The paradoxes of permission
An action based solution

Paper 485:proposed final version
revised December 2013

by
Dov Gabbay, Loïc Gammaitoni, Xin Sun

Abstract

The aim of this article is to construct a deontic logic in which the free choice postulate [11] would be consistent and all the implausible result mentioned in [5] will be blocked. To achieve this we first developed a new theory of action. Then we build a new deontic logic in which the deontic action operator and the deontic proposition operator are explicitly distinguished.

1 Background and orientation

1.1 Background discussion

Deontic logic is a field of logic that lets one reason about deontic concepts, such as obligations and permissions. SDL (Standard Deontic Logic) is a modal logic established by von Wright [18] to reason about such concepts. This logic has had difficulties and limitations and became outdated with the emergence of the DSDL (Dyadic Standard Deontic Logic) in 1969 by Hansson [4]. Nevertheless, our original idea to create a logic of action in order to reason about permissions and obligations comes from the analysis of paradoxes encountered while reasoning in SDL (Section 3.2). Traditional SDL uses the KD possible world semantics. Its models have the form (S, R, o) where $R \subseteq S^2$ is the accessibility relation, $o \in S$ is the actual world and we have that $\forall x \exists y(xRy)$ holds. *Figure 1* is a typical situation in the semantics. (In this figure, an arrow from x to y indicates xRy, and the labels on the arrows will be referred later, readers should ignore them here).

To evaluate a modal formula for example $o \vDash \Diamond\Diamond q$,[1] one has to find two worlds a,b such that $oRa \wedge aRb$ and $b \vDash q$. The point of view of SDL about this model is the following:

1. View the model from above. The model (S, R, o) is static and we evaluate deontic formulas in it.

2. The set of worlds $I_t = \{x|tRx\}$ is interpreted as a set of ideal worlds relative to t.

[1] In modal logic we use $\Box$ and $\Diamond := \neg\Box\neg$. In deontic logic, it is traditional to use $\bigcirc$ and $P := \neg \bigcirc \neg$

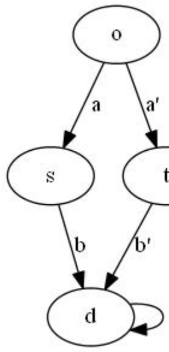

Figure 1: Typical representation in SDL

From this last point it follows that $\bigcirc q$ (read as obligatory q), holds at t if and only if q holds in all the ideal worlds in I_t.
The new action-based solution we are presenting in this paper is more dynamic. It is the following:
When we evaluate $\bigcirc q$ at t we actually consider ourselves as living in world t performing actions which enable us to go to ideal worlds in I_t.
Let the annotations $\mathbf{a}, \mathbf{a}', \mathbf{b}, \mathbf{b}'$ be the actions that can be taken in order to move from one world to another (from now on, bold faces symbols will be considered as actions). Then if we follow the model presented in *Figure 1*, from world o we could take for example the action $\mathbf{a}$ and move to world s. The action has the effect of taking us to world s.
One of the main problems with formalizing Px (read x is permitted) is the intuitive rule

$$P(x \vee y) \leftrightarrow Px \wedge Py$$

(The so called Free choice postulate). This together with another intuitive rule

$$\bigcirc x \rightarrow Px$$

and the rule

$$x \vdash y \text{ implies } \bigcirc x \vdash \bigcirc y$$

yield

$$\bigcirc x \vdash \bigcirc(x \vee y) \vdash P(x \vee y) \vdash Py$$

In other words, If there is any obligation, then anything is permitted! [2]
The above problem arises when we interpret x and y as variables ranging over formulas getting truth values in possible worlds, and we interpret x being obligatory as x being true in all ideal worlds.

The problem can disappear if apply deontic operators to actions instead of propositions. For two actions $\mathbf{a}$ and $\mathbf{b}$, intuitively $P(\mathbf{a} \vee \mathbf{b}) \leftrightarrow P\mathbf{a} \wedge P\mathbf{b}$ should hold, while $\bigcirc\mathbf{a} \rightarrow \bigcirc(\mathbf{a} \vee \mathbf{b})$ should not hold , here $\mathbf{a} \vee \mathbf{b}$ is the composite action which could be either $\mathbf{a}$ or $\mathbf{b}$. We could also assume

$$\bigcirc\mathbf{a} \wedge \bigcirc\mathbf{b} \rightarrow \bigcirc(\mathbf{a} \wedge \mathbf{b})$$

[2]Similar observation can be found in [14]

Where $\mathbf{a} \wedge \mathbf{b}$ means that both actions are taken in parallel.

This paper presents an action based deontic logic which solves the free choice paradox. For other deontic action logic the readers are refered to [12][2][15]

The structure of this paper is as following. In section 2 we introduce our theory of action. In section 3.2 we discuss the paradoxes of permission and in Section 3.3 we describe our solution to tackle the implausibility results presented in section 3.2. In Section 4 we present an action based approach to deontic logic proposed by van der Meyden and discuss its limitations, and we will see how our approach extends that the one of van der Meyden.

2 Action Based Deontic Model

2.1 A Theory of Action

2.1.1 Actions, the relational version

In the literature of action based deontic logic like [12, 13], [2] and [15], actions are treated as Boolean algebras. Here we develop a new theory of action which generalizes their approach. Our intuition is to view an action as a change from a state to another state, possibly via other intermediary states.

