UPSIDE DOWN WITH THE STRONG PARTIAL CLONES

ABSTRACT.

1. INTRODUCTION

Let A be an arbitrary finite set. In the case we deal with Boolean clones we have $A = \mathbf{2} := \{0, 1\}$.

A function $f : A^n \to A$ is called a total function on A. A function $f : S \to A$ with $S \subseteq A^n$ is called partial function on A and we denote the domain by $\operatorname{dom} f := S$. The set $\operatorname{Op}(A)$ is the set of all total functions on A, and $\operatorname{Par}(A)$ is the set of all partial functions on A.

The function $e_i^n : A^n \to A$ defined by $e_i^n(x_1, \dots, x_n) := x_i$ is called the n-ary *projection* onto the i-th coordinate. The set $\operatorname{Proj}(A)$ is the set of all projections on A, i.e., $\operatorname{Proj}(A) := \{e_i^n \mid i, n \in \mathbb{N}, 1 \le i \le n\}$.

Let $f \in \operatorname{Par}(A)$ be n-ary and let $g_1, \dots, g_n \in \operatorname{Par}(A)$ be m-ary. The *composition* $F := f(g_1, \dots, g_n)$ is an m-ary partial function defined by

$$F(x_1, \dots, x_m) := f(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m))$$

and

$$\operatorname{dom} F := \left\{ x \in \bigcap_{i=1}^{n} \operatorname{dom} g_i \,\middle|\, (g_1(x), \dots, g_n(x)) \in \operatorname{dom} f \right\}.$$

$C \subseteq \operatorname{Par}(A)$ is called a partial clone if it is composition closed and contains the projections. If additionally $C \subseteq \operatorname{Op}(A)$ then C is a total clone.

Let $f, g \in \operatorname{Par}(A)$. Then f is a restriction (or subfunction) of g if $\operatorname{dom} f \subseteq \operatorname{dom} g$ and $f(x) = g(x)$ for all $x \in \operatorname{dom} f$, short $f \le g$. Let $X \subseteq \operatorname{Par}(A)$. Then the set $\operatorname{Str}(X) \subseteq \operatorname{Par}(A)$ is defined by

$$\operatorname{Str}(X) := \{f \in \operatorname{Par}(A) \mid \exists g \in X : f \le g\}.$$

If $X = \operatorname{Str}(X)$ then X is called *strong*. That means, that X contains every restriction of every of its functions, i.e., $f \in C$ for every $f \in \operatorname{Par}(A)$ and $g \in C$ with $f \le g$.

The set $\operatorname{pProj}(A) := \operatorname{Str}(\operatorname{Proj}(A))$ contains all partial projections on A, i.e., all subfunctions of the projections on A.

Let $\operatorname{Rel}^{(h)}(A)$ be the set of all h-ary relations on A for some $h \ge 1$, i.e., $\operatorname{Rel}^{(h)}(A) := \{X \mid X \subseteq A^h\}$. Furthermore, let $\operatorname{Rel}(A) := \bigcup_{h \ge 1} \operatorname{Rel}^{(h)}(A)$.

Let $\varrho \in \operatorname{Rel}^{(h)}(A)$, and $f : S \to A$ with $S \subseteq A^n$ an n-ary partial function. Then f preserves ϱ iff $f(M) \in \varrho$ for any $h \times n$ matrix $M = (m_{ij})$ whose rows belong to the domain of f, i.e. $(m_{i1}, \dots, m_{in}) \in \operatorname{dom} f$ for all i, and whose columns belong to ϱ.

Let $\operatorname{pPol} R$ be the set of all partial functions preserving every relation $\varrho \in R$. Inversely, let $\operatorname{pInv} C$ be the set of all relations preserved by every

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partial function $f \in C$. Then $(\mathrm{pPol}, \mathrm{pInv})$ form a Galois connection, see Theorem 4.2.2. in [4].

