DETERMINANTS OVER GRADED-COMMUTATIVE ALGEBRAS, A CATEGORICAL VIEWPOINT

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Abstract. We investigate linear superalgebra to higher gradings and commutation factors, given by arbitrary abelian groups and bicharacters. Our central tool is an extension, to monoidal categories of modules, of the Nekludova-Scheunert faithful functor between the categories of graded-commutative and supercommutative algebras. As a result we generalize (super-)trace, determinant and Berezinian to graded matrices over graded-commutative algebras. For instance, on homogeneous quaternionic matrices, we obtain a lift of the Dieudonné determinant to the skew-field of quaternions.

Introduction

Superalgebra theory, which relies on $\mathbb{Z}_2$-grading and Koszul sign rule, admits many applications and turns out to be a non-trivial generalization of the non-graded case, regarding e.g. linear superalgebra, with the notion of Berezinian, or the classification of simple Lie superalgebras. Its success, both in mathematics and physics, prompted from the outset mathematicians to look for generalizations. Mirroring the superalgebras, were thence introduced first $(\mathbb{Z}_2)^n$-graded analogues [29, 30], originally called *color algebras*, and then more general Γ-graded versions, for an arbitrary abelian group Γ. These latter were introduced independently by Scheunert [31] in the Lie algebra case, and by Nekludova in the commutative algebra case (see [22]). The "color" character of the notions considered, is encoded in a pair (Γ, λ) consisting of a *grading group* Γ and a *commutation factor* (or *bicharacter*), i.e., a biadditive skew-symmetric map $\lambda : \Gamma \times \Gamma \to \mathbb{K}^\times$, valued in the multiplicative group of the base field $\mathbb{K}$. Then, a (Γ, λ)-commutative algebra over $\mathbb{K}$ is an associative unital $\mathbb{K}$-algebra, which is Γ-graded, i.e. $\mathcal{A} = \oplus_{\gamma\in\Gamma}\mathcal{A}^\gamma$ and $\mathcal{A}^\alpha \cdot \mathcal{A}^\beta \subset \mathcal{A}^{\alpha+\beta}$ for all $\alpha, \beta \in \Gamma$, and which satisfies the commutation rule

$$ab = \lambda(\alpha, \beta)\, ba\ ,$$

for any homogeneous elements $a \in \mathcal{A}^\alpha$, $b \in \mathcal{A}^\beta$.

The attention to higher gradings is not just a mere question of generalization. Besides the original motives of a possible application to particle physics along the lines of the coupling of superalgebra and SUSY (see [29, 31]), higher gradings appear naturally in different branches of mathematics. In geometry, they play a role in the theory of higher vector bundles [15]. For instance, the algebra of differential forms over a supermanifold happens to be a $((\mathbb{Z}_2)^2, \lambda)$-commutative algebra. Moreover, many classical non-commutative algebras, such as the algebra of quaternions [21, 23, 1, 2, 26] or the algebra of square matrices over $\mathbb{C}$ [5], can be regarded as (Γ, λ)-commutative algebras for appropriate choices of grading group Γ and commutation factor λ. Particularly interesting examples are the Clifford algebras [2]. Indeed, they are the only simple (Γ, λ)-commutative algebras, with Γ finitely generated and $\lambda : \Gamma \times \Gamma \to \{\pm 1\}$ [27].

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Moreover, the quaternion algebra is one of them, $\mathbb{H} \simeq \mathrm{Cl}(0,2)$.

We are interested in graded linear algebra over a graded commutative algebra. While a basic topic, this is the starting point for many developments. Thus, a well-suited notion of tensor product lies at the heart of a putative quaternionic algebraic geometry [18], while Moore determinant plays a central role in quaternionic analysis [3]. In the general graded-commutative setting, the color Lie algebra of traceless elements, with respect to a graded trace, is basic in the derivation based differential calculus [14]. Our aim is precisely to generalize trace, determinant and Berezinian to matrices with entries in a graded commutative algebra. For quaternionic matrices, this is an historical and tough problem, pursued by eminent mathematicians including Cayley, Moore and Dieudonné. We mention the Dieudonné determinant, which is the unique group morphism $\mathrm{Ddet} : \mathsf{GL}(n,\mathbb{H}) \to \mathbb{H}^\times/[\mathbb{H}^\times,\mathbb{H}^\times] \simeq \mathbb{R}_+^\times$, satisfying some normalization condition.

The notion of graded trace, for matrices with entries in a graded commutative algebra $\mathcal{A}$, was introduced in [32]. The introduction of graded determinant and Berezinian was done slightly after in [21]. These objects have been rediscovered in [8], where they received a unique characterization in terms of their properties. In particular, the graded determinant is constructed by means of quasi-determinants and UDL decomposition of matrices. Right after, a cohomological interpretation of the graded Berezinian has been given in [7], in a close spirit to the construction in [21]. The main motivations of the works [8, 7] was to lay the ground for a geometry based on quaternions, or more generally on Clifford algebras, and thus restrict to $((\mathbb{Z}_2)^n,\lambda)$-graded commutative algebras. Note that, so far, the graded determinant is only defined for matrices of degree zero, which is a strong restriction as we will see.

In the present paper, we follow another approach to the problem of defining trace, determinant and Berezinian in the graded setting. We use their formulation as natural transformations and the Nekludova-Scheunert functor between the categories of graded-commutative and supercommutative algebras. Its generalization to monoidal categories of graded and supermodules allows us to pull-back the wanted natural transformations to the graded setting. The morphisms properties of trace, determinant and Berezinian are then preserved if restricted to categorical endomorphisms $f \in \mathrm{End}_{\mathcal{A}}(M)$, of a free graded $\mathcal{A}$-module $M = \oplus_{\gamma\in\Gamma} M^\gamma$. These are $\mathcal{A}$-linear maps $f : M \to M$, which are homogeneous, $f(M^\gamma) \subset M^{\gamma+\tilde{f}}$ for all $\gamma \in \Gamma$, and of degree $\tilde{f} = 0$. If M is finitely generated, the space of all $\mathcal{A}$-linear maps, without assumption on degree, turns out to be equal to the space of internal morphisms $\mathcal{E}nd_{\mathcal{A}}(M)$. We prove that graded determinant and Berezinian admit a proper extension to all homogeneous endomorphisms and that the graded trace can be extended to any $\mathcal{A}$-linear maps, while keeping their defining properties. Note that, in the case of quaternionic matrices, this yields a lifting of the Dieudonné determinant from the quotient space $\mathbb{H}^\times/[\mathbb{H}^\times,\mathbb{H}^\times] \simeq \mathbb{R}_+^\times$ to homogeneous invertible quaternions. Besides, contrary to the previous works [21, 8, 7], we take full advantage of the well-developed theory of graded associative rings and their graded modules (see e.g. [28]). In particular, we clarify the notion of rank of a free graded $\mathcal{A}$-module and establish isomorphisms between various matrix algebras. This shows that the graded determinant of degree zero matrices, over quaternion or Clifford algebras, boils down to the determinant of a real matrix, after a change of basis.

We detail now the content of the paper and state the main theorems. The three first sections are of introductory nature, with one new result (Theorem A). Section 4 is dealing with the graded trace and constitutes a warm-up for the main subject of the paper, that is graded determinant and Berezinian. The grading group is always considered abelian and finitely generated.

In **Section 1**, we recall the basic notions of (Γ, λ)-commutative algebra. In particular, the commutation factor induces a splitting of the grading group in two parts $\Gamma = \Gamma_{\bar{0}} \cup \Gamma_{\bar{1}}$, called *even* and *odd*, and then provides an underlying $\mathbb{Z}_2$-grading $\varphi : \Gamma \to \mathbb{Z}_2$. This is the starting point of Nekludova-Scheunert equivalence [31, 22]. It is given by a family of invertible functors $I_\varsigma : (\Gamma, \lambda)\texttt{-Alg} \to (\Gamma, \lambda^\varsigma)\texttt{-Alg}$, where the modified commutation factor λ^ς factors through the underlying parity, i.e., such that $\lambda^\varsigma(\alpha, \beta) = (-1)^{\varphi(\alpha)\varphi(\beta)}$ for all $\alpha, \beta \in \Gamma$. Hence the *NS-functors* I_ς are valued in supercommutative algebras, with extra Γ-grading. These functors are parameterized by peculiar biadditive maps $\varsigma : \Gamma \times \Gamma \to \mathbb{K}^\times$, called *NS-multipliers*. For later reference, the set of such maps is denoted by $\mathfrak{S}(\lambda)$.

In **Section 2**, we introduce the closed symmetric monoidal category $\Gamma\texttt{-Mod}_{\mathcal{A}}$ of graded modules over a (Γ, λ)-commutative algebra $\mathcal{A}$, following partly [28]. We prove our first result: for every (Γ, λ)-commutative algebra $\mathcal{A}$ and every $\varsigma \in \mathfrak{S}(\lambda)$, there exists a functor

$$\widehat{I}_\varsigma : \quad \Gamma\texttt{-Mod}_{\mathcal{A}} \quad \to \quad \Gamma\texttt{-Mod}_{\underline{\mathcal{A}}},$$

which can be completed into a closed monoidal functor. Here, $\underline{\mathcal{A}} = I_{\varsigma(\mathcal{A})}$ is a supercommutative algebra. This yields in particular the following result, which is the starting point of later developments.

Theorem A. *Let M a graded module over a (Γ, λ)-commutative algebra $\mathcal{A}$. Then, for any map $\varsigma \in \mathfrak{S}(\lambda)$, the map η_ς defined in* (2.6) *is an $\mathcal{A}^0$-module isomorphism*

$$\eta_\varsigma : \quad \mathcal{E}nd_{\mathcal{A}}(M) \quad \to \quad \mathcal{E}nd_{\underline{\mathcal{A}}}\left(\widehat{I}_\varsigma(M)\right)$$

such that

$$\eta_\varsigma(f \circ g) = \varsigma\left(\widetilde{f}, \widetilde{g}\right)^{-1} \eta_\varsigma(f) \circ \eta_\varsigma(g) \ , \tag{0.1}$$

for any pair of homogeneous endomorphisms $f, g \in \mathcal{E}nd_{\mathcal{A}}(M)$.

In **Section 3**, we report on free graded modules and graded matrices, using [28] and [8] as references. In particular, the $\mathcal{A}$-module of $m \times n$ matrices receives a Γ-grading, induced by degrees $\boldsymbol{\mu} \in \Gamma^m$ and $\boldsymbol{\nu} \in \Gamma^n$ associated respectively to rows and columns of the matrices. This module is denoted by $\mathsf{M}(\boldsymbol{\mu} \times \boldsymbol{\nu}; \mathcal{A})$ or simply $\mathsf{M}(\boldsymbol{\nu}; \mathcal{A})$ if $\boldsymbol{\mu} = \boldsymbol{\nu}$. If a free graded $\mathcal{A}$-module M admits a basis of homogeneous elements (e_i) of degrees $(\nu_i) = \boldsymbol{\nu}$, then we have an isomorphism of Γ-algebras

$$\mathcal{E}nd_{\mathcal{A}}(M) \simeq \mathsf{M}(\boldsymbol{\nu}; \mathcal{A}) \ .$$

The statement of the previous section then rewrites in matrix form, the map η_ς becoming then an $\mathcal{A}^0$-modules isomorphism

$$J_\varsigma : \mathsf{M}(\boldsymbol{\nu}; \mathcal{A}) \to \mathsf{M}(\boldsymbol{\nu}; \underline{\mathcal{A}}).$$

To complete our understanding of matrix algebras, we determine conditions on degrees $\boldsymbol{\nu}, \boldsymbol{\mu} \in \Gamma^n$ such that $\mathsf{M}(\boldsymbol{\nu}; \mathcal{A}) \simeq \mathsf{M}(\boldsymbol{\mu}; \mathcal{A})$. This is equivalent to find bases of degrees $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$ in the same free graded $\mathcal{A}$-module. If $\mathcal{A}^0$ is a local ring, we prove that two such bases exist if and only if $\boldsymbol{\nu} \in (\Gamma_{\mathcal{A}^\times})^n + \boldsymbol{\mu}$ (up to permutation of components). Here, $\Gamma_{\mathcal{A}^\times}$ denotes the set of degrees $\gamma \in \Gamma$ such that $\mathcal{A}^\gamma$ contains at least one invertible element. This result is closely related to a classical Theorem of Dade [9] (cf. Proposition 3.3).

In **Section 4**, we study the *graded trace*. By definition, this is a degree-preserving $\mathcal{A}$-linear map $\Gamma\mathrm{tr} : \mathcal{E}nd_{\mathcal{A}}(M) \to \mathcal{A}$, which is also a (Γ, λ)-Lie algebra morphism. The latter property means that

$$\Gamma\mathrm{tr}(f \circ g) - \lambda(\tilde{f}, \tilde{g})\, \Gamma\mathrm{tr}(g \circ f) = 0$$

for any pair of homogeneous endomorphisms f, g of respective degrees $\tilde{f}, \tilde{g} \in \Gamma$. Using the map η_ς of Theorem A and the supertrace $\mathrm{str} : \mathcal{E}nd_{\mathcal{A}}(M) \to \underline{\mathcal{A}}$, we construct a graded trace and show it is essentially unique.

Theorem B. *Let $\mathcal{A}$ be a (Γ, λ)-commutative algebra, M be a free graded $\mathcal{A}$-module and $\varsigma \in \mathfrak{S}(\lambda)$. Up to multiplication by a scalar in $\mathcal{A}^0$, there exists a unique graded trace $\Gamma\mathrm{tr} : \mathcal{E}nd_{\mathcal{A}}(M) \to \mathcal{A}$. One is given by the map $\mathrm{str} \circ \eta_\varsigma$, which does not depend on $\varsigma \in \mathfrak{S}(\lambda)$.*

In matrix form, the graded trace reads as $\Gamma\mathrm{tr} = \mathrm{str} \circ J_\varsigma$, and its evaluation on a homogeneous matrix $X = (X^i{}_j)_{i,j} \in \mathsf{M}^x(\boldsymbol{\nu}; \mathcal{A})$ gives

$$\Gamma\mathrm{tr}(X) = \sum_i \lambda(\nu_i, x + \nu_i) X^i{}_i \ .$$

As a result our graded trace on matrices coincides with the one introduced in [32] and is a generalization of those studied in [21, 8, 14].

As for the graded determinant and Berezinian, the situation is more involved. For the sake of clarity, we first treat the case of purely-even algebras. These are (Γ, λ)-commutative algebras $\mathcal{A}$ with no odd elements, i.e., for which $\Gamma = \Gamma_{\bar{0}}$. In this case, each NS-functor I_ς sends $\mathcal{A}$ to a classical commutative algebra $\underline{\mathcal{A}} := I_\varsigma(\mathcal{A})$, and we make use of the classical determinant $\det_{\underline{\mathcal{A}}}$ over $\underline{\mathcal{A}}$.

In **Section 5**, we restrict to the subalgebra of 0-degree matrices, denoted by $\mathsf{M}^0(\boldsymbol{\nu}; \mathcal{A})$, and to the subgroup of invertible matrices, denoted by $\mathsf{GL}^0(\boldsymbol{\nu}; \mathcal{A})$. As already mentioned, the arrow function of the NS-functor $\widehat{I_\varsigma}$ allows then to pull-back the determinant to the graded side, while keeping its multiplicativity property. Following [25], we generalize the multiplicative characterization of the classical determinant to the graded setting.

Theorem C. *Let $\boldsymbol{\nu} \in \Gamma^n$, $n \in \mathbb{N}$. There exists a unique family of maps*

$$\Gamma\mathrm{det}^0_{\mathcal{A}} : \ \mathsf{GL}^0(\boldsymbol{\nu}; \mathcal{A}) \to (\mathcal{A}^0)^\times \ ,$$

parameterized by objects $\mathcal{A}$ in (Γ, λ)-$\mathtt{Alg}$, such that

AI. *it defines a natural transformation*

$$\begin{tikzcd}[column sep=huge] (\Gamma, \lambda)\text{-}\mathtt{Alg} \arrow[r, bend left=50, "\mathsf{GL}^0(\boldsymbol{\nu};-)", ""{name=U, below}] \arrow[r, bend right=50, "((-)^0)^\times"', ""{name=D, above}] & \mathtt{Grp} \arrow[Rightarrow, from=U, to=D, "\Gamma\mathrm{det}^0"] \end{tikzcd}$$

AII. *for any invertible $a \in \mathcal{A}^0$,*

$$\Gamma\mathrm{det}^0_{\mathcal{A}} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & a \end{pmatrix} = a \ .$$

The natural transformation $\Gamma\mathrm{det}^0$ is called the graded determinant *and satisfies*

$$\Gamma\mathrm{det}^0_{\mathcal{A}}(X) = \det_{\underline{\mathcal{A}}} \left(J_\varsigma(X) \right) \ , \tag{0.2}$$

for all $X \in \mathsf{GL}^0(\boldsymbol{\nu}; \mathcal{A})$ and all $\varsigma \in \mathfrak{S}(\lambda)$.

The axiom AI. implies that $\Gamma\mathrm{det}^0_{\mathcal{A}}$ is a group morphism. Equation (0.2) allows to extend $\Gamma\mathrm{det}^0$ to all matrices in $\mathsf{M}^0(\boldsymbol{\nu};\mathcal{A})$, and then $\Gamma\mathrm{det}^0(X)$ is invertible if and only if X is invertible. Our graded determinant coincides with the ones defined in [21, 8], and, as an advantage of our construction, we find an explicit formula for $\Gamma\mathrm{det}^0(X)$. This is the same polynomial expression in the entries of the matrix X that the classical determinant would be, but here the order of the terms in each monomial is very important: only with respect to a specific order does the formula retain such a nice form. This situation is analogous to that of Moore's determinant, defined for Hermitian quaternionic matrices (see e.g. [4]). In addition, if $\mathcal{A}$ admits invertible homogeneous elements of any degree, we provide a transition matrix P, such that PXP^{-1} is a matrix with entries in the commutative algebra $\mathcal{A}^0$ and $\Gamma\mathrm{det}^0_{\mathcal{A}}(X) = \det_{\mathcal{A}^0}(PXP^{-1})$, for all $X \in \mathsf{M}^0(\boldsymbol{\nu};\mathcal{A})$. This recovers a result in [21].

In **Section 6**, we investigate the natural extension of the graded determinant $\Gamma\mathrm{det}^0$ to arbitrary graded matrices, by defining maps in the same way

$$\Gamma\mathrm{det}_{\varsigma} = \det \circ J_{\varsigma} : \quad \mathsf{M}(\boldsymbol{\nu};\mathcal{A}) \quad \to \quad \mathcal{A}\,,$$

where $\varsigma \in \mathfrak{S}(\lambda)$. These functions share determinant-like properties and can be characterized as follows.

Theorem D. *Let $\mathfrak{s} : \Gamma \times \Gamma \to \mathbb{K}^{\times}$. If $\mathfrak{s} \in \mathfrak{S}(\lambda)$, then $\Gamma\mathrm{det}_{\mathfrak{s}}$ satisfies properties* I.–IV. *below. Conversely, if there exists, for all $\boldsymbol{\nu} \in \bigcup_{n\in\mathbb{N}^*} \Gamma^n$, a natural transformation*

$$\begin{tikzcd}[column sep=huge] (\Gamma,\lambda)\text{-}\mathtt{Alg} \arrow[r, bend left, "\mathsf{M}(\boldsymbol{\nu};-)"] \arrow[r, bend right, "\text{forget}"'] \arrow[r, phantom, "\Downarrow \Delta_{\mathfrak{s}}"] & \mathtt{Set} \end{tikzcd} \tag{0.3}$$

satisfying properties I.–IV.*, then $\mathfrak{s} \in \mathfrak{S}(\lambda)$ and $\Delta_{\mathfrak{s}} = \Gamma\mathrm{det}_{\mathfrak{s}}$. The properties are*

I. *extension of $\Gamma\mathrm{det}^0$, i.e., $\Delta_{\mathfrak{s}}(X) = \Gamma\mathrm{det}^0(X)$ for all $X \in \mathsf{GL}^0(\boldsymbol{\nu};\mathcal{A})$;*

II. *weak multiplicativity, i.e., $\Delta_{\mathfrak{s}}(XY) = \Delta_{\mathfrak{s}}(X)\cdot\Delta_{\mathfrak{s}}(Y)$ for all $X, Y \in \mathsf{M}(\boldsymbol{\nu};\mathcal{A})$ such that either X or Y is homogeneous of degree 0;*

III. *additivity in the rows, i.e., $\Delta_{\mathfrak{s}}(Z) = \Delta_{\mathfrak{s}}(X) + \Delta_{\mathfrak{s}}(Y)$ for all graded matrices*

$$X = \begin{bmatrix} \mathtt{x}^1 \\ \vdots \\ \mathtt{x}^n \end{bmatrix}, \; Y = \begin{bmatrix} \mathtt{y}^1 \\ \vdots \\ \mathtt{y}^n \end{bmatrix}, \; Z = \begin{bmatrix} \mathtt{z}^1 \\ \vdots \\ \mathtt{z}^n \end{bmatrix} \in \mathsf{M}(\boldsymbol{\nu};\mathcal{A})$$

whose rows satisfy, for a fixed index $1 \le k \le n$,

$$\mathtt{z}^i = \mathtt{x}^i = \mathtt{y}^i\,, \text{ if } i \neq k \quad \text{and} \quad \mathtt{z}^k = \mathtt{x}^k + \mathtt{y}^k\,; \tag{0.4}$$

IV. *heredity for diagonal matrices, i.e., for all $c \in \mathcal{A}^{\widetilde{c}}$,*

$$\Delta_{\mathfrak{s}}\left(\begin{array}{c|c} D & \\ \hline & \lambda(\widetilde{c},\nu_n)\,c \end{array}\right) = \mathfrak{s}\Big(\widetilde{c},\, \deg\left(\Delta_{\mathfrak{s}}\left(D\right)\right)\Big)\, c\cdot\Delta_{\mathfrak{s}}\left(D\right)\,, \tag{0.5}$$

where $D \in \mathsf{M}(\boldsymbol{\nu}';\mathcal{A})$, with $\boldsymbol{\nu}' = (\nu_1,\ldots,\nu_{n-1})$, is a diagonal matrix with homogeneous entries and $\deg\left(\Delta_{\mathfrak{s}}\left(D\right)\right)$ is the degree[1] *of $\Delta_{\mathfrak{s}}\left(D\right)$.*

[1]By induction on the rank, $\Delta_{\mathfrak{s}}\left(D\right)$ is homogeneous.

Despite the above characterization, the determinant-like functions $\Gamma\mathrm{det}_\varsigma$ are not proper determinants: they are not multiplicative and do not characterize invertibility of matrices in general. They depend both on the choices of NS-multiplier $\varsigma \in \mathfrak{S}(\lambda)$ and of grading $\boldsymbol{\nu}$ on matrices. We illustrate these drawbacks on 2×2 quaternionic matrices. Note that any multiplicative determinant satisfies properties I.-II., and property IV. is rather natural and satisfied, e.g., by Dieudonné determinant. Hence, Theorem D puts severe restriction on the existence of multiplicative and multiadditive determinants. On quaternionic matrices, as is well-known, no such determinant exists [12, 4]. Nevertheless, $\Gamma\mathrm{det}_\varsigma$ define proper determinants once restricted to homogeneous matrices. In particular, on homogeneous quaternionic matrices, $\Gamma\mathrm{det}_\varsigma$ provide lifts of the Dieudonné determinant to $\mathbb{H}$. Note that restriction to homogeneous elements is also needed for the recent extension of Dieudonné determinant to graded division algebras [16].

Finally, in **Section 7**, we deal with the general case of a (Γ, λ)-commutative algebra $\mathcal{A}$ with odd elements. In this case, each NS-functor I_ς sends $\mathcal{A}$ to a supercommutative algebra $\underline{\mathcal{A}} := I_\varsigma(\mathcal{A})$, and we use the classical Berezinian Ber over $\underline{\mathcal{A}}$ to define the *graded Berezinian* as $\Gamma\mathrm{Ber}_\varsigma := \mathrm{Ber} \circ J_\varsigma$, with $\varsigma \in \mathfrak{S}(\lambda)$. We get the following results.

