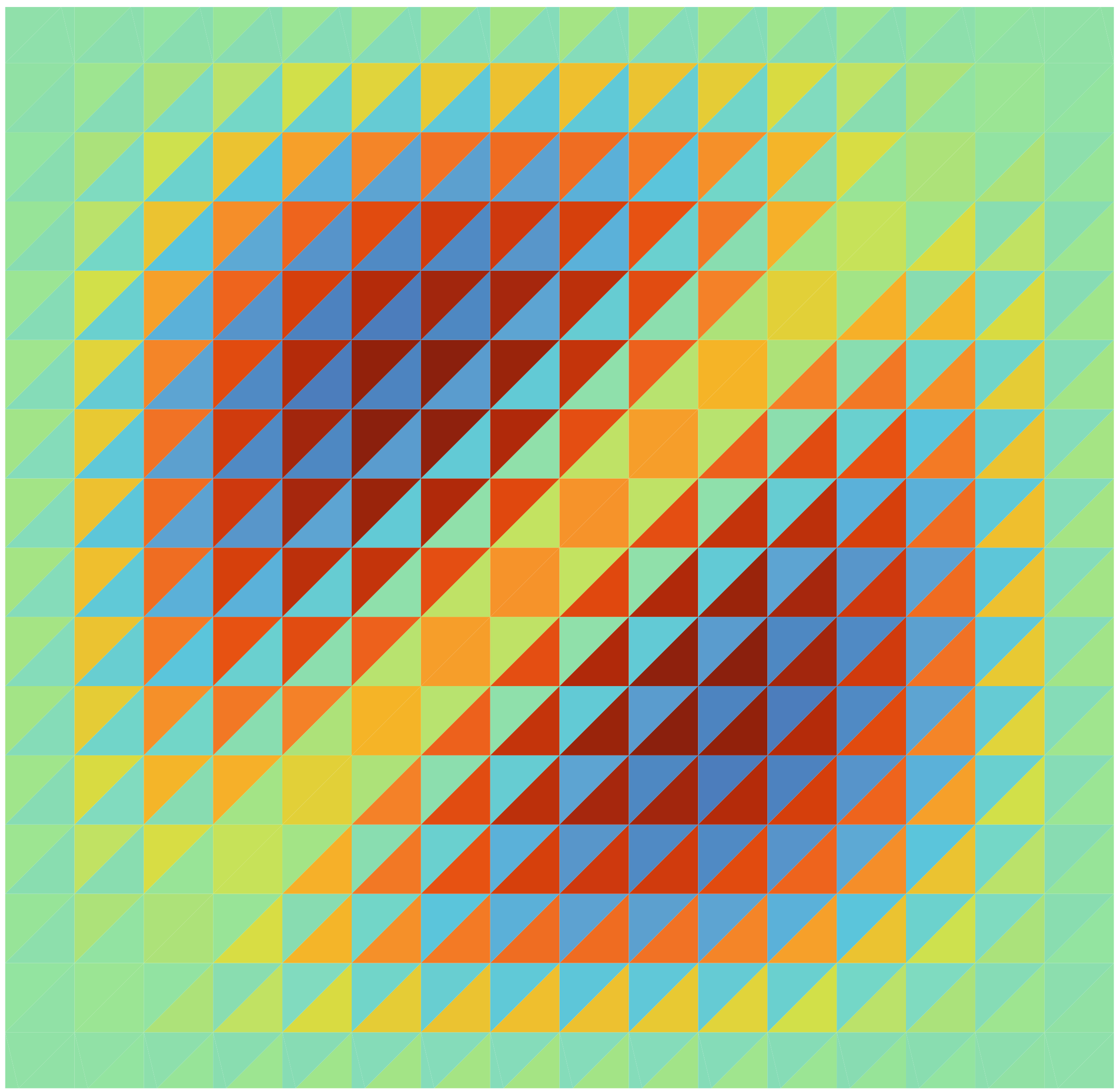
LBB Stability

$$\inf_{q \in \mathcal{P}_h} \sup_{v \in \mathcal{U}_h} \frac{\int_\Omega q \nabla \cdot \mathbf{v} d\Omega}{||v||_1 ||q||_{0/\mathbb{R}}} \geq \beta_h > 0$$

A question of **balance**

$$\mathcal{U}_h \qquad \qquad \mathcal{P}_h$$

Stability



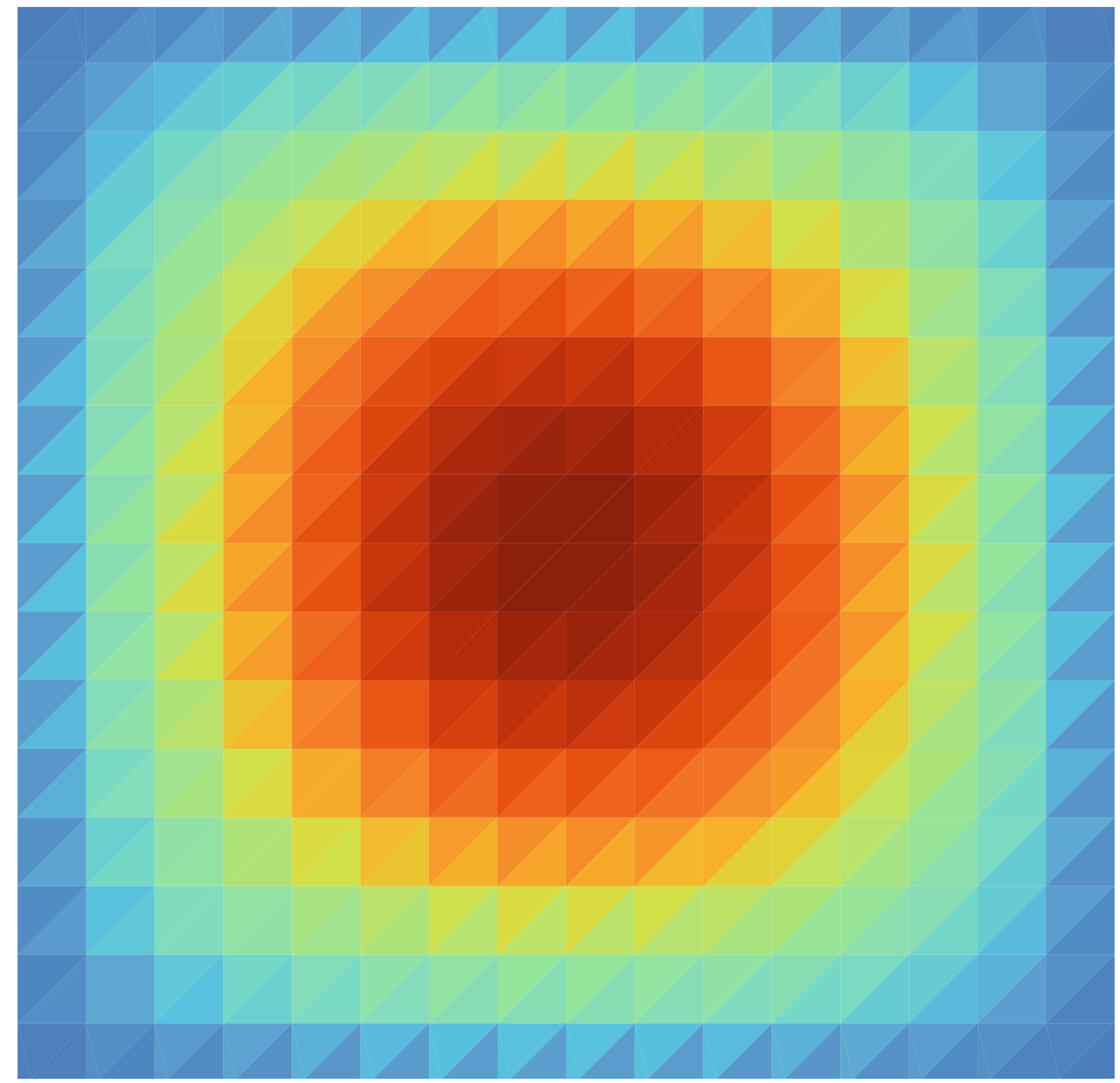
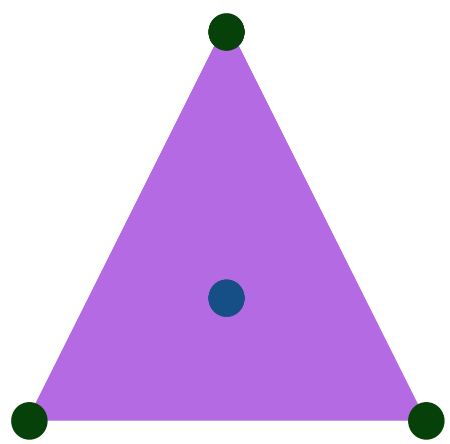


Image: Marie E. Rognes

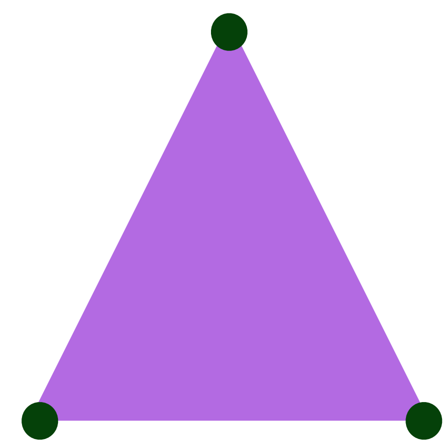
MINI element

Linear + Bubble



$\mathcal{U}_h$

Linear



$\mathcal{P}_h$

Arnold, Brezzi, Fortin 1984

The Volume Averaged Nodal Pressure Method (VANP)

Questions I will answer…

1. Can we produce a stable pair of spaces for the mixed formulation using *meshfree* approximation schemes?

2. Can we *eliminate* the pressure space to produce a generalised displacement method?

3. Can we produce a *general* scheme, which works for arbitrary spaces of meshfree basis functions and even finite element basis functions?

4. How does enrichment, in the manner of the MINI bubble, affect the convergence and stability?

5. Does it work in 3D?

Questions I can answer….

- Exactly how to construct meshfree basis functions, including high-order reproducing radial basis functions and maximum-entropy basis functions.