Given a set S of states, an action $\mathbf{a}$ is a set of sequences over S. In this section we concentrate on simple actions which is a set of sequences of length 2. We will represent it by a set of ordered pairs, that is, for a simple action $\mathbf{a}$, $\mathbf{a} \subseteq S \times S$.

The simplest action is a single ordered pair $\{(s_1, s_2)\}$, where s_1 and s_2 are states. We call such action particle action. The first component of a particle action $\mathbf{a} = \{(s_1, s_2)\}$ is called the pre-condition of $\mathbf{a}$, formally $pre(\mathbf{a}) = \{s_1\}$. The second component of an particle action $\mathbf{a}$ is called the post-condition of $\mathbf{a}$, formally $post(\mathbf{a}) = \{s_2\}$. Intuitively, a particle action is a deterministic change from a specific state. For example we can let s_1 represents "Israel" and s_2 represents "China", then the action $\{(s_1, s_2)\}$ is simply "go to China when you are in Israel".

With particle actions in hand, we build atomic action as a union of particle actions which share the same pre-condition. For particle actions $\mathbf{a}_1 = \{(s_1, s_2)\}$ and $\mathbf{a}_2 = \{(s_1, s_3)\}$, $\mathbf{a}_3 = \{(s_1, s_2), (s_1, s_3)\}$ is the atomic action consisted of $\mathbf{a}_1$ and $\mathbf{a}_2$. The pre-condition of an atomic action is the same as its consisted particle actions. The post condition of an atomic action is the union of the post conditions of its consisted particle actions. Therefore $post(\mathbf{a}_3) = \{s_2, s_3\}$. Intuitively, an atomic action is a nondeterministic change from a specific state. For example, if we let s_1 represents "China", s_2 represents "USA", and s_3 represents "Canada", then $\mathbf{a}_3$ means "go to north America when you are in China".

A non-atomic action is a union of atomic actions which bears different pre-conditions. For example, let s_1 be "England" and s_2 be "France", then $\mathbf{a} = \{(s_1, s_2), (s_2, s_1)\}$ is a non-atomic action which can be read as "go to France when you are in England, or, go to England when you are in France".(See Figure 2)

For a non-atomic action, its post-condition is sensitive to the starting point of its execution. More accurately, if $\mathbf{a}$ is executed in state t, the then post-

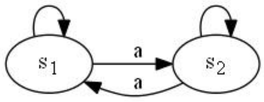

Figure 2: action $actiona = \{(s_1, s_2), (s_2, s_1)\}$

condition of **a** from the standpoint of t is the post-condition of the maximal atomic action of **a** with pre-condition t.

2.1.2 Operators on Actions

Let **a** be an atomic action. We can compute $post(\mathbf{a})$ and write $\mathbf{a} = \{(s, t)|t \in post(a)\}$. Define $\neg\mathbf{a} = \{(s, t)|t \in S - post(\mathbf{a})\}$[3] to be the action of refraining from doing **a**. Intuitively, $\neg$**a** has the same pre-condition as **a** but the post-condition is the set theoretic complement of $post(\mathbf{a})$. Therefore an action is atomic iff its negation is atomic.
If **a** is a non-atomic action, then we calculate $\neg$**a** in the following way: First we decompose **a** to its maximal atomic actions. Then for each of its maximal atomic action $\mathbf{a}_i$, we calculate $\neg\mathbf{a}_i$. Then we take the union of these $\neg\mathbf{a}_i$ to form $\neg$**a**.

Let **a**, **b** be two actions. Then $\mathbf{a} \vee \mathbf{b}$, read as "**a** or **b**" is the union of all their particle actions. For example, if $\mathbf{a} = \{(s_1, s_1)\}$ which means stay in s_1 when you are in s_1, $\mathbf{b} = \{(s_1, s_2)\}$ which means go to s_2 when you are in s_1, then $\mathbf{a} \vee \mathbf{b}$=$\{(s_1, s_1), (s_1, s_2)\}$ which means when you are in s_1, stay there or go to s_2. See Figure 3:

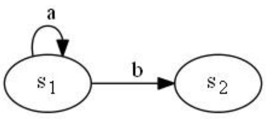

Figure 3: $\mathbf{a} \vee \mathbf{b} = \{(s_1, s_1), (s_1, s_2)\}$

We define $\mathbf{a} \wedge \mathbf{b}$, read as "do **a** and **b** in parallel", to be their intersection. For example if **a**=$\{(s_1, s_1), (s_1, s_2)\}$ and $\mathbf{b} = \{(s_1, s_2), (s_1, s_3)\}$, then $\mathbf{a} \wedge \mathbf{b} = \{(s_1, s_2)\}$. See Figure 4: A more concrete example is following: let **a** be "go to English speaking countries when you are in China" and let **b** be "go to Europe when you are in China", then $\mathbf{a} \wedge \mathbf{b}$ means "go to England when you are in China".
With the above definition we can prove **a** is not equivalent to $(\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \neg\mathbf{b})$. Fore example let $S = \{s_1, s_2\}$, $\mathbf{a} = \{(s_1, s_1)\}$, $\mathbf{b} = \{(s_2, s_1)\}$. Then $\neg\mathbf{b} = \{(s_2, s_2)\}$, $\mathbf{a} \wedge \mathbf{b}$ is empty and $\mathbf{a} \wedge \neg\mathbf{b}$ is also empty. See Figure 5:

[3]Note that we use the logical connective $\neg$,$\wedge$,$\vee$ as algebraic operators on actions. The role of these operator is always clear from context.