2. Further definitions

For some natural numbers $n, m \in \mathbb{N}$ with $n \le m$ we define the sets $[n, m] := \{n, n+1, \ldots, m\}$, and $[n] := [1, n]$. Tuples will be written with boldface small letters, and with the exception of $\mathbf{2} = \{0, 1\}$ a small boldface letter signifies a tuple. For a tuple $\mathbf{x} := (x_1, \ldots, x_n) \in A^n$ we define the set of its entries by $[\mathbf{x}] := \{x_1, \ldots, x_n\}$, and let $|\mathbf{x}| := |[\mathbf{x}]|$. For $I \subseteq [n]$ we let $\mathbf{x}_I := \{x_i \mid i \in I\}$. For $\mathbf{i} = (i_1, \ldots, i_l) \in [n]^l$ with $l \in \mathbb{N}$ we define $\mathbf{x}_{\mathbf{i}} := (x_{i_1}, \ldots, x_{i_l}) \in A^l$. We will often use the two special tuples $\mathbf{0} := (0, \ldots, 0)$ and $\mathbf{1} := (1, \ldots, 1)$.

Sometimes it will be more readable to omit some indices, but not necessarily the last one. Then for readability we write $\mathbf{x} \in A^I$ with $I \subseteq \mathbb{N}$, i.e., $\mathbf{x}$ is indexed by I in ascending order. That means $\mathbf{x} = (x_{i_1}, x_{i_2}, \ldots, x_{i_{|I|}})$ where $i_1 < i_2 < \cdots < i_{|I|}$ and $I = \{i_1, i_2, \ldots, i_{|I|}\}$. For example writing $\mathbf{x} \in A^{\{2\} \cup \{5,6\}}$ indicates that $\mathbf{x} = (x_2, x_5, x_6)$.

2.1. **Romov's definability lemma.**

The statement of Theorem 2.1 proven by Romov in [7] gives a nice characterization of the constructability of relations in the co-clone of a strong partial clone.

The relation $\rho \in \mathrm{Rel}^{(h)}(A)$ is called *irredundant* iff it fulfills the following two conditions:

(i) ρ has no duplicate rows, i.e., for all i, j with $1 \le i < j \le h$, there is a tuple $(a_1, \ldots, a_h) \in \rho$ with $a_i \ne a_j$;
(ii) ρ has no fictitious coordinates, i.e., there is no $i \in \{1, \ldots, h\}$, such that $(a_1, \ldots, a_h) \in \rho$ implies $(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_h) \in \rho$ for all $x \in A$.

For a relation $\sigma \in \mathrm{Rel}^{(h)}(A)$ we define $\mathrm{ar}(\sigma) := h$.

Theorem 2.1. *Let $\Sigma \subseteq \mathrm{Rel}(A)$ and $\rho \in \mathrm{Rel}^{(t)}(A)$ be relations. Furthermore let ρ be irredundant. Then*

$$\bigcap_{\sigma \in \Sigma} \mathrm{pPol}\, \sigma \subseteq \mathrm{pPol}\, \rho$$

iff there are some $\gamma_\sigma \subseteq [t]^{\mathrm{ar}(\sigma)}$ for all $\sigma \in \Sigma$ such that

$$\rho = \{\mathbf{x} \in A^t \mid \mathbf{x}_{\mathbf{i}} \in \sigma \text{ for all } \mathbf{i} \in \gamma_\sigma \text{ and } \sigma \in \Sigma\}$$

and

$$[t] = \bigcup_{\sigma \in \Sigma} \bigcup_{\mathbf{i} \in \gamma_\sigma} [\mathbf{i}].$$

3. Closure operators

For arbitrary relations $\rho \in \mathrm{Rel}^{(n)}(A)$ and $\sigma \in \mathrm{Rel}^{(m)}(A)$ we define the Maltsev-operations ζ, τ, Δ, ∇, and $\otimes$ by