- How to accurately integrate the weak form.

- How to implement the method in algorithmic form.

- And of course any other questions you may have…

1. Can we produce a stable pair of spaces for the mixed formulation using *meshfree* approximation schemes?

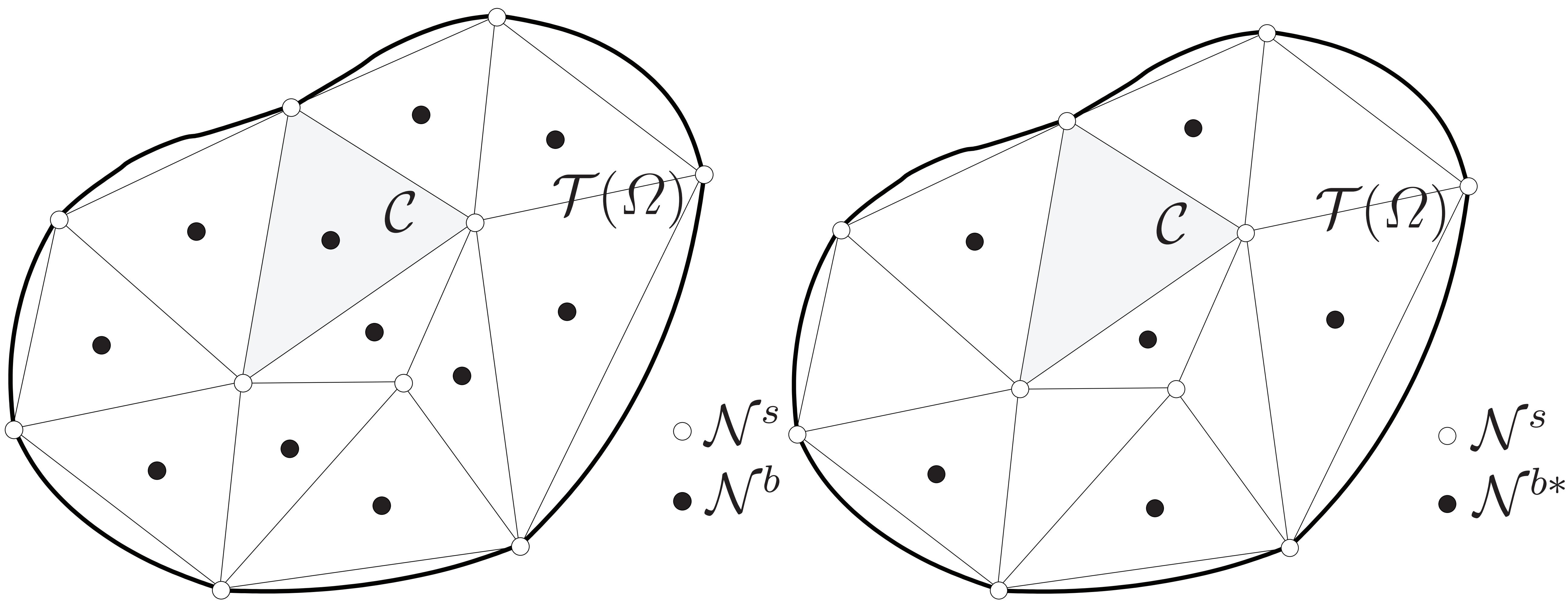


Fig. 1: Schematic representation of a two-dimensional simplicial tessellation for the `VANP` method. (a) Enhanced node set $\mathcal{N}^+$, and (b) enhanced node set $\mathcal{N}^*$.

Spaces

$$\mathcal{U}_h := [ME(\Omega; \mathcal{N}_h, \rho)]^2$$

$$\boldsymbol{u}_h(\boldsymbol{x}) = \sum_{i=1}^{N} \boldsymbol{\phi_i} \boldsymbol{u}_i$$

$$\mathcal{P}_h := CG_1(\Omega; \mathcal{T}_h)$$

For simplicity of the exposition, not required

$$p_h = \sum_{i=1}^{M} N_i p_i$$

Saddle-point problem

$$\int_{\Omega} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} \; d\Omega \; \boldsymbol{u} + \int_{\Omega} \boldsymbol{B}^T \mathbf{m} \boldsymbol{N}_p \; d\Omega \; \boldsymbol{p} = \int_{\Omega} \boldsymbol{\Phi}_{\boldsymbol{u}}^T \boldsymbol{f} \; d\Omega$$

$$\int_{\Omega} \boldsymbol{N}_p^T \mathbf{m}^T \boldsymbol{B} \; d\Omega \; \boldsymbol{u} - \frac{1}{\lambda_e} \int_{\Omega} \boldsymbol{N}_p^T \boldsymbol{N}_p \; d\Omega \; \boldsymbol{p} = 0$$

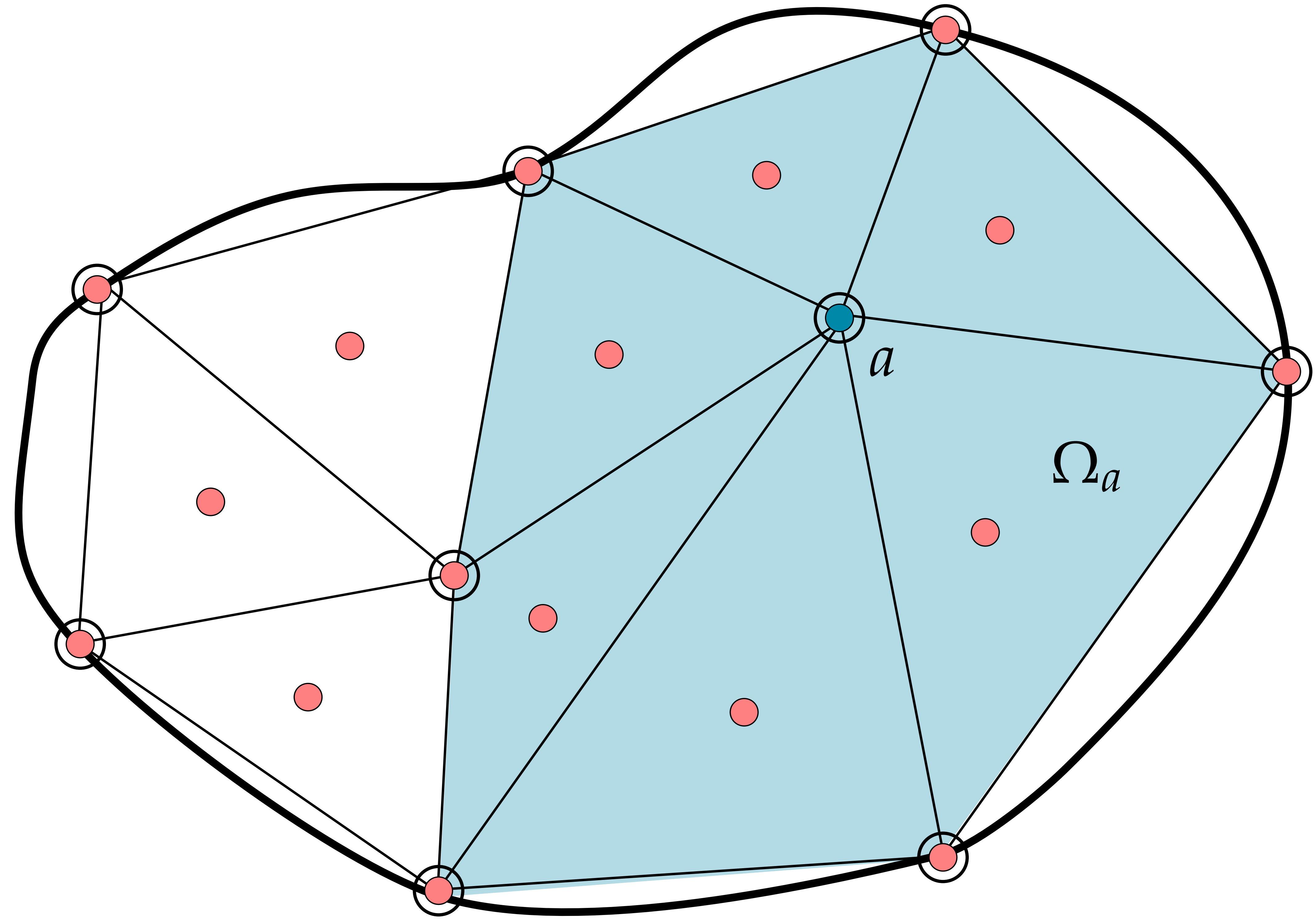
$$\mathbf{B} = \begin{bmatrix} \frac{\partial \phi_{u_x}}{\partial x_1} & 0 \\ 0 & \frac{\partial \phi_{u_y}}{\partial x_2} \\ \frac{\partial \phi_{u_x}}{\partial x_2} & \frac{\partial \phi_{u_y}}{\partial x_1} \end{bmatrix} \quad \mathbf{m} = \begin{Bmatrix} 1 & 1 & 0 \end{Bmatrix}^T \quad \mathbf{D} = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