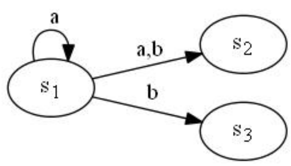

Figure 4: $\mathbf{a} \wedge \mathbf{b} = \{s_1, s_2)\}$

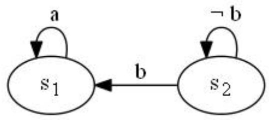

Figure 5: $\mathbf{a} \neq (\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \neg \mathbf{b})$

If we fix a state s, then the set of all atomic actions with s as their precondition, together with $\neg$, $\wedge$, $\vee$ forms a Boolean algebra, which is the approach of [12],[2] and [15].

2.2 Deontic Action/Proposition Logic

2.2.1 Language

Given a finite set $\mathbf{A} = \{\mathbf{a}_1 \ldots \mathbf{a}_n\}$ of atomic actions, and a countable set $Q = \{p, q, r, \ldots\}$ of proposition letters. The language of our deontic action/proposition logic (DAPL) can be defined in BNF in the following way:

1. $\mathbf{a} ::= \mathbf{a}_i | \neg \mathbf{a} | \mathbf{a} \wedge \mathbf{a} | \mathbf{a} \vee \mathbf{a}$
2. $\varphi_{\mathbf{a}} ::= \overline{\bigcirc}_{\mathbf{a}} | \overline{P}_{\mathbf{a}} | \neg \varphi_{\mathbf{a}} | \varphi_{\mathbf{a}} \wedge \varphi_{\mathbf{a}}$
3. $\varphi_p ::= \top | p | \neg \varphi_p | \varphi_p \wedge \varphi_p | \bigcirc \varphi_p | P \varphi_p$
4. $\varphi ::= \varphi_{\mathbf{a}} | \varphi_p | \neg \varphi | \varphi \wedge \varphi$

Intuitively, the symbols of type $\mathbf{a}$ represent actions build from the set $\mathbf{A}$ of atomic actions. Here we require $\mathbf{A}$ to be closed under $\neg$, i.e. for every $\mathbf{a} \in A$, there exist $\mathbf{b} \in A$ such that $\neg \mathbf{a} = \mathbf{b}$. We add this requirement because the negation of an atomic action is always an atomic action as we observed from the previous section.

The symbols of type $\varphi_{\mathbf{a}}$ are the deontic formula about actions and φ_p the factual and deontic formula about propositions. Symbols of type φ are formulas we will actually deal with. They are the Boolean combination of $\varphi_{\mathbf{a}}$ and φ_p. From now on when we state A is a formula we always mean A is of type φ.

Note although the symbol $\neg$, $\wedge$ appear several time in our notation, we can always avoid confusion from the context. When applied to actions, the meaning of $\neg$, $\wedge$, $\vee$ are as defined in the previous section. When applied to formulas, no matter for action or proposition, they are Boolean operators as usual.

2.2.2 Semantics

We now present our semantics for the language of DAPL. We will define the deontic operator for actions and propositions differently. Since we believe the source of normativity of actions and that of propositions are different. The normativity of an action can arise simply from a given normative system; While normativity of a proposition is usually related to ideal outcomes. But executing an obligatory/permitted action do not ensure the outcomes to be ideal. Meanwhile we can achieve ideal outcomes by prohibited actions.

Intuitively, we consider an action to be obligatory(permitted) if there is a legal document approving it to be obligatory(permitted). While a proposition is obligatory if it is true in those ideal worlds or outcomes defined through obligatory actions. We formalize this intuition as follows.

Let Q be a set of atomic propositions and let $\mathbf{A}$ be the set of symbols for atomic actions and their negation. Let $\boldsymbol{A}^+$ be the closure of $\mathbf{A}$ under $\neg, \wedge, \vee$. We use $q \in Q, \mathbf{a} \in \mathbf{A}^+$, $t \in S$ as notation.

Definition 1 *A deontic action/proposition model is a 4-tuple $M = \{S, IntAct, IntPro, V\}$, where S is the set of possible worlds, $IntAct$ is the interpretation of action, $IntPro$ is the interpretation of proposition, V is the valuation function of proposition letters. Here $IntAct = (S_{t,\boldsymbol{a}}, \mu(t, \boldsymbol{a}), \Gamma_t)_{t \in S, \boldsymbol{a} \in \boldsymbol{A}^+}$, $IntAct = (F_t, E_t)_{t \in S}$. They will be further defined below.*

1. For each $t \in S$ and $\mathbf{a} \in \mathbf{A}$, let $S_{t,\mathbf{a}} \subseteq S$, which can be understood as the post-condition of $\mathbf{a}$ when it is executed in t.

   We call the function $\mu : S \times \mathbf{A} \to 2^S$ such that $\mu(t, \mathbf{a}) \subseteq S_{t,\mathbf{a}}$ a *meaning assignment function*. Intuitively, the result of $\mu(t, \mathbf{a})$ is the set ideal outcomes of action $\mathbf{a}$ when it is executed at t.