$$\begin{aligned}
\zeta\rho &:= \{\mathbf{x} \in A^n \mid \mathbf{x}_{(2,3,\dots,n,1)} \in \rho\},\\
\tau\rho &:= \{\mathbf{x} \in A^n \mid \mathbf{x}_{(2,1,3,\dots,n)} \in \rho\},\\
\Delta\rho &:= \begin{cases} \{\mathbf{x} \in A^{n-1} \mid \mathbf{x}_{(1,1,2,\dots,n-1)} \in \rho\} & \text{for } n \geq 2,\\ \rho & \text{for } n = 1,\end{cases}\\
\nabla\rho &:= \{\mathbf{x} \in A^{n+1} \mid \mathbf{x}_{(2,3,\dots,n+1)} \in \rho\},\\
\rho \otimes \sigma &:= \{\mathbf{x} \in A^{n+m} \mid \mathbf{x}_{(1,\dots,n)} \in \rho, \mathbf{x}_{(n+1,\dots,n+m)} \in \sigma\}.
\end{aligned}$$

For arbitrary functions $f \in \mathrm{Par}^{(n)}(A)$ and $g \in \mathrm{Par}^{(m)}(A)$ we define the Maltsev-operations ζ, τ, Δ, ∇, $\otimes$, and $\star$ by

$$\begin{aligned}
\mathrm{dom}(\alpha f) &:= \alpha(\mathrm{dom}\, f) \qquad \text{for } \alpha \in \{\zeta, \tau, \Delta, \nabla\},\\
\mathrm{dom}(f \otimes g) &:= (\mathrm{dom}\, f) \otimes (\mathrm{dom}\, g),\\
\mathrm{dom}(f \star g) &:= \{\mathbf{x} \in A^{n+m-1} \mid \mathbf{x}_{(1,\dots,m)} \in \mathrm{dom}(g),\\
&\qquad\qquad (g(\mathbf{x}_{(1,\dots,m)}), \mathbf{x}_{(m+1,\dots,n+m-1)}) \in \mathrm{dom}(f)\},
\end{aligned}$$

and

$$\begin{aligned}
(\zeta f)(\mathbf{x}) &:= f(\mathbf{x}_{(2,3,\dots,n,1)}),\\
(\tau f)(\mathbf{x}) &:= f(\mathbf{x}_{(2,1,3,\dots,n)}),\\
(\Delta f)(\mathbf{x}) &:= \begin{cases} f(\mathbf{x}_{(1,1,2,\dots,n-1)}) & \text{for } n \geq 2,\\ f(\mathbf{x}) & \text{for } n = 1,\end{cases}\\
(\nabla f)(\mathbf{x}) &:= f(\mathbf{x}_{(2,3,\dots,n+1)}),\\
(f \otimes g)(\mathbf{x}) &:= f(\mathbf{x}_{(1,\dots,n)}),\\
(f \star g)(\mathbf{x}) &:= f(g(\mathbf{x}_{(1,\dots,m)}), \mathbf{x}_{(m+1,\dots,n+m-1)})\\
&\quad \text{for all } \mathbf{x} \in \mathrm{dom}(g \star f).
\end{aligned}$$

To enhance readability we denote by Ω the set of Maltsev operations (without $\star$), i.e., $\Omega := \{\zeta, \tau, \Delta, \nabla, \otimes\}$.

For a set $L \subseteq \{\zeta, \tau, \Delta, \nabla, \otimes, \star\}$ and some set $X \subseteq \mathrm{Par}(A)$, we denote by $\langle X \rangle_L$ the closure of X under the operations in L, i.e., the smallest set Y containing X, such that $p_n(f_1, \dots, f_n) \in Y$ for each $f_1, \dots, f_n \in Y$, and $p_n \in L$ (where p_n is an n-ary operation).

4. Closure operators beneath $\mathrm{pProj}(A)$

The following statement about partial clones has been shown long ago, or was used as the definition:

reference, more explanations

Theorem 4.1. *Let $C \subseteq \mathrm{Par}^{(n)}(A)$. Then C is a partial clone on A if and only if A is closed under the Maltsev-operations ζ, τ, Δ, ∇, and $\star$, and* $\mathrm{Proj}(A) \subseteq C$.

We now want to show that we can replace $\star$ by $\otimes$ if C contains only partial projections, i.e., if $C \subseteq \mathrm{pProj}(A)$ holds.