2. Can we *eliminate* the pressure space to produce a generalised displacement method?

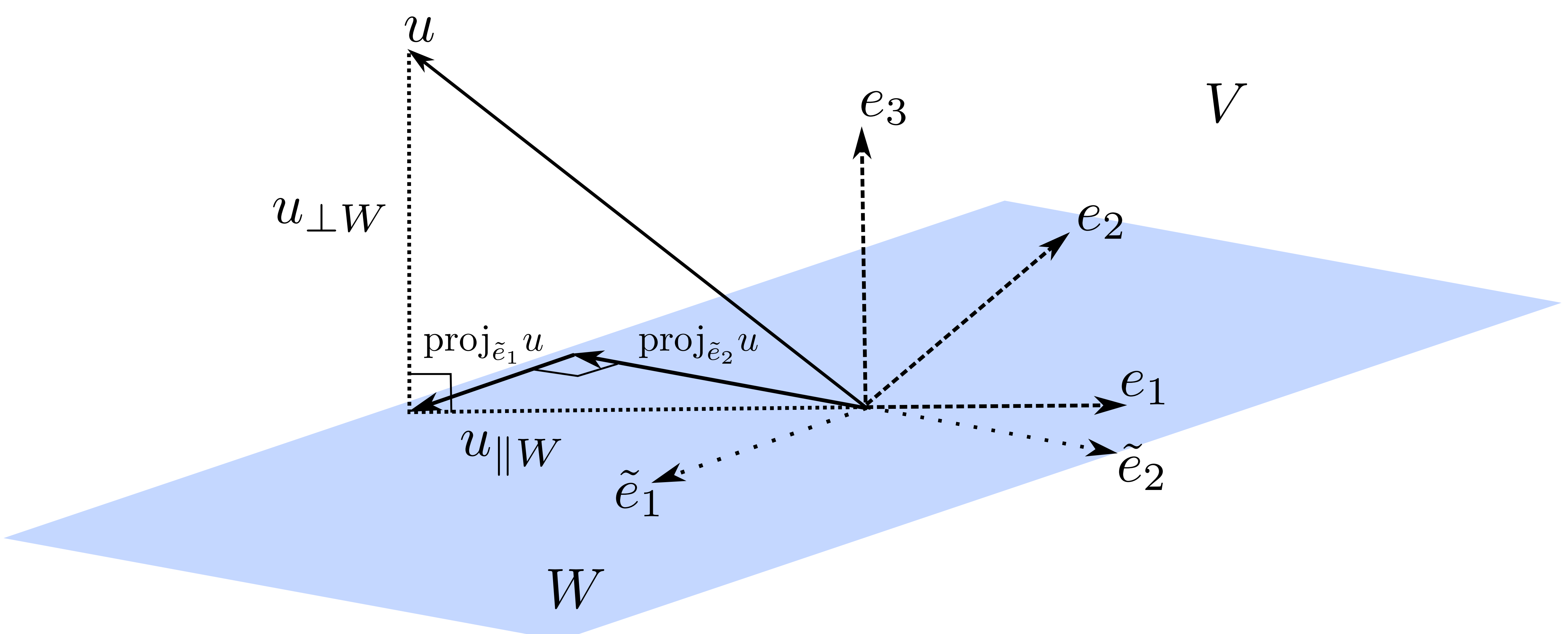
Saddle-point problem

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & -\mathbf{C} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{0} \end{Bmatrix}$$

How can we get rid of p?


a
Ω_a

Projection



u
V
e_3
$u_{\perp W}$
e_2
$\mathrm{proj}_{\tilde{e}_1} u$
$\mathrm{proj}_{\tilde{e}_2} u$
e_1
$u_{\parallel W}$
$\tilde{e}_2$
$\tilde{e}_1$
W

Many methods include a projection or ‘softening’

Enhanced Assumed Strains (EAS)

Reduced Integration

Smoothed Finite Element Method (SFEM)

Mixed Interpolation of Tensorial Components

For every pressure node:

$$\sum_{b=1}^{N} \int_{\Omega} \mathbf{N}_{pa} \mathbf{m}^T \mathbf{B}_b \, d\Omega \, \mathbf{u}_b - \frac{1}{\lambda} \sum_{a=1}^{M} \int_{\Omega} \mathbf{N}_{pa} \, d\Omega \, p_a = 0$$

Restrict integration domain to local domain:

$$\sum_{b=1}^{N} \int_{\Omega_a} N_{pa} \mathbf{m}^T \mathbf{B}_b \, d\Omega \, \mathbf{u}_b - \frac{1}{\lambda} \sum_{a=1}^{M} \int_{\Omega_a} N_{pa} \, d\Omega \, p_a = 0$$

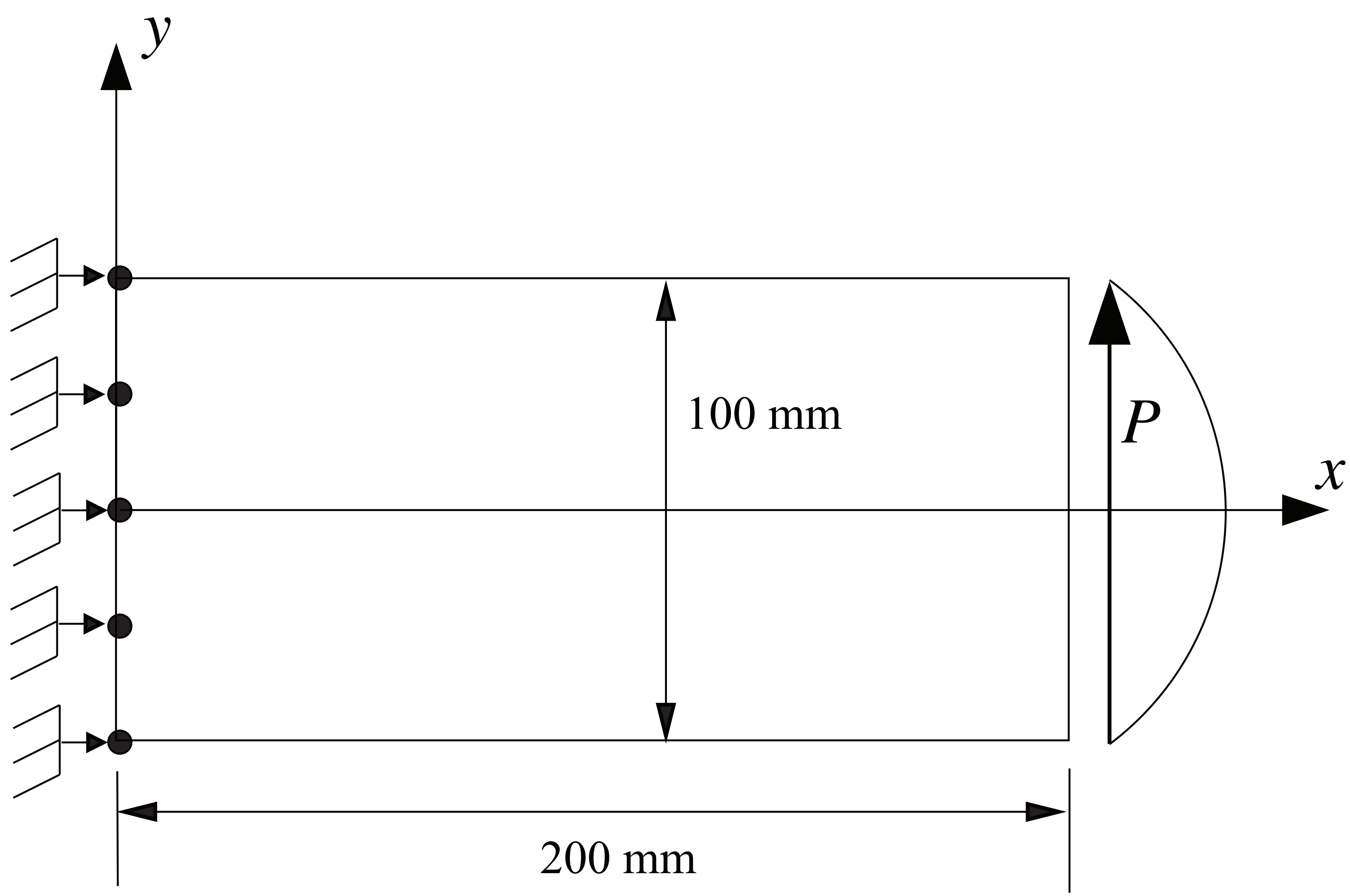
Re-arrange to get equation for every pressure dof:

$$p_a = -\lambda \sum_{b=1}^{N} \left\{ \frac{\int_{\Omega_a} N_{pa} \mathbf{m}^T \mathbf{B}_b \, d\Omega}{\int_{\Omega_a} N_{pa} \, d\Omega} \right\} \mathbf{u}_b$$

