Associative and preassociative aggregation functions

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Associative functions

Let X be a nonempty set

$G\colon X^2 \to X$ is *associative* if

$$G(G(a,b),c) \;=\; G(a,G(b,c))$$

Examples: $G(a,b) = a + b$ on $X = \mathbb{R}$
$G(a,b) = a \wedge b$ on $X = L$ (lattice)

Associative functions

$$G(G(a,b),c) \;=\; G(a,G(b,c))$$

Extension to *n*-ary functions

$$\begin{aligned} G(a,b,c) &= G(G(a,b),c) \;=\; G(a,G(b,c)) \\ G(a,b,c,d) &= G(G(a,b,c),d) \;=\; G(a,G(b,c),d) \;=\; \cdots \end{aligned}$$

etc.

Associative functions with indefinite arity

Let

$$X^* = \bigcup_{n \in \mathbb{N}} X^n$$

$F\colon X^* \to X$ is *associative* if

$$\begin{aligned} & F(x_1, \ldots, x_p, \quad y_1, \ldots, y_q, \quad z_1, \ldots, z_r) \\ = \; & F(x_1, \ldots, x_p, F(y_1, \ldots, y_q), z_1, \ldots, z_r) \end{aligned}$$

Example: $F(x_1, \ldots, x_n) = x_1 + \cdots + x_n$ on $X = \mathbb{R}$

$F(x_1, \ldots, x_n) = x_1 \wedge \cdots \wedge x_n$ on $X = L$ (lattice)

Notation

We regard n-tuples $\mathbf{x}$ in X^n as *n-strings* over X

0-string: ε
1-strings: $x, y, z, \ldots$
n-strings: $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$

X^* is endowed with concatenation

Example: $\mathbf{x} \in X^n$, $y \in X$, $\mathbf{z} \in X^m \quad \Rightarrow \quad \mathbf{x}y\mathbf{z} \in X^{n+1+m}$

$$|\mathbf{x}| = \text{length of } \mathbf{x}$$

$$F(\mathbf{x}) = \varepsilon \quad \Longleftrightarrow \quad \mathbf{x} = \varepsilon$$

Associative functions with indefinite arity

$F\colon X^* \to X$ is *associative* if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

Equivalent definitions

$$F(F(\mathbf{xy})\,\mathbf{z}) \;=\; F(\mathbf{x}\,F(\mathbf{yz})) \qquad \forall\ \mathbf{xyz} \in X^*$$

$$F(\mathbf{xy}) \;=\; F(F(\mathbf{x})F(\mathbf{y})) \qquad \forall\ \mathbf{xy} \in X^*$$

Associative functions with indefinite arity

$F\colon X^* \to X$ is *associative* if

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

Theorem

We can assume that $|\mathbf{xz}| \leqslant 1$ in the definition above

That is, $F\colon X^* \to X$ is associative if and only if

$$\begin{aligned} F(\mathbf{y}) &= F(F(\mathbf{y})) \\ F(x\mathbf{y}) &= F(xF(\mathbf{y})) \\ F(\mathbf{y}z) &= F(F(\mathbf{y})z) \end{aligned}$$

Associative functions with indefinite arity

$$F(\mathbf{y}z) \;=\; F(F(\mathbf{y})z)$$

$$F_n \;=\; F|_{X^n}$$

$$\boxed{F_n(x_1\cdots x_n) \;=\; F_2(F_{n-1}(x_1\cdots x_{n-1})\,x_n) \qquad n\geqslant 3}$$

Associative functions are completely determined
by their unary and binary parts

Proposition

Let $F\colon X^*\to X$ and $G\colon X^*\to X$ be two associative functions such that $F_1=G_1$ and $F_2=G_2$. Then $F=G$.

Associative functions with indefinite arity

Link with binary associative functions ?

Proposition

A binary function $G\colon X^2 \to X$ is associative if and only if there exists an associative function $F\colon X^* \to X$ such that $F_2 = G$.