2. We can extend the definitions we gave in the above point involving elements of $\mathbf{a}$ to be applied to elements of $\boldsymbol{A}^+$ as follows.
   $S_{t,\mathbf{a} \vee \mathbf{b}} = S_{t,\mathbf{a}} \cup S_{t,\mathbf{b}}$
   $S_{t,\mathbf{a} \wedge \mathbf{b}} = S_{t,\mathbf{a}} \cap S_{t,\mathbf{b}}$
   We also define :
   $\mu(t, \mathbf{a} \vee \mathbf{b}) = \mu(t, \mathbf{a}) \cup \mu(t, \mathbf{b})$
   $\mu(t, \mathbf{a} \wedge \mathbf{b}) = \mu(t, \mathbf{a}) \cap \mu(t, \mathbf{b})$
   Here we need not define $S_{t,\neg\mathbf{a}}$ and $\mu(t, \neg\mathbf{a})$ since the negation of atomic actions are in $\mathbf{A}$, which has been defined previously. And the negation of every non-atomic action can be decomposed to the union of atomic actions.

3. Let $\Gamma_t \subseteq \{\overline{\bigcirc}_{\mathbf{a}} | \mathbf{a} \in \mathbf{A}^+\} \cup \{\overline{P}_{\mathbf{a}} | \mathbf{a} \in \mathbf{A}^+\}$. Intuitively Γ_t can be viewed as the collection legal documents informing which actions are obligatory and what actions are permitted.

   We require that Γ_t satisfies the following[4]

   - for every $\mathbf{a} \in \mathbf{A}^+$, it is not the case that both $\overline{\bigcirc}_{\mathbf{a}}$ and $\overline{\bigcirc}_{\neg\mathbf{a}}$ are in Γ_t.

[4]These are axioms governing the atomic propositional symbols formed for all instances from $\mathbf{A}^+$

- if $\overline{\bigcirc}_{\mathbf{a}}$ and $\overline{\bigcirc}_{\mathbf{b}}$ are in Γ_t, then $\overline{\bigcirc}_{\mathbf{a}\wedge\mathbf{b}}$ is in Γ_t.
- $\overline{P}_{\mathbf{a}\vee\mathbf{b}}$ is in Γ_t iff both $\overline{P}_{\mathbf{a}}$ and $\overline{P}_{\mathbf{b}}$ are in Γ_t.
- if $\overline{\bigcirc}_{\mathbf{a}}$ is in Γ_t, then $\overline{P}_{\mathbf{a}}$ is in in Γ_t.

Note that we do not require $\overline{\bigcirc}_{\mathbf{a}} \to \overline{\bigcirc}_{\mathbf{a}\vee\mathbf{b}}$. This will help us tackle some of the paradoxes that will be discussed in a latter section.

4. For every state t, we define a neighborhood F_t using Γ_t and the meaning assignment function μ as follows. Here a neighborhood of S is a family of subsets of S, readers can consult [3] for further details.
   (a) For each $\mathbf{a} \in \boldsymbol{A}^+$, if $\overline{\bigcirc}_{\mathbf{a}} \in \Gamma_t$, let $\mu(t, \mathbf{a}) \in F_t$.
   (b) if $x \in F_t$ and $y \in F_t$, let $x \cap y \in F_t$.

Now we are ready to define the truth value of formulas of DAPL:

- $t \vDash q$ iff $t \in V(q), q$ being atomic.
- $t \vDash \overline{\bigcirc}_{\mathbf{a}}$ iff $\overline{\bigcirc}_{\mathbf{a}} \in \Gamma_t$.
- $t \vDash \overline{P}_{\mathbf{a}}$ iff $\overline{P}_{\mathbf{a}} \in \Gamma_t$.
- $t \vDash \bigcirc A$ iff $||A|| \in F_t$ and $||A|| \neq \emptyset$
- $t \vDash PA$ iff $||A|| \subseteq \bigcup F_t = \bigcup_{S' \in F_t} S'$
- $t \vDash \neg A$ iff it is not the case that $t \vDash A$
- $t \vDash A \wedge B$ iff $t \vDash A$ and $t \vDash B$

Here $||A|| = \{t \in S | t \vDash A\}$. Intuitively, an action $\mathbf{a}$ is obligatory or permitted if we have a legal document in Γ_t to approve it. A proposition is obligatory iff there exist an obligatory action $\mathbf{a}$ such that the proposition is true in exactly the ideal outcomes of $\mathbf{a}$.[5]

<u>Remark R_1</u>

1. The following holds in the semantics:
   (a) $\bigcirc A \wedge \bigcirc B \to \bigcirc(A \wedge B)$
   (b) $\overline{\bigcirc}_{\mathbf{a}} \wedge \overline{\bigcirc}_{\mathbf{b}} \to \overline{\bigcirc}_{\mathbf{a}\wedge\mathbf{b}}$
   (c) $\bigcirc A \to PA$
   (d) $\overline{\bigcirc}_{\mathbf{a}} \to \overline{P}_{\mathbf{a}}$
   (e) $P(A \vee B) \leftrightarrow PA \wedge PB$
   (f) $\overline{P}_{\mathbf{a}\vee\mathbf{b}} \to \overline{P}_{\mathbf{a}} \wedge \overline{P}_{\mathbf{b}}$

[5]This reminds us of what the Bible says, “Deuteronomy 4:2 Do not add to what I command you and do not subtract from it, but keep the commands of the LORD your God that I give you”

2. We interpret $t \vDash \bigcirc A$ to mean you have to take an action at t which will result in an approved set of possible states.(i.e $||A|| \in F_t$). All the possible resulting states have to be desirable and approved. Permission means you are permitted to take an action for which the set of results is tolerable.