Lemma 4.2. *Let $f \in \mathrm{pProj}^{(n)}(A)$ and $g \in \mathrm{pProj}^{(m)}(A)$.*
Then $f \otimes g = e_1^2(\zeta^m \nabla^m f, \nabla^n g)$.

Proof. Since

$$\begin{aligned} \mathrm{dom}(\zeta^m \nabla^m f) &= \{\mathbf{x} \in A^{n+m} \mid \mathbf{x}_{(1,\dots,n)} \in \mathrm{dom}\, f\}, \text{ and} \\ \mathrm{dom}(\nabla^n g) &= \{\mathbf{x} \in A^{n+m} \mid \mathbf{x}_{(n+1,\dots,n+m)} \in \mathrm{dom}\, g\}, \end{aligned}$$

we see that

$$\mathrm{dom}(e_1^2(\zeta^m \nabla^m f, \nabla^n g)) = \mathrm{dom}(\zeta^m \nabla^m f) \cap \mathrm{dom}(\nabla^n g) = \mathrm{dom}(f \otimes g).$$

Let $\mathbf{x} \in \mathrm{dom}(f \otimes g)$ be arbitrary. Since $(\zeta^m \nabla^m f)(\mathbf{x}) = f(\mathbf{x}_{(1,\dots,n)})$ we get

$$(e_1^2(\zeta^m \nabla^m f, \nabla^n g))(\mathbf{x}) = f(\mathbf{x}_{(1,\dots,n)}) = f \otimes g.$$

Thus the equality holds. □

Lemma 4.3. *Let $f \in \mathrm{pProj}^{(n)}(A)$ and $g \in \mathrm{pProj}^{(m)}(A)$.*
Then $f \star g = \Delta_{j,m+1} \zeta^m (f \otimes g)$ for some $j \in [m]$.

Proof. Since $g \in \mathrm{pProj}^{(m)}(A)$ there is some $j \in [m]$ with $g \leq e_j^m$, i.e., $g(\mathbf{x}) = x_j$ for all $\mathbf{x} \in \mathrm{dom}\, g$.

We have

$$\begin{aligned} &\mathrm{dom}(\Delta_{j,m+1} \zeta^m (f \otimes g)) \\ &= \Delta_{j,m+1} \zeta^m \,\mathrm{dom}(f \otimes g) \\ &= \Delta_{j,m+1} \zeta^m \{\mathbf{x} \in A^{n+m} \mid \mathbf{x}_{(n+1,\dots,n+m)} \in \mathrm{dom}\, g, \mathbf{x}_{(1,\dots,n)} \in \mathrm{dom}\, f\} \\ &= \Delta_{j,m+1} \{\mathbf{x} \in A^{n+m} \mid \mathbf{x}_{(1,\dots,m)} \in \mathrm{dom}\, g, \mathbf{x}_{(m+1,\dots,m+n)} \in \mathrm{dom}\, f\} \\ &= \{\mathbf{x} \in A^{n+m-1} \mid \mathbf{x}_{(1,\dots,m)} \in \mathrm{dom}\, g, \mathbf{x}_{(j,m+1,\dots,m+n-1)} \in \mathrm{dom}\, f\} \\ &= \{\mathbf{x} \in A^{n+m-1} \mid \mathbf{x}_{(1,\dots,m)} \in \mathrm{dom}\, g, (g(\mathbf{x}_{(1,\dots,m)}), \mathbf{x}_{(m+1,\dots,m+n-1)}) \in \mathrm{dom}\, f\} \\ &= \mathrm{dom}(f \star g) \end{aligned}$$

Let $\mathbf{x} \in \mathrm{dom}(f \star g)$ be arbitrary. Then

$$\begin{aligned} &(\Delta_{j,m+1} \zeta^m (f \otimes g))(\mathbf{x}) \\ &= (\zeta^m (f \otimes g))(\mathbf{x}_{(1,\dots,m,j,m+1,\dots,m+n-1)}) \\ &= (f \otimes g))(\mathbf{x}_{(j,m+1,\dots,m+n-1,1,\dots,m)}) \\ &= f(\mathbf{x}_{(j,m+1,\dots,m+n-1)}) \\ &= f(g(\mathbf{x}_{(1,\dots,m)}), \mathbf{x}_{(m+1,\dots,m+n-1)}) \\ &= (f \star g)(\mathbf{x}) \end{aligned}$$