Theorem E. *Let $r_{\bar{0}}, r_{\bar{1}} \in \mathbb{N}$ and $\boldsymbol{\nu} = (\boldsymbol{\nu}_{\bar{0}}, \boldsymbol{\nu}_{\bar{1}}) \in \Gamma_{\bar{0}}^{r_{\bar{0}}} \times \Gamma_{\bar{1}}^{r_{\bar{1}}}$.*

1. *The maps $\Gamma\mathrm{Ber}_\varsigma : \mathsf{GL}^0(\boldsymbol{\nu}; \mathcal{A}) \to (\mathcal{A})^\times$ do not depend on the choice of $\varsigma \in \mathfrak{S}(\lambda)$ and define a group morphism, denoted by $\Gamma\mathrm{Ber}^0$.*
2. *The maps $\Gamma\mathrm{Ber}_\varsigma$ are well-defined on matrices $X \in \mathsf{GL}^x(\boldsymbol{\nu}; \mathcal{A})$, $x \in \Gamma_{\bar{0}}$, and given by*

$$\Gamma\mathrm{Ber}_\varsigma(X) = \varsigma(x,x)^{-r_{\bar{1}}(r_{\bar{0}} - r_{\bar{1}})}\ \Gamma\mathrm{det}_\varsigma(\mathcal{X}_{00} - \mathcal{X}_{01}\mathcal{X}_{11}^{-1}\mathcal{X}_{10}) \cdot \Gamma\mathrm{det}_\varsigma(\mathcal{X}_{11})^{-1}\ , \tag{0.6}$$

where $\begin{pmatrix} \mathcal{X}_{00} & \mathcal{X}_{01} \\ \mathcal{X}_{10} & \mathcal{X}_{11} \end{pmatrix} = X$ *is the block-decomposition of X according to parity.*

Note that $\varsigma(0,0) = 1$. The graded Berezinian $\Gamma\mathrm{Ber}^0$ coincides with the ones introduced formerly, and independently, in [21] and [8, 7]. For a characterization of $\Gamma\mathrm{Ber}^0$, we refer to [8]. The formula $\Gamma\mathrm{Ber}_\varsigma = \mathrm{Ber} \circ J_\varsigma$, defining the graded Berezinian, cannot be extended further to inhomogeneous matrices. Indeed, the map J_ς does not preserves invertibility of inhomogeneous matrices in general and the Berezinian is only defined over even invertible matrices.

For the sake of self-consistency of the paper, we present some basic notions of category theory in an appendix.

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Notation. In the whole paper, $\mathbb{K}$ is a field of characteristic zero (e.g. $\mathbb{R}$ or $\mathbb{C}$) and $\mathbb{K}^\times$ denotes its group of invertible elements. The grading group is a finitely generated abelian group, denoted by $(\Gamma,+)$, with neutral element 0.

1. Categories of Graded Commutative Algebras

In this section, we review the basic notions of graded-commutative algebra and present the Nekludova-Scheunert Theorem, following [31, 22]. For general notions on graded rings and algebras, we refer to [6, 9, 28].

1.1. **Definitions and examples.** A Γ-*algebra* is an associative unital algebra $\mathcal{A}$ over $\mathbb{K}$, with the structure of a Γ-graded vector space $\mathcal{A} = \oplus_{\alpha\in\Gamma}\mathcal{A}^\alpha$, in which the multiplication respects the grading,

$$\mathcal{A}^\alpha \cdot \mathcal{A}^\beta \subset \mathcal{A}^{\alpha+\beta}\,, \quad \forall \alpha, \beta \in \Gamma\,. \tag{1.1}$$

If $\mathcal{A}^\alpha \cdot \mathcal{A}^\beta = \mathcal{A}^{\alpha+\beta}$ for all $\alpha, \beta \in \Gamma$, we say that $\mathcal{A}$ is *strongly graded*. The elements in $\mathcal{A}^\gamma$ are called *homogeneous elements* of degree γ. The Γ-algebra $\mathcal{A}$ is called a *crossed product* if there exists invertible elements of each degree $\gamma \in \Gamma$ (see Remark 1.11 for an alternative, more usual, definition). It is a *graded division algebra* if all non-zero homogeneous elements are invertible. Clearly, if a Γ-algebra is a crossed product, then it is strongly graded. The converse holds if $\mathcal{A}^0$ is a local ring [9]. *Morphisms of* Γ-*algebras* are morphisms of algebras $f : \mathcal{A} \to \mathcal{B}$ that preserve the degree, $f(\mathcal{A}^\gamma) \subset \mathcal{B}^\gamma$, for all $\gamma \in \Gamma$. The kernel of such morphism is an *homogeneous ideal* of $\mathcal{A}$, i.e., an ideal I such that

$$I = \bigoplus_{\gamma\in\Gamma} (I \cap \mathcal{A}^\gamma)\ .$$

The quotient of a Γ-algebra by a homogeneous ideal is again a Γ-algebra. The class of Γ-algebras (over a fixed field $\mathbb{K}$) and corresponding morphisms form the category Γ-`Alg`.

Let λ be a map $\lambda : \Gamma \times \Gamma \to \mathbb{K}^\times$. A (Γ, λ)-*commutative algebra* is a Γ-algebra in which the multiplication is λ-commutative, namely

$$a \cdot b = \lambda\big(\widetilde{a}, \widetilde{b}\,\big)\, b \cdot a\ , \tag{1.2}$$

for all homogeneous elements $a, b \in \mathcal{A}$ of degrees $\widetilde{a}, \widetilde{b} \in \Gamma$. *Morphisms of* (Γ, λ)-*commutative algebras* are morphisms of Γ-algebras. The class of (Γ, λ)-commutative algebras (over a fixed field $\mathbb{K}$) and corresponding morphisms form a full subcategory (Γ, λ)-`Alg` of Γ-`Alg`.

Example 1.1. If $\mathcal{A}^0$ is a unital associative and commutative algebra, then the group algebra $\mathcal{A}^0[\Gamma]$ is a $(\Gamma, \mathbf{1})$-commutative algebra, with $\mathbf{1} : \Gamma \times \Gamma \to \mathbb{K}$ the constant map equal to $1 \in \mathbb{K}$. This is a crossed product in general and a graded division algebra if and only if $\mathcal{A}^0$ is a field.

Example 1.2. *Supercommutative algebras* are $(\mathbb{Z}_2, \lambda^{\mathrm{super}})$-commutative algebras, with

$$\lambda^{\mathrm{super}}(x, y) = (-1)^{xy}\ ,$$

for all $x, y \in \mathbb{Z}_2$. The category $(\mathbb{Z}_2, \lambda^{\mathrm{super}})$-`Alg` is denoted for short by `SAlg`.

Example 1.3. Let $n \in \mathbb{N}$. The local commutative algebra

$$\mathbb{R}[\varepsilon_1, \ldots, \varepsilon_n]/(\varepsilon_1^2, \ldots, \varepsilon_n^2)\ ,$$

which generalizes dual numbers, receives a $(\mathbb{Z}_2)^n$-grading by setting $\tilde{\varepsilon}_1 = (1, 0, \ldots, 0),\ \ldots\ , \tilde{\varepsilon}_n = (0, \ldots, 0, 1)$. This is a $((\mathbb{Z}_2)^n, \lambda)$-commutative algebra with commutation factor

$$\lambda(x, y) := (-1)^{\langle\, x,\, y\,\rangle}\ , \tag{1.3}$$

where $\langle -, - \rangle$ denotes the standard scalar product of binary n-vectors. This algebra is not strongly graded if $n \geq 1$.

Example 1.4. Let $p, q \in \mathbb{N}$ and $n = p + q$. The real *Clifford algebra* $\mathrm{Cl}(p, q)$ is the real algebra with n generators $(e_i)_{i=1,\ldots,n}$, satisfying the relations $e_ie_j + e_je_i = \pm 2\delta_{ij}$, with plus for $i \leq p$ and minus otherwise. Setting $\tilde{e}_1 = (1, 0, \ldots, 0, 1),\ \ldots\ , \tilde{e}_n = (0, \ldots, 0, 1, 1)$, it turns into a $\big((\mathbb{Z}_2)^{n+1}, \lambda\big)$-commutative algebra with commutation factor of the form (1.3). For such a grading, the real Clifford algebras are graded division algebras.

Example 1.5. The algebra of $n \times n$ matrices $\mathsf{M}(n;\mathbb{C})$ turns into a graded division algebra with respect to the grading group $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_n$ and is graded commutative for a commutation factor $\lambda : \Gamma \times \Gamma \to \mathbb{U}_n$, with $\mathbb{U}_n$ the group of n-th roots of unity (see [5]).

1.2. **Basic properties.** Let $\mathcal{A}$ be a (Γ, λ)-commutative algebra. Clearly, the unit of $\mathcal{A}$, denoted by $1_{\mathcal{A}}$ or simply 1, is necessarily of degree 0. The commutation relation (1.2), together with the associativity of $\mathcal{A}$, implies that the *commutation factor* λ on Γ satisfies the three following conditions (see [31]):

(C1) $\lambda(x,y)\lambda(y,x) = 1$,
(C2) $\lambda(x+y,z) = \lambda(x,z)\lambda(y,z)$,
(C3) $\lambda(z,x+y) = \lambda(z,x)\lambda(z,y)$,

for all $x,y,z \in \Gamma$. Conversely, for any commutation factor λ, there exists a (Γ, λ)-commutative algebra. An easy consequence of the conditions (C2) and (C3) is that

$$\lambda(0,x) = \lambda(x,0) = 1 ,$$

hence $\mathcal{A}^0 \subseteq Z(\mathcal{A})$, the center of $\mathcal{A}$. As for the condition (C1), it implies $(\lambda(x,x))^2 = 1$, for all $x \in \Gamma$. Therefore the commutation factor induces a splitting of the grading group into an "even" and an "odd" part,

$$\Gamma = \Gamma_{\bar{0}} \cup \Gamma_{\bar{1}} \tag{1.4}$$
$$\text{with} \quad \Gamma_{\bar{0}} := \{x \in \Gamma \,:\, \lambda(x,x) = 1\} \quad \text{and} \quad \Gamma_{\bar{1}} := \{x \in \Gamma \,:\, \lambda(x,x) = -1\} .$$

This is equivalently encoded into a morphism of additive groups

$$\phi_\lambda : \quad \Gamma \quad \to \quad \mathbb{Z}_2 ,$$

with kernel $\Gamma_{\bar{0}}$. The map ϕ_λ provides a $\mathbb{Z}_2$-grading on every (Γ, λ)-commutative algebra $\mathcal{A}$, also called *parity* and denoted by $\bar{\gamma} := \phi_\lambda(\gamma) \in \mathbb{Z}_2$. Homogeneous elements of $\mathcal{A}$ are named *even* or *odd* depending on the parity of their degree. Notice that odd elements of $\mathcal{A}$ are nilpotent. Hence, if $\mathcal{A}$ is strongly graded, we have $\Gamma_{\bar{1}} = \emptyset$. The group morphism ϕ_λ induces a *regrading functor*

$$\Phi_\lambda : \quad (\Gamma,\lambda)\text{-}\mathtt{Alg} \quad \to \quad \mathbb{Z}_2\text{-}\mathtt{Alg} , \tag{1.5}$$

equal to the identity on arrows and such that $\Phi_\lambda(\mathcal{A})$ is the algebra $\mathcal{A}$ with $\mathbb{Z}_2$-grading

$$(\Phi_\lambda(\mathcal{A}))_{\bar{0}} = \bigoplus\nolimits_{\gamma \in \Gamma_{\bar{0}}} \mathcal{A}^\gamma \quad \text{and} \quad (\Phi_\lambda(\mathcal{A}))_{\bar{1}} = \bigoplus\nolimits_{\gamma \in \Gamma_{\bar{1}}} \mathcal{A}^\gamma .$$

The $\mathbb{Z}_2$-graded algebra $\Phi_\lambda(\mathcal{A})$ is supercommutative if and only if λ can be factorized through the parity, as stated below.

Proposition 1.6. [22] *Let λ a commutation factor over Γ. Then, the two following statements are equivalent:*

(i) for all $x, y \in \Gamma$, $\lambda(x,y) = \lambda^{\mathrm{super}}(\phi_\lambda(x), \phi_\lambda(y))$;
(ii) the functor Φ_λ takes values in `SAlg`.

1.3. **Change of commutation factor.** Let $\varsigma : \Gamma \times \Gamma \to \mathbb{K}^\times$ be a map. One can twist the multiplication of a (Γ, λ)-commutative algebra $\mathcal{A} = (A, \cdot)$ by the map ς in the following way:

$$a \star b := \varsigma(\widetilde{a}, \widetilde{b})\, a \cdot b . \tag{1.6}$$

The resulting graded algebra $\underline{\mathcal{A}} := (A, \star)$ is an associative deformation of $\mathcal{A}$ if and only if ς satisfies the cocycle condition

$$\varsigma(x, y+z)\varsigma(y,z) = \varsigma(x,y)\varsigma(x+y,z) ,$$

for all $x, y, z \in \Gamma$. If in addition $\varsigma(0,0) = 1$, then $\underline{\mathcal{A}}^0 = \mathcal{A}^0$ as algebras and ς is called *multiplier* (see [31]). Note that biadditive maps are multipliers.

If ς is a multiplier, the deformed algebra $\underline{\mathcal{A}}$ is a $(\Gamma, \lambda^\varsigma)$-commutative algebra, the commutation factor $\lambda^\varsigma : \Gamma \times \Gamma \to \mathbb{K}^\times$ being given by

$$\lambda^\varsigma(a,b) := \lambda(a,b)\varsigma(a,b)(\varsigma(b,a))^{-1} .$$

Several multipliers can lead to the same change of commutation factor. More precisely,

Lemma 1.7. [31] *Let ς, ς' be two multipliers on Γ. Then, $\lambda^\varsigma = \lambda^{\varsigma'}$ if and only if there exists a biadditive symmetric map $\mathfrak{b} : \Gamma \times \Gamma \to \mathbb{K}^\times$ such that $\varsigma'(x,y) = \varsigma(x,y)\,\mathfrak{b}(x,y)$ for all $x, y \in \Gamma$.*

For a classification of multipliers and commutation factors, we refer to [14].

1.4. **The Nekludova-Scheunert Theorem.** Let ς be a multiplier on Γ. In general, the (Γ, λ)-commutative algebra $\mathcal{A} = (A, \cdot)$ is not isomorphic to the $(\Gamma, \lambda^\varsigma)$-commutative algebra $\underline{\mathcal{A}} := (A, \star)$ obtained by change of commutation factor. The link between these two types of algebra is encompassed by a functor, defined below.

Proposition 1.8. [22] *Let $\varsigma : \Gamma \times \Gamma \to \mathbb{K}^\times$ be a multiplier. There exists an isomorphism of categories*

$$\begin{array}{rcl} I_\varsigma : (\Gamma, \lambda)\text{-}\mathtt{Alg} & \to & (\Gamma, \lambda^\varsigma)\text{-}\mathtt{Alg} \\ \mathcal{A} = (A, \cdot) & \mapsto & \underline{\mathcal{A}} = (A, \star) \end{array} \tag{1.7}$$

equal to the identity on morphisms.

The proof is obvious, the inverse functor is given by I_δ with $\delta(x,y) := (\varsigma(x,y))^{-1}$.

According to Scheunert [31], for any commutation factor λ, there exists a multiplier ς such that λ^ς factorizes through the parity. This uses the fact that Γ is a finitely generated abelian group. By Lemma 1.7, the set of such *NS-multipliers*,

$$\mathfrak{S}(\lambda) := \{\varsigma : \Gamma \times \Gamma \to \mathbb{K}^\times \,|\, \lambda^\varsigma = \lambda^{\text{super}} \circ (\phi_\lambda \times \phi_\lambda)\} ,$$

is parameterized by symmetric biadditive maps from $\Gamma \times \Gamma$ to $\mathbb{K}^\times$. It turns out that all $\varsigma \in \mathfrak{S}(\lambda)$ are biadditive [31]. This result, combined with Propositions 1.6 and 1.8, yields the following Theorem, due to Nekludova [22, p. 280]. Scheunert has proved a similar theorem, in the Lie algebra setting [31].

Theorem 1.9 (The Nekludova-Scheunert Theorem)**.** *Let λ be a commutation factor on Γ. There exists a biadditive map ς such that the functor Φ_{λ^ς} (defined in* (1.5)*) takes values in* $\mathtt{SAlg}$*, and the composite*

$$(\Gamma, \lambda)\text{-}\mathtt{Alg} \xrightarrow[I_\varsigma]{\sim} (\Gamma, \lambda^\varsigma)\text{-}\mathtt{Alg} \xrightarrow[\Phi_{\lambda^\varsigma}]{} \mathtt{SAlg}$$

is a faithful functor.

If ς is a NS-multiplier, the functors I_ς satisfy automatically Theorem 1.9 and we call them *NS-functors*.

The Γ-algebras which are crossed product are characterized in [28]. This specifies as follows for (Γ, λ)-commutative algebras.

Corollary 1.10. *An algebra $\mathcal{A}$ is (Γ, λ)-commutative and a crossed product if and only if $\mathcal{A} \simeq I_\varsigma^{-1}(\mathcal{A}^0[\Gamma])$ for some $\varsigma \in \mathfrak{S}(\lambda)$.*

Proof. Let $\mathcal{A}$ be a (Γ,λ)-commutative algebra. Since odd elements are nilpotent, $\mathcal{A}$ is a crossed product implies $\Gamma=\Gamma_{\bar 0}$. For such Γ, the NS-functor I_ς, with $\varsigma\in\mathfrak{S}(\lambda)$, takes values in the category $(\Gamma,\mathbf{1})$-`Alg` of commutative Γ-graded algebras. Moreover, it clearly sends crossed products to crossed products. Thence, from the crossed product $\mathcal{A}^0[\Gamma]$, we get a (Γ,λ)-commutative algebra $I_\varsigma(\mathcal{A}^0[\Gamma])$ which is also a crossed product. Conversely, if $\mathcal{A}$ is (Γ,λ)-commutative and a crossed product, it is easy to prove that $I_\varsigma^{-1}(\mathcal{A})$ is isomorphic to $\mathcal{A}^0[\Gamma]$, and the result follows. □

Remark 1.11. *By definition, the algebra $I_\varsigma^{-1}(\mathcal{A}^0[\Gamma])$ is the $\mathcal{A}^0$-module generated by the group elements $(e_\alpha)_{\alpha\in\Gamma}$, with product law $e_\alpha e_\beta=\varsigma(\alpha,\beta)e_{\alpha+\beta}$ for all $\alpha,\beta\in\Gamma$. This algebra is usually denoted by $\mathcal{A}^0\rtimes_\varsigma\Gamma$ and called* crossed product of $\mathcal{A}$ by Γ relatively to ς*.*

We provide now some examples of NS-functors.

Example 1.12. The algebra of differential forms Ω over a smooth supermanifold is a $\mathbb{Z}_2\times\mathbb{Z}_2$-algebra, where the $\mathbb{Z}_2\times\mathbb{Z}_2$-degree of a homogeneous differential form α is provided by

$$\widetilde{\alpha}=(\bar\alpha,|\alpha|)\ ,$$

with $\bar\alpha$ the "super" degree and $|\alpha|$ the cohomological degree modulo 2. There exist two conventions for the commutation relation of homogeneous differential forms [10],

$$\begin{aligned} \alpha\wedge\beta&=(-1)^{\bar\alpha\bar\beta+|\alpha||\beta|}\beta\wedge\alpha\ , &&\text{Deligne sign rule,}\\ \alpha\wedge\beta&=(-1)^{(\bar\alpha+|\alpha|)(\bar\beta+|\beta|)}\beta\wedge\alpha\ , &&\text{Bernstein-Leites sign rule.}\end{aligned}$$

They correspond to two distinct commutation factors on $\mathbb{Z}_2\times\mathbb{Z}_2$, which are related by a NS-functor I_ς (see [10, p. 64]), with NS-multiplier given by $\varsigma(\widetilde\alpha,\widetilde\beta)=(-1)^{\bar\alpha|\beta|}$. Moreover, the morphism

$$\phi:\ \mathbb{Z}_2\times\mathbb{Z}_2\ni\widetilde\alpha\ \mapsto\ \bar\alpha+|\alpha|\in\mathbb{Z}_2\ ,$$

induces a parity on Ω, which turns Ω into a supercommutative algebra for the Bernstein-Leites sign rule.

Example 1.13. Consider the algebra of quaternions $\mathbb{H}\simeq\mathrm{Cl}(0,2)$, with multiplication law

$\cdot$	i	j	k
i	-1	k	$-$ j
j	$-$ k	-1	i
k	$+$ j	$-$ i	-1

(1.8)

Setting $\widetilde{\mathrm{i}},\widetilde{\mathrm{j}},\widetilde{\mathrm{k}}\in(\mathbb{Z}_2)^3$ as follows,

$$\widetilde{\mathrm{i}}:=(0,1,1)\ ,\quad \widetilde{\mathrm{j}}:=(1,0,1)\ ,\quad \widetilde{\mathrm{k}}:=(1,1,0)\ ,$$

the algebra $\mathbb{H}$ turns into a real $((\mathbb{Z}_2)^3,\lambda)$-commutative algebra with commutation factor $\lambda(x,y):=(-1)^{\langle x,y\rangle}=(-1)^{x_1y_1+x_2y_2+x_3y_3}$, where $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)\in(\mathbb{Z}_2)^3$.

The NS-multiplier $\varsigma(x,y) := (-1)^{x_1(y_2+y_3)+x_2y_3}$ yields the following twisted multiplication:

$\star$	i	j	k
i	$+1$	$-\,$k	$-\,$j
j	$-\,$k	$+1$	$-\,$i
k	$-\,$j	$-\,$i	$+1$

Another choice of NS-multiplier, e.g., $\varsigma(x,y) := (-1)^{x_1y_3+x_2(y_1+y_2+y_3)}$, leads to a different product:

$\star$	i	j	k
i	-1	k	$-\,$j
j	k	$+1$	i
k	$-\,$j	i	-1

Since the two above tables are symmetric, both products "$\star$" are classically commutative.

Example 1.14. According to Example 1.4, the Clifford algebra $\mathrm{Cl}(p,q)$ is a $\big((\mathbb{Z}_2)^{n+1},\lambda\big)$-commutative algebra ($n=p+q$), with $\lambda(-,-)=(-1)^{\langle\,-,\,-\rangle}$. Choosing as multiplier the biadditive map

$$\varsigma(x,y) := (-1)^{\sum_{i<j} x_iy_j},$$

we obtain the new commutation factor $\lambda^\varsigma = \mathbf{1}$, and the algebra $I_\varsigma\left(\mathrm{Cl}(p,q)\right)$ is then commutative.

2. Categories of Graded Modules

From now on, $\mathcal{A}$ denotes a (Γ,λ)-commutative algebra.We prove an analogue of Nekludova-Scheunert Theorem for the monoidal category of graded $\mathcal{A}$-modules. For graded modules over Γ-algebras we refer to [28], as for basic notions of category theory, we refer to the appendix based on [24, 17, 19].

2.1. **Definitions.** A *left* (resp. *right*) *graded $\mathcal{A}$-module* M is a Γ-graded vector space over $\mathbb{K}$, $M=\oplus_{\beta\in\Gamma}M^\beta$, endowed with a compatible $\mathcal{A}$-module structure, $\mathcal{A}^\alpha M^\beta \subset M^{\alpha+\beta}$ (resp. $M^\beta\mathcal{A}^\alpha \subset M^{\alpha+\beta}$), for all $\alpha,\beta\in\Gamma$. As $\mathcal{A}$ is a (Γ,λ)-commutative algebra, a left graded $\mathcal{A}$-module structure on M also defines a right graded $\mathcal{A}$-module structure on M, e.g., by setting

$$m\cdot a = \lambda(\widetilde{m},\widetilde{a})\, a\cdot m\ , \tag{2.1}$$

for any $a\in\mathcal{A}$, $m\in M$. A graded $\mathcal{A}$-module is a graded module over $\mathcal{A}$ with compatible left and right structures, in the sense of (2.1).

A *morphism of graded $\mathcal{A}$-modules* is a map $\ell : M\to N$ which is $\mathcal{A}$-linear and of degree 0,

$$\ell(m+m'a) = \ell(m)+\ell(m')a \quad \text{and} \quad \widetilde{\ell(m)} = \widetilde{m},$$

for all homogeneous $m,m'\in M$ and $a\in\mathcal{A}$. Notice that "$\mathcal{A}$-linear", here and thereafter, means right $\mathcal{A}$-linear, as it is of common use in superalgebra theory. We denote the set of such morphisms by $\mathrm{Hom}_{\mathcal{A}}(M,N)$. Graded $\mathcal{A}$-modules and corresponding morphisms form a category denoted $\Gamma\text{-}\mathtt{Mod}_{\mathcal{A}}$.