3. It may be the case that in F_t there exists a smallest non-empty set H_t such that $H_t = \bigcap F_t$, in which case this suggests that we can take a stronger semantics, which is a simplified version of our semantics in Definition 1, whose models look like (S, R_1, R_2) where:

   - tR_1s iff $s \in H_t$
   - tR_2s iff $s \in E_t = \bigcup F_t$
   - We have $R_1 \subseteq R_2$.
   - $t \vDash \bigcirc A$ iff for all s, if tR_2s then (tR_1s *iff* $s \vDash A$).(This means that H_t is exactly the subset of E_t in which A holds)
   - $t \vDash PA$ iff ($\exists s \in S$ such that tR_1s and $s \vDash A$), and ($\forall s, tR_2s$ and $s \vDash A$ imply tR_1s).
     iff ($\exists s, tR_1s$ and $s \vDash A$), and (if $\forall s, \neg tR_1s \wedge tR_2s$ then $s \nvDash A$)
     iff ($\exists s, tR_1s$ and $s \vDash A$), and ($\forall s$, if $t(R_2 - R_1)s$ then $s \vDash \neg A$) (This means the subset of E_t in which A holds is not empty and it is indeed a subset of H_t).

   Let NA be the modality for the relation $R_2 - R_1$, then we have a bimodal logic with two modalities, $\bigcirc$ for R_1 and N for $R_N = R_2 - R_1$, such that $R_1 \subseteq \overline{R_N}$ or equivalently $R_1 \bigcap R_N = \emptyset$. We have:

   $$\neg \bigcirc \neg A \wedge \neg \bigcirc \neg B \to (P(A \vee B) \equiv PA \wedge PB)$$

   .

4. The worlds of H_t are considered all good worlds, relative to t. The worlds of $E_t - H_t$ are considered bad worlds from the point of view of t. $\bigcirc A$ says A is true exactly in all the good worlds and NA says A is true in all the bad worlds. PA says A is permitted and it means that A is true in some good world and it can be tolerated because $N\neg A$ holds, i.e. no world in which A is true can be bad. We have $\bigcirc A \to PA$.
   This should be compared with the subideal idea of Jones and Pörn [7]. They defined (using our notation) **ought** q by q is true in all R_1 accessible worlds as well as q being false in at least one ($R_2 - R_1$) accessible world. We have then:

   $$\bigcirc q \to \textbf{ought } q$$

   However, their two relations R_1 and R_2 also satisfy $uR_1u \vee uR_2u$ and they use their logic to deal with the Chisholm Paradox. See the Two dimensional Deontic Logic paper [1] [6]

[6]In models in which $R_2 - R_1$ has only one accessible world for each t, then our definition is the same as Jones-Pörn. In fact in Section 3.3 of the Two dimensional Deontic Logic paper , the authors do consider the same operator as our $\bigcirc q$ of Definition D1 , which they call ***ought****q. The authors of that paper, however, had no thematic motivation for their ***ought***, but only local technical advantages.

5. Note that the definition of satisfaction for $\bigcirc A$ is :
$t \vDash \bigcirc A \;\; iff \;\; ||A| = H_t$
Thus we are using the neighborhood $F_t = \{H_t\}$

3 Free choice paradox in SDL

3.1 Free choice permissions

The semantic of a disjunctive permission generally refers to a choice that an agent can make between different actions. As Hansson[5] pointed out in his Handbook chapter (section 5), there's two different usage of disjunctive permission.
The most common usage implies that one have the permission to do every actions included in the given disjunction.

$$P(a \vee b) \rightarrow Pa \wedge Pb$$

For example:

"you may either eat the cake or drink the cup of coffee"

could be understand as : *"There is some cake, and there is some coffee, so feel free to chose one of them"* (implicitly both of them or none if you want).

Now suppose that the person who has formulated the latter permission was told that one may eat the cake only. It may be the case that this person unfortunately forgot whether it was the cake, or the coffee that was available for consumption.

"you may either eat the cake or drink the cup coffee. But I unfortunately forgot which."

Then , the postulate $P(a \vee b) \rightarrow Pa \wedge Pb$ doesn't hold anymore, as we know that the agent doesn't have both permission. But we still know that the agent can do at least one of the two actions. Thus a weaker postulate holds

$$P(a \vee b) \rightarrow Pa \vee Pb$$

The first usage (the one generally implied while using common language) is called by von Wright a Free choice permission [17]. We will see in the next subsection that this free choice postulate taken together with SDL leads to implausible results.

3.2 Paradoxes

Free choices as described in the previous subsection are something one would like to incorporates in his logic, as it's the most natural way a human will interpret a disjunctive permission. Unfortunately several implausibility results where highlighted by different authors when the free choice postulate $P(a \vee b) \rightarrow Pa \wedge Pb$, is taken into consideration with SDL. In this subsection we will list those implausibility results.[5]

Implausibility result 1: $\bigcirc a \to \bigcirc(a \wedge b)$ [8]
Requirements: Extensionality and interdefinability ($\bigcirc a \leftrightarrow \neg P \neg a$)
Derivation:
$P(\neg a \vee \neg b) \to P\neg a$
$P\neg(a \wedge b) \to P\neg a$
$\neg P\neg a \to \neg P\neg(a \wedge b)$
$\bigcirc a \to \bigcirc(a \wedge b)$

Implausibility result 2: $\bigcirc a \to Pb$ ([18])
Requirements: Extensionality, $\bigcirc a \to Pa$ and $\bigcirc a \to \bigcirc(a \vee b)$
Derivation:
$P(a \vee b) \to Pb$
$\bigcirc(a \vee b) \to Pb$ (since $\bigcirc(a \vee b) \to P(a \vee b)$)
$\bigcirc a \to Pb$ (since $\bigcirc a \to \bigcirc(a \vee b)$