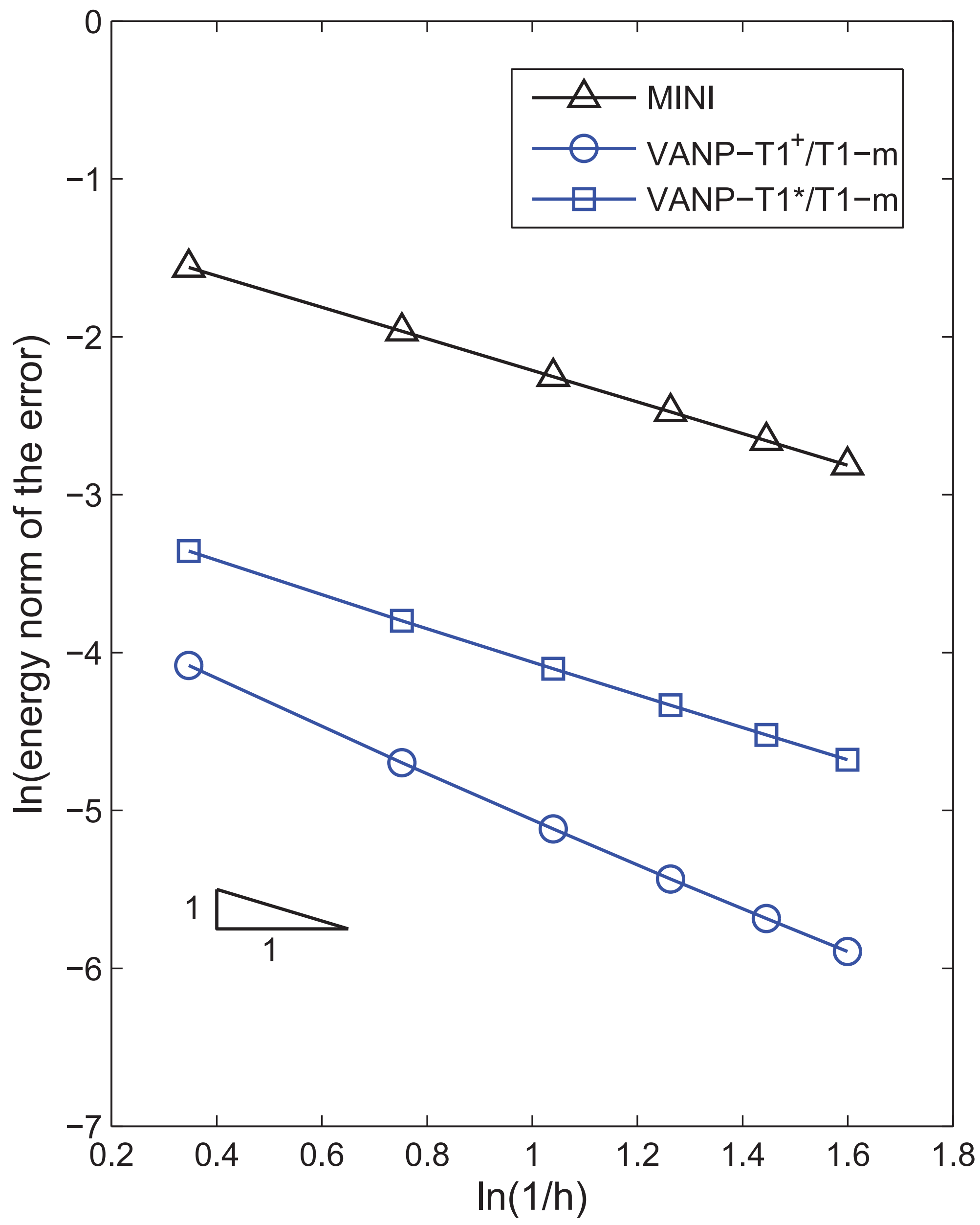
Results

1. Can we produce a stable pair of spaces for the mixed formulation using *meshfree* approximation schemes?

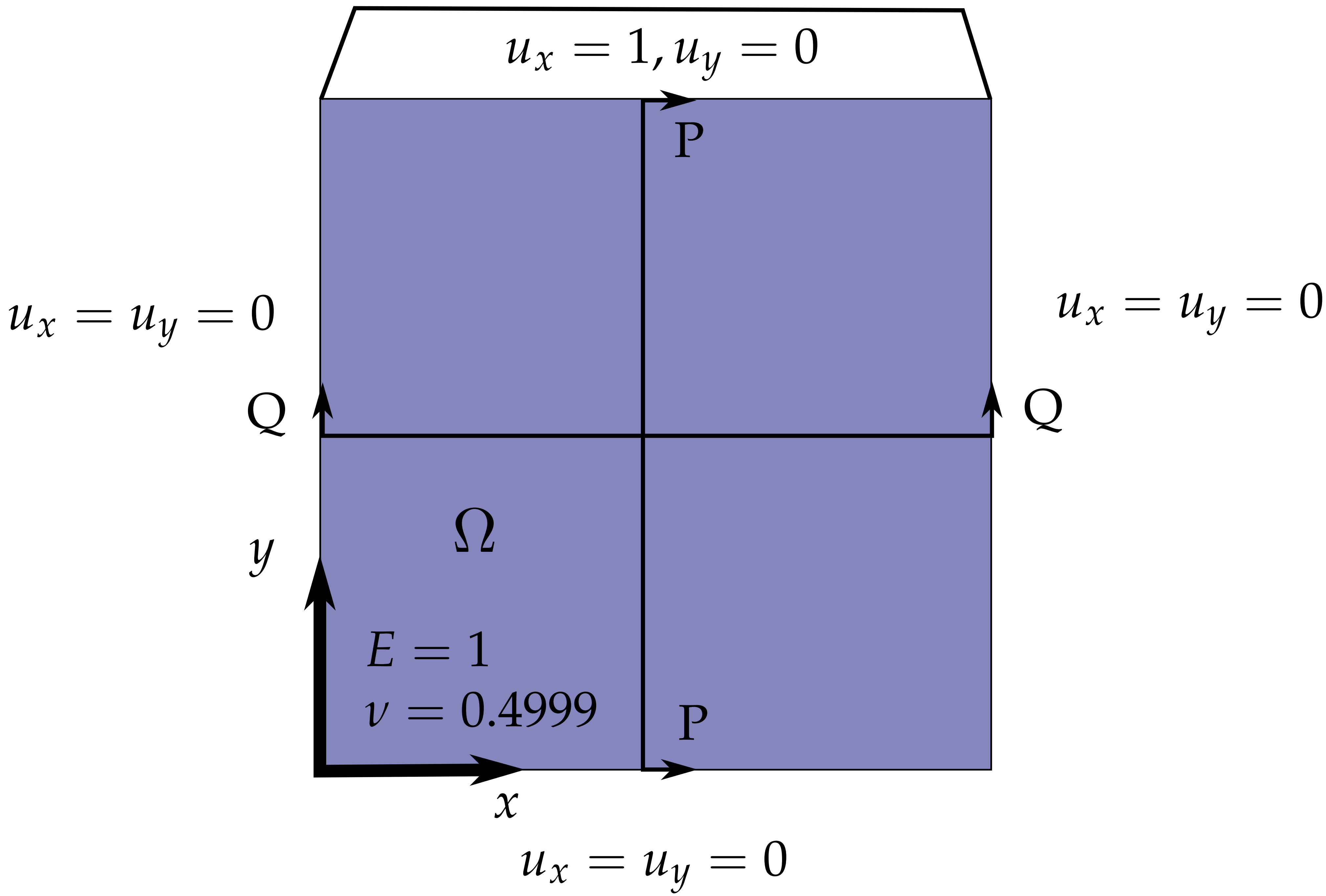
Timoshenko Beam

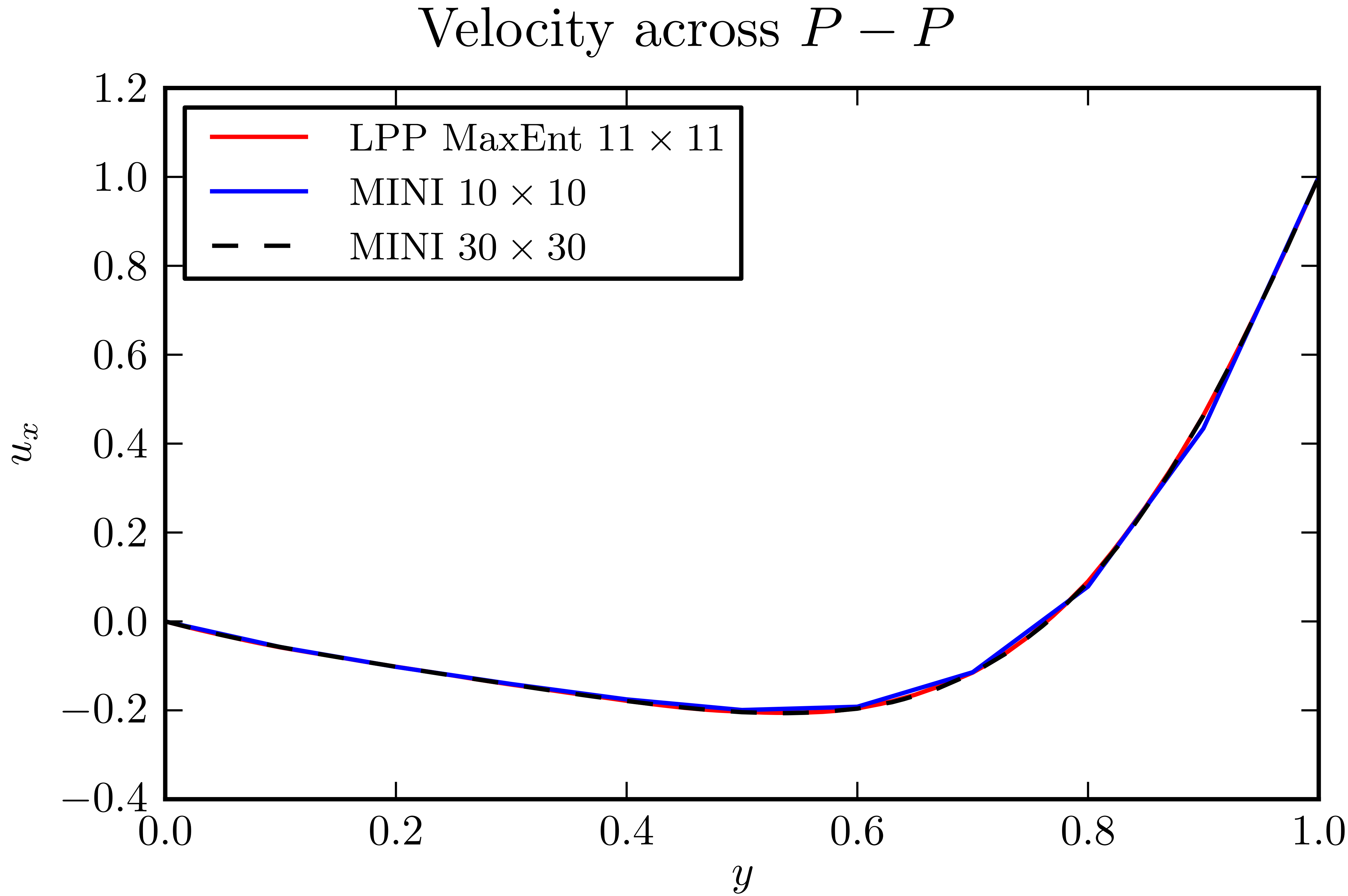


Timoshenko Beam

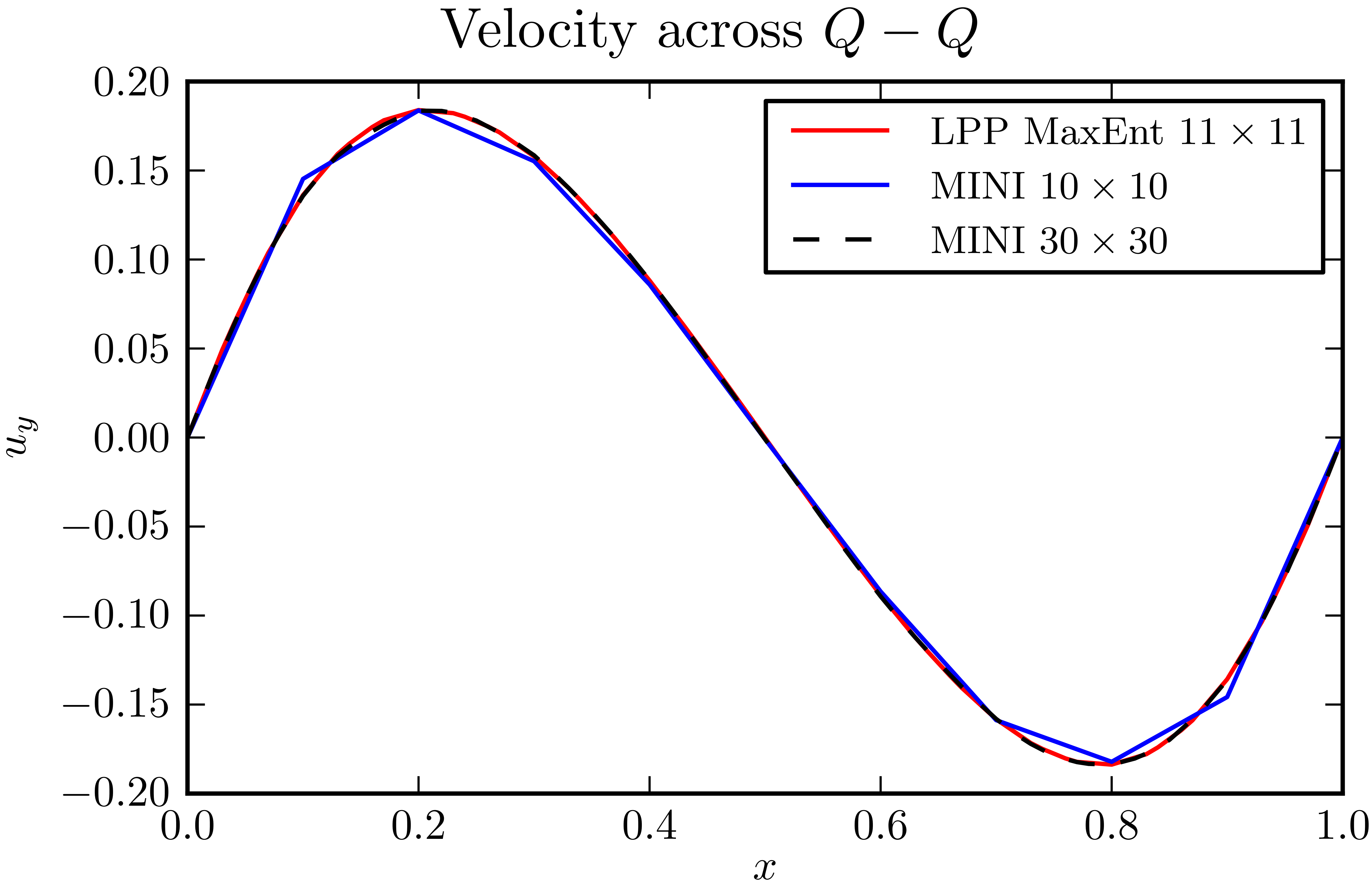


Leaky-lid cavity flow





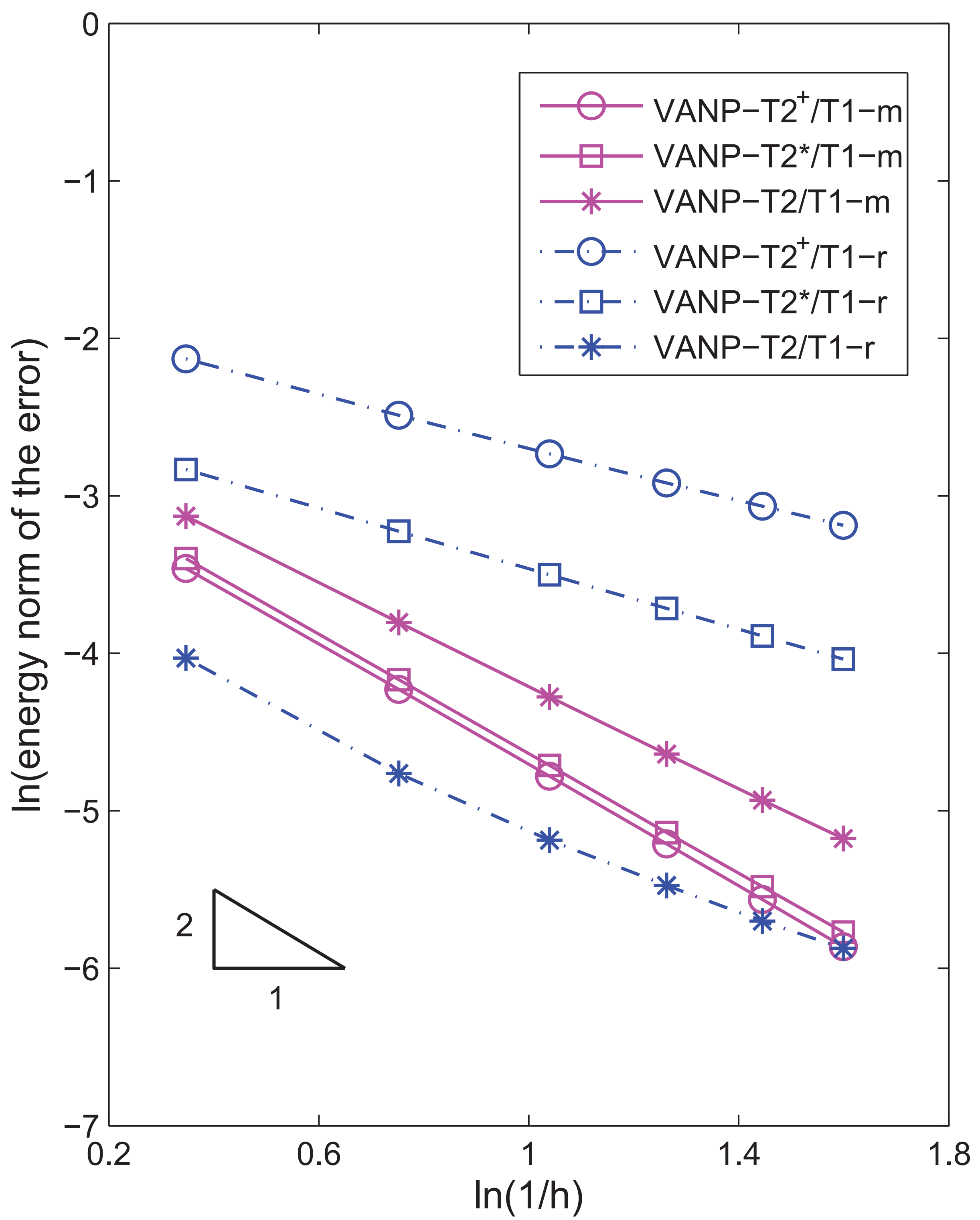
Velocity across $P - P$