Example 2.1. If $\mathcal{A}$ is a supercommutative algebra (see Example 1.2), the graded $\mathcal{A}$-modules are usually called supermodules (see, e.g., [10]). Accordingly, the category $(\mathbb{Z}_2)$-$\mathtt{Mod}_{\mathcal{A}}$ is denoted by $\mathtt{SMod}_{\mathcal{A}}$.

2.2. **Closed Symmetric Monoidal Structure.** The tensor product of graded bimodules over a Γ-algebra and the internal $\mathcal{H}om$ functor (see [28]) particularizes in Γ-$\mathtt{Mod}_{\mathcal{A}}$ as follows.

2.2.1. *Tensor product.* Let M, N be two graded $\mathcal{A}$-modules. Their tensor product is defined as the quotient $\mathbb{Z}$-module

$$M \otimes_{\mathcal{A}} N := (M \otimes_{\mathbb{Z}} N)/_I$$

where $I = \operatorname{span}\{ma \otimes n - m \otimes an \mid m \in M, n \in N, a \in \mathcal{A}\}$. The Γ-grading

$$(M \otimes_{\mathcal{A}} N)^{\gamma} := \bigoplus\nolimits_{\alpha+\beta=\gamma} \left\{ \sum m \otimes n \;\middle|\; m \in M^{\alpha}, n \in N^{\beta} \right\} \tag{2.2}$$

together with the following $\mathcal{A}$-module structures,

$$a(m \otimes n) := (am) \otimes n \qquad \text{and} \qquad (m \otimes n)\, a := m \otimes (na)\ ,$$

turns $M \otimes_{\mathcal{A}} N$ into a graded $\mathcal{A}$-module. The tensor product $\otimes_{\mathcal{A}}$ endows the category Γ-$\mathtt{Mod}_{\mathcal{A}}$ with a monoidal structure. The isomorphism

$$\begin{array}{rccc} \beta_{M,N}: & M \otimes_{\mathcal{A}} N & \to & N \otimes_{\mathcal{A}} M \\ & m \otimes n & \mapsto & \lambda(\widetilde{m}, \widetilde{n})\, n \otimes m \end{array}$$

defines a braiding, which satisfies $\beta_{N,M} \circ \beta_{M,N} = \mathrm{id}_{M \otimes_{\mathcal{A}} N}$ thanks to the properties of the commutation factor λ. Hence, the category Γ-$\mathtt{Mod}_{\mathcal{A}}$ is a symmetric monoidal category.

Remark 2.2. *If $\mathcal{A}$ and $\mathcal{B}$ are two (Γ, λ)-commutative algebras over $\mathbb{K}$, their tensor product $\mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$ is Γ-graded according to* (2.2)*. The natural product*

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = \lambda(\widetilde{b_1}, \widetilde{a_2})\, (a_1 a_2) \otimes (b_1 b_2)$$

turns $\mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$ into a (Γ, λ)-commutative algebra.

2.2.2. *Internal $\mathcal{H}om$.* A monoidal category is *closed* if the binary operation $\otimes$ defining the monoidal structure admits an adjoint operation, the so-called *internal $\mathcal{H}om$ functor*. By definition, this latter satisfies the natural isomorphism

$$\operatorname{Hom}(N \otimes M, P) \simeq \operatorname{Hom}(N, \mathcal{H}om(M, P)),$$

for all objects M, N and P. The next result is an easy adaptation of the ungraded case and appears, e.g., in [28, Prop. 2.4.9].

Proposition 2.3. *The category Γ-$\mathtt{Mod}_{\mathcal{A}}$ of graded $\mathcal{A}$-modules is a closed symmetric monoidal category. The internal $\mathcal{H}om$ is given by*

$$\mathcal{H}om_{\mathcal{A}}(M, N) := \bigoplus\nolimits_{\gamma \in \Gamma} \mathcal{H}om^{\gamma}_{\mathcal{A}}(M, N)$$

where, for each $\gamma \in \Gamma$,

$$\mathcal{H}om^{\gamma}_{\mathcal{A}}(M, N) := \{ f : M \to N |\ f \text{ is } \mathcal{A}\text{-linear and } f(M^{\alpha}) \subset N^{\alpha+\gamma}, \text{ for all } \alpha \in \Gamma \}\ .$$

The $\mathcal{A}$-module structure of $\mathcal{H}om_{\mathcal{A}}(M, N)$ reads as

$$(af)(m) := a \cdot f(m) \quad \text{and} \quad (fa)(m) := \lambda(\widetilde{a}, \widetilde{m}) f(m) \cdot a\ ,$$

where f is a morphism and $m \in M$, $a \in \mathcal{A}$ are homogeneous.

By abuse of notation, we will refer to elements in $\mathcal{H}om_{\mathcal{A}}(M,N)$ as *graded morphisms* and to elements in the subsets $\mathcal{H}om^{\gamma}_{\mathcal{A}}(M,N)$ as *homogeneous morphisms* or more precisely as *morphisms of degree* γ. Elements in $\mathcal{H}om^{0}_{\mathcal{A}}(M,N) = \mathrm{Hom}_{\mathcal{A}}(M,N)$ will be called either *morphisms* or *morphisms of degree* 0.

2.3. **An extension of the Nekludova-Scheunert Theorem.** Assume $\varsigma \in \mathfrak{S}(\lambda)$ is a NS-multiplier, so that Nekludova-Scheunert Theorem applies. The isomorphism of categories I_ς, defined in (1.7), yields an analogous isomorphism between categories of graded modules,

$$\begin{array}{cccc} \widehat{I_\varsigma}: & \Gamma\text{-}\mathtt{Mod}_{\mathcal{A}} & \stackrel{\sim}{\rightarrow} & \Gamma\text{-}\mathtt{Mod}_{\underline{\mathcal{A}}} \\ & M & \mapsto & \underline{M} \\ & \mathrm{Hom}_{\mathcal{A}}(M,N) \ni f & \mapsto & f \in \mathrm{Hom}_{\underline{\mathcal{A}}}(\underline{M},\underline{N}) \end{array}$$

Here, $\underline{M} := \widehat{I_\varsigma}(M)$ is the Γ-graded vector space M endowed with the $\underline{\mathcal{A}}$-module structure

$$a \star m := \varsigma(\widetilde{a},\widetilde{m})\, a \cdot m\ , \quad \forall a \in \underline{\mathcal{A}}, \forall m \in \underline{M}\ . \tag{2.3}$$

Moreover, the regrading functor $\Phi_{\lambda^\varsigma} : (\Gamma,\lambda^\varsigma)\text{-}\mathtt{Alg} \to \mathtt{SAlg}$ induces analogously a regrading functor on graded modules

$$\widehat{\Phi}_{\lambda^\varsigma} : \Gamma\text{-}\mathtt{Mod}_{\underline{\mathcal{A}}} \to \mathtt{SMod}_{\underline{\mathcal{A}}} \tag{2.4}$$

for all $(\Gamma,\lambda^\varsigma)$-commutative algebra $\underline{\mathcal{A}}$. Hence, we get a faithful functor

$$\Gamma\text{-}\mathtt{Mod}_{\mathcal{A}} \xrightarrow{\widehat{I_\varsigma}} \Gamma\text{-}\mathtt{Mod}_{\underline{\mathcal{A}}} \xrightarrow{\widehat{\Phi}_{\lambda^\varsigma}} \mathtt{SMod}_{\underline{\mathcal{A}}}\ .$$

To investigate the properties of the above functors with respect to the symmetric monoidal structures on $\Gamma\text{-}\mathtt{Mod}_{\mathcal{A}}$, $\Gamma\text{-}\mathtt{Mod}_{\underline{\mathcal{A}}}$ and $\mathtt{SMod}_{\underline{\mathcal{A}}}$, we need further notions of category theory.

2.3.1. *Closed Monoidal Functors.* In this section, we use notation from Appendix. For example, monoidal categories are written as quintuple: $(\mathtt{C},\otimes,\mathbb{I},\alpha,r,l)$ refers to a monoidal category C with tensor product $\otimes$, identity object $\mathbb{I}$ and structural maps (α,r,l).

Definition 2.4. *A* Lax monoidal functor *between two monoidal categories* $(\mathtt{C},\otimes,\mathbb{I},\alpha,r,l)$ *and* $(\mathtt{D},\boxtimes,\mathbb{I}',\alpha',r',l')$ *is a triple* (F,u,τ) *which consists of*

(i) a functor $F:\mathtt{C}\to\mathtt{D}$*,*
(ii) a morphism $u:\mathbb{I}'\to F(\mathbb{I})$ *in* $\mathtt{D}$*,*
(iii) a natural morphism τ*, i.e., a family of morphisms in* $\mathtt{D}$ *natural in* $X,Y\in\mathrm{Ob}(\mathtt{C})$*,*

$$\tau_{X,Y} : F(X)\boxtimes F(Y) \to F(X\otimes Y)\ ,$$

such that the following diagrams commute.

$$
\begin{tikzcd}
\Big(F(X)\boxtimes F(Y)\Big)\boxtimes F(Z) \arrow[rr, "\alpha'_{F(X),F(Y),F(Z)}"] \arrow[d, "\tau_{X,Y}\boxtimes \mathrm{id}_{F(Z)}"'] & & F(X)\boxtimes\Big(F(Y)\boxtimes F(Z)\Big) \arrow[d, "\mathrm{id}_{F(X)}\boxtimes\tau_{Y,Z}"] \\
F(X\otimes Y)\boxtimes F(Z) \arrow[d, "\tau_{X\otimes Y,Z}"'] & & F(X)\boxtimes F(Y\otimes Z) \arrow[d, "\tau_{X,Y\otimes Z}"] \\
F\Big((X\otimes Y)\otimes Z\Big) \arrow[rr, "F\alpha_{X,Y,Z}"] & & F\Big(X\otimes(Y\otimes Z)\Big)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{I}'\boxtimes F(X) \arrow[r, "l'_{F(X)}"] \arrow[d, "u\boxtimes \mathrm{id}_{F(X)}"'] & F(X) \\
F(\mathbb{I})\boxtimes F(X) \arrow[r, "\tau_{\mathbb{I},X}"'] & F(\mathbb{I}\otimes X) \arrow[u, "Fl_X"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X)\boxtimes\mathbb{I}' \arrow[r, "r'_{F(X)}"] \arrow[d, "\mathrm{id}_{F(X)}\boxtimes u"'] & F(X) \\
F(X)\boxtimes F(\mathbb{I}) \arrow[r, "\tau_{X,\mathbb{I}}"'] & F(X\otimes\mathbb{I}) \arrow[u, "Fr_X"']
\end{tikzcd}
$$

A Lax monoidal functor is called a monoidal functor *if u and all $\tau_{M,N}$ are isomorphisms.*

Definition 2.5. *A (Lax) monoidal functor between symmetric monoidal categories*

$$(F,u,\tau):\ (\mathtt{C},\otimes,\mathbb{I},\alpha,r,l,\beta)\to(\mathtt{D},\boxtimes,\mathbb{I}',\alpha',r',l',\beta')$$

is called symmetric *if it commutes with the braidings, i.e. if the following diagram commutes.*

$$
\begin{tikzcd}
F(X)\boxtimes F(Y) \arrow[r, "\beta'_{F(X),F(Y)}"] \arrow[d, "\tau_{X,Y}"'] & F(Y)\boxtimes F(X) \arrow[d, "\tau_{Y,X}"] \\
F(X\otimes Y) \arrow[r, "F\beta_{X,Y}"'] & F(Y\otimes X)
\end{tikzcd}
$$

Given a (Lax) monoidal functor (F,u,τ) between two closed monoidal categories, one can construct a natural transformation η via the internal $\mathcal{H}om_{\mathtt{D}}$ adjunction as follows (we omit indices of natural transformations):

$$
\begin{array}{rcl}
\mathrm{Hom}_{\mathtt{D}}\Big(F\big(\mathcal{H}om_{\mathtt{C}}(X,Y)\big)\boxtimes F(X)\,,\ F(Y)\Big) & \simeq & \mathrm{Hom}_{\mathtt{D}}\Big(F\big(\mathcal{H}om_{\mathtt{C}}(X,Y)\big)\,,\ \mathcal{H}om_{\mathtt{D}}\big(F(X),F(Y)\big)\Big) \\
F(\mathrm{ev})\circ\tau & \leftrightarrow & \eta
\end{array}
\tag{2.5}
$$

where

$$\mathrm{ev}:\ \mathcal{H}om_{\mathtt{C}}(X,Y)\otimes X\to Y$$

is the evaluation map. The triple (F, u, η) is a *Lax closed functor* between the closed categories $(\mathtt{C}, \mathcal{H}om_{\mathtt{C}}, \mathbb{I}_{\mathtt{C}})$ and $(\mathtt{D}, \mathcal{H}om_{\mathtt{D}}, \mathbb{I}_{\mathtt{D}})$ (see [13] for the definition). It is a *closed functor* if moreover η is an isomorphism. Note that, even if u and τ are isomorphisms, η may not be one.

2.3.2. *NS-Functors on Modules.* Obviously, from the regrading functor $\widehat{\Phi}_{\lambda^\varsigma}$ defined in (2.4), we obtain a symmetric monoidal functor,

$$(\widehat{\Phi}_{\lambda^\varsigma}, \mathrm{id}, \mathrm{id}) : \Gamma\text{-}\mathtt{Mod}_{\underline{\mathcal{A}}} \to \mathtt{SMod}_{\underline{\mathcal{A}}} \ ,$$

which induces via (2.5) a closed functor $(\widehat{\Phi}_{\lambda^\varsigma}, \mathrm{id}, \mathrm{id})$.

Theorem 2.6. *Let $\mathcal{A}$ be a (Γ, λ)-commutative algebra, $\varsigma \in \mathfrak{S}(\lambda)$ and $\underline{\mathcal{A}} = I_\varsigma(\mathcal{A})$ the associated Γ-graded supercommutative algebra. For every $u \in \mathrm{Aut}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}})$, there exists a natural transformation τ, such that*

$$(\widehat{I}_\varsigma, u, \tau) : \ \Gamma\text{-}\mathtt{Mod}_{\mathcal{A}} \to \Gamma\text{-}\mathtt{Mod}_{\underline{\mathcal{A}}}$$

is a symmetric monoidal functor, which induces a closed functor $(\widehat{I}_\varsigma, u, \eta)$. By composition with the regrading monoidal functor $(\widehat{\Phi}_{\lambda^\varsigma}, \mathrm{id}, \mathrm{id})$, we get a symmetric monoidal functor

$$\left(\widehat{\Phi}_{\lambda^\varsigma} \circ \widehat{I}_\varsigma, \widehat{\Phi}_{\lambda^\varsigma}(u), \widehat{\Phi}_{\lambda^\varsigma}(\tau)\right) : \Gamma\text{-}\mathtt{Mod}_{\mathcal{A}} \to \mathtt{SMod}_{\underline{\mathcal{A}}}$$

which induces a closed functor $\left(\widehat{\Phi}_{\lambda^\varsigma} \circ \widehat{I}_\varsigma, \widehat{\Phi}_{\lambda^\varsigma}(u), \widehat{\Phi}_{\lambda^\varsigma}(\eta)\right)$.

Proof. The map $u : \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ being an $\underline{\mathcal{A}}$-module morphism, it is completely characterized by $u(1)$, its value on the unit element $1 \in \underline{\mathcal{A}}$. By construction, $u(1)$ is necessarily of degree 0 and invertible. For any pair M, N of $\mathcal{A}$-modules, we set

$$\begin{array}{rccc} \tau : & \underline{M} \boxtimes \underline{N} & \to & \underline{M \otimes N} \\ & m \boxtimes n & \mapsto & \varsigma(m,n)\, u(1)^{-1} \star (m \otimes n) \end{array}$$

where $\underline{M} = \widehat{I}_\varsigma(M)$. This family of morphisms of graded $\mathcal{A}$-modules is natural both in M and N and $(\widehat{I}_\varsigma, u, \tau)$ is easily checked to be a symmetric monoidal functor. The induced natural transformation η (see (2.5)) is then given by

$$\begin{array}{rccc} \eta : & \underline{\mathcal{H}om_{\mathcal{A}}(M,N)} & \to & \mathcal{H}om_{\underline{\mathcal{A}}}\left(\underline{M}, \underline{N}\right) \\ & f & \mapsto & \big(\eta(f) : \ m \mapsto \varsigma(f,m)\, u(1)^{-1} \star f(m)\big) \end{array}$$

which is clearly invertible.

The remaining statement follows from the rule of composition of monoidal and closed functors (see [13]). □

As a consequence of Theorem 2.6 and its proof, we have a closed functor $(\widehat{I}_\varsigma, \mathrm{id}, \eta_\varsigma)$, with $\underline{\mathcal{A}}$-module isomorphisms

$$\begin{array}{rccc} \eta_\varsigma : & \widehat{I}_\varsigma\left(\mathcal{E}nd_{\mathcal{A}}(M)\right) & \to & \mathcal{E}nd_{\underline{\mathcal{A}}}(\widehat{I}_\varsigma(M)) \\ & f & \mapsto & (\eta_\varsigma(f) : \ m \mapsto \varsigma(f,m)\, f(m)) \end{array} \tag{2.6}$$

where f and m are homogeneous. Note that we have $\mathcal{E}nd_{\mathcal{A}}(M) = \widehat{I}_\varsigma\left(\mathcal{E}nd_{\mathcal{A}}(M)\right)$ as algebras and as Γ-graded $\mathcal{A}^0$-modules, but they admit distinct $\mathcal{A}$- and $\underline{\mathcal{A}}$-module structures (see (2.3)). A direct computation shows that Equation (0.1) holds and **Theorem A** in the Introduction follows.

3. Free graded modules and graded matrices

Throughout this section, $\mathcal{A}$ is a (Γ,λ)-commutative algebra over $\mathbb{K}$. We focus on free graded modules, investigate the notion of rank and transfer the NS-functors $\widehat{I_\varsigma}$ to algebras of graded matrices.

3.1. **Free graded structures.**

We define and build free graded modules and free graded algebras. The latter are constructed from graded tensor algebras.

3.1.1. *Free graded Modules.* Let $S=\sqcup_{\gamma\in\Gamma}S^\gamma$ be a Γ-graded set. We denote by $\langle S\rangle_{\mathbb{K}}$ the Γ-graded vector space generated by S over $\mathbb{K}$.

The *free graded $\mathcal{A}$-module generated by S* is the tensor product $\langle S\rangle_{\mathcal{A}}=\mathcal{A}\otimes_{\mathbb{K}}\langle S\rangle_{\mathbb{K}}$, with $\mathcal{A}$-module structure given by $a\cdot(b\otimes m)=(ab)\otimes m$, for all $a,b\in\mathcal{A}$ and $m\in\langle S\rangle_{\mathbb{K}}$. The Γ-grading of $\langle S\rangle_{\mathcal{A}}$ is defined by

$$(\langle S\rangle_{\mathcal{A}})^\gamma=\bigoplus_{\alpha+\beta=\gamma}\mathcal{A}^\alpha\otimes_{\mathbb{K}}(\langle S\rangle_{\mathbb{K}})^\beta\ ,$$

for all $\gamma\in\Gamma$. A *free graded $\mathcal{A}$-module* M is a graded $\mathcal{A}$-module freely generated by a subset $S\subset\sqcup_{\gamma\in\Gamma}M^\gamma$, called a *basis* of M (cf. [28] for the notions of free, injective and projective graded modules over Γ-algebras). An element $m\in M$ decomposes in a given basis $(e_i)_{i\in I}$ as follows,

$$m=\sum_{i\in I}e_i\cdot m^i=\sum_{i\in I}n^i\cdot e_i\,,$$

where the map $i\mapsto m^i$ has finite support. Right and left components are linked via the relation (2.1) between left and right $\mathcal{A}$-module structures. By convention, we only consider right components. Note that $m\in M$ is homogeneous of degree γ if and only if its components m^i are of degree $\widetilde{m^i}=\gamma-\widetilde{e}_i$ for all $i\in I$.

Lemma 3.1. *Let M be a free graded $\mathcal{A}$-module with basis S, and ς a NS-multiplier. The image of M under the NS-functor $\widehat{I_\varsigma}$ is a free graded $I_\varsigma(\mathcal{A})$-module with the same basis S.*

Proof. By construction, $I_\varsigma:\mathcal{A}=(A,\cdot)\mapsto\underline{\mathcal{A}}=(A,\star)$ changes the product, $a\star b:=\varsigma(\widetilde{a},\widetilde{b})\,a\cdot b$, and $\widehat{I_\varsigma}:M\mapsto\widehat{I_\varsigma}(M)$ changes the graded $\mathcal{A}$-module structure into the graded $\underline{\mathcal{A}}$-module structure $a\star m:=\varsigma(\widetilde{a},\widetilde{m})\,a\cdot m$. The decomposition of an element $m\in M$ in a basis $(e_i)_{i\in I}$ varies accordingly:

$$m=\sum_{i\in I}e_i\cdot m^i=\sum_{i\in I}e_i\star\left(\varsigma(\widetilde{e}_i,\widetilde{m}^i)^{-1}m^i\right)\ ,$$

so that $(e_i)_{i\in I}$ is also a basis of the module $\widehat{I_\varsigma}(M)$. □

3.1.2. *Tensor algebras.* In this paragraph, we follow [31, 32]. The tensor algebra of a graded $\mathcal{A}$-module M is an $\mathcal{A}$-module and an algebra,

$$\mathrm{T}(M):=\bigoplus_{k\in\mathbb{N}}M^{\otimes k}=\mathcal{A}\oplus M\oplus(M\otimes_{\mathcal{A}}M)\oplus(M\otimes_{\mathcal{A}}M\otimes_{\mathcal{A}}M)\oplus\dots\ ,$$

with multiplication given by the tensor product over $\mathcal{A}$. The algebra $\mathrm{T}(M)$ is more precisely a $(\mathbb{N}\times\Gamma)$-algebra with the following gradings:

- the $\mathbb{N}$-grading, called *weight*, given by the number of factors in M;
- the Γ-grading, called *degree*, induced by the Γ-grading of the module M (see (2.2)).

The *graded symmetric algebra* on M is the $\mathcal{A}$-module and (Γ,λ)-commutative algebra

$$\bigvee M := \mathrm{T}(M)/_{\mathcal{I}}\ ,$$

where $\mathcal{I}$ is the homogeneous ideal of $\mathrm{T}(M)$ generated by the Γ-graded commutators

$$[v,w]_\lambda := v\otimes_{\mathcal{A}} w - \lambda(\widetilde{v},\widetilde{w})\, w\otimes_{\mathcal{A}} v\ ,\quad v,w\in M\ .$$

Analogously, the *graded exterior algebra* on M is the $\mathcal{A}$-module and Γ-algebra

$$\bigwedge M := \mathrm{T}(M)/_{\mathcal{J}}\ ,$$

where $\mathcal{J}$ is the homogeneous ideal of $\mathrm{T}(M)$ generated by the Γ-graded anti-commutators

$$v\otimes_{\mathcal{A}} w + \lambda(\widetilde{v},\widetilde{w})\, w\otimes_{\mathcal{A}} v\ ,\quad v,w\in M\ .$$

3.1.3. *Free graded algebras.* Let $S=\sqcup_{\gamma\in\Gamma}S^\gamma$ be a Γ-graded set and λ be a commutation factor over the grading group Γ.

We view $\mathbb{K}$ as a (Γ,λ)-commutative algebra concentrated in degree 0 so that definitions of the previous section apply to the Γ-graded $\mathbb{K}$-vector space $\langle S\rangle_{\mathbb{K}}$. The *free* Γ*-algebra* generated by S is the tensor algebra $\mathrm{T}(\langle S\rangle_{\mathbb{K}})$. The *free* (Γ,λ)*-commutative algebra* generated by S is the graded symmetric algebra

$$\mathbb{K}[S] := \bigvee\langle S\rangle_{\mathbb{K}}\ , \tag{3.1}$$

which is the algebra of polynomials in the graded variables $(X_i)_{i\in S}$.