Thus $f \star g = \Delta_{j,m+1} \zeta^m (f \otimes g)$ holds. □

Lemma 4.4. *Let $C \subseteq \mathrm{pProj}(A)$. Then C is a partial clone if and only if $C = \langle C \rangle_\Omega$ and $\mathrm{Proj}(A) \subseteq C$.*

Proof. If C is a partial clone, then C is closed under ζ, τ, Δ, ∇ and $\star$, and $\mathrm{Proj}(A) \subseteq C$. By Lemma 4.2 the operation $\otimes$ is definable with these operations, and $\mathrm{Proj}(A)$. Thus C is closed under $\otimes$.

Let $C \subseteq \mathrm{pProj}(A)$ be closed under ζ, τ, Δ, ∇ and $\otimes$, and $\mathrm{Proj}(A) \subseteq C$. By Lemma 4.3 we see that C is also closed under $\star$, and therefore C is a partial clone. □

4.1. **Partial clones of projections and their domains.** The following material is similar to the *weak systems of relations* from the work of Börner, Haddad, and Pöschel [1] to describe the minimal partial clones. The approach here handles the relations with Maltsev operations. Furthermore, the empty relation and the equality relation are handled specially since they are required to get the connection to strong partial clones later on.

In this subsection we show that a clone $C \subseteq \mathrm{pProj}(A)$ of partial projections is completely specified by the set of domains $\mathrm{dom}(C)$ of its members. These are relations on A, i.e., $\mathrm{dom}(C) := \{\mathrm{dom}\, f \mid f \in C\} \subseteq \mathrm{Rel}(A)$. We define a kind of inverse operation dom^* for $\rho \in \mathrm{Rel}^{(n)}(A)$ with $n \geq 1$ by

$$\mathrm{dom}^* \rho := \{f \mid \mathrm{dom}\, f = \rho, f \leq e_i^n \text{ for } 1 \leq i \leq n\}.$$

This can then be used to define it for a set $\mathcal{R} \subseteq \mathrm{Rel}(A)$ by

$$\mathrm{dom}^* \mathcal{R} := \bigcup_{\rho \in \mathcal{R}} \mathrm{dom}^* \rho.$$

We call $\mathcal{R} \subseteq \mathrm{Rel}(A)$ a *domain clone* if $\mathcal{R}$ is closed under ζ, τ, Δ, ∇, $\otimes$, and $A^n \in \mathcal{R}$ for all $n \geq 1$.

Lemma 4.5. *Let $C \subseteq \mathrm{Par}(A)$ be a partial clone. Then $\mathrm{dom}(C)$ is a domain clone.*

Proof. Let $\rho, \sigma \in \mathrm{dom}(C)$. Then there are $f, g \in C$ with $\rho = \mathrm{dom}\, f$ and $\sigma = \mathrm{dom}\, g$.

Since $\alpha f \in C$ for all $\alpha \in \{\zeta, \tau, \nabla, \Delta\}$, and $\alpha(\mathrm{dom}\, f) = \mathrm{dom}(\alpha f)$ we obtain $\alpha\rho = \mathrm{dom}(\alpha f) \in \mathrm{dom}(C)$. Similar $f \otimes g \in C$, and thus

$$\rho \otimes \sigma = (\mathrm{dom}\, f) \otimes (\mathrm{dom}\, g) = \mathrm{dom}(f \otimes g) \in \mathrm{dom}(C).$$

Furthermore, $A^n = \mathrm{dom}\, e_1^n \in \mathrm{dom}\, C$ for all $n \geq 1$ since $\mathrm{Proj}(A) \subseteq C$. Thus $\mathrm{dom}(C)$ is a domain clone. □

Lemma 4.6. *Let $\mathcal{R} \subseteq \mathrm{Rel}(A)$ be a domain clone. Then $\mathrm{dom}^*(\mathcal{R}) \subseteq \mathrm{pProj}(A)$ is a partial clone.*

Proof. Let $f, g \in \mathrm{dom}^*(\mathcal{R})$. Then there are $\rho, \sigma \in \mathcal{R}$ with $\rho = \mathrm{dom}\, f$ and $\sigma = \mathrm{dom}\, g$.