Implausibility result 3: $Pa \to Pb$[9]
Requirements: $\bigcirc(a \wedge b) \to \bigcirc a$ and interdefinability $\bigcirc a \leftrightarrow \neg P \neg a$
Derivation:
$\bigcirc(\neg a \wedge \neg b) \to \bigcirc\neg a$
$\bigcirc\neg(a \vee b) \to \bigcirc\neg a$
$\neg \bigcirc \neg a \to \neg \bigcirc \neg(a \vee b)$
$Pa \to P(a \vee b)$
$Pa \to Pb$ (since $P(a \vee b) \to Pb$

Implausibility result 4: $Pa \to P(a \wedge b)$ [6]
Requirements: Extensionality
Derivation:
$P((a \wedge b) \vee (a \wedge \neg b)) \to P(a \wedge b) \wedge P(a \wedge \neg b)$
$Pa \to P(a \wedge b) \wedge P(a \wedge \neg b)$ (Extensionality)
$Pa \to P(a \wedge b)$

Several authors have tried to solve those implausibility results by introducing a new permission operator P_c. This operator is defined in such a way that $P_c a \leftrightarrow \neg\bigcirc\neg a$ is not necessarily holding in order to block some of the previously exposed paradoxes.
Free choices would then be expressed as follow

$$P_c(a \vee b) \leftrightarrow Pa\&Pb$$

Implausibility results from 5 to 7 are paradoxes that occurs while using this Free choice operator.

Implausibility result 5: $\bigcirc a \wedge Pb \to P_c(a \vee b)$ [19]
Requirements: $P_c(a \vee b) \leftrightarrow Pa \wedge Pb$ and $\bigcirc a \to Pa$
Derivation:
$\bigcirc a \wedge Pb$ (assumption)
$Pa \wedge Pb$ (Postulate $\bigcirc a \to Pa$)
$P_c(a \vee b)$ (definition of P_c)

Implausibility result 6: $\bigcirc a \wedge \bigcirc b \rightarrow P_c(a \vee b)$ [4]
Requirements: $P_c(a \vee b) \leftrightarrow Pa \wedge Pb$ and $\bigcirc a \rightarrow Pa$
Derivation:
$\bigcirc a \wedge \bigcirc b$ (assumption)
$Pa \wedge Pb$ (Postulate $\bigcirc a \rightarrow Pa$)
$P_c(a \vee b)$ (definition of P_c)

Implausibility result 7: $Pa \rightarrow P_c(a \vee b)$ [4]
Requirements: Extensionality, $P_c(a \vee b) \leftrightarrow Pa \wedge Pb$ and $Pa \rightarrow P(a \vee b)$
Derivation:
Pa (assumption)
$Pa \wedge P(a \vee b)$ (Postulate $Pa \rightarrow P(a \vee b)$)
$P_c(a \vee (a \vee b))$ (definition of P_c) $P_c(a \vee b))$ (Extensionality)

To avoid those latter problems, von Wright proposed a system in which

$$P_c(a \vee b) \leftrightarrow P_c a \wedge P_c b$$

whenever both a and b are contingent.
But this property leads to the following implausible results:

Implausibility result 8: $P_c(a \vee b) \wedge P_c(c \vee d) \rightarrow P_c(a \vee c)$ if $a, b, c,$ and d are contingent. [5]
Requirements: Extensionality, $P_c(a \vee b) \leftrightarrow P_c a \wedge P_c b$ when a and b are contingent.
Derivation:
Let a,b,c and d be contingent.
$P_c(a \vee b) \wedge P_c(c \vee d)$ (assumption)
$P_c(a) \wedge P_c(c)$
$P_c(a \vee c)$

Implausibility result 9: $P_c(a \vee b) \leftarrow P_c((a \wedge c) \vee (b \wedge c))$ if $a, b, a \wedge c, a \wedge \neg c, b \wedge c$ and $b \wedge \neg c$ are contingent. [5]
Requirements: Extensionality, $P_c(a \vee b) \leftrightarrow P_c a \wedge P_c b$ when a and b are contingent.
Derivation:
Let a,b and c be contingent.
$P_c(a \vee b)$ (assumption)
$P_c((a \wedge c) \vee (a \wedge \neg c)) \wedge P_c((b \wedge c) \vee (b \wedge \neg c))$
$P_c(a \wedge c) \wedge P_c(b \wedge c)$
$P_c((a \wedge c) \vee (b \wedge c))$

In the next section, we will introduce a new concept that will hopefully allow us to block all those implausibility results.

3.3 Paradoxes in our semantics

We now examine the status of the paradoxes (mentioned in section 3.2) in our semantics. Can we derive these paradoxes in our semantics?

We now check the validity of the paradoxes in our model.

Implausibility result 1 does not hold in our semantics. We can block it in two ways because we have neither $\bigcirc a \leftrightarrow \neg P \neg a$ nor $\overline{\bigcirc}_{\mathbf{a}} \leftrightarrow \neg \overline{P}_{\neg \mathbf{a}}$.

Implausibility result 2 doesn't hold since we do have neither $\bigcirc a \rightarrow \bigcirc(a \vee b)$ nor $\overline{\bigcirc}_{\mathbf{a}} \rightarrow \overline{\bigcirc}_{\mathbf{a} \vee \mathbf{b}}$.