We denote by $\overline{\langle S\rangle}_{\mathbb{K}}$ the vector space $\langle S\rangle_{\mathbb{K}}$ with reversed grading

$$\left(\overline{\langle S\rangle}_{\mathbb{K}}\right)^\gamma := (\langle S\rangle_{\mathbb{K}})^{-\gamma}\ ,\quad \text{for all } \gamma\in\Gamma\ .$$

If (Γ,λ) is such that $\Gamma_{\bar 1}=\emptyset$, we define the algebra of Laurent polynomials in graded variables $X_i\in S$ by

$$\mathbb{K}[S,S^{-1}]_\lambda := \left(\bigvee\langle S\rangle_{\mathbb{K}}\otimes_{\mathbb{K}}\bigvee\overline{\langle S\rangle}_{\mathbb{K}}\right)/\,\mathcal{I} \tag{3.2}$$

where $\mathcal{I}$ is the ideal generated by the elements $(X_i\otimes\overline{X}_i)-1$, for $i=1,\dots,r$. According to Remark 2.2, the space $\mathbb{K}[S,S^{-1}]_\lambda$ is a (Γ,λ)-commutative algebra, which is clearly a crossed product.

3.2. **Basis of free graded modules.** Let $(e_i)_{i=1,\dots,n}$ be a basis of a free and finitely generated graded $\mathcal{A}$-module M. The *rank* of (e_i) is $n\in\mathbb{N}$, the *degree* of (e_i) is the n-vector

$$\boldsymbol{\nu} := (\widetilde{e}_1,\dots,\widetilde{e}_n)\in\Gamma^n,$$

which depends on the order of the basis. The sequence $(r_\gamma)_{\gamma\in\Gamma}\in\mathbb{N}^{|\Gamma|}$, where

$$r_\gamma = \sharp\{e_i \mid \widetilde{e}_i=\gamma\,,\ i=1,\dots,n\}\,,$$

is called the Γ*-rank* of (e_i). Obviously, we have $\sum_{\gamma\in\Gamma}r_\gamma=n$. Two bases have same Γ-rank if and only if they have the same degree up to permutation.

The underlying $\mathbb{Z}_2$-grading of Γ (see (1.4)), induces a $\mathbb{Z}_2$-rank, also called *superrank*,

$$(r_{\bar 0},r_{\bar 1}) = \left(\sum\nolimits_{\gamma\in\Gamma_{\bar 0}} r_\gamma\ ,\ \sum\nolimits_{\gamma\in\Gamma_{\bar 1}} r_\gamma\right)\ .$$

The rank and superrank are invariants of the graded module M.

Proposition 3.2. *Let M be a free and finitely generated graded $\mathcal{A}$-module. Any two bases of M have the same rank and superrank, henceforth defining the rank and superrank of M.*

Proof. Let M be a free graded $\mathcal{A}$-module with two bases (e_i) and (e'_j). Denote by $\varsigma \in \mathfrak{S}(\lambda)$ a NS-multiplier. By Lemma 3.1, (e_i) and (e'_j) are then bases of the graded $I_\varsigma(A)$-module $\widehat{I_\varsigma}(M)$, with unchanged superranks. Since $\widehat{I_\varsigma}(M)$ is a supermodule over the supercommutative algebra $I_\varsigma(\mathcal{A})$, the superrank of all the bases of $\widehat{I_\varsigma}(M)$ are equal (see e.g. [33, p. 114]). Hence, (e_i) and (e'_j) have the same superrank. $\square$

For the Γ-rank, the situation is more involved and depends on the algebra $\mathcal{A}$. A Theorem of Dade [9] yields the following result.

Proposition 3.3. *Let M and N be two free and finitely generated graded $\mathcal{A}$-modules, with $\mathcal{A}$ a strongly graded algebra. Then, M and N are isomorphic if and only if they have same rank.*

Equivalently, if $\mathcal{A}$ is strongly graded, a free graded $\mathcal{A}$-module of rank $n \in \mathbb{N}$ admits bases of any degree $\boldsymbol{\nu} \in \Gamma^n$. As the grading group of a strongly graded algebra satisfies $\Gamma_{\bar{1}} = \emptyset$, rank and superrank are equal for its graded modules. Set

$$\Gamma_{\mathcal{A}^\times} := \{\gamma \in \Gamma \mid \mathcal{A}^\gamma \cap \mathcal{A}^\times \neq \emptyset\},$$

where $\mathcal{A}^\times$ is the subgroup of invertible elements. If the algebra $\mathcal{A}$ is strongly graded and $\mathcal{A}^0$ is a local ring, then $\mathcal{A}$ is a crossed product (see [9]) and $\Gamma_{\mathcal{A}^\times} = \Gamma$. In general, $\Gamma_{\mathcal{A}^\times}$ is a subgroup of Γ and $\Gamma_{\mathcal{A}^\times} \cap \Gamma_{\bar{1}} = \emptyset$, as odd elements of $\mathcal{A}$ are nilpotent. Admissible degrees and Γ-ranks of bases of M can be determined by studying the translation action of $(\Gamma_{\mathcal{A}^\times})^n$ on Γ^n. To our knowledge, the point (ii) in the following proposition is new.

Proposition 3.4. *Let M be a free graded $\mathcal{A}$-module of rank n, admitting a basis of degree $\boldsymbol{\mu} \in \Gamma^n$,*

- *(i) For all $\boldsymbol{\nu} \in (\Gamma_{\mathcal{A}^\times})^n + \boldsymbol{\mu}$, there exists a basis (f_i) of M of degree $\boldsymbol{\nu}$;*
- *(ii) Up to reordering its elements, the degree $\boldsymbol{\nu}$ of any basis of M satisfies $\boldsymbol{\nu} \in (\Gamma_{\mathcal{A}^\times})^n + \boldsymbol{\mu}$, if $\mathcal{A}^0$ is a local ring.*

In order to prove this, we need a definition and a lemma.

A *maximal homogeneous* ideal is a proper homogeneous ideal I such that any other homogeneous ideal containing I is equal to $\mathcal{A}$.

Lemma 3.5. *Let $\varsigma \in \mathfrak{S}(\lambda)$, and $\underline{\mathcal{A}}$ be a $(\Gamma, \lambda^\varsigma)$-commutative algebra with $\underline{\mathcal{A}}^0$ a local ring. If I is a maximal homogeneous ideal of $\underline{\mathcal{A}}$ and $I^\gamma = I \cap \mathcal{A}^\gamma$, then we get $\underline{\mathcal{A}}^\gamma \setminus I^\gamma = \underline{\mathcal{A}}^\gamma \cap \underline{\mathcal{A}}^\times$ for all $\gamma \in \Gamma$.*

Proof of Lemma 3.5. Let I be a maximal homogeneous ideal of $\underline{\mathcal{A}}$. As I is homogeneous, the algebra $\underline{\mathcal{A}}/_I$ is Γ-graded and $(\underline{\mathcal{A}}/_I)^\gamma = \underline{\mathcal{A}}^\gamma/_{I^\gamma}$. Assume that $a \in \underline{\mathcal{A}}^\gamma \setminus I^\gamma$. The ideal $(a) + I$ is homogeneous, hence $(a) + I = \underline{\mathcal{A}}$ by maximality of I. This means there exists $b = \sum_{\alpha \in \Gamma} b_\alpha \in \underline{\mathcal{A}}$ such that $ba + I = 1 + I$. Projecting onto the 0-degree part, we get $1 - (b_{-\gamma})a \in I^0$. Since $\underline{\mathcal{A}}^0$ is a local ring, $1 - (b_{-\gamma})a$ is also contained in the maximal ideal of $\mathcal{A}^0$ and then a is invertible in $\mathcal{A}$. This means $\underline{\mathcal{A}}^\gamma \setminus I^\gamma \subset \underline{\mathcal{A}}^\gamma \cap \underline{\mathcal{A}}^\times$ and the converse inclusion is obvious. Moreover, as a is invertible in $\mathcal{A}$, $a + I$ is invertible in $\underline{\mathcal{A}}/I$. $\square$

Proof of Proposition 3.4. (i) Let (e_i) be a basis of M of degree $\boldsymbol{\mu} \in \Gamma^n$ and $\boldsymbol{\mu}' = (\mu'_i) \in (\Gamma_{\mathcal{A}^\times})^n$. We can choose elements $a_i \in \mathcal{A}^{\mu'_i} \cap \mathcal{A}^\times$ and set $f_i := e_i a_i$ for every $i = 1, \ldots, n$. Then, (f_i) form a basis of M of degree $\boldsymbol{\nu} = \boldsymbol{\mu} + \boldsymbol{\mu}'$.

(ii) Let (e_i) and (f_i) be two bases of M of degrees $\boldsymbol{\mu}, \boldsymbol{\nu} \in \Gamma^n$ respectively. Let $\varsigma \in \mathfrak{S}(\lambda)$. According to Lemma 3.1, the module $\underline{M} := \widehat{I_\varsigma}(M)$ is a free graded module over the $(\Gamma, \lambda^\varsigma)$-commutative algebra $\underline{\mathcal{A}} := I_\varsigma(\mathcal{A})$, with same bases (e_i) and (f_i).

Let I be a maximal homogeneous ideal of $\underline{\mathcal{A}}$. Then, the quotient $\underline{M}/_{I\underline{M}}$ is a graded $(\underline{\mathcal{A}}/_I)$-module. Furthermore, the bases (e_i) and (f_i) of $\underline{M}$ induce bases $([e_i])$ and $([f_i])$ of $\underline{M}/_{I\underline{M}}$, with unchanged degrees. We have $[f_i]=\sum_j a_{ij}[e_i]$, with at least one non-vanishing coefficient. Up to reordering, we can suppose $a_{ii}\in(\underline{\mathcal{A}}/_I)\setminus\{0\}$. Since a_{ii} is homogeneous, of degree $\widetilde{a}_{ii}=\widetilde{f}_i-\widetilde{e}_i$, there is an invertible element of same degree in $\underline{\mathcal{A}}$, by Lemma 3.5. This means $\widetilde{f}_i-\widetilde{e}_i\in\Gamma_{\underline{\mathcal{A}}^\times}$. As homogeneous invertible elements of $\mathcal{A}$ and $\underline{\mathcal{A}}$ coincide, we get $\boldsymbol{\nu}\in (\Gamma_{\mathcal{A}^\times})^n+\boldsymbol{\mu}$, up to reordering entries of $\boldsymbol{\nu}$. □

As a direct consequence of (i) in Proposition 3.4, we get the following result.

Corollary 3.6. *Let $\mathcal{A}$ be a (Γ,λ)-commutative algebra whose even part is a crossed product, i.e. $\Gamma_{\mathcal{A}^\times}=\Gamma_{\bar{0}}$. Then, two free graded $\mathcal{A}$-modules are isomorphic if and only if they have same superrank.*

Note that Clifford algebras (Example 1.4) satisfy the condition $\Gamma_{\mathcal{A}^\times}=\Gamma_{\bar{0}}$, so that their free graded modules are characterized by the rank. For the algebras introduced in Example 1.3, which generalize the dual numbers, the situation is opposite: $\Gamma_{\mathcal{A}^\times}=\{0\}$ and all bases of a free graded module have same Γ-rank. Over such an algebra , two free graded modules are isomorphic if and only if they have the same Γ-rank.

3.3. **Graded matrix algebras.** In this paragraph, we introduce graded matrices over (Γ,λ)-commutative algebras, partly following [21, 8]. See also [28, 16] for graded matrices over Γ-algebras, without commutativity assumption. Let $m,n\in\mathbb{N}$ and $\boldsymbol{\mu}\in\Gamma^m$, $\boldsymbol{\nu}\in\Gamma^n$. The space $\mathsf{M}(\boldsymbol{\mu}\times\boldsymbol{\nu};\mathcal{A})$ of $\boldsymbol{\mu}\times\boldsymbol{\nu}$ *graded matrices* is the $\mathbb{K}$-vector space of $m\times n$ matrices, over the (Γ,λ)-commutative algebra $\mathcal{A}$, endowed with the Γ-grading:

$$\mathsf{M}(\boldsymbol{\mu}\times\boldsymbol{\nu};\mathcal{A})=\bigoplus_{\gamma\in\Gamma}\mathsf{M}^\gamma(\boldsymbol{\mu}\times\boldsymbol{\nu};\mathcal{A})\ ,$$

where, for every $\gamma\in\Gamma$,

$$\mathsf{M}^\gamma(\boldsymbol{\mu}\times\boldsymbol{\nu};\mathcal{A}):=\left\{X=(X^i{}_j)_{i,j}\mid X^i{}_j\in\mathcal{A}^{\gamma-\mu_i+\nu_j}\right\}\ .$$

The elements of $\mathsf{M}^\gamma(\boldsymbol{\mu}\times\boldsymbol{\nu};\mathcal{A})$ are called *homogeneous matrices of degree* γ, or simply γ-*degree matrices*.

Let $p\in\mathbb{N}$ and $\boldsymbol{\pi}\in\Gamma^p$. The product of $m\times n$ matrices with $n\times p$ matrices defines a product

$$\mathsf{M}(\boldsymbol{\mu}\times\boldsymbol{\nu};\mathcal{A})\times\mathsf{M}(\boldsymbol{\nu}\times\boldsymbol{\pi};\mathcal{A})\to\mathsf{M}(\boldsymbol{\mu}\times\boldsymbol{\pi};\mathcal{A})$$

which is compatible with the Γ-grading, i.e., $\mathsf{M}^\alpha(\boldsymbol{\mu}\times\boldsymbol{\nu};\mathcal{A})\cdot\mathsf{M}^\beta(\boldsymbol{\nu}\times\boldsymbol{\pi};\mathcal{A})\subset\mathsf{M}^{\alpha+\beta}(\boldsymbol{\mu}\times\boldsymbol{\pi};\mathcal{A})$. It turns the space $\mathsf{M}(\boldsymbol{\nu};\mathcal{A}):=\mathsf{M}(\boldsymbol{\nu}\times\boldsymbol{\nu};\mathcal{A})$ into a Γ-algebra. Moreover, $\mathsf{M}(\boldsymbol{\nu};\mathcal{A})$ admits graded $\mathcal{A}$-module structures. We choose the following one, defined for all $\alpha\in\Gamma$ by

$$\begin{array}{rcl}\mathcal{A}^\alpha\times\mathsf{M}(\boldsymbol{\nu};\mathcal{A}) & \to & \mathsf{M}(\boldsymbol{\nu};\mathcal{A})\\ (a,X) & \mapsto & a\cdot X=\begin{pmatrix}\lambda(\alpha,\nu_1)a & & \\ & \ddots & \\ & & \lambda(\alpha,\nu_n)a\end{pmatrix}X\end{array}\tag{3.3}$$

From the point of view of category theory, we can see $\mathsf{M}(\boldsymbol{\nu};-)$ as the functor

$$\mathsf{M}(\boldsymbol{\nu};-):\ (\Gamma,\lambda)\text{-}\mathtt{Alg}\to\Gamma\text{-}\mathtt{Alg}$$

which assigns to each graded algebra $\mathcal{A}$ the graded algebra of $\boldsymbol{\nu}$-square graded matrices, and to each Γ-algebra homomorphism $f : \mathcal{A} \to \mathcal{B}$ the map

$$\begin{array}{rccc} \mathsf{M}(f): & \mathsf{M}(\boldsymbol{\nu};\mathcal{A}) & \to & \mathsf{M}(\boldsymbol{\nu};\mathcal{B}) \\ & \left(X^i{}_j\right)_{i,j} & \mapsto & \left(f(X^i{}_j)\right)_{i,j} \end{array} \tag{3.4}$$

The group of invertible $\boldsymbol{\nu}$-square graded matrices is denoted by $\mathsf{GL}(\boldsymbol{\nu};\mathcal{A})$, and analogously, we have the functor

$$\mathsf{GL}(\boldsymbol{\nu};-) : (\Gamma,\lambda)\text{-}\mathtt{Alg} \to \mathtt{Grp}\ . \tag{3.5}$$

The subset of invertible homogeneous matrices of degree γ is denoted by $\mathsf{GL}^{\gamma}(\boldsymbol{\nu};\mathcal{A})$. Similarly to the ungraded case, one gets

Proposition 3.7. *Let M, N be two free graded $\mathcal{A}$-modules with bases (e_i) and (e'_j) of degrees $\boldsymbol{\mu} \in \Gamma^m$ and $\boldsymbol{\nu} \in \Gamma^n$ respectively. Graded morphisms can be represented by graded matrices via the following isomorphism of Γ-graded $\mathbb{K}$-vector spaces,*

$$\begin{array}{ccc} \mathcal{H}om_{\mathcal{A}}(M,N) & \xrightarrow{\sim} & \mathsf{M}(\boldsymbol{\nu}\times\boldsymbol{\mu};\mathcal{A}) \\ f & \mapsto & (F^i{}_j)_{ij} \end{array} \tag{3.6}$$

where $F^i{}_j$ are determined by $f(e_j) = \sum_i e'_i \cdot F^i{}_j$. If $M = N$ and $(e_i) = (e'_i)$, then (3.6) is an isomorphism of Γ-algebras and graded $\mathcal{A}$-modules, namely $\mathcal{E}nd_{\mathcal{A}}(M) \simeq \mathsf{M}(\boldsymbol{\nu};\mathcal{A})$.

It is necessary to write matrix coefficients on the right of the basis vectors to get a morphism of algebra. This leads to an atypical matrix representation for diagonal endomorphisms, but which fits with the $\mathcal{A}$-module structure defined in (3.3). Namely, the endomorphism f, specified by $f(e_i) = a_i e_i$ with $a_i \in \mathcal{A}^{\widetilde{a}_i}$, reads in matrix form as

$$F = \begin{pmatrix} \lambda(\widetilde{a}_1,\widetilde{e}_1)\,a_1 & & \\ & \ddots & \\ & & \lambda(\widetilde{a}_n,\widetilde{e}_n)\,a_n \end{pmatrix} .$$

3.4. **Change of basis.** Let M be a free graded $\mathcal{A}$-module of rank n, with bases (e_i) to (e'_i) of degree $\boldsymbol{\mu} \in \Gamma^n$ and $\boldsymbol{\nu} \in \Gamma^n$ respectively. A basis transformation matrix from (e_i) to (e'_i) is a graded matrix $P \in \mathsf{M}^0(\boldsymbol{\nu}\times\boldsymbol{\mu};\mathcal{A})$ with inverse $P^{-1} \in \mathsf{M}^0(\boldsymbol{\mu}\times\boldsymbol{\nu};\mathcal{A})$. The graded matrices $F \in \mathsf{M}(\boldsymbol{\mu};\mathcal{A})$ and $F' \in \mathsf{M}(\boldsymbol{\nu};\mathcal{A})$ representing the same graded endomorphism $f \in \mathcal{E}nd_{\mathcal{A}}(M)$ are, as usual, linked through the equality

$$F = P^{-1}F'P\ .$$

This provides the isomorphism of Γ-algebra, $\mathsf{M}(\boldsymbol{\mu};\mathcal{A}) \simeq \mathsf{M}(\boldsymbol{\nu};\mathcal{A})$.

We use now particular change of basis to get graded matrices of specific forms.

3.4.1. *Permutation of basis and block graded matrices.* Assume Γ is of finite order N. We choose an ordering on Γ, i.e., a bijective map

$$\begin{array}{ccc} \{1,\ldots,N\} & \to & \Gamma \\ u & \mapsto & \gamma_u \end{array}$$

By permuting the elements of a basis (e_i), of degree $\boldsymbol{\nu} \in \Gamma^n$ and Γ-rank $\mathbf{r} = (r_{\gamma_u})_{u=1,\ldots,n}$, we obtain a basis (e'_i) with ordered degree $\boldsymbol{\mu} \in \Gamma^n$. This means $\mu_i \leq \mu_{i+1}$ if $i = 1,\ldots,n-1$. In this new basis, a homogeneous graded matrix $X \in \mathsf{M}^x(\boldsymbol{\mu};\mathcal{A})$ is a block matrix

$$X = \left(\begin{array}{c|c|c} \mathcal{X}_{11} & \dots & \mathcal{X}_{1N} \\ \hline \dots & \dots & \dots \\ \hline \mathcal{X}_{N1} & \dots & \mathcal{X}_{NN} \end{array}\right), \tag{3.7}$$

where each block is a matrix of the form $\mathcal{X}_{uv} \in \mathsf{M}(r_{\gamma_u} \times r_{\gamma_v};\mathcal{A})$, with entries in $\mathcal{A}^{x-\gamma_u+\gamma_v}$. This representation of the algebra of graded matrices is the one used in [8, 7], where it is referred to as $\mathsf{M}(\mathbf{r};\mathcal{A})$.

3.4.2. *Rescaling of basis and classical matrices.* Assume that the graded algebra $\mathcal{A}$ contains an invertible element $\mathsf{t}_\alpha \in \mathcal{A}^\alpha$ of each even degree α. Then, one can change the degree of a matrix through rescaling matrices

$$P = \begin{pmatrix} \mathsf{t}_{\alpha_1} & & \\ & \ddots & \\ & & \mathsf{t}_{\alpha_n} \end{pmatrix}$$

with $\alpha_1, \dots, \alpha_n \in \Gamma_{\bar 0}$. This leads to the following result.

Proposition 3.8. *Assume $\Gamma_{\mathcal{A}^\times} = \Gamma_{\bar 0}$ and $(r_{\bar 0}, r_{\bar 1}) \in \mathbb{N}^2$. For all $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$ in $\Gamma_{\bar 0}^{r_{\bar 0}} \times \Gamma_{\bar 1}^{r_{\bar 1}}$, we have an isomorphism of Γ-algebras*

$$\mathsf{M}(\boldsymbol{\mu};\mathcal{A}) \simeq \mathsf{M}(\boldsymbol{\nu};\mathcal{A}) .$$

If more specifically $\boldsymbol{\mu} \in \Gamma_{\bar 0}^{r_{\bar 0}}$, the 0-degree matrices over $\mathcal{A}$ identify to the usual matrices over the commutative algebra $\mathcal{A}^0$, namely, we have an isomorphism of Γ-algebras

$$\mathsf{M}^0(\boldsymbol{\mu};\mathcal{A}) \simeq \mathsf{M}(r_{\bar 0};\mathcal{A}^0) . \tag{3.8}$$

Proof. Let M a free graded $\mathcal{A}$-module with a basis of degree $\boldsymbol{\mu} \in \Gamma_{\bar 0}^{r_{\bar 0}} \times \Gamma_{\bar 1}^{r_{\bar 1}}$. By Proposition 3.7, we have $\mathsf{M}(\boldsymbol{\mu};\mathcal{A}) \simeq \mathcal{E}nd_{\mathcal{A}}(M)$. In view of Corollary 3.6, for all $\boldsymbol{\nu} \in \Gamma_{\bar 0}^{r_{\bar 0}} \times \Gamma_{\bar 1}^{r_{\bar 1}}$, there exists a basis of M of degree $\boldsymbol{\nu}$. Applying again Proposition 3.7, we deduce

$$\mathsf{M}(\boldsymbol{\nu};\mathcal{A}) \simeq \mathcal{E}nd_{\mathcal{A}}(M) \simeq \mathsf{M}(\boldsymbol{\mu};\mathcal{A}) .$$

As a direct consequence, if $\boldsymbol{\mu} \in \Gamma_{\bar 0}^{r_{\bar 0}}$ we have $\mathsf{M}(\boldsymbol{\mu};\mathcal{A}) \simeq \mathsf{M}(\mathbf{0};\mathcal{A})$, with $\mathbf{0} = (0,\dots,0) \in \Gamma^{r_{\bar 0}}$. By definition of the degree of a graded matrix, this leads to (3.8). ☐

The second part of the Proposition 3.8 is well-known. This is a consequence of a Theorem of Dade [9], which holds for matrix algebras over any strongly graded Γ-algebra.