LPP MaxEnt 11×11
MINI 10×10
MINI 30×30
u_x
y
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0.0
0.2
0.4
0.6
0.8
1.0



Velocity across $Q - Q$
0.20
0.15
0.10
0.05
0.00
−0.05
−0.10
−0.15
−0.20
u_y
0.0
0.2
0.4
0.6
0.8
1.0
x
LPP MaxEnt 11×11
MINI 10×10
MINI 30×30

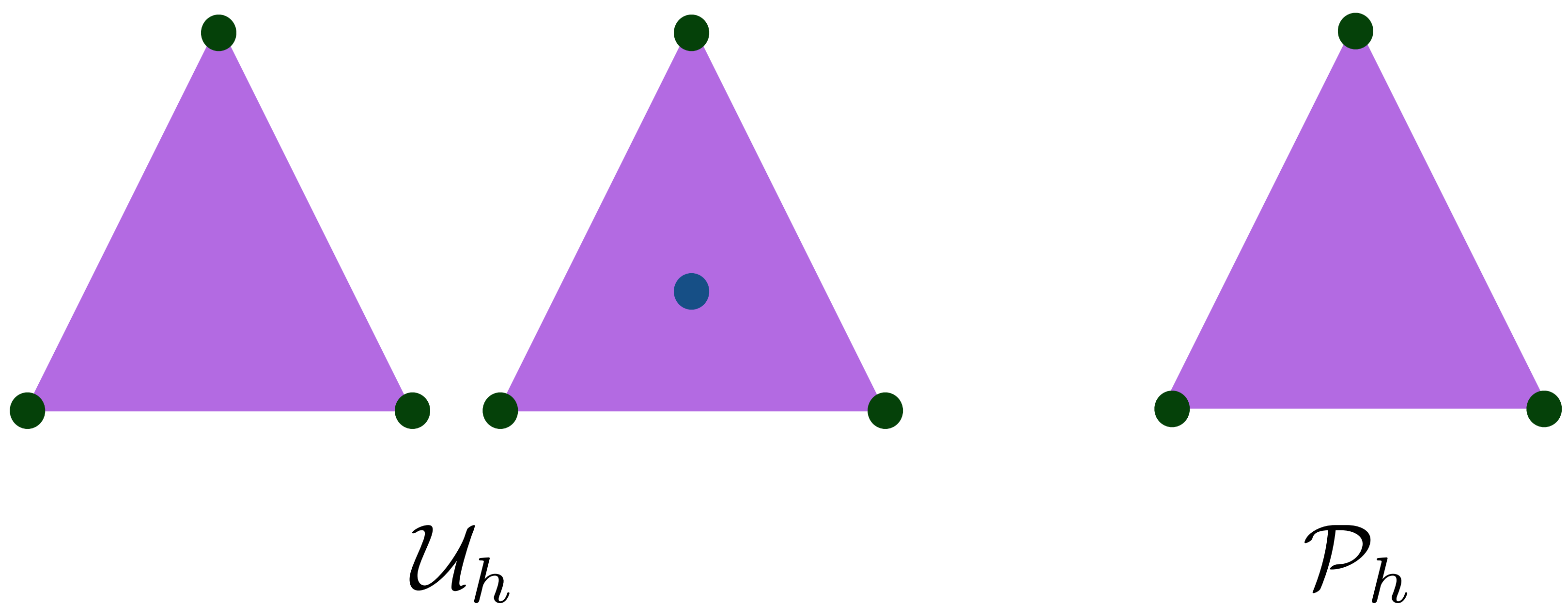
3. Can we produce a *general* scheme, which works for arbitrary spaces of meshfree basis functions?