Does F_1 really play a role ?

$$F_1(F(\mathbf{x})) = F(\mathbf{x})$$

$$F(\mathbf{x}y\mathbf{z}) = F(\mathbf{x}F_1(y)\mathbf{z})$$

Associative functions with indefinite arity

$$F_1(F(\mathbf{x})) = F(\mathbf{x})$$

$$F(\mathbf{x}y\mathbf{z}) = F(\mathbf{x}F_1(y)\mathbf{z})$$

Theorem

$F\colon X^* \to X$ is associative if and only if

(i) $F_1(F_1(x)) = F_1(x), \quad F_1(F_2(xy)) = F_2(xy)$

(ii) $F_2(xy) = F_2(F_1(x)\,y) = F_2(x\,F_1(y))$

(iii) $F_2(F_2(xy)\,z) = F_2(x\,F_2(yz))$

(iv) $F_n(x_1 \cdots x_n) = F_2(F_{n-1}(x_1 \cdots x_{n-1})\,x_n) \qquad n \geqslant 3$

Associative functions with indefinite arity

Theorem

$F\colon X^* \to X$ is associative if and only if

(i) $F_1(F_1(x)) = F_1(x), \quad F_1(F_2(xy)) = F_2(xy)$

(ii) $F_2(xy) = F_2(F_1(x)\, y) = F_2(x\, F_1(y))$

(iii) $F_2(F_2(xy)\, z) = F_2(x\, F_2(yz))$

(iv) $F_n(x_1 \cdots x_n) = F_2(F_{n-1}(x_1 \cdots x_{n-1})\, x_n) \qquad n \geqslant 3$

Suppose F_2 satisfying (iii) is given. What could be F_1 ?

Example: $F_2(xy) = x + y$

$$\Rightarrow \quad F_1(x+y) \;=\; F_1(F_2(xy)) \;\overset{(i)}{=}\; F_2(xy) \;=\; x+y$$

$$\Rightarrow \quad F_1(x) = x$$

Associative functions with indefinite arity

Theorem

$F\colon X^* \to X$ is associative if and only if

(i) $F_1(F_1(x)) = F_1(x), \quad F_1(F_2(xy)) = F_2(xy)$

(ii) $F_2(xy) = F_2(F_1(x)\,y) = F_2(x\,F_1(y))$

(iii) $F_2(F_2(xy)\,z) = F_2(x\,F_2(yz))$

(iv) $F_n(x_1 \cdots x_n) = F_2(F_{n-1}(x_1 \cdots x_{n-1})\,x_n) \qquad n \geqslant 3$

Example: $F_n(x_1 \cdots x_n) = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$

$$F_1(x) \;=\; x$$

$$F_1(x) \;=\; |x|$$

Preassociative functions

Let Y be a nonempty set

Definition. We say that $F\colon X^* \to Y$ is *preassociative* if

$$F(\mathbf{y}) = F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{xyz}) = F(\mathbf{xy}'\mathbf{z})$$

Examples: $F_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2 \quad (X = Y = \mathbb{R})$

$F_n(\mathbf{x}) = |\mathbf{x}| \quad (X \text{ arbitrary}, \ Y = \mathbb{N})$

Preassociative functions

$$F(\mathbf{y}) \;=\; F(\mathbf{y}') \quad\Rightarrow\quad F(\mathbf{x}\mathbf{y}\mathbf{z}) \;=\; F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

Equivalent definition

$$F(\mathbf{x}) \;=\; F(\mathbf{x}') \quad\text{and}\quad F(\mathbf{y}) \;=\; F(\mathbf{y}')$$
$$\Downarrow$$
$$F(\mathbf{x}\mathbf{y}) \;=\; F(\mathbf{x}'\mathbf{y}')$$

Preassociative functions

$$F(\mathbf{y}) \ = \ F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{xyz}) \ = \ F(\mathbf{xy}'\mathbf{z})$$

Fact. If $F\colon X^* \to X$ is associative, then it is preassociative

Proof. Suppose $F(\mathbf{y}) = F(\mathbf{y}')$
Then $F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}) = F(\mathbf{x}F(\mathbf{y}')\mathbf{z}) = F(\mathbf{xy}'\mathbf{z})$ □

Preassociative functions

$$F(\mathbf{y}) = F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}\mathbf{y}'\mathbf{z})$$

Proposition

$F\colon X^* \to X$ is associative if and only if it is preassociative and $F_1(F(\mathbf{x})) = F(\mathbf{x})$

Proof. (Necessity) OK.
(Sufficiency) We have $F(\mathbf{y}) = F(F(\mathbf{y}))$
Hence, by preassociativity, $F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z})$ □

Preassociative functions

Proposition

If $F\colon X^* \to Y$ is preassociative, then so is the function

$$x_1 \cdots x_n \ \mapsto F_n(g(x_1) \cdots g(x_n))$$

for every function $g\colon X \to X$

Example: $F_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2 \qquad (X = Y = \mathbb{R})$