3.5. **Natural isomorphisms on graded matrices.** Let $\varsigma \in \mathfrak{S}(\lambda)$ and M be a free graded $\mathcal{A}$-module of rank n, with basis (e_i) of degree $\boldsymbol{\nu} \in \Gamma^n$. We denote by $\underline{\mathcal{A}} = I_\varsigma(\mathcal{A})$ the Γ-graded supercommutative algebra given by the NS-functor I_ς. Theorem A, together with the isomorphism (3.6), yields an isomorphism of Γ-graded $\mathcal{A}^0$-modules

$$\begin{array}{rccc} J_{\varsigma,\mathcal{A}}: & \mathsf{M}(\boldsymbol{\nu};\mathcal{A}) & \to & \mathsf{M}(\boldsymbol{\nu};\underline{\mathcal{A}}) \\ & X & \mapsto & J_{\varsigma,\mathcal{A}}(X) \end{array} \tag{3.9}$$

which explicitly reads, on a homogeneous matrix X of degree x, as

$$\left(J_{\varsigma,\mathcal{A}}(X)\right)_{i,j} := \varsigma(x,\nu_j)\varsigma(\nu_i, x-\nu_i+\nu_j)^{-1} X^i_j , \tag{3.9a}$$

for all $i, j \in \{1, 2, \ldots, n\}$. The inverse of $J_{\varsigma,\mathcal{A}}$ is of the same form, namely $(J_{\varsigma,\mathcal{A}})^{-1} = J_{\delta,\mathcal{A}}$, with $\delta(x, y) := \varsigma(x, y)^{-1}$ for all $x, y \in \Gamma$. From Theorem A, we deduce the following

Proposition 3.9. *The map $J_{\varsigma,\mathcal{A}}$ defined in (3.9) has the following properties:*

(1) for any homogeneous matrices $X, Y \in \mathsf{M}(\boldsymbol{\nu}; \mathcal{A})$, of degree respectively x and y,

$$J_\varsigma(XY) = \varsigma(x, y)^{-1}\, J_\varsigma(X) J_\varsigma(Y)\ ;$$

(2) for any graded matrices $X, Y \in \mathsf{M}(\boldsymbol{\nu}; \mathcal{A})$, with either X or Y homogeneous of degree 0,

$$J_\varsigma(XY) = J_\varsigma(X) J_\varsigma(Y)\ ;$$

(3) for any homogeneous invertible matrix $X \in \mathsf{GL}^x(\boldsymbol{\nu}; \mathcal{A})$, $J_\varsigma(X)$ is invertible and

$$J_\varsigma(X^{-1}) = \varsigma(x, -x)\ (J_\varsigma(X))^{-1}\ .$$

The family of maps $J_{\varsigma,\mathcal{A}}$, parameterized by $\mathcal{A} \in \mathrm{Ob}((\Gamma, \lambda)\text{-}\mathtt{Alg})$, is a natural isomorphism

$$\begin{tikzcd}
(\Gamma,\lambda)\text{-}\mathtt{Alg} \arrow[rr, bend left, "\mathsf{M}\left(\boldsymbol{\nu};-\right)"] \arrow[rr, bend right, "\mathsf{M}\left(\boldsymbol{\nu};I_\varsigma(-)\right)"'] \arrow[rr, phantom, "\Downarrow J_\varsigma"] & & \Gamma\text{-}\mathtt{Mod}_{\mathbb{K}}
\end{tikzcd}$$

where the functor $\mathsf{M}\left(\boldsymbol{\nu}; I_\varsigma(-)\right)$ is the composition of the functor $\mathsf{M}\left(\boldsymbol{\nu}; -\right)$ (recall (3.4)) with the NS-functor I_ς. The restriction of $J_{\varsigma,\mathcal{A}}$ to 0-degree invertible matrices has further properties.

Proposition 3.10. *Let $\varsigma \in \mathfrak{S}(\lambda)$ and $\boldsymbol{\nu} \in \Gamma^n$. The family of maps*

$$J_{\varsigma,\mathcal{A}} :\ \mathsf{GL}^0(\boldsymbol{\nu}; \mathcal{A}) \xrightarrow{\sim} \mathsf{GL}^0(\boldsymbol{\nu}; \underline{\mathcal{A}})\ , \tag{3.10}$$

parameterized by $\mathcal{A} \in \mathrm{Ob}((\Gamma, \lambda)\text{-}\mathtt{Alg})$, defines a natural isomorphism

$$\begin{tikzcd}
(\Gamma,\lambda)\text{-}\mathtt{Alg} \arrow[rr, bend left, "\mathsf{GL}^0\left(\boldsymbol{\nu};-\right)"] \arrow[rr, bend right, "\mathsf{GL}^0\left(\boldsymbol{\nu};I_\varsigma(-)\right)"'] \arrow[rr, phantom, "\Downarrow J_\varsigma"] & & \mathtt{Grp}
\end{tikzcd}$$

Proof. By Proposition 3.9, the map (3.10) is a group isomorphism. The naturality of $J_{\varsigma,\mathcal{A}}$ follows from the diagram of groups

$$\begin{tikzcd}
\mathsf{GL}^0(\boldsymbol{\nu}; \mathcal{A}) \arrow[r, "\mathsf{GL}(f)"] \arrow[d, "J_{\varsigma,\mathcal{A}}"'] & \mathsf{GL}^0(\boldsymbol{\nu}; \mathcal{B}) \arrow[d, "J_{\varsigma,\mathcal{B}}"] \\
\mathsf{GL}^0(\boldsymbol{\nu}; \underline{\mathcal{A}}) \arrow[r, "\mathsf{GL}\left(I_\varsigma(f)\right)"'] & \mathsf{GL}^0(\boldsymbol{\nu}; \underline{\mathcal{B}})
\end{tikzcd}$$

which is commutative for all (Γ, λ)-commutative algebras $\mathcal{A}, \mathcal{B}$ and any Γ-algebra morphism $f : \mathcal{A} \to \mathcal{B}$. $\square$

4. Graded Trace

Throughout this section, $\mathcal{A}$ is a (Γ,λ)-commutative algebra over $\mathbb{K}$ and M is a free graded $\mathcal{A}$-module of finite rank. We use the closed monoidal NS-functors to pull-back the supertrace to graded endomorphisms of M. This yields the *graded trace* introduced in [32] and extends the one introduced in [8]. We prove its characterization (Theorem B) and provide a matrix formula for it.

4.1. **Definitions.** A *Lie color algebra*, or more precisely of (Γ,λ)-*Lie algebra* (see e.g. [31]), is a Γ-graded $\mathbb{K}$-vector space, $\mathcal{L}=\bigoplus_{\gamma\in\Gamma}\mathcal{L}^\gamma$, endowed with a bracket $[-\,,-]$, which satisfies the following properties, for all homogeneous $a,b,c\in\mathcal{L}$,

- λ-*skew symmetry :* $\quad [a,b]=-\lambda\big(\widetilde{a},\widetilde{b}\,\big)\,[b,a]$;
- *Graded Jacobi identity :* $\quad [a,[b,c]]=[[a,b],c]+\lambda\big(\widetilde{a},\widetilde{b}\,\big)\,[b,[a,c]]$.

A *morphism of* (Γ,λ)-*Lie algebras* is a degree-preserving $\mathbb{K}$-linear map $f:\mathcal{L}\to\mathcal{L}'$ which satisfies $f([a,b])=[f(a),f(b)]$, for every $a,b\in\mathcal{L}$.

Examples of (Γ,λ)-Lie algebras are Γ-algebras, e.g. the algebras $\mathcal{A}$ or $\mathcal{E}nd_{\mathcal{A}}(M)$, endowed with the Γ-graded commutator

$$[a,b]_\lambda := ab-\lambda\big(\widetilde{a},\widetilde{b}\,\big)\,ba\ .$$

Note that Lie superalgebras are $(\mathbb{Z}_2,\lambda^{\mathrm{super}})$-Lie algebras. By definition, a *graded trace* is a degree-preserving $\mathcal{A}$-linear map $\Gamma\mathrm{tr}:\mathcal{E}nd_{\mathcal{A}}(M)\to\mathcal{A}$, which is also a (Γ,λ)-Lie algebra morphism.

4.2. **Proof of Theorem B.** Let $\varsigma\in\mathfrak{S}(\lambda)$ and I_ς be the corresponding NS-functor. We use the notation $\underline{\mathcal{A}}=I_\varsigma(\mathcal{A})$, $\lambda^\varsigma=\lambda^{\mathrm{super}}$, and $\underline{M}=\widehat{I_\varsigma}(M)$. In view of Theorem A, the linear isomorphism $\eta_\varsigma:\mathcal{E}nd_{\mathcal{A}}(M)\to\mathcal{E}nd_{\underline{\mathcal{A}}}(\underline{M})$ satisfies

$$\eta_\varsigma([f,g]_\lambda)=\varsigma\left(\widetilde{f},\widetilde{g}\right)^{-1}\,[\eta_\varsigma(f),\eta_\varsigma(g)]_{\lambda^{\mathrm{super}}}\ ,$$

for any pair f,g of homogeneous endomorphisms. Hence, a graded trace $t:\ \mathcal{E}nd_{\mathcal{A}}(M)\to\mathcal{A}$ induces a Lie superalgebra morphism, $t\circ(\eta_\varsigma)^{-1}:\mathcal{E}nd_{\underline{\mathcal{A}}}(\underline{M})\to\underline{\mathcal{A}}$. The latter is $\underline{\mathcal{A}}$-linear, since t is $\mathcal{A}$-linear. As a consequence, the map $t\circ(\eta_\varsigma)^{-1}$ is a multiple of the supertrace str by a non-zero element of $\underline{\mathcal{A}}_{\bar{0}}$. Since t respects the Γ-degree, it is then equal to $\Gamma\mathrm{tr}=\mathrm{str}\circ\eta_\varsigma$ up to multiplication by an element of $\mathcal{A}^0$.

4.3. **Graded trace on matrices.** Let $n\in\mathbb{N}$ and M be a free graded $\mathcal{A}$-module of rank n with a basis (e_i) of degree $\boldsymbol{\nu}\in\Gamma^n$. Through the isomorphism (3.6), the graded trace $\Gamma\mathrm{tr}$ automatically defines an $\mathcal{A}$-linear (Γ,λ)-Lie algebra morphism

$$\Gamma\mathrm{tr}:\ \mathsf{M}(\boldsymbol{\nu};\mathcal{A})\to\mathcal{A}\ .$$

With the notation of Section 3.5, we have $\Gamma\mathrm{tr}=\mathrm{str}\circ J_\varsigma$. Hence, for a homogeneous matrix $X=(X^i{}_j)_{i,j}\in\mathsf{M}^x(\boldsymbol{\nu};\mathcal{A})$, we find that

$$\Gamma\mathrm{tr}(X)=\sum_i\lambda(\nu_i,x+\nu_i)X^i{}_i\ .$$

By linearity, the graded trace of an inhomogeneous matrix is the sum of the graded traces of its homogeneous components. The above formula shows that $\Gamma\mathrm{tr}$ does not depend on the NS-multiplier ς chosen for its construction.

In the categorical language, the graded trace $\Gamma\mathrm{tr}$ defines a natural transformation

$$\begin{tikzcd}[column sep=huge]
(\Gamma,\lambda)\text{-}\mathtt{Alg} \arrow[r, bend left=50, "\mathsf{M}(\boldsymbol{\nu};-)"{name=U}] \arrow[r, bend right=50, "\mathrm{Lie}(-)"'{name=D}] & (\Gamma,\lambda)\text{-}\mathtt{LieAlg} \arrow[Rightarrow, from=U, to=D, "\Gamma\mathrm{tr}"]
\end{tikzcd}$$

where (Γ,λ)-$\mathtt{LieAlg}$ is the category of (Γ,λ)-Lie algebras and the functor Lie associates to each (Γ,λ)-commutative algebra $\mathcal{A}=(A,\cdot)$ the corresponding abelian (Γ,λ)-Lie algebra $(A,[\cdot,\cdot]_\lambda)$.

5. Graded Determinant of 0-degree matrices

Throughout this section, $\mathcal{A}$ is a (Γ,λ)-commutative algebra over $\mathbb{K}$. Moreover, we restrict ourselves to the *purely even* case, i.e., $\Gamma=\Gamma_{\bar{0}}$. The NS-functors allow us to pull-back the determinant to 0-degree invertible matrices over $\mathcal{A}$. This yields the *graded determinant* introduced in [21] and extends the one introduced in [8]. We prove its characterization (Theorem C), provide a new formula for it (Theorem 5.4) and establish some of its properties.

5.1. **Proof of Theorem C.** We first prove existence of the graded determinant $\Gamma\mathrm{det}^0$ and then its uniqueness. We end up with a remark.

5.1.1. *Existence.* We define the natural transformation $\Gamma\mathrm{det}^0$ as the vertical composition (see Appendix A.3.1)

$$\begin{tikzcd}[column sep=8em]
(\Gamma_{\bar{0}},\lambda)\text{-}\mathtt{Alg}
\arrow[r, bend left=70, "\mathsf{GL}^0\left(\boldsymbol{\nu};-\right)"{name=A}]
\arrow[r, bend left=20, "\mathsf{GL}^0\left(\boldsymbol{\nu};I_\varsigma(-)\right)"{name=B}]
\arrow[r, bend right=20, "\left((I_\varsigma(-))^0\right)^\times"'{name=C}]
\arrow[r, bend right=70, "\left((-)^0\right)^\times"'{name=D}]
& \mathtt{Grp}
\arrow[Rightarrow, from=A, to=B, "J_\varsigma"]
\arrow[Rightarrow, from=B, to=C, "\det_{I_\varsigma(-)}"]
\arrow[equal, from=C, to=D]
\arrow[Rightarrow, from=A, to=D, bend right, "\Gamma\mathrm{det}^0"']
\end{tikzcd}$$

where J_ς is the natural isomorphism introduced in Proposition 3.10 and $\det_{I_\varsigma(-)}$ denotes the natural transformation obtained by whiskering (see Appendix A.3.3) as follows

$$\begin{tikzcd}[column sep=huge]
(\Gamma_{\bar 0},\lambda)\text{-}\mathtt{Alg} \arrow[r, "I_\varsigma"'] \arrow[rr, bend left=60, "\mathsf{GL}^0(\boldsymbol{\nu};I_\varsigma(-))"] \arrow[rr, bend right=60, "\big((I_\varsigma(-))^0\big)^\times"'] \arrow[r, phantom, bend left=40, "\det_{I_\varsigma(-)}\Downarrow"] & (\Gamma_{\bar 0},\mathbf{1})\text{-}\mathtt{Alg} \arrow[r, bend left, "\mathsf{GL}^0(\boldsymbol{\nu};-)"] \arrow[r, bend right, "\big((-)^0\big)^\times"'] \arrow[r, phantom, "\Downarrow\det"] & \mathtt{Grp}
\end{tikzcd}$$

Here, $(\Gamma_{\bar 0},1)\text{-}\mathtt{Alg}$ denotes the category of commutative algebras which are $\Gamma_{\bar 0}$-graded.

By construction, $\Gamma\!\det^0$ satisfies the first axiom of Theorem C and reads as (0.2). The second axiom of Theorem C follows then from the explicit expression of J_ς given in (3.9a).

5.1.2. *Uniqueness.* We need a preliminary result, which generalizes a Theorem of McDonald [25].

Lemma 5.1. *If $(\Delta_{\mathcal{A}})_{\mathcal{A}\in(\Gamma,\lambda)\text{-}\mathtt{Alg}}$ is a family of maps*

$$\Delta_{\mathcal{A}}:\ \mathsf{GL}^0(\boldsymbol{\nu};\mathcal{A})\to(\mathcal{A}^0)^\times$$

satisfying axiom AI *of Theorem C, then there exists $t\in\mathbb{Z}$ and $\varepsilon\in\{\pm1\}$ such that*

$$\Delta_{\mathcal{A}}(X)=\varepsilon\cdot\left(\Gamma\!\det{}^0_{\mathcal{A}}(X)\right)^t\ , \tag{5.1}$$

for all $X\in\mathsf{GL}^0(\boldsymbol{\nu};\mathcal{A})$, with $\Gamma\!\det^0_{\mathcal{A}}$ as in (0.2).

Proof. We generalize the proof in [25].

According to (3.1), we denote the (Γ,λ)-commutative algebra of graded polynomials, with homogeneous indeterminates ${X^i}_j$ of degrees

$$\widetilde{{X^i}_j}:=\nu_j-\nu_i\ ,\quad\text{for all } i,j=1,\dots,n\,,$$

by $\mathbb{K}[{X^i}_j\,|\,i,j=1,\dots n]$. Thus, $X:=({X^i}_j)_{i,j}$ is a formal $\boldsymbol{\nu}\times\boldsymbol{\nu}$ graded matrix of degree 0.

From (0.2) and (3.9a), it follows that $\Gamma\!\det^0(X)$ is an element of $\mathbb{K}[{X^i}_j\,|\,i,j=1,\dots n]$, and by localizing this algebra at the powers of $\Gamma\!\det^0(X)$, we get the (Γ,λ)-commutative algebra

$$T(\boldsymbol{\nu}):=\mathbb{K}\left[{X^1}_1,{X^1}_2,\dots,{X^n}_n,\frac{1}{\Gamma\!\det^0(X)}\right]\ .$$

For every object $\mathcal{A}\in\operatorname{Ob}\big((\Gamma,\lambda)\text{-}\mathtt{Alg}\big)$, we define the map

$$\begin{array}{rccc}\theta(\boldsymbol{\nu})_{\mathcal{A}}: & \mathsf{GL}^0(\boldsymbol{\nu};\mathcal{A}) & \to & \operatorname{Hom}\left(T(\boldsymbol{\nu}),\mathcal{A}\right)\\ & ({a^i}_j)_{ij} & \mapsto & \Big(\mathfrak{a}:\ {X^i}_j\mapsto{a^i}_j\Big)\end{array}$$

This is an isomorphism of groups, which is natural in $\mathcal{A}$. Indeed, for every Γ-algebra morphism $f \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$, the following diagram of groups commutes.

$$\begin{tikzcd}
\mathsf{GL}^0(\boldsymbol{\nu};\mathcal{A}) \arrow[r, "\mathsf{GL}(f)"] \arrow[d, "\theta(\boldsymbol{\nu})_{\mathcal{A}}"', "\wr"] & \mathsf{GL}^0(\boldsymbol{\nu};\mathcal{B}) \arrow[d, "\theta(\boldsymbol{\nu})_{\mathcal{B}}", "\wr"'] \\
\mathrm{Hom}(T(\boldsymbol{\nu}), \mathcal{A}) \arrow[r, "f \circ -"'] & \mathrm{Hom}\left(T(\boldsymbol{\nu}), \mathcal{B}\right)
\end{tikzcd}$$

This means that the covariant functor $\mathsf{GL}^0(\boldsymbol{\nu}; -)$ is represented by the (Γ, λ)-commutative algebra $T(\boldsymbol{\nu})$. In particular, the functor $\left((-)^0\right)^\times$, which is equal to $\mathsf{GL}^0(\boldsymbol{\mu}; -)$ for $\boldsymbol{\mu} = 0 \in \Gamma$, is represented by $T(1) = \mathbb{K}\left[Y, \frac{1}{Y}\right]$, with indeterminate Y of degree 0. We deduce the following bijection

$$\begin{array}{ccc}
\mathrm{Nat}\Big(\,\mathsf{GL}^0(\boldsymbol{\nu}; -)\,,\, \left((-)^0\right)^\times\Big) & \xrightarrow{\sim} & \mathrm{Nat}\Big(\,\mathrm{Hom}(T(\boldsymbol{\nu}), -)\,,\, \mathrm{Hom}(T(1), -)\Big) \\
\Delta & \leftrightarrow & \kappa := \theta(1) \circ \Delta \circ \theta(\boldsymbol{\nu})^{-1}
\end{array}$$

where $\mathrm{Nat}(-,-)$ denotes the set of natural transformations between two functors.

On the other hand, thanks to Yoneda Lemma, there exists a bijection

$$\begin{array}{ccc}
\mathrm{Nat}\Big(\,\mathrm{Hom}(T(\boldsymbol{\nu}), -)\,,\, \mathrm{Hom}(T(1), -)\Big) & \xrightarrow{\sim} & \mathrm{Hom}\left(T(1), T(\boldsymbol{\nu})\right) \\
\kappa = \{\kappa_{\mathcal{A}} \,|\, \mathcal{A} \in (\Gamma, \lambda)\text{-}\mathtt{Alg}\} & \leftrightarrow & \varkappa
\end{array}$$

This correspondence goes as follows: any natural transformation κ defines a Γ-algebra morphism $\varkappa := \kappa_{T(\boldsymbol{\nu})}\left(\mathrm{id}_{T(\boldsymbol{\nu})}\right)$ and conversely, given an algebra morphism $\varkappa$, a natural transformation is completely defined by

$$\kappa_{\mathcal{A}}(f) := f \circ \varkappa\,,$$

for all $f \in \mathrm{Hom}(T(\boldsymbol{\nu}), \mathcal{A})$ and all object $\mathcal{A} \in \mathrm{Ob}\left((\Gamma, \lambda)\text{-}\mathtt{Alg}\right)$.

Since $T(1) = \mathbb{K}\left[Y, \frac{1}{Y}\right]$, an algebra morphism $\varkappa \in \mathrm{Hom}\left(T(1), T(\boldsymbol{\nu})\right)$ is characterized by $\varkappa(Y)$, which is necessarily an invertible element of $T(\boldsymbol{\nu})$. By construction, the only possibilities are

$$\varkappa(Y) = \pm\left(\Gamma\!\det{}^0(X)\right)^t\ ,$$

with $t \in \mathbb{Z}$. Hence, via the two above bijections, we finally obtain that any natural transformation $\Delta : \mathsf{GL}^0(\boldsymbol{\nu}; -) \Rightarrow \left((-)^0\right)^\times$ satisfies (5.1). □

Let Δ be a natural transformation satisfying axioms AI and AII of Theorem C. The preceding lemma forces Δ to satisfy (5.1), and axiom AII yields then $\Delta = \Gamma\!\det^0$. This concludes the proof of Theorem C.

Remark 5.2. *The graded determinant introduced in* [21] *satisfies axioms* AI *and* AII *of Theorem C. Hence it coincides with* $\Gamma\!\det^0$.

5.2. **Explicit formula.** Let $n \in \mathbb{N}$, S_n be the group of permutations of $\{1, \ldots, n\}$ and $\sigma \in \mathrm{S}_n$. The decomposition of the permutation σ into disjoint cycles can be written in terms of an auxiliary permutation $\widehat{\sigma} \in \mathrm{S}_n$ as follows,

$$\sigma = \Big(\widehat{\sigma}(1)\,\widehat{\sigma}(2)\,\ldots\widehat{\sigma}(i_1)\Big)\Big(\widehat{\sigma}(i_1{+}1)\,\widehat{\sigma}(i_1{+}2)\,\ldots\widehat{\sigma}(i_2)\Big)\cdots\Big(\widehat{\sigma}(i_{m-1}{+}1)\,\widehat{\sigma}(i_{m-1}{+}2)\,\ldots\widehat{\sigma}(i_m)\Big)\,, \tag{5.2}$$

with $i_1 < \ldots < i_m = n$. This means

$$\begin{cases} \sigma\left(\widehat{\sigma}(k)\right) = \widehat{\sigma}(k+1) & \text{if } k \notin \{i_1, \ldots, i_m\} \\ \sigma\left(\widehat{\sigma}(i_j)\right) = \widehat{\sigma}(i_{j-1}+1) & \text{for } j = 1, \ldots, m \end{cases} \tag{5.3}$$

with $i_0 = 0$ by convention. Such a permutation $\widehat{\sigma}$ is called an *ordering* associated to σ.

Example 5.3. *The permutation*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix} \in \mathrm{S}_5$$

can be written in cycle notation as

$$\sigma = (14)(253) \quad \textit{or} \quad \sigma = (532)(14).$$

The corresponding orderings are respectively

$$\widehat{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix} \quad \textit{and} \quad \widehat{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$$

Using the above notion of ordering, we derive a formula for the graded determinant $\Gamma\mathrm{det}^0$, extended to $\mathsf{M}^0(\boldsymbol{\nu}; \mathcal{A})$ via its defining equation (0.2).

Theorem 5.4. *Let $\boldsymbol{\nu} \in \Gamma^n$. The graded determinant of a 0-degree matrix $X = (X^i{}_j) \in \mathsf{M}^0(\boldsymbol{\nu}; \mathcal{A})$ is given by the following formula*

$$\Gamma\mathrm{det}^0(X) = \sum_{\sigma \in \mathrm{S}_n} \mathrm{sgn}(\sigma)\, X^{\widehat{\sigma}(1)}{}_{\sigma(\widehat{\sigma}(1))} \cdot X^{\widehat{\sigma}(2)}{}_{\sigma(\widehat{\sigma}(2))} \cdots X^{\widehat{\sigma}(n)}{}_{\sigma(\widehat{\sigma}(n))}\,, \tag{5.4}$$

where the right-hand side does not depend on the chosen orderings $\widehat{\sigma}$ associated to each $\sigma \in \mathrm{S}_n$.