Implausibility result 3 is blocked for the same reason we block implausibility result 1.

Implausibility result 4 is blocked since we do not have $\overline{P}_{\mathbf{a}} \rightarrow \overline{P}_{(\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \neg \mathbf{b})}$. This can be illustrated in by Figure 5 in section 2.
Implausibility results 5 and 6 are not a problem to us as we think they are acceptable.
Implausibility result 7 does not hold in our model because the essential assumption $\overline{P}_{\mathbf{a}} \rightarrow \overline{P}_{\mathbf{a} \wedge \mathbf{b}}$ does hold in our model.
Implausibility results 8 and 9 do not concern us because they involve P_c alone, a connective not in our language. These do not hold for our P.

4 Extension and related work

One influential work on deontic action logic with the free choice axiom is van der Meyden [16]. Quite different from our work up to now, van der Meyden adopt sequential semantics [10] for dynamic logic. In this section, we will first recapitulate van der Meydem's work then extend our dynamic logic from relational semantics to sequential semantics and build deontic logic on it. Such logic will also solve the free choice paradox and extends van der Meyden's work.

4.1 Van der Meyden's Action Based Approach

Meyden's paper uses a language with actions set $A = \{a, b, c, ...\}$ and atomic propositions set Q.
The operations on actions are $\alpha \cup \beta$ (disjunction) , $\alpha;\beta$ (concatenation of α and β) and α^* (finite repeated concatenation of α), for α and β being action expressions.
The connectives available are the classical Boolean connectives and the following modal connectives.

$$< \alpha > A, \Diamond(\alpha, A), \Pi(\alpha, A)$$

Intuitively, $< \alpha > A$ means that starting from the current state, some execution of the action α ends in a state in which A is true. $\Diamond(\alpha, A)$ asserts that A holds after some execution of α which is not forbidden. $\Pi(\alpha, A)$ asserts that each execution of α which ends in a state satisfying A is permitted.

A model as the form (S, R, V, τ), where (S, R, V) is an ordinary Kripke model with valuation V and τ is an assignment to atomic actions. τ gives to each atomic action a a set $\tau(a)$ of finite sequences of states. These sequences are of the form $(t_1, ..., t_n) \in \tau(a)$. When the sequence satisfies : $t_1 R t_2 R t_3 R ... R t_n$, we say that the sequence is permitted. τ can be extended in the obvious way to the rest of the actions:

$$\tau(\alpha;\beta) = \tau(\alpha);\tau(\beta)$$

$$\tau(\alpha \cup \beta) = \tau(\alpha) \cup \tau(\beta)$$

$$\tau(\alpha^*) = \tau(\alpha)^* = \tau(\alpha) \cup (\tau(\alpha);\tau(\alpha)) \cup (\tau(\alpha);\tau(\alpha);\tau(\alpha)) \cup \ldots$$

We write $\tau_t(\alpha)$ for all the sequences in $\tau(\alpha)$ of the form $(t, t_1, ..., t_k), k \geqslant 0$
We now define satisfaction in the model according to Meyden.

- $t \vDash q$, for q atomic if $t \in V(q)$
- The usual definition for the Boolean connectives.
- $t \vDash < \alpha > A$ if for some sequence $((t, s_1, ..., s_k, r) \in \tau(\alpha)$ we have $r \vDash A$
- $t \vDash \Diamond(\alpha, A)$ if for some permitted sequence $(t, s_1, ..., s_k, r) \in \tau(\alpha)$we have $r \vDash A$
- $t \vDash \Pi(\alpha, A)$ if whenever we have a sequence $(t, s_1, ..., s_k, r)$ such that $r \vDash A$, then this sequence is permitted.

4.2 Sequential Semantics of Dynamic Logic

In section 2 we introduced a relational interpretation to actions, where an action is interpreted by a set of ordered pairs of states. An ordered pair of states can be understood as a sequence of state with length 2. In this section we extend to general sequential interpretation without restrict the length of sequence to be 2. In the sequential semantic of dynamic logic [10], an action is interpreted by a set of sequence of states. Formally,

Definition 2 (Sequential model) *A sequential model* $\mathbb{M} = (W, R^{\mathbb{A}}, V)$ *is a triple:*

- W *is a non-empty set of possible states.*
- $R^{\mathbb{A}} : \mathbb{A} \to 2^{W\times W} \cup 2^{W\times W\times W} \cup 2^{W\times \ldots \times W}$ *is an action interpretation function, assigning a set of sequences over* W *to each action generator* $a \in \mathbb{A}$*. For technical reasons, we require the length of each sequence to be at least 2.*
- V *is the valuation function for propositional letters.*

The action interpretation function $R^{\mathbb{A}}$ is extended to a new function R to interpret arbitrary actions as in the relational version, except for the negation operator.

For a sequence $s =< s_1, \ldots, s_n >$ with each $s_i \in W$, we define the pre-condition of s to its first element s_1, and post-condition to be its last elements s_n. The length of s, denote as $|s|$, is the number of appearance of states. We call an action α molecular if it is interpreted by a set of sequences with the same length. More formally, α is molecular if for every $s, t \in R(\alpha)$, $|s| = |t|$. The action α is atomic if it is molecular and for all $s, t \in R(\alpha)$, the pre-condition of s is the same as the pre-condition of t. For every atomic action α, we define $pre(\alpha)$ to be the singular set of the pre-condition of its consisted sequences; we let the post-condition of α to be the set of the post-conditions of its consisted sequence; we let the length of α to be the length of its consisted sequence.