Preassociative functions

Proposition

If $F\colon X^* \to Y$ is preassociative, then so is

$$g \circ F \;\; : \;\; \mathbf{x} \mapsto g(F(\mathbf{x}))$$

for every function $g\colon Y \to Y$ such that $g|_{\mathrm{ran}(F)}$ is one-to-one

Example: $F_n(\mathbf{x}) = \exp(x_1^2 + \cdots + x_n^2) \qquad (X = Y = \mathbb{R})$

Preassociative functions

Proposition

Assume $F\colon X^* \to Y$ is preassociative
If F_n is constant, then so is F_{n+1}

Proof. If $F_n(\mathbf{y}) = F_n(\mathbf{y}')$ for all $\mathbf{y}, \mathbf{y}' \in X^n$, then $F_{n+1}(x\mathbf{y}) = F_{n+1}(x\mathbf{y}')$ and hence F_{n+1} depends only on its first argument... □

Preassociative functions

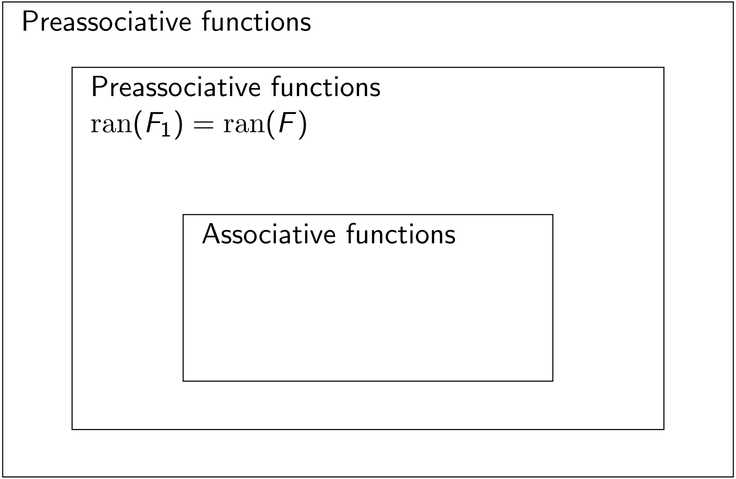
We have seen that $F\colon X^* \to X$ is associative if and only if it is preassociative and $F_1(F(\mathbf{x})) = F(\mathbf{x})$

Relaxation of $F_1(F(\mathbf{x})) = F(\mathbf{x})$:

$$\boxed{\mathrm{ran}(F_1) \;=\; \mathrm{ran}(F)}$$

$$\begin{aligned} \mathrm{ran}(F_1) &= \{F_1(x) : x \in X\} \\ \mathrm{ran}(F) &= \{F(\mathbf{x}) : \mathbf{x} \in X^*\} \end{aligned}$$

Preassociative functions



Preassociative functions

We now focus on preassociative functions $F\colon X^* \to Y$ satisfying $\mathrm{ran}(F_1) = \mathrm{ran}(F)$

Proposition

These functions are completely determined by their unary and binary parts

Preassociative functions

Theorem

Let $F\colon X^* \to Y$. The following assertions are equivalent:

(i) F is preassociative and satisfies $\mathrm{ran}(F_1) = \mathrm{ran}(F)$

(ii) F can be factorized into

$$F \;=\; f \circ H$$

where $H\colon X^* \to X$ is associative
$f\colon \mathrm{ran}(H) \to Y$ is one-to-one.

Axiomatizations of function classes

Theorem (Aczél 1949)

$H\colon \mathbb{R}^2 \to \mathbb{R}$ is

- continuous
- one-to-one in each argument
- associative

if and only if

$$H(xy) \;=\; \varphi^{-1}(\varphi(x) + \varphi(y))$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous and strictly monotone

$$H_n(\mathbf{x}) \;=\; \varphi^{-1}(\varphi(x_1) + \cdots + \varphi(x_n))$$

Axiomatizations of function classes

Theorem

Let $F\colon \mathbb{R}^* \to \mathbb{R}$. The following assertions are equivalent:

(i) F is preassociative and satisfies $\mathrm{ran}(F_1) = \mathrm{ran}(F)$, F_1 and F_2 are continuous and one-to-one in each argument

(ii) we have

$$F_n(\mathbf{x}) \;=\; \psi\big(\varphi(x_1) + \cdots + \varphi(x_n)\big)$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $\psi\colon \mathbb{R} \to \mathbb{R}$ are continuous and strictly monotone