Timoshenko Beam

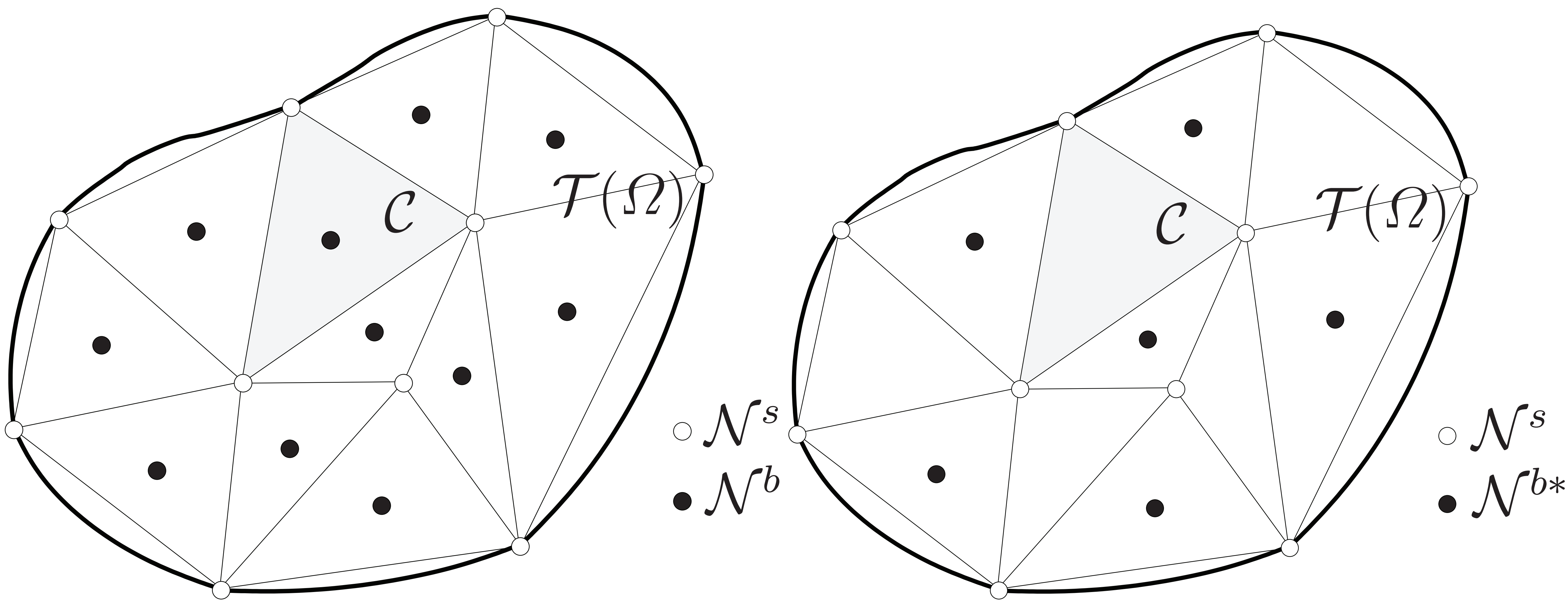


4. How does enrichment, in the manner of the MINI bubble, affect the convergence and stability?

MINI* element



Kim and Lee, 2000, 10.1023/A:1018973303935



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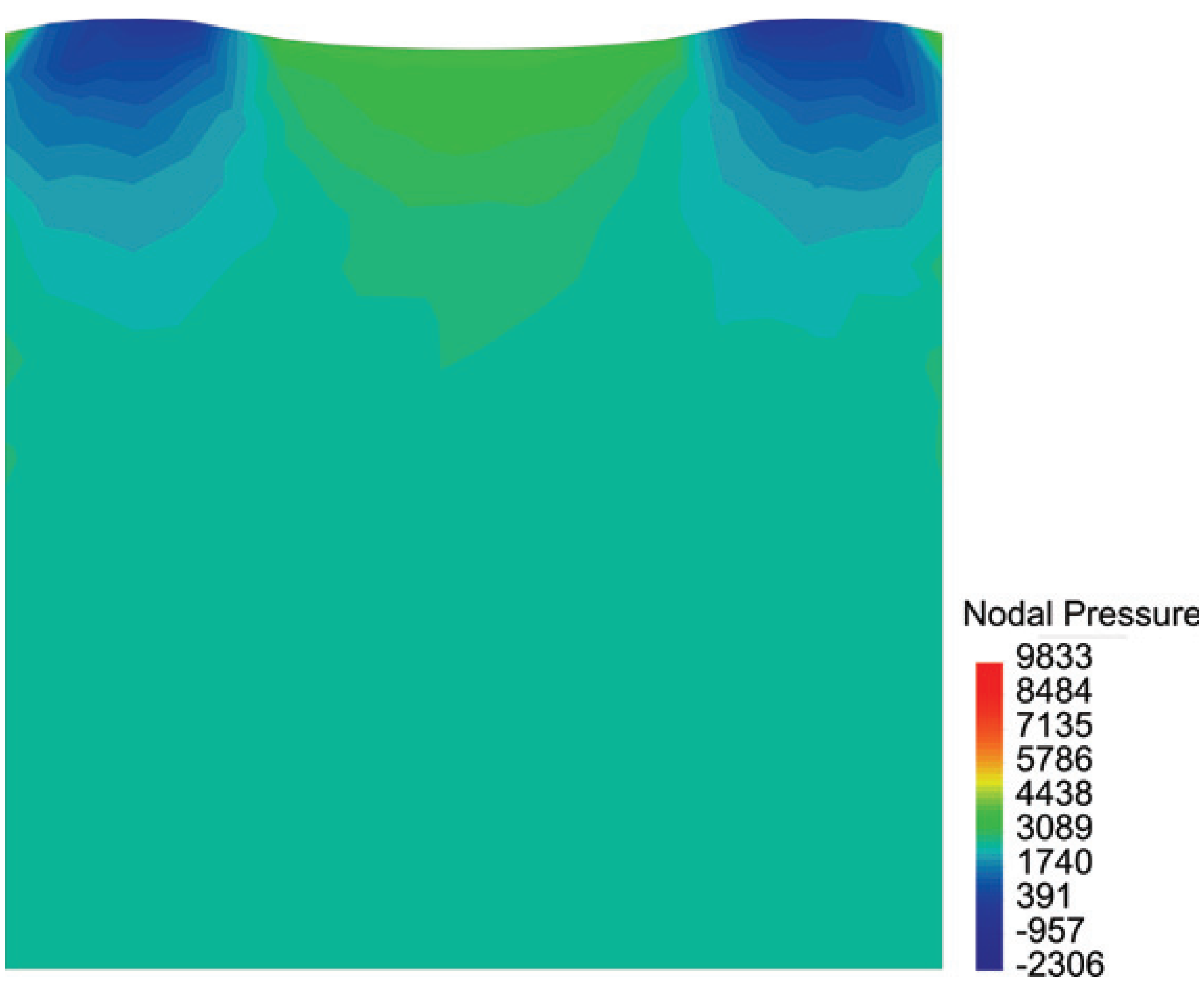
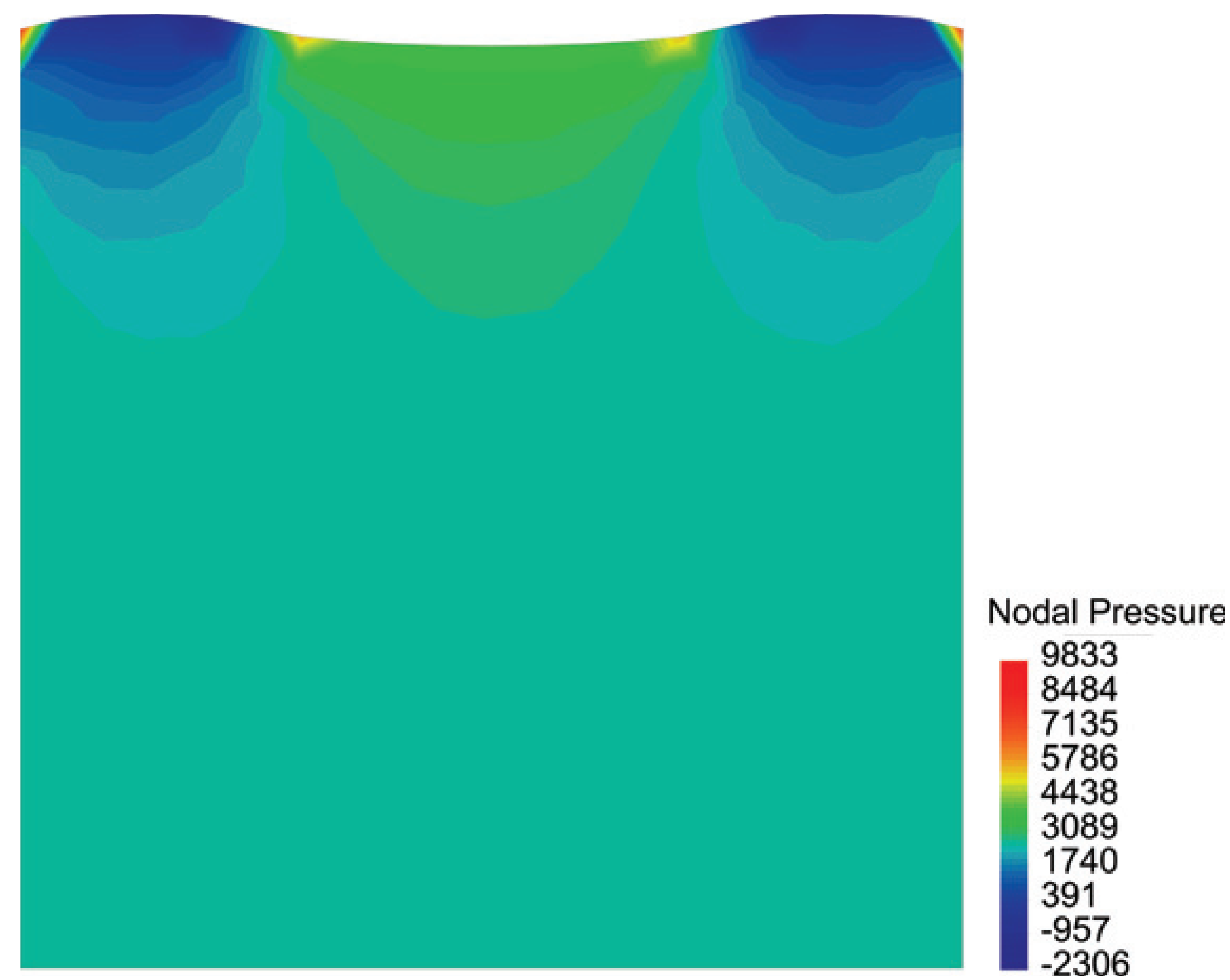
Constrained Block