Proof. Let $\varsigma \in \mathfrak{S}(\lambda)$, and define $(\underline{X}^i{}_j) = J_\varsigma(X)$. By Theorem C, the graded determinant satisfies $\Gamma\mathrm{det}^0_{\mathcal{A}}(X) = \mathrm{det}_{\underline{\mathcal{A}}}(\underline{X})$. Since the product "$\star$" in $\underline{\mathcal{A}}$ is commutative, we obtain

$$\Gamma\mathrm{det}^0_{\mathcal{A}}(X) = \sum_{\sigma \in \mathrm{S}_r} \mathrm{sgn}(\sigma)\, \underline{X}^{\widehat{\sigma}(1)}{}_{\sigma(\widehat{\sigma}(1))} \star \underline{X}^{\widehat{\sigma}(2)}{}_{\sigma(\widehat{\sigma}(2))} \star \cdots \star \underline{X}^{\widehat{\sigma}(r)}{}_{\sigma(\widehat{\sigma}(r))}\,,$$

for any ordering $\widehat{\sigma}$ associated to σ.

Let us now consider one monomial $\underline{X}^{\widehat{\sigma}(1)}{}_{\sigma(\widehat{\sigma}(1))} \star \underline{X}^{\widehat{\sigma}(2)}{}_{\sigma(\widehat{\sigma}(2))} \star \cdots \star \underline{X}^{\widehat{\sigma}(r)}{}_{\sigma(\widehat{\sigma}(r))}$. The permutation σ admits a decomposition into disjoint cycles as in (5.2), therefore

$$\prod_{j=1}^{n} \underline{X}^{\widehat{\sigma}(i)}{}_{\sigma(\widehat{\sigma}(i))} \;=\; \left(\prod_{j=1}^{i_1} \underline{X}^{\widehat{\sigma}(j)}{}_{\sigma(\widehat{\sigma}(j))}\right) \star \left(\prod_{j=i_1+1}^{i_2} \underline{X}^{\widehat{\sigma}(j)}{}_{\sigma(\widehat{\sigma}(j))}\right) \star \cdots \star \left(\prod_{j=i_{m-1}+1}^{n} \underline{X}^{\widehat{\sigma}(j)}{}_{\sigma(\widehat{\sigma}(j))}\right)$$

with $\prod$ denoting the product with respect to "$\star$".

Since $J_\varsigma(X)$ is a matrix of degree 0, the degree of its entries is given by $\deg\left(\underline{X}^{\widehat{\sigma}(i)}{}_{\sigma(\widehat{\sigma}(i))}\right) = \nu_{\sigma(\widehat{\sigma}(i))} - \nu_{\widehat{\sigma}(i)}$. In view of (5.3), the m products above are then of degree 0 and we deduce

$$\prod_{j=1}^{n} \underline{X}^{\widehat{\sigma}(i)}{}_{\sigma(\widehat{\sigma}(i))} \;=\; \left(\prod_{j=1}^{i_1} \underline{X}^{\widehat{\sigma}(j)}{}_{\sigma(\widehat{\sigma}(j))}\right) \cdot \left(\prod_{j=i_1+1}^{i_2} \underline{X}^{\widehat{\sigma}(j)}{}_{\sigma(\widehat{\sigma}(j))}\right) \cdot \cdots \cdot \left(\prod_{j=i_{m-1}+1}^{n} \underline{X}^{\widehat{\sigma}(j)}{}_{\sigma(\widehat{\sigma}(j))}\right)\,.$$

Using $\underline{X}^i{}_j = \varsigma(\nu_i, \nu_i - \nu_j)\, X^i{}_j$ (see (3.9a)) and the definition of "$\star$", we get

$$\underline{X}^{\widehat{\sigma}(j)}{}_{\widehat{\sigma}(j+1)} \star \left(\underline{X}^{\widehat{\sigma}(j+1)}{}_{\widehat{\sigma}(j+2)} \star \cdots \star \underline{X}^{\widehat{\sigma}(i_1)}{}_{\widehat{\sigma}(1)}\right) =$$

$$= \varsigma\left(\nu_{\widehat{\sigma}(j+1)}, \nu_{\widehat{\sigma}(1)} - \nu_{\widehat{\sigma}(j+1)}\right) \varsigma\left(\nu_{\widehat{\sigma}(j)}, \nu_{\widehat{\sigma}(1)} - \nu_{\widehat{\sigma}(j)}\right)^{-1} X^{\widehat{\sigma}(j)}_{\widehat{\sigma}(j+1)} \cdot \left(\underline{X}^{\widehat{\sigma}(j+1)}_{\widehat{\sigma}(j+2)} \star \cdots \star \underline{X}^{\widehat{\sigma}(i_1)}_{\widehat{\sigma}(1)}\right) ,$$

for any index $1 \leq j < i_1$. By induction, we then obtain

$$\underline{X}^{\widehat{\sigma}(1)}_{\sigma(\widehat{\sigma}(1))} \star \underline{X}^{\widehat{\sigma}(2)}_{\sigma(\widehat{\sigma}(2))} \star \ldots \star \underline{X}^{\widehat{\sigma}(i_1)}_{\sigma(\widehat{\sigma}(i_1))} = X^{\widehat{\sigma}(1)}_{\sigma(\widehat{\sigma}(1))} \cdot X^{\widehat{\sigma}(2)}_{\sigma(\widehat{\sigma}(2))} \cdot \ldots \cdot X^{\widehat{\sigma}(i_1)}_{\sigma(\widehat{\sigma}(i_1))} .$$

The same holds for the monomials built from the other cycles of σ. The claim follows. □

Remark 5.5. *The use of specific orderings associated to permutations also appears in the Moore determinant, defined for Hermitian quaternionic matrices (see e.g.* [4]*).*

5.3. **Graded determinant on endomorphisms.** Let M be a free graded $\mathcal{A}$-module of finite rank. The graded determinant carry over to endomorphisms.

Proposition 5.6. *Let (e_i) be a basis of M of degree $\boldsymbol{\nu}$. Then, the composite*

$$\mathcal{E}nd^0_{\mathcal{A}}(M) \xrightarrow{\sim} \mathsf{M}^0(\boldsymbol{\nu};\mathcal{A}) \xrightarrow{\Gamma\mathrm{det}^0} \mathcal{A}^0$$

does not depend on the choice of basis of M and thus defines the graded determinant on 0-degree endomorphisms of M. It satisfies

$$\Gamma\mathrm{det}^0(f) = \det\left(\widehat{I}_\varsigma(f)\right) ,$$

for all $\varsigma \in \mathfrak{S}(\lambda)$, where $\widehat{I}_\varsigma$ is the NS-functor associated to ς.

Proof. By definition of J_ς and of the classical determinant, the following diagram commutes.

$$\begin{tikzcd} \mathsf{M}^0(\boldsymbol{\nu};\mathcal{A}) \arrow[r, "J_\varsigma", "\sim"'] \arrow[d, phantom, "\wr|"] & \mathsf{M}^0(\boldsymbol{\nu};\underline{\mathcal{A}}) \arrow[r, "\det"] \arrow[d, phantom, "\wr|"] & \underline{\mathcal{A}}^0 = \mathcal{A}^0 \\ \mathcal{E}nd^0_{\mathcal{A}}(M) \arrow[r, "\sim", "\widehat{I}_\varsigma"'] & \mathcal{E}nd^0_{\underline{\mathcal{A}}}(\underline{M}) \arrow[ur, "\det"'] & \end{tikzcd}$$

This means that $\Gamma\mathrm{det}^0(f) = \det\left(\widehat{I}_\varsigma(f)\right)$, for $f \in \mathcal{E}nd^0_{\mathcal{A}}(M)$. Hence, $\Gamma\mathrm{det}^0(f)$ is independent of the chosen basis. □

5.4. **Properties.** Most of the classical properties of the determinant over commutative rings still hold true for $\Gamma\mathrm{det}^0$. We provide some of them, see also [21].

Proposition 5.7. *Let M be a free graded $\mathcal{A}$-module of rank n, admitting bases of degrees $\boldsymbol{\mu}, \boldsymbol{\nu} \in \Gamma^n$.*

(1) $u \in \mathcal{E}nd^0(M)$ is bijective if and only if $\Gamma\mathrm{det}^0(u)$ is invertible in $\mathcal{A}^0$;
(2) $\Gamma\mathrm{det}^0(P^{-1}XP) = \Gamma\mathrm{det}^0(X)$ for all $X \in \mathsf{M}^0(\boldsymbol{\nu};\mathcal{A})$ and all invertible $P \in \mathsf{M}^0(\boldsymbol{\nu}\times\boldsymbol{\mu};\mathcal{A})$;
(3) $\Gamma\mathrm{det}^0$ is $\mathcal{A}^0$-multilinear with respect to rows and columns.

Proof. (1) Invertibility of 0-degree elements is preserved by the arrow functor $\widehat{I}_\varsigma$. Hence, the statement follows from the analogous result over commutative algebras.

(2) This is a direct consequence of Proposition 5.6.

(3) This follows from the explicit formula (5.4) of the graded determinant.

□

The permutation of the rows of a graded matrix $X \in \mathsf{M}^0(\boldsymbol{\nu};\mathcal{A})$ by $\sigma \in \mathrm{S}_n$ produces a matrix $P_\sigma \cdot X \in \mathsf{M}^0(\boldsymbol{\mu}\times\boldsymbol{\nu};\mathcal{A})$, where $P_\sigma = (\delta^i_{\sigma(j)})_{i,j}$ and $\mu_i = \nu_{\sigma(i)}$. In general, the matrix $P_\sigma \cdot X$ is not homogeneous as a matrix in $\mathsf{M}(\boldsymbol{\nu};\mathcal{A})$. We circumvent this difficulty below, by modifying the permutation matrices.

Let us consider the auxiliary algebra $\mathcal{B} := \mathcal{A}\otimes_{\mathbb{K}}\mathbb{K}[S,S^{-1}]_\lambda$ (see (3.2)), with S a finite generating set of Γ. Then, we can choose invertible homogeneous elements $\mathfrak{t}_\gamma \in \mathcal{B}^\times \cap \mathcal{B}^\gamma$ for all $\gamma\in\Gamma$.

Lemma 5.8. *The following map is a group morphism,*

$$\begin{array}{rcl} \mathrm{S}_n & \to & \mathsf{GL}^0(\boldsymbol{\nu};\mathcal{B}) \\ \sigma & \mapsto & P(\sigma) := \left(\delta^i_{\sigma(j)}\,\mathfrak{t}_{\nu_i}^{-1}\mathfrak{t}_{\nu_j}\right)_{i,j} \end{array}$$

which satisfies $\Gamma\mathrm{det}^0(P(\sigma)) = \mathrm{sgn}(\sigma)$.

Proof. The statements follow from direct computations. □

From this lemma and multiplicativity of $\Gamma\mathrm{det}^0$, we deduce the effect of permutation of rows or columns on $\Gamma\mathrm{det}^0$.

Proposition 5.9. *Let $\boldsymbol{\nu}\in\Gamma^n$ and $X\in\mathsf{M}^0(\boldsymbol{\nu};\mathcal{A})$. We have for every $\sigma\in\mathrm{S}_n$,*

$$\Gamma\mathrm{det}^0_{\mathcal{B}}(P(\sigma)\cdot X) = \Gamma\mathrm{det}^0_{\mathcal{B}}(X\cdot P(\sigma)) = \mathrm{sgn}(\sigma)\ \Gamma\mathrm{det}^0_{\mathcal{A}}(X)\ .$$

5.5. **Graded determinant in particular bases.** Assume Γ is of finite order N. According to Section 3.4.1, one can order a degree $\boldsymbol{\mu}\in\Gamma^n$, so that graded matrices in $\mathsf{M}^0(\boldsymbol{\mu};\mathcal{A})$ are in block form (3.7). The graded determinant $\Gamma\mathrm{det}^0$ coincides with the one introduced in [8] on such matrices.

Proposition 5.10. *Let $\boldsymbol{\mu}\in\Gamma^n$ be an ordered degree.The graded determinant $\Gamma\mathrm{det}^0$ has the following properties:*

(1) for any block-diagonal matrix $D = (D_{uu})_{u=1,\dots,N}\in\mathsf{GL}^0(\boldsymbol{\mu};\mathcal{A})$,

$$\Gamma\mathrm{det}^0\left(\begin{array}{c|c|c} D_{11} & & \\ \hline & \ddots & \\ \hline & & D_{NN}\end{array}\right) = \prod_{u=1}^N \det(D_{uu})\ ;$$

(2) for any upper or lower block-unitriangular matrix

$$T = \left(\begin{array}{c|c|c} \mathbb{I} & * & * \\ \hline & \ddots & * \\ \hline & & \mathbb{I}\end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|c|c} \mathbb{I} & & \\ \hline * & \ddots & \\ \hline * & * & \mathbb{I}\end{array}\right)\ ,$$

we have

$$\Gamma\mathrm{det}^0(T) = 1\ ;$$

(3) if $p\in\mathbb{N}$ and $\mathcal{A}$ is a $\big((\mathbb{Z}_2)^p,(-1)^{\langle -,-\rangle}\big)$-commutative algebra, the morphism $\Gamma\mathrm{det}^0_{\mathcal{A}}$, from $\mathsf{GL}^0(\boldsymbol{\mu};\mathcal{A})$ to $(\mathcal{A}^0)^\times$, coincides with the graded determinant introduced in [8] .

Proof. (1) If $D = (D_{uu})_{u=1,\dots,N}$ is a block-diagonal matrix of degree 0, then we deduce from the definition (3.9a) of J_ς, that $J_\varsigma(D)$ is also block diagonal and $\big(J_\varsigma(D)\big)_{uu} = D_{uu}$, for all $u = 1,\dots,N$. Hence, the formula $\Gamma\mathrm{det}^0(X) = \det\left(J_\varsigma(X)\right)$ implies the result.

(2) Similarly, if T is a block unitriangular matrix of degree 0, then $J_\varsigma(T)$ is also one and we get $\Gamma\!\det^0(T) = 1$.

(3) In [8], the authors introduce a notion of graded determinant over $((\mathbb{Z}_2)^p, (-1)^{\langle\,\cdot\,,\,\cdot\,\rangle})$-commutative algebras $\mathcal{A}$. This graded determinant is the unique group morphism from $\mathsf{GL}^0(\boldsymbol{\mu};\mathcal{A})$ to $(\mathcal{A}^0)^\times$ which satisfies the properties (1) and (2) above. Hence, it coincides with $\Gamma\!\det^0_{\mathcal{A}}$ on $\mathsf{GL}^0(\boldsymbol{\mu};\mathcal{A})$. □

Assume that the graded algebra $\mathcal{A}$ is a crossed product, i.e., it admits invertible elements $\mathsf{t}_\alpha \in \mathcal{A}^\alpha$ for each degree α. In this case, Proposition 3.8 provides an algebra isomorphism, $\mathsf{M}^0(\boldsymbol{\nu};\mathcal{A}) \simeq \mathsf{M}(n;\mathcal{A}^0)$, via the change of basis $X \mapsto PXP^{-1}$ with

$$P = \begin{pmatrix} \mathsf{t}_{\nu_1} & & \\ & \ddots & \\ & & \mathsf{t}_{\nu_n} \end{pmatrix} .$$

Then, Proposition 5.7 leads to the graded determinant in this particular basis.

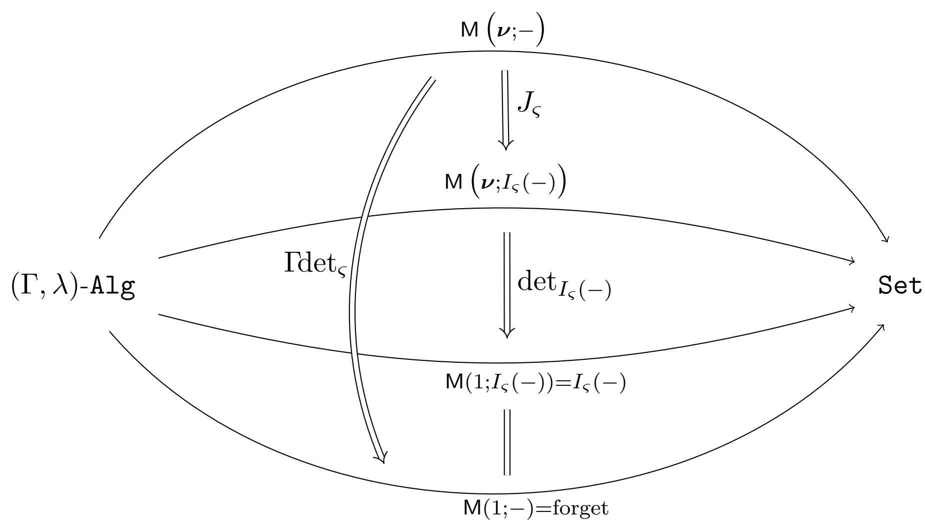
Proposition 5.11. *If $\mathcal{A}$ is a crossed product, then, for all $X \in \mathsf{M}^0(\boldsymbol{\nu};\mathcal{A})$, we have*

$$\Gamma\!\det^0(X) = \det(PXP^{-1}) .$$

6. A family of determinant-like functions on graded matrices

As in the previous section, all the considered pairs (Γ,λ) are such that $\Gamma = \Gamma_{\bar{0}}$.

Let $\mathcal{A}$ be a (Γ,λ)-commutative algebra, $n \in \mathbb{N}$, $\boldsymbol{\nu} \in \Gamma^n$ and ς a NS-multiplier. Using the NS-functor I_ς and the induced isomorphisms J_ς of Γ-graded $\mathcal{A}^0$-modules, we extend the graded determinant to all graded matrices via the formula $\Gamma\!\det_\varsigma := \det \circ J_\varsigma$. Hence, the maps $\Gamma\!\det_\varsigma$ are natural transformations defined as the following vertical composition



As $\Gamma\!\det^0$, the natural transformations $\Gamma\!\det_\varsigma$ can be equivalently defined on endomorphisms, via the formula $\Gamma\!\det_\varsigma := \det \circ \eta_\varsigma$, with η_ς defined in (2.6). In the case of a free graded module admitting bases of degrees $\boldsymbol{\nu},\boldsymbol{\mu} \in \Gamma^n$, this yields the following equality

$$\Gamma\!\det_\varsigma(P^{-1}XP) = \Gamma\!\det_\varsigma(X) ,$$

for all $X \in \mathsf{M}(\boldsymbol{\nu};\mathcal{A})$ and all invertible $P \in \mathsf{M}^0(\boldsymbol{\nu}\times\boldsymbol{\mu};\mathcal{A})$.

6.1. **Proof of Theorem D.** The proof is in three steps.

Step 1. We prove that $\Gamma\mathrm{det}_\varsigma$ satisfies the properties I.–IV., using its defining formula $\Gamma\mathrm{det}_\varsigma = \det\circ J_\varsigma$.

Property I. is obvious.

Property II. follows from Proposition 3.9.

Assume the matrices X, Y, Z satisfy (0.4). Using their homogeneous decomposition and the definition of J_ς (see (3.9a)), we see that the rows of matrices $J_\varsigma(X), J_\varsigma(Y), J_\varsigma(Z)$ satisfy again the relation (0.4). Property III. is then deduced from multi-additivity of the determinant.

Property IV. follows from the definitions of J_ς and of "$\star$" (see (1.6)), namely

$$\begin{aligned}\Gamma\mathrm{det}_\varsigma\left(\begin{array}{c|c} D & \\ \hline & \lambda(\widetilde{c},\nu_n)\,c\end{array}\right) = \det\left(\begin{array}{c|c} J_\varsigma(D) & \\ \hline & c\end{array}\right) &= c\star\det\left(J_\varsigma(D)\right)\\ &= \varsigma\Big(\widetilde{c},\deg\left(\Gamma\mathrm{det}_\varsigma(D)\right)\Big)\,c\cdot\Gamma\mathrm{det}_\varsigma(D)\,.\end{aligned}$$

Step 2. Let us consider a map $\mathfrak{s}:\Gamma\times\Gamma\to\mathbb{K}^\times$. Assume there exists a natural transformation $\Delta_\mathfrak{s}$ as in Theorem D. We prove that $\mathfrak{s}\in\mathfrak{S}(\lambda)$, or equivalently, that $\mathfrak{s}$ satisfies the two following equations

$$\lambda(x,y)\mathfrak{s}(x,y)\mathfrak{s}(y,x)^{-1} = 1\,, \tag{6.1}$$

$$\mathfrak{s}(x+y,z)\mathfrak{s}(x,y) = \mathfrak{s}(x,y+z)\mathfrak{s}(y,z)\,, \tag{6.2}$$

for all $x,y,z\in\Gamma$. To that end, we work over the algebra $\mathcal{B}=\mathcal{A}\otimes\mathbb{K}[S,S^{-1}]_\lambda$ (see (3.2)), with S a finite generating set of Γ, and use the 0-degree matrix permutations $P(\sigma)\in\mathsf{GL}^0(\boldsymbol{\nu};\mathcal{B})$, defined in Lemma 5.8.

We first prove (6.1). Let $\boldsymbol{\nu}\in\Gamma^2$ and $a,b\in\mathcal{B}$ be two homogeneous invertible elements. Then, using the transposition $\sigma=(1\,2)$, we get

$$\begin{pmatrix}\lambda(\widetilde{a},\nu_1)\,a & \\ & \lambda(\widetilde{b},\nu_2)\,b\end{pmatrix} = P(\sigma)\begin{pmatrix}\lambda(\widetilde{b},\nu_1)\,b & \\ & \lambda(\widetilde{a},\nu_2)\,a\end{pmatrix}P(\sigma)\,.$$

From Properties I., II. and Lemma 5.8, we deduce that

$$\Delta_\mathfrak{s}\begin{pmatrix}\lambda(\widetilde{a},\nu_1)\,a & \\ & \lambda(\widetilde{b},\nu_2)\,b\end{pmatrix} = \Delta_\mathfrak{s}\begin{pmatrix}\lambda(\widetilde{b},\nu_1)\,b & \\ & \lambda(\widetilde{a},\nu_2)\,a\end{pmatrix}$$

Applying IV. to the left and right hand side of the above equation, we obtain

$$\mathfrak{s}(\widetilde{b},\widetilde{a})ba = \mathfrak{s}(\widetilde{a},\widetilde{b})\lambda(\widetilde{a},\widetilde{b})\,ba\,,$$

by λ-commutativity. Since a and b are invertible, this implies (6.1).

We now prove (6.2). Let $\boldsymbol{\nu}\in\Gamma^3$ and $D\in\mathsf{M}(\boldsymbol{\nu};\mathcal{B})$ be a diagonal matrix with homogeneous invertible entries $D^i{}_i=\lambda(\widetilde{a}_i,\nu_i)\,a_i\in\mathcal{B}^{\widetilde{a}_i}$, for $i=1,2,3$. Using the permutation $\sigma=(123)$, we define

$$D' := P(\sigma^{-1})DP(\sigma) = \begin{pmatrix}\lambda(\widetilde{a}_2,\nu_1)\,a_2 & & \\ & \lambda(\widetilde{a}_3,\nu_2)\,a_3 & \\ & & \lambda(\widetilde{a}_1,\nu_3)\,a_1\end{pmatrix}.$$

Reasoning as above, we obtain that $\Delta_{\mathfrak{s}}(D') = \Delta_{\mathfrak{s}}(D)$. Furthermore, by means of IV., we can compute explicitly both $\Delta_{\mathfrak{s}}(D')$ and $\Delta_{\mathfrak{s}}(D)$. Using moreover the equality (6.1), we get

$$\begin{aligned}\Delta_{\mathfrak{s}}(D') &= \mathfrak{s}(\widetilde{a}_1, \widetilde{a}_3 + \widetilde{a}_2)\mathfrak{s}(\widetilde{a}_3, \widetilde{a}_2)\, a_1 a_3 a_2 \\ &= \mathfrak{s}(\widetilde{a}_1, \widetilde{a}_3 + \widetilde{a}_2)\mathfrak{s}(\widetilde{a}_3, \widetilde{a}_2)\lambda(\widetilde{a}_1, \widetilde{a}_3 + \widetilde{a}_2)\, a_3 a_2 a_1 \\ &= \mathfrak{s}(\widetilde{a}_3 + \widetilde{a}_2, \widetilde{a}_1)\mathfrak{s}(\widetilde{a}_3, \widetilde{a}_2)\, a_3 a_2 a_1 \,,\end{aligned}$$

and similarly,

$$\Delta_{\mathfrak{s}}(D) = \mathfrak{s}(\widetilde{a}_3, \widetilde{a}_2 + \widetilde{a}_1)\mathfrak{s}(\widetilde{a}_2, \widetilde{a}_1)\, a_3 a_2 a_1 \ .$$

Since a_1, a_2, a_3 are invertible, we finally obtain (6.2). In conclusion, we have $\mathfrak{s} \in \mathfrak{S}(\lambda)$.