For an atomic action α of length n, we define the negation operators as following:

- $R(\overline{\alpha}) = pre(\alpha) \times W \times \ldots \times W - R(\alpha)$, here the appearance of W is $n-1$

We can now define the negation of arbitrary actions. For every action α, we calculate $R(\overline{\alpha})$ following those steps: (for notational convenience we use $\overline{\alpha}$ to represent the negation of action α)

1. decompose α to its maximal molecular sub-actions. That is, decompose α to its sub-actions $\alpha_1, \ldots, \alpha_m$ such that $R(\alpha) = R(\alpha_1) \cup \ldots \cup R(\alpha_m)$, each α_i is molecular and there is no α_i and α_j with the same length.
2. for each molecular α_i, divide it to its maximal atomic sub-actions. That is, decompose α_i to $\alpha_i^1, \ldots, \alpha_i^n$ such that $R(\alpha_i) = R(\alpha_i^1) \cup \ldots \cup R(\alpha_i^n)$, each α_i^j is atomic and there is no α_i^j and α_j^k with the same pre-condition.
3. for each atomic action α_i^j, calculate $R(\overline{\alpha_i^j})$.
4. let $R(\overline{\alpha_i}) = R(\overline{\alpha_i^1}) \cup \ldots \cup R(\overline{\alpha_i^n})$
5. let $R(\alpha) = R(\overline{\alpha_1}) \cup \ldots \cup R(\overline{\alpha_m})$

It can be verified that the above approach coincides with the relational approach for those actions α of length 2

Recall the meaning assignment function μ from Definition 1 in section 2.2.2, here we define a simplified version.

Definition 3 (Simplified Meaning assignment function) *Let $\mu' : W \rightarrow 2^{2^W}$ be a simplified meaning assignment function such that for every $w \in W$, every element of $\mu'(w)$ is an ideal set worlds.*

A general deontic action model $M =< W, R^{\mathbb{A}}, \mu', V >$ is a deontic action model with the interpretation function $R^{\mathbb{A}}$ interprets actions to sequences of state. We define

- $M, w \models P\alpha$ iff for all $s \in R(\alpha)$ with $pre(s) = w$, if $s = (s_1, \ldots, s_n)$ then $s_{i+1} \in \bigcup \mu'(s_i)$
- $M, w \models O\alpha$ iff $M, w \models P\alpha$ and $post(\alpha) \in \mu'(w)$

Our logic extends van der Meyeden's logic in the sense that the action negation operator and the obligation operator are involved.

5 Concluding Discussion

In this paper we develop a new theory of action and based on that we build a new deontic logic in which deontic action statement and deontic proposition statement are explicitly distinguished. Our new logic validates the free choice axiom, both for action and proposition, but all the implausible results sketched in Hansson's survey paper [5] are blocked.
For the futher work, a natural direction is to axiomatize our logic.

References

[1] M. Boer, D. Gabbay, X. Parent, and M. Slavkovic. Two dimensional standard deontic logic. *Synthese*, 187:623–660, 2012.

[2] P. Castro and T.S.E. Maibaum. Deontic action logic, atomic boolean algebras and fault-tolerance. *Journal of Applied Logic*, 7(4):441–466, 2009.

[3] B. F. Chellas. *Modal logic: an Introduction.* Cambridge University Press.

[4] B. Hansson. An analysis of some deontic logics. *Nous*, 1969.

[5] S. Hansson. The varieties of permissons. In *Handbook of deontic logic and normative systems.* to appear in College Publication.

[6] R. Hilpinen. Disjunctive permissions and conditionals with disjunctive antecedent. *Acta Philosophica Fennica*, 35:175–194, 1982.

[7] A. Jones and I. Porn. Ideality, sub-ideality and deontic logic. *Synthese*, 1985.

[8] H. Kamp. Free choice permission. In *Porceedings of the Aristotelian Society*, pages 53–74, 1973.

[9] D. Makinson. Stenius' approach to disjunctive permission. *Theoria*, 50:138–147, 1984.

[10] V.R. Pratt. Process logic: Preliminary report. In *Proceedings of the 6th Symposium on Principles of Programming Languages*, pages 93–100. ACM, 1979.

[11] A. Ross. Imperatives and logic. *Theoria*, 7:53–71, 1941.

[12] K. Segerberg. A deontic logic of action. *Studia Logica*, 41(2-3):269–282, 1982.

[13] K. Segerberg. Getting started: Beginnings in the logic of action. *Studia Logica*, 51(3-4):347–378, 1992.

[14] R. Trypuz and P. Kulicki. Towards metalogical systematisation of deontic action logics based on boolean algebra. In Guido Governatori and Giovanni Sartor, editors, *DEON*, volume 6181 of *Lecture Notes in Computer Science*, pages 132–147. Springer, 2010.

[15] R. Trypuz and R. Kulicki. A systematics of deontic action logics based on boolean algebra. *Logic and Logical Philosophy*, 18(3-4):253–270, 2009.

[16] R. van der Meyden. The dynamic logic of permission. *Journal of Logic and Computation*, 6:465–479, 1996.

[17] G.H. von Wright. *Norm and Action.* Routledge and Kegan, 1963.

[18] G.H. von Wright. *An essay in deontic logic and the general theory of action.* North Holland, Amsterdam, 1968.

[19] J. Wolènski. A note on free choice permission. 66:507–510, 1980.