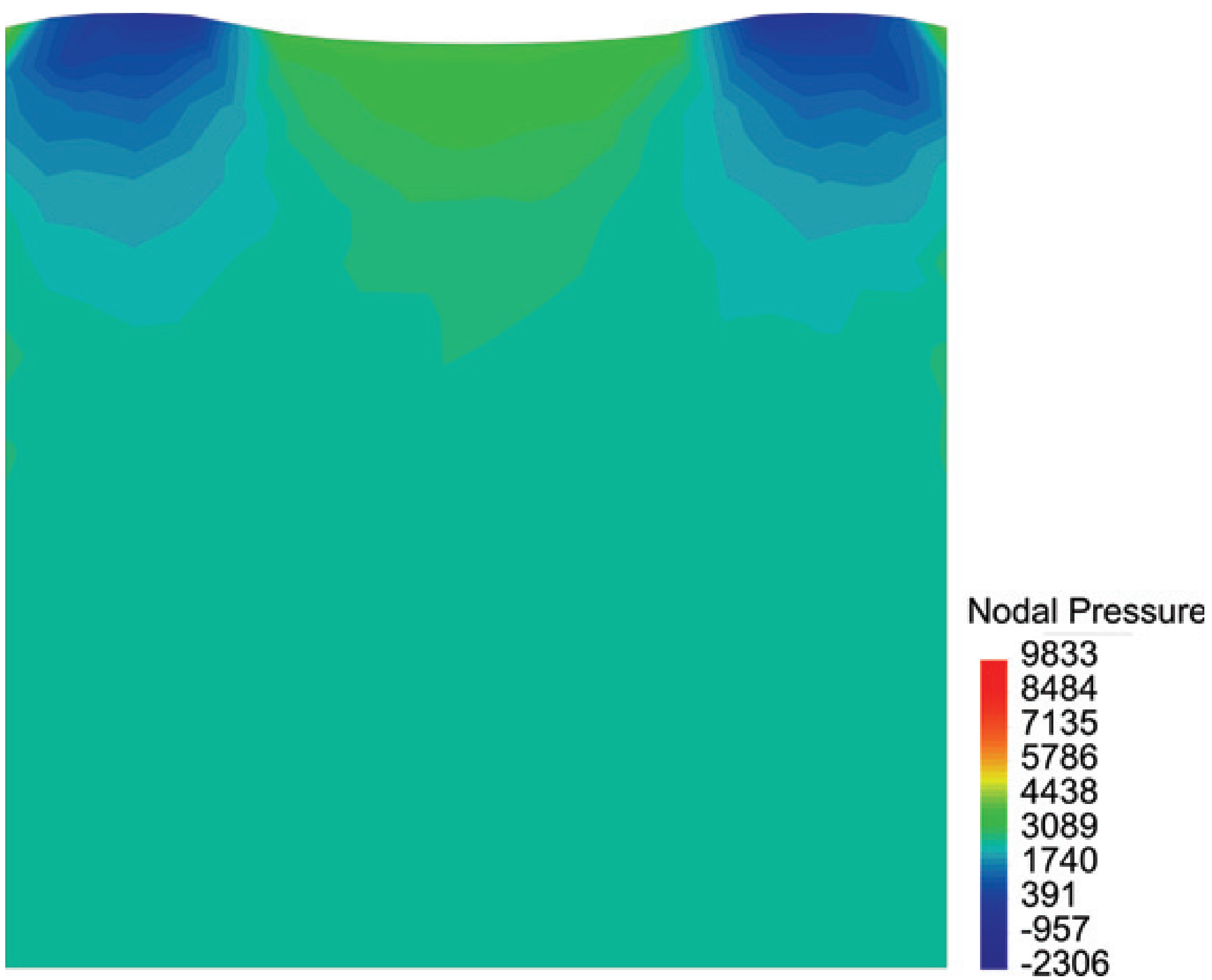


MINI

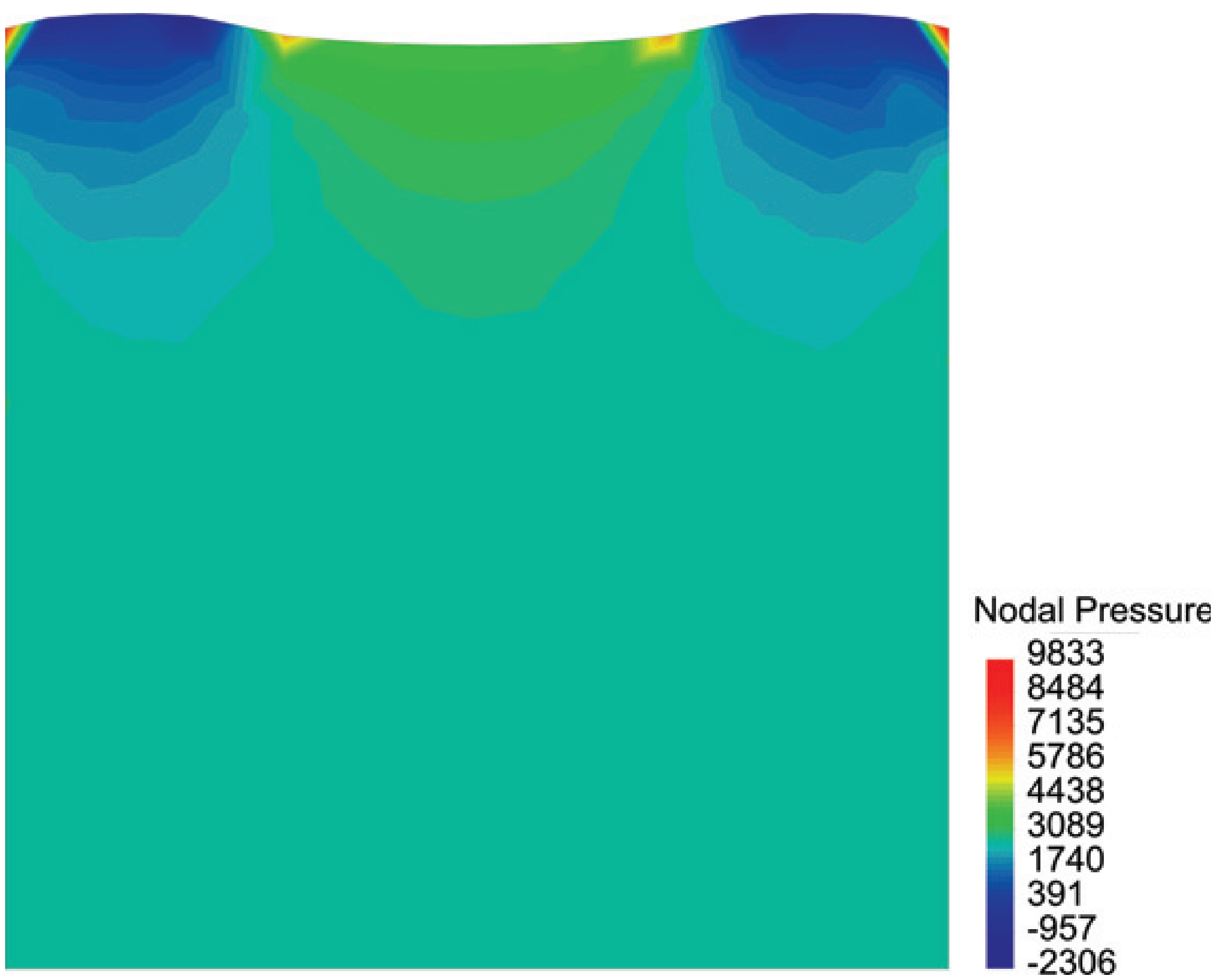


First-order MaxEnt Full Bubbles

Constrained Block

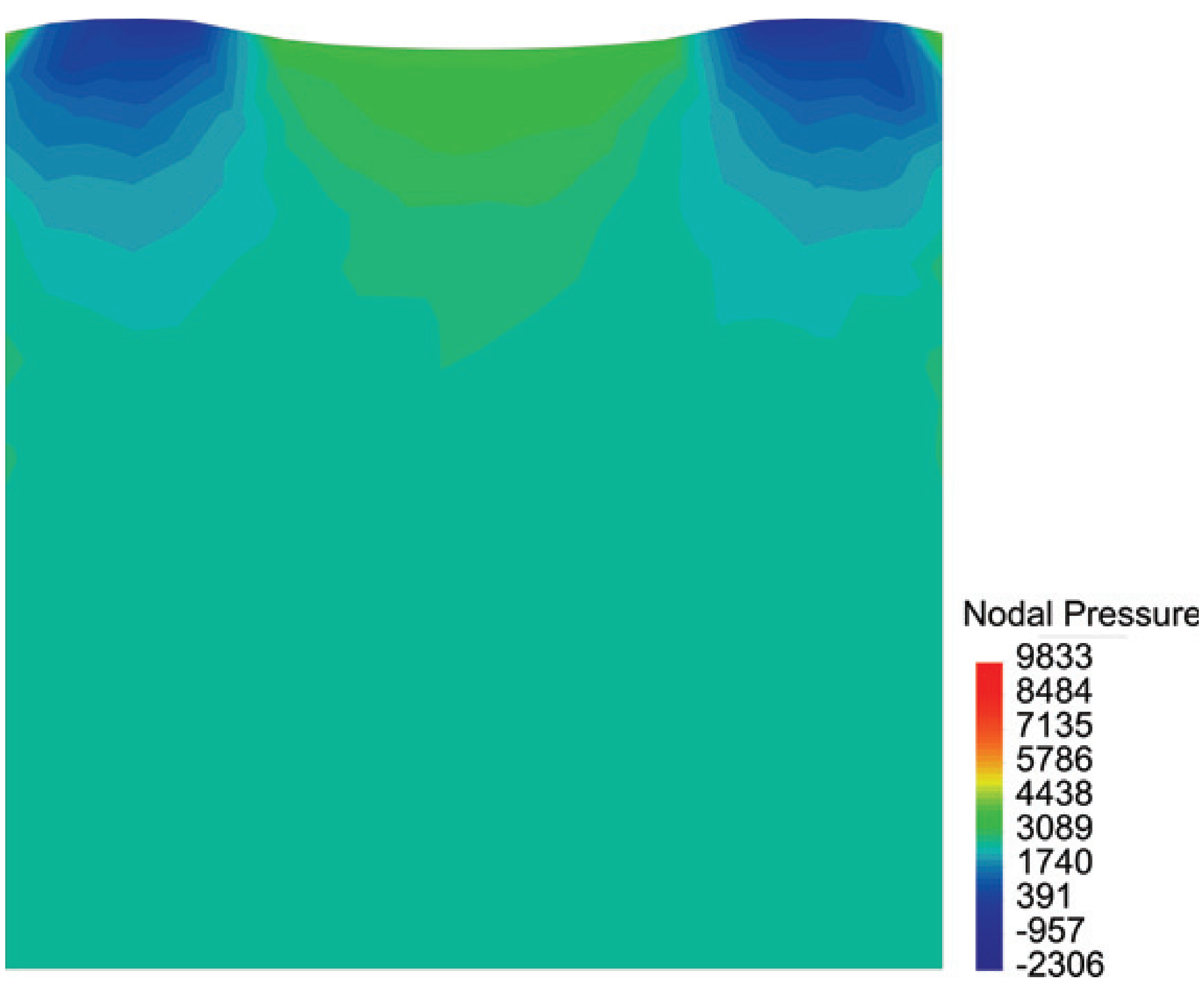


MINI

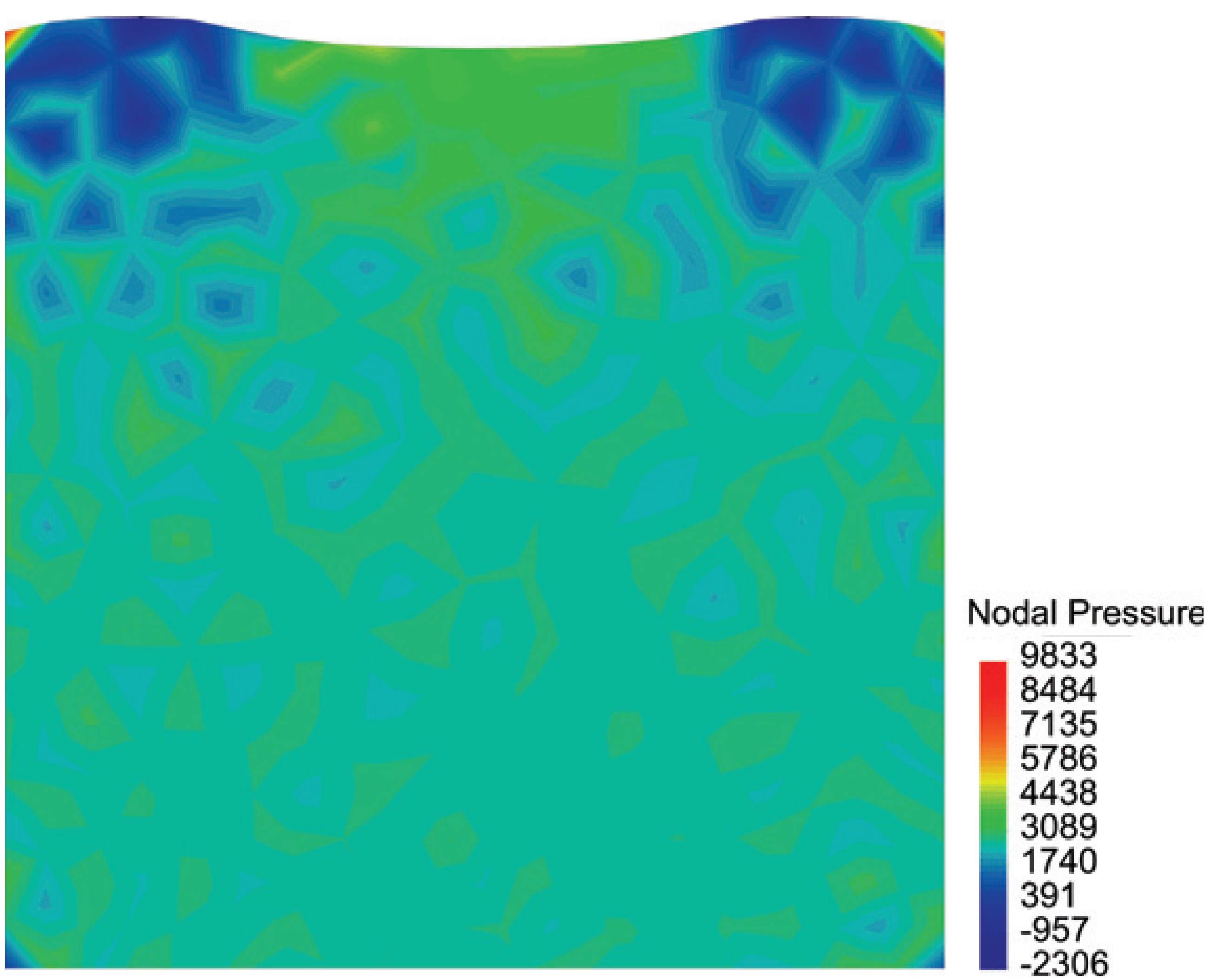


First-order MaxEnt Half Bubbles

Constrained Block



MINI



No bubbles

5. Does it work in 3D?

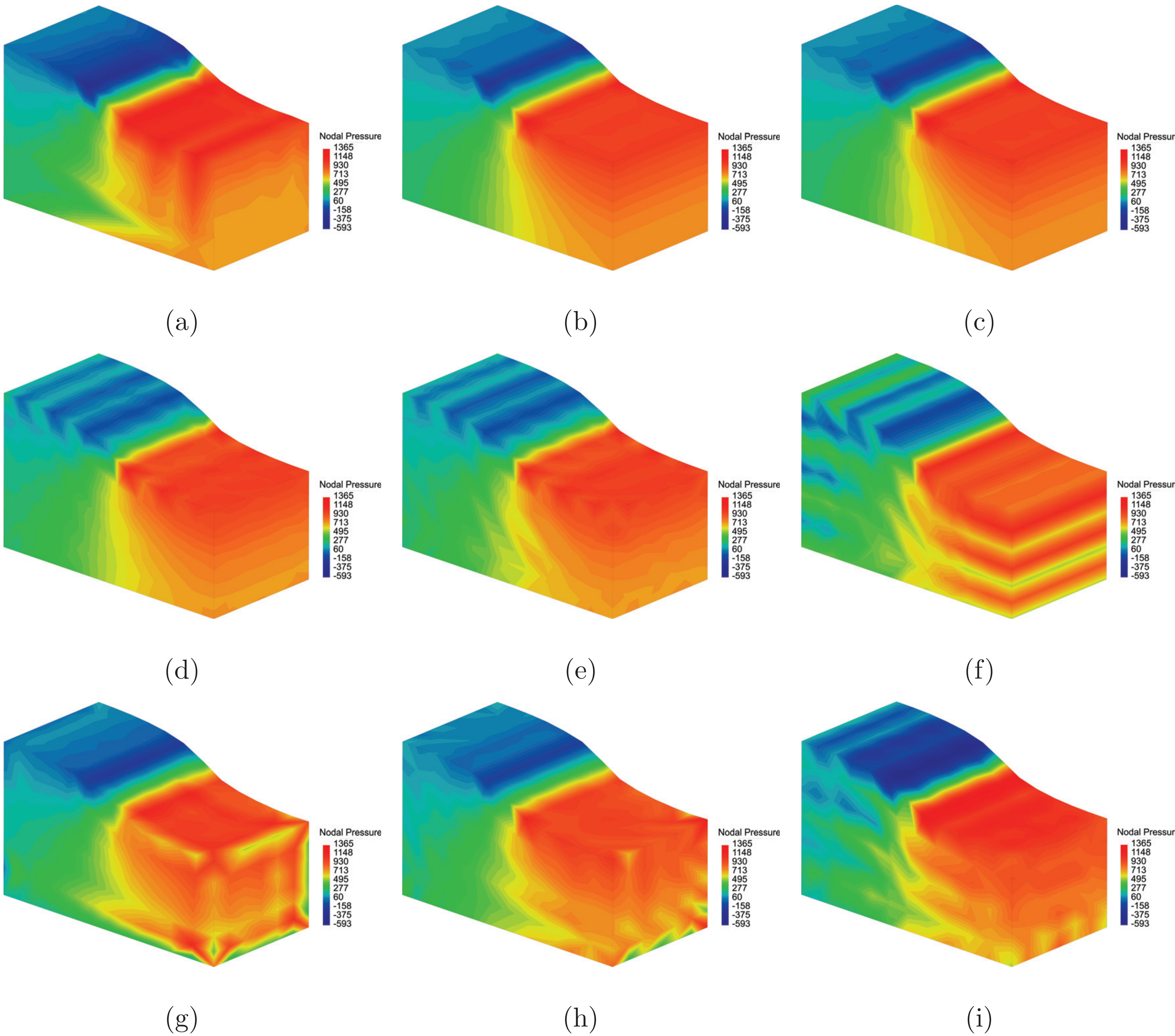


Fig. 17: Compression of a constrained block: Nodal pressure variable for (a) `MINI` element, (b) VANP-T_1^+/T_1-m, (c) VANP-T_1^*/T_1-m, (d) VANP-T_2^+/T_1-m, (e) VANP-T_2^*/T_1-m, (f) VANP-T_2/T_1-m, (g) VANP-T_2^+/T_1-r, (h) VANP-T_2^*/T_1-r, and (i) VANP-T_2/T_1-r.

Summary

- Stable meshfree methods are developed by mimicking existing inf-sup stable finite element methods.

- The auxiliary pressure variable is eliminated using a volume-averaged nodal pressure technique.

- Currently extending to hyperelasticity, where the robust nature of the meshfree shape functions will be a great advantage.

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Questions?