Step 3. Let us consider a map $\mathfrak{s} \in \mathfrak{S}(\lambda)$. Assume that $\Delta_{\mathfrak{s}}$ is as in Theorem D. We prove that $\Delta_{\mathfrak{s}}$ is equal to $\Gamma\mathrm{det}_{\mathfrak{s}}$. For that, we show that $\Delta_{\mathfrak{s}}$ is uniquely determined by Properties I. - IV., first over algebras of the form $\mathcal{B} = \mathcal{A} \otimes_{\mathbb{K}} \mathbb{K}[S, S^{-1}]_\lambda$, with S a finite generating set of Γ, and then over all (Γ, λ)-commutative algebras.

a) Let $n \in \mathbb{N}^*$, $\boldsymbol{\nu} \in \Gamma^n$ and $X \in \mathsf{M}(\boldsymbol{\nu}; \mathcal{B})$. The rows of X can be decomposed into homogeneous parts,

$$X = \begin{bmatrix} \mathrm{x}^1 \\ \vdots \\ \mathrm{x}^n \end{bmatrix} = \begin{bmatrix} \sum_{\alpha_1 \in \Gamma} \mathrm{x}^{1,\alpha_1} \\ \vdots \\ \sum_{\alpha_n \in \Gamma} \mathrm{x}^{n,\alpha_n} \end{bmatrix} .$$

Applying III. inductively to such a decomposition, we obtain that

$$\Delta_{\mathfrak{s}}(X) = \sum_{\boldsymbol{\alpha} \in \Gamma^n} \Delta_{\mathfrak{s}}(X^{\boldsymbol{\alpha}}) \,, \quad \text{where} \quad X^{\boldsymbol{\alpha}} := \begin{bmatrix} \mathrm{x}^{1,\alpha_1} \\ \vdots \\ \mathrm{x}^{n,\alpha_n} \end{bmatrix} .$$

Hence, $X^{\boldsymbol{\alpha}}$ is the matrix whose k-th row is the α_k-degree component of the k-th row of X. Let us now consider an arbitrary map $\xi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and denote by $X^{\boldsymbol{\alpha}}(\xi)$ the matrix whose entries are given by

$$\left(X^{\boldsymbol{\alpha}}(\xi)\right)^i{}_j = (X^{\boldsymbol{\alpha}})^i{}_j\, \delta_{i,\xi(j)} \,,$$

where δ is the Kronecker delta. Thanks to III., we then have for any multi-index $\boldsymbol{\alpha} \in \Gamma^n$,

$$\Delta_{\mathfrak{s}}(X^{\boldsymbol{\alpha}}) = \sum_{\xi} \Delta_{\mathfrak{s}}(X^{\boldsymbol{\alpha}}(\xi)) \,.$$

If the map ξ is not bijective, the matrix $X^{\boldsymbol{\alpha}}(\xi)$ presents a whole row of zeros, and then $\Delta_{\mathfrak{s}}(X^{\boldsymbol{\alpha}}(\xi)) = 0$ by Property III.. As a consequence, we get

$$\Delta_{\mathfrak{s}}(X) = \sum_{\boldsymbol{\alpha} \in \Gamma^n} \sum_{\sigma \in S_n} \Delta_{\mathfrak{s}}(X^{\boldsymbol{\alpha}}(\sigma)) \,. \tag{6.3}$$

Let $P(\sigma) \in \mathsf{GL}^0(\boldsymbol{\nu}; \mathcal{B})$ be the permutation matrix associated to $\sigma \in \mathrm{S}_n$, as introduced in Lemma 5.8. By definition of $X^{\boldsymbol{\alpha}}(\sigma)$ we have

$$X^{\alpha}(\sigma) = P(\sigma) \cdot \mathcal{D}(\alpha, \sigma) \,,$$

where $\mathcal{D}(\alpha, \sigma)$ is a diagonal matrix with homogeneous entries. By Lemma 5.8 and Properties I. and II., we end up with

$$\Delta_{\mathfrak{s}}(X) = \sum_{\boldsymbol{\alpha} \in \Gamma^n} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, \Delta_{\mathfrak{s}}(\mathcal{D}(\alpha, \sigma)) \,. \tag{6.4}$$

By induction, Property IV. fixes the values of $\Delta_{\mathfrak{s}}(\mathcal{D}(\alpha,\sigma))$, and the uniqueness of the above map $\Delta_{\mathfrak{s}}$ follows.

b) Let $\mathcal{A}$ be an arbitrary (Γ,λ)-commutative algebra, S a finite generating set of Γ, and $\iota : \mathcal{A} \to \mathcal{B} = \mathcal{A} \otimes_{\mathbb{K}} \mathbb{K}[S, S^{-1}]_\lambda$ the canonical embedding. Since $\Delta_{\mathfrak{s}}$ is a natural transformation, we have

$$\iota(\Delta_{\mathfrak{s}}(X)) = \Delta_{\mathfrak{s}}(\mathsf{GL}(\iota)(X)) ,$$

for all $X \in \mathsf{M}(\boldsymbol{\nu};\mathcal{A})$. The right-hand side of the equation is fixed by point a) above and ι is injective, hence the map $\Delta_{\mathfrak{s}} : \ \mathsf{M}(\boldsymbol{\nu};\mathcal{A}) \to \mathcal{A}$ is unique and $\Delta_{\mathfrak{s}} = \Gamma\mathrm{det}_{\mathfrak{s}}$. This concludes the proof of Theorem D.

6.2. $\Gamma\mathrm{det}_\varsigma$ **on homogeneous graded matrices.** The restriction of $\Gamma\mathrm{det}_\varsigma$ to homogeneous matrices has additional properties, which turns it into a proper determinant.

First of all, by construction, $\Gamma\mathrm{det}_\varsigma$ preserves homogeneity. Indeed, for any $\gamma \in \Gamma$ and any $\boldsymbol{\nu} \in \Gamma^n$, we have

$$\Gamma\mathrm{det}_\varsigma(\mathsf{M}^\gamma(\boldsymbol{\nu};\mathcal{A})) \subset \mathcal{A}^{n\gamma} .$$

To go further, we need the following preliminary results.

Proposition 6.1. *Let $\varsigma \in \mathfrak{S}(\lambda)$ and $\boldsymbol{\nu} \in \Gamma^n$.*

(1) For any couple of homogeneous matrices $X, Y \in \mathsf{M}(\boldsymbol{\nu};\mathcal{A})$, of degrees x and y respectively, we have

$$\Gamma\mathrm{det}_\varsigma(XY) = \varsigma(x,y)^{n(n-1)}\ \Gamma\mathrm{det}_\varsigma(X) \cdot \Gamma\mathrm{det}_\varsigma(Y) .$$

(2) For any invertible homogeneous matrix $X \in \mathsf{GL}^x(\boldsymbol{\nu};\mathcal{A})$, $\Gamma\mathrm{det}_\varsigma(X)$ is invertible in $\mathcal{A}$,

$$\Gamma\mathrm{det}_\varsigma(X^{-1}) = \varsigma(x,x)^{n(n-1)}\ (\Gamma\mathrm{det}_\varsigma(X))^{-1} .$$

(3) For any homogeneous element $a \in \mathcal{A}$ and homogeneous matrix $X \in \mathsf{M}^x(\boldsymbol{\nu};\mathcal{A})$, we have

$$\Gamma\mathrm{det}_\varsigma(a \cdot X) = \varsigma(\widetilde{a},\widetilde{a})^{\frac{n(n-1)}{2}} \varsigma(\widetilde{a},x)^{n(n-1)}\ a^n \cdot \Gamma\mathrm{det}_\varsigma(X) .$$

In particular, if $X = \mathbb{I}$ is the identity matrix, this reduces to $\Gamma\mathrm{det}_\varsigma(a\cdot\mathbb{I}) = \varsigma(\widetilde{a},\widetilde{a})^{\frac{n(n-1)}{2}}\ a^n$.

Proof. The first result follows from the formula $\Gamma\mathrm{det}_\varsigma = \det \circ J_\varsigma$ and from the second point of Proposition 3.9.

The second result is a consequence of the first one.

The third result relies on the first point and on the computation of $\Gamma\mathrm{det}_\varsigma(a \cdot \mathbb{I})$. Using displays (3.3) and (3.9a) we get

$$\begin{aligned}\Gamma\mathrm{det}_\varsigma(a \cdot \mathbb{I}) &= \det \begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix} = a \star a \star \ldots \star a \\ &= \left(\prod_{i<j} \varsigma(\widetilde{a},\widetilde{a}) \right)\ a \cdot a \cdot \ldots \cdot a = \varsigma(\widetilde{a},\widetilde{a})^{\frac{n(n-1)}{2}}\ a^n .\end{aligned}$$

□

The next two propositions show that $\Gamma\mathrm{det}_\varsigma$ satisfies the two fundamental properties of a determinant on homogeneous matrices.

Proposition 6.2. *Let $\varsigma \in \mathfrak{S}(\lambda)$, $\boldsymbol{\nu} \in \Gamma^n$ and $X \in \mathsf{M}^x(\boldsymbol{\nu};\mathcal{A})$. The homogeneous matrix X is invertible if and only if $\Gamma\mathrm{det}_\varsigma(X)$ is invertible in $\mathcal{A}$.*

Proof. By Proposition 3.9, a homogeneous matrix X is invertible if and only if $J_\varsigma(X)$ is invertible. Since $\Gamma\mathrm{det}_\varsigma(X) = \det(J_\varsigma(X))$, the proposition follows from the analogous result over commutative algebras. $\square$

We introduce the set of homogeneous invertible matrices

$$\mathsf{hGL}(\boldsymbol{\nu};\mathcal{A}) := \bigcup_{\gamma\in\Gamma} \mathsf{GL}^\gamma(\boldsymbol{\nu};\mathcal{A})\ .$$

Since multiplication of two homogeneous matrices gives a homogeneous matrix, $\mathsf{hGL}(\boldsymbol{\nu};\mathcal{A})$ is a group. In particular, $\cup_{\gamma\in\Gamma}(\mathcal{A}^\times\cap\mathcal{A}^\gamma)$ is a group. Recall that $\mathbb{K}$ is the ground field of $\mathcal{A}$.

Proposition 6.3. *For all $\varsigma\in\mathfrak{S}(\lambda)$, there exists a finitely generated subgroup $\mathbb{U}\leq\mathbb{K}^\times$ such that*

$$\Gamma\mathrm{det}_\varsigma:\quad \mathsf{hGL}(\boldsymbol{\nu};\mathcal{A}) \quad\to\quad \bigcup_{\gamma\in\Gamma}(\mathcal{A}^\times\cap\mathcal{A}^\gamma)\,/\,\mathbb{U}$$

is a group morphism for all $\nu\in\bigcup_{n\in\mathbb{N}^*}\Gamma^n$. Moreover, $\mathbb{U}$ can be chosen as the subgroup of $\mathbb{K}^\times$ generated by $\{\varsigma(x,y)^2\,|\,x,y\in\Gamma\}$.

Proof. Let $\mathbb{U}$ be the subgroup of $\mathbb{K}^\times$ generated by $\{\varsigma(x,y)^2\,|\,x,y\in S\}$ in $\mathbb{K}^\times$, with S a finite generating set of Γ. The group $\mathbb{U}$ is finitely generated and contains $\{\varsigma(x,y)^2\,|\,x,y\in\Gamma\}$, since ς is biadditive. Hence we have $\varsigma(x,y)^{n(n-1)}\in\mathbb{U}$, for all $x,y\in\Gamma$, $n\in\mathbb{N}^*$, and the result follows from point (1) in Proposition 6.1. $\square$

If Γ is a finite group, $\mathbb{U}$ can be chosen as a finite group of roots of unity in $\mathbb{K}^\times$.

If ς takes values in $\{\pm1\}$, the preceding results simplify. In particular, one can take $\mathbb{U}=\{1\}$ and then obtain group morphisms

$$\Gamma\mathrm{det}_\varsigma:\quad \mathsf{hGL}(\boldsymbol{\nu};\mathcal{A}) \quad\to\quad \bigcup_{\gamma\in\Gamma}(\mathcal{A}^\times\cap\mathcal{A}^\gamma)\ .$$

Besides, the statement (3) in Proposition 6.1 reduces then to

$$\Gamma\mathrm{det}_\varsigma(a\cdot X) = \varsigma(\widetilde{a},\widetilde{a})^{\frac{n(n-1)}{2}}\,a^n\cdot\Gamma\mathrm{det}_\varsigma(X)\ . \tag{6.5}$$

If $\mathcal{A}$ admits homogeneous elements of each degree, then $X\in\mathsf{M}^x(\boldsymbol{\nu};\mathcal{A})$ can be written as $X = a\cdot X_0$, with $a\in\mathcal{A}^x$ and $X_0\in\mathsf{M}^0(\boldsymbol{\nu};\mathcal{A})$. The above equation can then be used as an ansatz to generalize $\Gamma\mathrm{det}^0$ to homogeneous matrices. If $n\equiv 0,1\mod 4$, this ansatz simplifies into the naive one $\Gamma\mathrm{det}(a\cdot X_0) = a^n\ \Gamma\mathrm{det}^0(X_0)$, used in [8], which does not depend on ς. But, if $n\equiv 2,3\mod 4$, such a naive ansatz is not coherent with multiplication by scalars (see [8]) and one should use (6.5).

Remark 6.4. *Assume $\mathcal{A}$ is a graded division ring and a graded commutative algebra, and write $\mathcal{A}_h^\times := \bigcup_{\gamma\in\Gamma}(\mathcal{A}^\times\cap\mathcal{A}^\gamma)$. The construction of a Dieudonné determinant in* [16] *specifies then as a group morphism*

$$\mathsf{hGL}(\boldsymbol{\nu};\mathcal{A}) \to \mathcal{A}_h^\times/[\mathcal{A}_h^\times,\mathcal{A}_h^\times],$$

where $[\mathcal{A}_h^\times,\mathcal{A}_h^\times]$ is generated by the products $aba^{-1}b^{-1} = \lambda(\tilde{a},\tilde{b})$ of homogeneous elements. By Proposition 6.3, such a morphism is provided by any graded determinant $\Gamma\mathrm{det}_\varsigma$.

6.3. $\Gamma\mathrm{det}_\varsigma$ **on quaternionic matrices: the good, the bad and the ugly.** Recall that the quaternion algebra is the real algebra $\mathbb{H} = \{x + \mathrm{i}\,y + \mathrm{j}\,z + \mathrm{k}\,t \,|\, x, y, z, t \in \mathbb{R}\}$ with multiplication law given in table (1.8). According to Example 1.13, $\mathbb{H}$ is a purely even (Γ, λ)-commutative algebra, with $\Gamma = \left\{0, \widetilde{\mathrm{i}}, \widetilde{\mathrm{j}}, \widetilde{\mathrm{k}}\right\} \leq (\mathbb{Z}_2)^3$ and $\lambda = (-1)^{\langle\cdot,\cdot\rangle}$, the grading being given by

$$\widetilde{\mathrm{i}} := (0,1,1)\,, \quad \widetilde{\mathrm{j}} := (1,0,1)\,, \quad \text{and} \quad \widetilde{\mathrm{k}} := (1,1,0)\,.$$

The graded $\mathbb{H}$-modules structures on $\mathbb{H}^n$ are in bijection with subspaces $V \subset \mathbb{H}^n$ of real dimension n, the grading being

$$\mathbb{H}^n = V \oplus \mathrm{i}\,V \oplus \mathrm{j}\,V \oplus \mathrm{k}\,V\,,$$

where, e.g., $(\mathbb{H}^n)^{\widetilde{\mathrm{i}}} = \mathrm{i}\,V$.

Let $X \in \mathsf{M}(n;\mathbb{H})$ be a quaternionic matrix, representing an endomorphism of $\mathbb{H}^n$ in the basis (e_i). For every $\boldsymbol{\nu} \in \Gamma^n$, there exists a grading $V \subset \mathbb{H}^n$ such that the basis vectors e_i are homogeneous and (e_i) has degree $\boldsymbol{\nu}$. Such a choice of grading turns X into a graded matrix, $X \in \mathsf{M}(\boldsymbol{\nu};\mathbb{H})$. This allows us to apply our determinant-like functions to quaternionic matrices.

6.3.1. *The good: homogeneous matrices.* The Dieudonné determinant over $\mathbb{H}$ is the unique group morphism

$$\mathrm{Ddet}: \quad \mathsf{GL}(n;\mathbb{H}) \quad \to \quad \mathbb{R}^\times/\{\pm 1\}\,,$$

which satisfies

$$\mathrm{Ddet}\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & a \end{pmatrix} = a \cdot \{\pm 1\}\,,$$

for all $a \in \mathbb{R}^\times$ (see [11]). In particular, this determinant defines a group morphism on $\mathsf{GL}(\boldsymbol{\nu};\mathbb{H})$ which is independent of $\boldsymbol{\nu} \in \Gamma^n$. We use the following quotient map

$$\pi: \quad \mathbb{R}^\times \oplus \mathrm{i}\,\mathbb{R}^\times \oplus \mathrm{j}\,\mathbb{R}^\times \oplus \mathrm{k}\,\mathbb{R}^\times \quad \to \quad (\mathbb{R}^\times \oplus \mathrm{i}\,\mathbb{R}^\times \oplus \mathrm{j}\,\mathbb{R}^\times \oplus \mathrm{k}\,\mathbb{R}^\times)/\{\pm 1, \pm\,\mathrm{i}, \pm\,\mathrm{j}, \pm\,\mathrm{k}\}\,,$$

the last group being isomorphic to $\mathbb{R}^\times/\{\pm 1\}$.

Proposition 6.5. *For all $\varsigma \in \mathfrak{S}(\lambda)$ and all $\boldsymbol{\nu} \in \Gamma^n$, the following diagram of groups commutes*

$$\begin{tikzcd} \mathsf{hGL}(\boldsymbol{\nu};\mathbb{H}) \arrow[r, "\Gamma\mathrm{det}_\varsigma"] \arrow[dr, "\mathrm{Ddet}"'] & \mathbb{R}^\times \oplus \mathrm{i}\,\mathbb{R}^\times \oplus \mathrm{j}\,\mathbb{R}^\times \oplus \mathrm{k}\,\mathbb{R}^\times \arrow[d, "\pi"] \\ & \mathbb{R}^\times/\{\pm 1\} \end{tikzcd} \tag{6.6}$$

Proof. Let $X_0 \in \mathsf{GL}^0(\boldsymbol{\nu};\mathbb{H})$. According to Proposition 5.11, there exists $P \in \mathsf{M}^0(\mathbf{0} \times \boldsymbol{\nu};\mathbb{H})$ such that $PX_0P^{-1} \in \mathsf{M}(n;\mathbb{R})$ and $\Gamma\mathrm{det}^0(X_0) = \det(PX_0P^{-1})$. By multiplicativity of the Dieudonné determinant, we get $\mathrm{Ddet}(X_0) = \mathrm{Ddet}(PX_0P^{-1})$. Since Ddet is equal to the classical determinant (modulo the sign) on real matrices, the diagram (6.6) commutes if restricted to the subgroup $\mathsf{GL}^0(\boldsymbol{\nu};\mathbb{H}) \leq \mathsf{hGL}(\boldsymbol{\nu};\mathbb{H})$.

If $X \in \mathsf{GL}^{\widetilde{\mathrm{i}}}(\boldsymbol{\nu};\mathbb{H})$, then $X = \mathrm{i} \cdot X_0$ with $X_0 \in \mathsf{GL}^0(\boldsymbol{\nu};\mathbb{H})$. By Proposition 6.1, we have

$$\Gamma\mathrm{det}_\varsigma(X) = \varsigma(\widetilde{\mathrm{i}},\widetilde{\mathrm{i}})^{\frac{n(n-1)}{2}}\ \mathrm{i}^n \cdot \Gamma\mathrm{det}_\varsigma(X_0)\,,$$

and $\varsigma(\widetilde{\mathrm{i}},\widetilde{\mathrm{i}})^{\frac{n(n-1)}{2}}\ \mathrm{i}^n \in \{\pm 1, \pm\mathrm{i}, \pm\mathrm{j}, \pm\mathrm{k}\}$. Besides, it is known that

$$\mathrm{Ddet}(X) = \mathrm{Ddet}(\mathrm{i}\cdot\mathbb{I})\mathrm{Ddet}(X_0) = \mathrm{Ddet}(X_0)\ .$$

Analogous results hold for $X \in \mathsf{GL}^{\widetilde{\mathrm{j}}}(\boldsymbol{\nu};\mathbb{H})$ and $X \in \mathsf{GL}^{\widetilde{\mathrm{k}}}(\boldsymbol{\nu};\mathbb{H})$. This conclude the proof. □

6.3.2. *The bad: non-uniqueness of* $\Gamma\mathrm{det}_\varsigma$. Let $X \in \mathsf{M}(n,\mathbb{H})$ be a quaternionic matrix representing an endomorphism of $\mathbb{H}^n$ in a basis (e_i) . Once the basis (e_i) receives a degree $\boldsymbol{\nu} \in \Gamma^n$, the matrix X becomes a graded matrix $X \in \mathsf{M}(\boldsymbol{\nu};\mathbb{H})$.

The value of $\Gamma\mathrm{det}_\varsigma(X)$ depends both on $\varsigma \in \mathfrak{S}(\lambda)$ and $\boldsymbol{\nu} \in \Gamma^n$. By Lemma 1.7, the multipliers $\varsigma \in \mathfrak{S}(\lambda)$ are characterized by the values of $\varsigma(\widetilde{\mathrm{i}},\widetilde{\mathrm{i}})$, $\varsigma(\widetilde{\mathrm{i}},\widetilde{\mathrm{j}})$ and $\varsigma(\widetilde{\mathrm{j}},\widetilde{\mathrm{j}})$, which can be either 1 or -1. Hence, for each of the 4^n possible degrees $\boldsymbol{\nu} \in \Gamma^n$, there are 8 determinant-like functions $\Gamma\mathrm{det}_\varsigma$ on $\mathsf{M}(\boldsymbol{\nu};\mathbb{H})$.

6.3.3. *The ugly:* $\Gamma\mathrm{det}_\varsigma$ *on inhomogeneous matrices.* We compute the values of $\Gamma\mathrm{det}_\varsigma(X)$ for a quaternionic matrix $X \in \mathsf{M}(2,\mathbb{H})$. This shows that, indeed, the value of $\Gamma\mathrm{det}_\varsigma(X)$, and even its vanishing, strongly depends on both choices: of multiplier $\varsigma \in \mathfrak{S}(\lambda)$ and of degree $\boldsymbol{\nu} \in \Gamma^2$ of the basis.

First, we work with the invertible matrix

$$X := \begin{pmatrix} 1 & \mathrm{j} \\ \mathrm{j} & 1 \end{pmatrix} \in \mathsf{GL}(2;\mathbb{H})\ ,$$

with inverse $X^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -\mathrm{j} \\ -\mathrm{j} & 1 \end{pmatrix}$. The matrix X is a homogeneous graded matrix of $\mathsf{GL}(\boldsymbol{\nu};\mathbb{H})$ if and only if the chosen degree $\boldsymbol{\nu} \in \Gamma^2$ is of the form $\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \boldsymbol{\nu}_1 + \widetilde{j})$, with $\boldsymbol{\nu}_1 \in \Gamma$. For such a degree, the matrix X is of degree 0 and, for all $\varsigma \in \mathfrak{S}(\lambda)$, we get

$$\Gamma\mathrm{det}_\varsigma(X) = \Gamma\mathrm{det}^0(X) = 2\ .$$

However, for a different $\boldsymbol{\nu}$, the result is completely different. For instance, if $\boldsymbol{\nu} = (0,0)$, we then obtain $J_\varsigma(X) = X$ and

$$\Gamma\mathrm{det}_\varsigma(X) = 1 - \mathrm{j}\star\mathrm{j} = 1 + \varsigma(\widetilde{\mathrm{j}},\widetilde{\mathrm{j}})\ ,$$

which means that

$$\Gamma\mathrm{det}_\varsigma(X) = \begin{cases} 0 & \text{if } \varsigma(\mathrm{j},\mathrm{j}) = -1\ , \\ 2 & \text{if } \varsigma(\mathrm{j},\mathrm{j}) = 1\ . \end{cases}$$

Each case happens for half of the choices of $\varsigma \in \mathfrak{S}(\lambda)$.