Axiomatizations of function classes

Recall that a *triangular norm* is a function $T\colon [0,1]^2 \to [0,1]$ which is nondecreasing in each argument, symmetric, associative, and such that $T(1x) = x$

Theorem

Let $F\colon [0,1]^* \to \mathbb{R}$ be such that F_1 is strictly increasing. The following assertions are equivalent:

(i) F is preassociative and $\mathrm{ran}(F_1) = \mathrm{ran}(F)$, F_2 is symmetric, nondecreasing, and $F_2(1x) = F_1(x)$

(ii) we have

$$F = f \circ T$$

where $f\colon [0,1] \to \mathbb{R}$ is strictly increasing and $T\colon [0,1]^* \to [0,1]$ is a triangular norm

Strongly preassociative functions

Definition. We say that $F\colon X^* \to Y$ is *strongly preassociative* if

$$F(\mathbf{xz}) = F(\mathbf{x}'\mathbf{z}') \quad \Rightarrow \quad F(\mathbf{xyz}) = F(\mathbf{x}'\mathbf{y}\mathbf{z}')$$

Theorem

$F\colon X^* \to Y$ is strongly preassociative if and only if
F is preassociative and F_n is symmetric for every $n \in \mathbb{N}$

Open problems

(1) Find new axiomatizations of classes of preassociative functions from existing axiomatizations of classes of associative functions

(2) Find interpretations of preassociativity in fuzzy logic, artificial intelligence,...

Back to the factorization theorem

Theorem

Let $F\colon X^* \to Y$. The following assertions are equivalent:

(i) F is preassociative and $\mathrm{ran}(F_1) = \mathrm{ran}(F)$

(ii) F can be factorized into

$$F = f \circ H$$

where $H\colon X^* \to X$ is associative
$f\colon \mathrm{ran}(H) \to Y$ is one-to-one.

String functions

A *string function* if a function

$$F\colon X^* \to X^*$$

$F\colon X^* \to X^*$ is *associative* (E. Lehtonen) if

$$\boxed{F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*}$$

(same equivalent definitions)

Associative string functions

$F\colon X^* \to X^*$ is *associative* if

$$F(\mathbf{xyz}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}) \qquad \forall\, \mathbf{xyz} \in X^*$$

Examples

- $F = \mathrm{id}$
- $F =$ sorting data in alphabetic order
- $F =$ transforming a string of letters into upper case
- $F =$ removing a given letter, say 'a'
- $F =$ removing all repeated occurrences of letters

$$F(\text{mathematics}) = \text{matheics}$$

Preassociative functions

Theorem

Let $F\colon X^* \to Y$. The following assertions are equivalent:

(i) F is preassociative

(ii) F can be factorized into

$$F \;=\; f \circ H$$

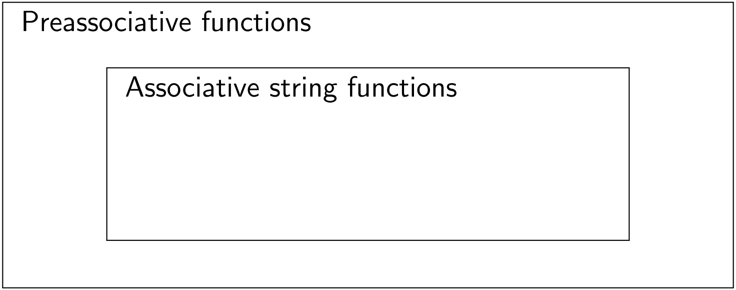
where $H\colon X^* \to X^*$ is associative
$f\colon \mathrm{ran}(H) \to Y$ is one-to-one.

We can add:

(i) $\mathrm{ran}(F) \;=\; \mathrm{ran}(F_1) \cup \cdots \cup \mathrm{ran}(F_m)$

(ii) $H\colon X^* \to X^1 \cup \cdots \cup X^m$

Preassociative functions



Open question:

Find characterizations of classes of associative string functions

Barycentrically associative functions

Notation

$$\mathbf{x}^n \ = \ \mathbf{x} \cdots \mathbf{x} \qquad (n \text{ times})$$

$$|\mathbf{x}| \ = \ \text{length of } \mathbf{x}$$

$F\colon X^* \to X$ is *B-associative* if

$$F(\mathbf{xyz}) \ = \ F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

Alternative names: decomposability, associativity of means.