Let $\alpha \in \{\zeta, \tau, \nabla, \Delta\}$. Then $\alpha f \leq e_i^n$ for some $1 \leq i \leq n$, and $\mathrm{dom}(\alpha f) = \alpha(\mathrm{dom}\, f) \in \mathcal{R}$. Thus $\alpha f \in \mathrm{dom}^*(\mathcal{R})$.

Similarly, $f \otimes g \leq e_i^n$ for some (possibly different) $1 \leq i \leq n$, and $\mathrm{dom}(f \otimes g) = (\mathrm{dom}\, f) \otimes (\mathrm{dom}\, g) \in \mathcal{R}$. Thus $f \otimes g \in \mathrm{dom}^*(\mathcal{R})$.

Furthermore, $e_i^n \leq e_i^n$ for all $1 \leq i \leq n$, and $\mathrm{dom}(e_i^n) = A^n \in \mathcal{R}$. Thus $e_i^n \in \mathrm{dom}^*(\mathcal{R})$. □

We have shown that dom and dom^* map partial clones to domain clones, and reversely. For our purposes we need a stronger result, namely that dom and dom^* are the inverse operations of each other, i.e., that $C = \mathrm{dom}^* \mathrm{dom}\, C$ for each partial clone $C \subseteq \mathrm{pProj}(A)$, and $\mathcal{R} = \mathrm{dom}\, \mathrm{dom}^* \mathcal{R}$ for each domain clone $\mathcal{R} \subseteq \mathrm{Rel}(A)$. But these are straight forward as given in the following two lemmas.

Lemma 4.7. *Let $C \subseteq \mathrm{pProj}(A)$ be a partial clone. Then $C = \mathrm{dom}^* \mathrm{dom}\, C$.*

Proof. We first show that $C \subseteq \operatorname{dom}^* \operatorname{dom} C$ holds. Let $f \in C$. Then $f \subseteq e_i^n$ for some $1 \leq i \leq n$, and $\operatorname{dom} f \in \operatorname{dom} C$. Thus $f \in \operatorname{dom}^*(\operatorname{dom} C)$.

Now we show $C \supseteq \operatorname{dom}^* \operatorname{dom} C$. Let $f \in \operatorname{dom}^* \operatorname{dom} C$. Then $f \subseteq e_i^n$ for some $1 \leq i \leq n$, and there is some $g \in C$ with $\operatorname{dom} f = \operatorname{dom} g$. Let $f' := e_1^2(e_i^n, g)$. Then $\operatorname{dom} f' = A^n \cap \operatorname{dom} g = \operatorname{dom} f$ and $f' \leq e_i^n$. Thus $f = f' \in C$. □

Lemma 4.8. *Let $\mathcal{R} \subseteq \operatorname{Rel}(A)$ be a domain clone. Then $\mathcal{R} = \operatorname{dom} \operatorname{dom}^* \mathcal{R}$.*

Proof. We first show $\mathcal{R} \subseteq \operatorname{dom} \operatorname{dom}^* \mathcal{R}$. Let $\rho \in \mathcal{R}$. Then there is some $f \in \operatorname{dom}^* \mathcal{R}$ with $f \leq e_1^n$ and $\operatorname{dom} f = \rho$. Thus $\rho = \operatorname{dom} f \in \operatorname{dom}(\operatorname{dom}^* \mathcal{R})$.