Second, we work with a rank one matrix

$$Y := \begin{pmatrix} 1 & \mathrm{j} \\ -\mathrm{j} & 1 \end{pmatrix} \in \mathsf{M}(2;\mathbb{H})\ ,$$

whose kernel is given by $\mathbb{H}\cdot(1,\mathrm{j})$. Again, the matrix Y is homogeneous of degree 0 as a graded matrix in $\mathsf{M}(\boldsymbol{\nu};\mathbb{H})$ if and only if $\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \boldsymbol{\nu}_1 + \widetilde{j})$ for some $\boldsymbol{\nu}_1 \in \Gamma$. We get then

$$\Gamma\mathrm{det}_\varsigma(Y) = \Gamma\mathrm{det}^0(Y) = 0\ ,$$

for all $\varsigma \in \mathfrak{S}(\lambda)$. However, if $\boldsymbol{\nu} = (0,0)$, we then obtain

$$\Gamma\mathrm{det}_\varsigma(Y) = 1 - \varsigma(\mathrm{j},\mathrm{j}) = \begin{cases} 2 & \text{if } \varsigma(\mathrm{j},\mathrm{j}) = -1\ , \\ 0 & \text{if } \varsigma(\mathrm{j},\mathrm{j}) = 1\ . \end{cases}$$

As a conclusion, the non-uniqueness of the functions $\Gamma\mathrm{det}_\varsigma$ prevents them to characterize invertible matrices.

7. Graded Berezinian

In this section, we go back to the general case of a (Γ,λ)-commutative algebra $\mathcal{A}$, for Γ a finitely generate abelian group with non-zero odd part $\Gamma_{\bar{1}}$. Applying a Nekludova-Scheunert functor I_ς to $\mathcal{A}$, we obtain a supercommutative algebra $\underline{\mathcal{A}}$.

For matrices with supercommutative entries, the notion of determinant is replaced by the Berezinian, which is a supergroup morphism

$$\mathrm{Ber} : \mathsf{GL}^{\bar{0}}((n,m);\underline{\mathcal{A}}) \to \underline{\mathcal{A}}^{\times} ,$$

with $(n,m)\in\mathbb{N}[\mathbb{Z}_2]$. The supergroup $\mathsf{GL}^{\bar{0}}((n,m);\underline{\mathcal{A}})$ of even invertible supermatrices is often written as $\mathsf{GL}(n|m;\underline{\mathcal{A}})$. Pulling back the Berezinian to the graded case, we obtain the notion of graded Berezinian.

7.1. **Preliminaries.** Let $\boldsymbol{\nu}=(\boldsymbol{\nu}_{\bar{0}},\boldsymbol{\nu}_{\bar{1}})\in\Gamma_{\bar{0}}^{r_{\bar{0}}}\times\ \Gamma_{\bar{1}}^{r_{\bar{1}}}$, where $(r_{\bar{0}},r_{\bar{1}})\in\mathbb{N}^2$. We introduce the group of even homogeneous invertible matrices

$$\mathsf{hGL}_{\bar{0}}(\boldsymbol{\nu};\underline{\mathcal{A}}):=\bigcup_{\gamma\in\Gamma_{\bar{0}}}\mathsf{GL}^{\gamma}(\boldsymbol{\nu};\underline{\mathcal{A}}).$$

Assume J_ς is the map (3.9), associated to a NS-multiplier $\varsigma\in\mathfrak{S}(\lambda)$. The composites

$$\mathsf{hGL}_{\bar{0}}(\boldsymbol{\nu};\mathcal{A})\xrightarrow[J_\varsigma]{}\mathsf{GL}(r_{\bar{0}}|r_{\bar{1}};\underline{\mathcal{A}})\xrightarrow[\mathrm{Ber}]{}(\underline{\mathcal{A}}_{\bar{0}})^{\times}\xrightarrow[J_\varsigma^{-1}]{\sim}(\mathcal{A}_{\bar{0}})^{\times}$$

define a family of maps parameterized by $\varsigma\in\mathfrak{S}(\lambda)$,

$$\Gamma\mathrm{Ber}_\varsigma:\ \ \mathsf{hGL}_{\bar{0}}(\boldsymbol{\nu};\mathcal{A})\ \ \to\ \ (\mathcal{A}_{\bar{0}})^{\times}\ .$$

Considering the last arrow, labeled with J_ς^{-1}, is a matter of taste as this is the identity map. This only changes the algebra structure and makes clear that the product and inverse involved in Formula (0.6) are taken in $\mathcal{A}$ and not in $\underline{\mathcal{A}}$.

Any matrix $X\in\mathsf{hGL}_{\bar{0}}(\boldsymbol{\nu};\mathcal{A})$ reads as

$$X=\begin{pmatrix}\mathcal{X}_{00} & \mathcal{X}_{01}\\ \mathcal{X}_{10} & \mathcal{X}_{11}\end{pmatrix},$$

where, in particular, $\mathcal{X}_{00}\in\mathsf{hGL}_{\bar{0}}(\boldsymbol{\nu}_{\bar{0}};\mathcal{A}_{\bar{0}})$ and $\mathcal{X}_{11}\in\mathsf{hGL}_{\bar{0}}(\boldsymbol{\nu}_{\bar{1}};\mathcal{A}_{\bar{0}})$. The Formula (0.6), giving the graded Berezinian of X, involves the graded determinant of $\mathcal{X}_{11}$. We define it just like for matrices of even degree $\boldsymbol{\nu}_{\bar{0}}$, via the formula $\Gamma\mathrm{det}_\varsigma=\det\circ J_\varsigma$.

Remark 7.1. *Let $\pi\in\Gamma$ and $\boldsymbol{\pi}:=(\pi,\ldots,\pi)\in\Gamma^{r_{\bar{1}}}$. The identity map $T_\pi:\mathsf{M}(\boldsymbol{\nu}_{\bar{1}};\mathcal{A})\to\mathsf{M}(\boldsymbol{\nu}_{\bar{1}}+\boldsymbol{\pi};\mathcal{A})$ is a morphism of Γ-algebra and of graded $\mathcal{A}$-module. From the Formula* (3.9a), *defining J_ς, we deduce that $\Gamma\mathrm{det}_\varsigma(\mathcal{X}_{11})=\lambda(x,\pi)^{r_{\bar{1}}}\,\Gamma\mathrm{det}_\varsigma(T_\pi(\mathcal{X}_{11}))$ for all $\pi\in\Gamma$.*

7.2. **Proof of Theorem E.** According to Proposition (3.10), the map J_ς restricts to $\mathsf{GL}^0(\boldsymbol{\nu};\mathcal{A})$ as a group morphism. Hence, the map

$$\Gamma\mathrm{Ber}^0 := \mathrm{Ber}\circ J_\varsigma : \mathsf{GL}^0(\boldsymbol{\nu};\mathcal{A}) \to (\mathcal{A})^\times$$

defines a group morphism. The independence of $\Gamma\mathrm{Ber}^0$ in $\varsigma \in \mathfrak{S}(\lambda)$ follows from the Formula (0.6) and the equality $\Gamma\!\det_\varsigma = \Gamma\!\det^0$ on 0-degree matrices.

It remains to prove Formula (0.6). Let us consider a homogeneous matrix $X \in \mathsf{GL}^x((\boldsymbol{\nu}_{\bar 0}, \boldsymbol{\nu}_{\bar 1});\mathcal{A})$ of arbitrary even degree $x \in \Gamma_{\bar 0}$. Decomposing X in block matrices with respect to parity, it reads as a supermatrix,

$$X = (X^i{}_j)_{i,j=1,\dots,n} = \left(\begin{array}{c|c} \mathcal{X}_{00} & \mathcal{X}_{01} \\ \hline \mathcal{X}_{10} & \mathcal{X}_{11} \end{array}\right) .$$

Since moding out by the ideal $\mathcal{A}_{\bar 1}$ of odd elements preserves invertibility, we deduce that the block $\mathcal{X}_{11}$ is invertible. This allows for an UDL decomposition of X, with respect to the parity block subdivision,

$$X = UDL = \left(\begin{array}{c|c} \mathbb{I} & \mathcal{X}_{01}\mathcal{X}_{11}^{-1} \\ \hline & \mathbb{I} \end{array}\right) \left(\begin{array}{c|c} \mathcal{X}_{00} - \mathcal{X}_{01}\mathcal{X}_{11}^{-1}\mathcal{X}_{10} & \\ \hline & \mathcal{X}_{11} \end{array}\right) \left(\begin{array}{c|c} \mathbb{I} & \\ \hline \mathcal{X}_{11}^{-1}\mathcal{X}_{10} & \mathbb{I} \end{array}\right) .$$

By construction, we also see that the three graded matrices in the decomposition are homogeneous: the block triangular matrices U and L are of degree 0, whereas the diagonal matrix D is of the same degree as the original matrix X. Thanks to the properties of J_ς, we then have

$$J_\varsigma(X) = J_\varsigma(U)J_\varsigma(D)J_\varsigma(L)$$

which is again a UDL decomposition. Hence, applying the Berezinian, we finally obtain

$$\mathrm{Ber}(J_\varsigma(X)) = \mathrm{Ber}(J_\varsigma(D)), \tag{7.1}$$

since the Berezinian is multiplicative and equal to 1 on a block unitriangular matrices. Let us recall that the Berezinian of a block diagonal invertible supermatrix is equal to

$$\mathrm{Ber}\left(\begin{array}{c|c} \mathcal{Y}_{00} & \\ \hline & \mathcal{Y}_{11} \end{array}\right) = \det(\mathcal{Y}_{00})\det(\mathcal{Y}_{11})^{-1} .$$

Hence, using (7.1), the properties of J_ς and the equality $(J_\varsigma)^{-1} = J_\delta$ with $\delta(x,y) := \varsigma(x,y)^{-1}$, for all $x, y \in \Gamma$, we get

$$\begin{aligned}
\Gamma\mathrm{Ber}(X) &= J_\varsigma^{-1}\left(\mathrm{Ber}\left(J_\varsigma(X)\right)\right) \\
&= J_\varsigma^{-1}\Big(\det\left(J_\varsigma\left(\mathcal{X}_{00} - \mathcal{X}_{01}\mathcal{X}_{11}^{-1}\mathcal{X}_{10}\right)\right) \star \det\left(J_\varsigma(\mathcal{X}_{11})\right)^{-1}\Big) \\
&= \varsigma(r_{\bar 0}x, -r_{\bar 1}x)\, J_\varsigma^{-1}\Big(\det\left(J_\varsigma\left(\mathcal{X}_{00} - \mathcal{X}_{01}\mathcal{X}_{11}^{-1}\mathcal{X}_{10}\right)\right)\Big) \cdot J_\varsigma^{-1}\Big(\det\left(J_\varsigma(\mathcal{X}_{11})\right)^{-1}\Big) \\
&= \varsigma(r_{\bar 0}x, -r_{\bar 1}x)\varsigma(-r_{\bar 1}x, -r_{\bar 1}x)\ \Gamma\!\det_\varsigma\left(\mathcal{X}_{00} - \mathcal{X}_{01}\mathcal{X}_{11}^{-1}\mathcal{X}_{10}\right) \cdot \Gamma\!\det_\varsigma(\mathcal{X}_{11})^{-1} ,
\end{aligned}$$

which recovers Formula (0.6).

Appendix A. Basic Notions of Category Theory

In this section we recall basic notions of category theory used in the paper. The main references are [24], [17] and [19].

A.1. **Categories.** A *locally small* (respectively *small*) *category* $\mathtt{C}$ consists of

- a class (respectively a set) of *objects* $\mathrm{Ob}(\mathtt{C})$;
- for every pair of objects $X, Y \in \mathrm{Ob}(\mathtt{C})$, a set of *morphisms* $\mathrm{Hom}_{\mathtt{C}}(X, Y)$ (if no confusion is possible the subscript is usually dropped);
- for any triple $X, Y, Z \in \mathrm{Ob}(\mathtt{C})$, a *composition of morphisms*

$$\circ : \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \ \to \ \mathrm{Hom}(X, Z)$$

which satisfy the following axioms

Associativity: For any given morphisms $f \in \mathrm{Hom}(Z, W)$, $g \in \mathrm{Hom}(Y, Z)$ and $h \in \mathrm{Hom}(X, Y)$, the equality $f \circ (g \circ h) = (f \circ g) \circ h$ holds.

Identity: For every $X \in \mathrm{Ob}(\mathtt{C})$, there exists a morphism $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ such that, for any morphisms $f \in \mathrm{Hom}(X, Z)$ and $g \in \mathrm{Hom}(Y, X)$, the following equalities hold

$$f \circ \mathrm{id}_X = f \quad \text{and} \quad \mathrm{id}_X \circ g = g \ .$$

All the categories encountered in this paper are *concrete categories*, i.e., the objects are sets with additional structure, and morphisms are functions. This translates into the existence of a forgetful functor to the category of sets, denoted by "forget".

A.2. **Functors.** A *functor* $F : \mathtt{C} \to \mathtt{D}$ between categories consist of

- an *object function* $F : \mathrm{Ob}(\mathtt{C}) \to \mathrm{Ob}(\mathtt{D})$,
- for each pair $X, Y \in \mathrm{Ob}(\mathtt{C})$, an *arrow function* $F : \mathrm{Hom}_{\mathtt{C}}(X, Y) \to \mathrm{Hom}_{\mathtt{D}}(F(X), F(Y))$,

such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathtt{C})$, and, when the composition is meaningful, $F(f \circ g) = F(f) \circ F(g)$.

The *composite of two functors* $F : \mathtt{B} \to \mathtt{C}$ and $G : \mathtt{A} \to \mathtt{B}$ is a functor $F \circ G : \mathtt{A} \to \mathtt{C}$ given by usual composition of the corresponding object functions and arrow functions. Composition of functors is associative.

A particular example of functor is the *identity functor* $\mathbb{I}_{\mathtt{C}} : \mathtt{C} \to \mathtt{C}$ which assigns to every object, respectively to every morphism of $\mathtt{C}$, itself. It acts as the identity element for the composition of functors.

A functor $F : \mathtt{A} \to \mathtt{B}$ is said *invertible* if there exists a second functor $G : \mathtt{B} \to \mathtt{A}$ such that

$$F \circ G = \mathbb{I}_{\mathtt{B}} \quad \text{and} \quad G \circ F = \mathbb{I}_{\mathtt{A}} \, .$$

A.3. **Natural Transformations.** If a functor is intuitively a morphism in the category of (small/locally small) categories, a natural transformation is a morphism between functors.

More precisely, if F and G are two functors between the same categories $\mathtt{C}$ and $\mathtt{D}$, a *natural transformation* η between the functors F and G is represented by the diagram

$$\begin{tikzcd} \mathtt{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=L}] & \mathtt{D} \arrow[Rightarrow, from=U, to=L, "\eta"] \end{tikzcd} \tag{A.1}$$

and η consists of a family of maps

$$\eta_X : F(X) \to G(X) \ ,$$

which is natural in $X \in \mathrm{Ob}(\mathtt{C})$. This means that, for every morphism $h \in \mathrm{Hom}_{\mathtt{C}}(X, Y)$, the following diagram commutes,

$$\begin{tikzcd} F(X) \arrow[r, "F(h)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(h)"'] & G(Y) \end{tikzcd}$$

There are two ways of composing natural transformation: "vertically" or "horizontally".

A.3.1. *Vertical composition of transformations.* Given two natural transformations

$$\begin{tikzcd}[column sep=huge] \mathtt{C} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, "G"{name=M}] \arrow[r, bend right=60, "H"'{name=L}] & \mathtt{D} \arrow[Rightarrow, from=U, to=M, "\eta"] \arrow[Rightarrow, from=M, to=L, "\theta"] \end{tikzcd}$$

their vertical composition is the natural transformation

$$\begin{tikzcd}[column sep=huge] \mathtt{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "H"'{name=L}] & \mathtt{D} \arrow[Rightarrow, from=U, to=L, "\theta \bullet \eta"] \end{tikzcd}$$

defined component-wise by $(\theta \bullet \eta)_X := \theta_X \circ \eta_X$, $X \in \mathrm{Ob}(\mathtt{C})$.

Vertical composition is the law of composition of the *functor category* $\mathtt{D}^{\mathtt{C}}$ (also denoted by $\mathrm{Fun}(\mathtt{C}, \mathtt{D})$), whose objects are functors between $\mathtt{C}$ and $\mathtt{D}$ and whose arrows are the natural transformations between such functors. For every functor F, the identity map Id_F is the natural transformation of components $\mathrm{Id}_{F,X} = \mathrm{id}_{F(X)}$.

A.3.2. *Horizontal composition of transformations.* Given two natural transformations

$$\begin{tikzcd}[column sep=huge] \mathtt{A} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=L}] & \mathtt{B} \arrow[Rightarrow, from=U, to=L, "\eta"] \arrow[r, bend left=50, "F'"{name=U2}] \arrow[r, bend right=50, "G'"'{name=L2}] & \mathtt{C} \arrow[Rightarrow, from=U2, to=L2, "\eta'"] \end{tikzcd}$$

their horizontal composition is the natural transformation

$$\begin{tikzcd} \mathtt{A} \arrow[rr, bend left, "F' \circ F"] \arrow[rr, bend right, "G' \circ G"'] \arrow[rr, phantom, "\Downarrow \eta' \circ \eta"] & & \mathtt{C} \end{tikzcd}$$

defined component-wise by $(\eta' \circ \eta)_X := \eta'_{G(X)} \circ F'(\eta_X) = G'(\eta_X) \circ \eta'_{F(X)}$, for all $X \in \mathrm{Ob}(\mathtt{A})$.

A.3.3. *Whiskering.* The horizontal composition allows to define the composition of a natural transformation with a functor, also called *whiskering*. Given a natural transformation as in (A.1) and a functor $H : \mathtt{B} \to \mathtt{C}$, their composite

$$\begin{tikzcd} \mathtt{B} \arrow[r, "H"] & \mathtt{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathtt{D} \end{tikzcd}$$

is defined as the horizontal composition of natural transformations $\mathrm{Id}_H \circ \eta$, i.e.

$$\begin{tikzcd} \mathtt{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow \mathrm{Id}_H"] & \mathtt{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathtt{D} \end{tikzcd} = \begin{tikzcd} \mathtt{B} \arrow[r, bend left, "H \circ F"] \arrow[r, bend right, "H \circ G"'] \arrow[r, phantom, "\Downarrow \mathrm{Id}_H \circ \eta"] & \mathtt{D} \end{tikzcd}$$

Clearly, one can do the same for right composition of η with a functor $H' : \mathtt{D} \to \mathtt{A}$,

$$\begin{tikzcd} \mathtt{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathtt{D} \arrow[r, "H'"] & \mathtt{A} \end{tikzcd}$$

A.4. **Adjoint Functors.** A functor $F : \mathtt{C} \to \mathtt{D}$ is the *left-adjoint* of a functor $G : \mathtt{D} \to \mathtt{C}$ (or equivalently G is the *right-adjoint* of F) if there exists a bijection between the hom-sets,

$$\mathrm{Hom}_{\mathtt{D}}(F(X), U) \simeq \mathrm{Hom}_{\mathtt{C}}(X, G(U))$$

which is natural in both $X \in \mathrm{Ob}(\mathtt{C})$ and $U \in \mathrm{Ob}(\mathtt{D})$. This means that

$$\mathrm{Hom}_{\mathtt{D}}(F(X), -) \overset{\sim}{\Rightarrow} \mathrm{Hom}_{\mathtt{C}}(X, G(-)) \quad \text{and} \quad \mathrm{Hom}_{\mathtt{D}}(F(-), U) \overset{\sim}{\Rightarrow} \mathrm{Hom}_{\mathtt{C}}(-, G(U))$$

are natural transformations (for X and U fixed respectively).

A.5. **Closed Monoidal Categories.** A *monoidal category* is a (locally small) category $\mathtt{C}$ endowed with a bifunctor $\otimes$, a unit object $\mathbb{I}$, and three natural isomorphisms

$$\begin{aligned} \alpha = \alpha_{X,Y,Z} &: (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \qquad (\textit{associator} \text{ or } \textit{associativity constraint}) \\ l = l_X &: \mathbb{I} \otimes X \xrightarrow{\sim} X \\ r = r_X &: X \otimes \mathbb{I} \xrightarrow{\sim} X \end{aligned}$$

such that $l_{\mathbb{I}} = r_{\mathbb{I}} : \mathbb{I} \to \mathbb{I}$, and they satisfy the following coherence laws:

$$\begin{tikzcd} & (X \otimes Y) \otimes (Z \otimes W) \arrow[dr, "\alpha_{X,Y,Z\otimes W}"] & \\ ((X \otimes Y) \otimes Z) \otimes W \arrow[ur, "\alpha_{X\otimes Y,Z,W}"] \arrow[d, "\alpha_{X,Y,Z} \otimes \mathrm{id}_W"'] & & X \otimes (Y \otimes (Z \otimes W)) \\ (X \otimes (Y \otimes Z)) \otimes W \arrow[rr, "\alpha_{X,Y\otimes Z,W}"'] & & X \otimes ((Y \otimes Z) \otimes W) \arrow[u, "\mathrm{id}_X \otimes \alpha_{Y,Z,W}"'] \end{tikzcd}$$

$$\begin{tikzcd} (X \otimes \mathbb{I}) \otimes Y \arrow[rr, "\alpha_{X,\mathbb{I},Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] & & X \otimes (\mathbb{I} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes l_Y"] \\ & X \otimes Y & \end{tikzcd}$$

A monoidal category is called *braided* if it is endowed with a natural isomorphism

$$\beta = \beta_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X ,$$

(usually called *commutativity constraint* or simply *braiding*) satisfying the following coherence laws:

$$\begin{tikzcd} & X \otimes (Y \otimes Z) \arrow[r, "\beta_{X,Y\otimes Z}"] & (Y \otimes Z) \otimes X \arrow[dr, "\alpha_{Y,Z,X}"] & \\ (X \otimes Y) \otimes Z \arrow[ur, "\alpha_{X,Y,Z}"] \arrow[dr, "\beta_{X,Y} \otimes \mathrm{id}_Z"'] & & & Y \otimes (Z \otimes X) \\ & (Y \otimes X) \otimes Z \arrow[r, "\alpha_{Y,X,Z}"'] & Y \otimes (X \otimes Z) \arrow[ur, "\mathrm{id}_Y \otimes \beta_{X,Z}"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{I}\otimes X \arrow[rr, "\beta_{\mathbb{I},X}"] \arrow[dr, "l_X"'] & & X\otimes\mathbb{I} \arrow[dl, "r_X"] \\ & X & \end{tikzcd}$$

If the braiding also satisfies

$$\beta_{Y,X}\circ\beta_{X,Y} = \mathrm{id}_{X\otimes Y}\ ,$$

for all $X, Y \in \mathrm{Ob}(\mathtt{C})$, then $(\mathtt{C}, \otimes, \mathbb{I}, \alpha, l, r, \beta)$ is a *symmetric monoidal category.*

A monoidal category $(\mathtt{C}, \otimes, \mathbb{I}, \alpha, l, r)$ is a *right* (resp. *left*) *closed monoidal category* if there exists a bifunctor

$$\mathcal{H}om^{right}_{\mathtt{C}}(-,-) : \mathtt{C}^{\mathrm{op}}\times\mathtt{C}\to\mathtt{C} \quad (\text{resp.}\quad \mathcal{H}om^{left}_{\mathtt{C}}(-,-) : \mathtt{C}^{\mathrm{op}}\times\mathtt{C}\to\mathtt{C}\ )\ ,$$

such that for every $X\in\mathrm{Ob}(\mathtt{C})$, the functor $\mathcal{H}om^{right}_{\mathtt{C}}(X,-)$ (resp. $\mathcal{H}om^{left}_{\mathtt{C}}(-,X)$) is the right adjoint of the functor $-\otimes X$ (resp. of the functor $X\otimes -$). A *closed monoidal category* is a left and right closed monoidal category.

Note that if a closed monoidal category is symmetric, the right and left $\mathcal{H}om$ are naturally isomorphic thanks to the braiding. Hence, we identify them and designate the *internal* $\mathcal{H}om$ by $\mathcal{H}om(-,-)$.

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