Barycentrically associative functions

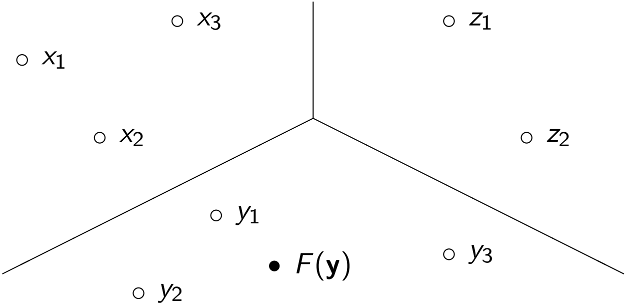


Figure : Barycentric associativity

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z}) \qquad \forall\ \mathbf{xyz} \in X^*$$

Barycentrically associative functions

Theorem (Kolomogoroff-Nagumo, 1930)

$F\colon \mathbb{R}^* \to \mathbb{R}$ is B-associative,

every F_n is

- symmetric
- continuous
- idempotent (i.e., $F_n(x^n) = x$)
- str. increasing in each argument

if and only if

$$F_n(\mathbf{x}) \;=\; \varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i)\right)$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous and strictly monotone

B-preassociative functions

Let Y be a nonempty set

Definition. We say that $F\colon X^* \to Y$ is *B-preassociative* if

$$|\mathbf{y}| = |\mathbf{y}'| \text{ and } F(\mathbf{y}) = F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{xyz}) = F(\mathbf{xy}'\mathbf{z})$$

Examples: $F_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2 \quad (X = Y = \mathbb{R})$

$F_n(\mathbf{x}) = |\mathbf{x}| \quad (X \text{ arbitrary}, \ Y = \mathbb{N})$

Fact. Preassociative functions are B-preassociative

B-preassociative functions

Proposition

$F\colon X^* \to X$ is B-associative if and only if it is B-preassociative and $F(F(\mathbf{x})^{|\mathbf{x}|}) = F(\mathbf{x})$

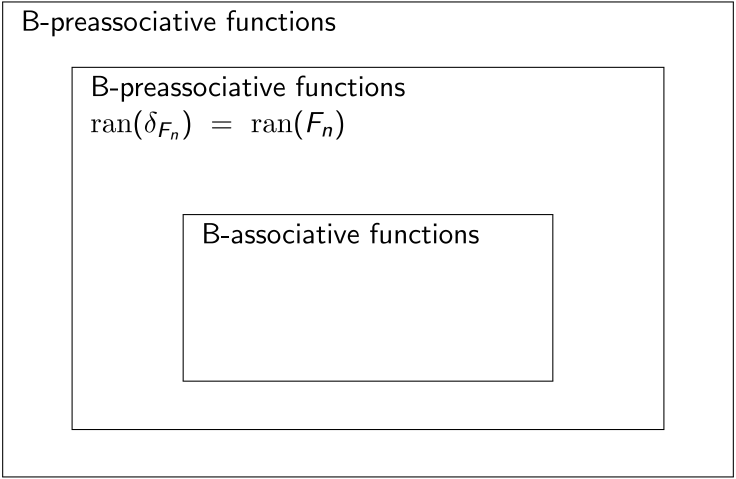
$$F(F(\mathbf{x})^{|\mathbf{x}|}) = F(\mathbf{x}) \qquad \Longleftrightarrow \qquad \delta_{F_n} \circ F_n = F_n \quad (n \in \mathbb{N})$$

$$\delta_{F_n}(x) \ = \ F_n(x^n)$$

Relaxation:

$$\mathrm{ran}(\delta_{F_n}) \ = \ \mathrm{ran}(F_n) \quad (n \in \mathbb{N})$$

B-preassociative functions



B-preassociative functions

Theorem

Let $F\colon X^* \to Y$. The following assertions are equivalent:

(i) F is B-preassociative and $\mathrm{ran}(\delta_{F_n}) = \mathrm{ran}(F_n)$ for all $n \in \mathbb{N}$

(ii) F can be factorized into

$$F_n \;=\; f_n \circ H_n$$

where $H\colon X^* \to X$ is B-associative
$f_n\colon \mathrm{ran}(H_n) \to Y$ is one-to-one.

Open question:

Describe the class of B-preassociative functions

Extension of Kolmogoroff-Nagumo theorem

Theorem

$F\colon \mathbb{R}^* \to \mathbb{R}$ is B-preassociative,

every F_n is

- symmetric
- continuous
- strictly increasing in each argument

if and only if

$$F_n(\mathbf{x}) \;=\; \psi_n\left(\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i)\right)$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $\psi_n\colon \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{N})$ are continuous and strictly increasing

Thank you for your attention !