Now we show $\mathcal{R} \supseteq \operatorname{dom} \operatorname{dom}^* \mathcal{R}$. Let $\rho \in \operatorname{dom} \operatorname{dom}^* \mathcal{R}$. Then there is some $f \in \operatorname{dom}^* \mathcal{R}$ with $\operatorname{dom} f = \rho$. By the definition of $\operatorname{dom}^* \mathcal{R}$ follow $\rho = \operatorname{dom} f \in \mathcal{R}$. □

Let $\mathcal{L}_{\mathrm{pProj}(A)}$ be the lattice of all partial clones in the interval $\mathcal{I}(\operatorname{Proj}(A), \operatorname{pProj}(A))$ ordered by set inclusion. Let $\mathcal{L}_{\mathrm{Dom}(A)}$ be the lattice of all domain clones on A also ordered by set inclusion.

We now proof that dom and dom^* are lattice homomorphisms between $\mathcal{L}_{\mathrm{pProj}(A)}$ and $\mathcal{L}_{\mathrm{Dom}(A)}$, and reversely.

Lemma 4.9. *Let $C, C' \subseteq \operatorname{Par}(A)$ be partial clones with $C \subseteq C'$. Then $\operatorname{dom} C \subseteq \operatorname{dom} C'$.*

Proof. Let $\rho \in \operatorname{dom} C$. Then there is some $f \in C$ with $\rho = \operatorname{dom} f$. Since $f \in C'$ we also have $\rho = \operatorname{dom} f \in \operatorname{dom} C'$. □

Lemma 4.10. *Let $\mathcal{R}, \mathcal{R}' \subseteq \operatorname{Rel}(A)$ be domain clones with $\mathcal{R} \subseteq \mathcal{R}'$. Then $\operatorname{dom}^* \mathcal{R} \subseteq \operatorname{dom}^* \mathcal{R}'$.*

Proof. Let $f \in \operatorname{dom}^* \mathcal{R}$. Then $f \leq e_i^n$ for some $1 \leq i \leq n$, and there is some $\rho \in \mathcal{R}$ with $\rho = \operatorname{dom} f$. Then $\rho \in \mathcal{R}'$ and thus there is some $f' \in \operatorname{dom}^* \mathcal{R}'$ with $f \leq e_i^n$ and $\operatorname{dom} f' = \rho = \operatorname{dom} f$. Thus $f = f' \in \operatorname{dom}^* \mathcal{R}'$. □

Theorem 4.11. *The map* dom *is a lattice isomorphism from $\mathcal{L}_{\mathrm{pProj}(A)}$ to $\mathcal{L}_{\mathrm{Dom}(A)}$, and the map* dom^* *is its inverse lattice isomorphism from $\mathcal{L}_{\mathrm{Dom}(A)}$ to $\mathcal{L}_{\mathrm{pProj}(A)}$.*

4.2. **Shuffling operators.** Some preparations for the next section:

Lemma 4.12. *Let $\mathcal{R} \subseteq \operatorname{Rel}(A)$. Then $\langle \mathcal{R} \rangle_{\zeta,\tau,\Delta,\otimes} = \langle \langle \mathcal{R} \rangle_{\otimes} \rangle_{\zeta,\tau,\Delta}$.*

Proof. Let $\rho \in \operatorname{Rel}^{(n)}(A)$ and $\sigma \in \operatorname{Rel}^{(m)}(A)$. Then

$$
\begin{aligned}
&(\tau\rho) \otimes \sigma = \tau(\rho \otimes \sigma);\\
&\rho \otimes (\tau\sigma) = \zeta^n \tau \zeta^{-n}(\rho \otimes \sigma);\\
&(\zeta\rho) \otimes \sigma = \pi_{(1\ldots n)}(\rho \otimes \sigma);\\
&\rho \otimes (\zeta\sigma) = \pi_{(n+1\ldots m)}(\rho \otimes \sigma);\\
&(\Delta\rho) \otimes \sigma = \begin{cases} \Delta(\rho \otimes \sigma) & \text{if } n \geq 2,\\ \rho \otimes \sigma & \text{if } n = 1; \end{cases}\\
&\rho \otimes (\Delta\sigma) = \begin{cases} \zeta^n \Delta \zeta^{-n}(\rho \otimes \sigma) & \text{if } m \geq 2,\\ \rho \otimes \sigma & \text{if } m = 1; \end{cases}
\end{aligned}
$$

The π's are just permutations of coordinates, and thus representable with ζ and τ. $\square$

Let $\rho \in \mathrm{Rel}^{(n)}(A)$. We define a variant of ∇ by $\nabla_l \in \mathrm{Rel}^{(n+1)}(A)$

$$\nabla_l \rho := \{\mathbf{x} \in A^{n+1} \mid \mathbf{x}_{(1,\dots,l,l+2,\dots,n+1)} \in \rho\}$$

for $0 \leq l \leq n$, i.e., we add a fictious coordinate after the first l coordinates of ρ. For $l > n$ we define $\nabla_l \rho := \nabla_n \rho$. Clearly, $\nabla = \nabla_0$.

Lemma 4.13. *Let $\mathcal{R} \subseteq \mathrm{Rel}(A)$. Then $\langle \mathcal{R} \rangle_\Omega = \langle \langle \mathcal{R} \rangle_{\zeta,\tau,\Delta,\otimes} \rangle_{(\nabla_l)_{l \geq 0}}$.*

Proof. Let $\rho \in \mathrm{Rel}^{(n)}(A)$, $\sigma \in \mathrm{Rel}^{(m)}(A)$, and $0 \leq l \leq n$. Then

$$\tau(\nabla_l \rho) = \begin{cases} \nabla_1 \rho & \text{if } l = 0, \\ \nabla_0 \rho & \text{if } l = 1, \\ \nabla_l(\tau\rho) & \text{otherwise;} \end{cases}$$

$$\zeta(\nabla_l \rho) = \begin{cases} \nabla_n \rho & \text{if } l = 0, \\ \nabla_{l-1}(\zeta\rho) & \text{otherwise;} \end{cases}$$

$$\Delta(\nabla_l \rho) = \begin{cases} \rho & \text{if } l \in \{0,1\}, \\ \nabla_{l-1}(\Delta\rho) & \text{otherwise;} \end{cases}$$

$$(\nabla_l \rho) \otimes \sigma = \nabla_l(\rho \otimes \sigma)$$

$$\sigma \otimes (\nabla_l \rho) = \nabla_{l+m}(\sigma \otimes \rho)$$

$\square$

Let $\rho \in \mathrm{Rel}^{(n)}(A)$. We define another operation $\overline{\delta}_{i,j}$ for $i, j \in [n]$ by

$$\overline{\delta}_{i,j}\rho = \{\mathbf{x} \in \rho \mid x_i = x_j\}.$$

Furthermore, let $\delta \in \mathrm{Rel}^{(2)}(A)$ be defined by $\delta := \{(x,x) \mid x \in A\}$. Then we see that, $\delta = \overline{\delta}_{1,2} A^2$, and $\overline{\delta}_{i,i}\rho = \rho$.

Lemma 4.14. *Let $\mathcal{R} \subseteq \mathrm{Rel}(A)$.*
Then $\langle \mathcal{R} \cup \{\delta\} \rangle_\Omega = \langle \langle \mathcal{R} \cup \{A\} \rangle_{\zeta,\tau,\Delta,\otimes} \rangle_{(\nabla_l)_{l \geq 0}, (\overline{\delta}_{i,j})_{i,j \geq 1}}$.

Proof. Let $\rho \in \mathrm{Rel}^{(n)}(A)$, $\sigma \in \mathrm{Rel}^{(m)}(A)$, and $i, j \in [n]$. Then

$$\tau(\overline{\delta}_{i,j}\rho) = \overline{\delta}_{\tau(i),\tau(j)}\tau\rho$$

$$\zeta(\overline{\delta}_{i,j}\rho) = \overline{\delta}_{\zeta(i),\zeta(j)}\zeta\rho$$

$$\Delta(\overline{\delta}_{i,j}\rho) = \overline{\delta}_{\max(1,i-1),\max(1,j-1)}\Delta\rho$$

$$(\overline{\delta}_{i,j}\rho) \otimes \sigma = \overline{\delta}_{i,j}(\rho \otimes \sigma)$$

$$\sigma \otimes (\overline{\delta}_{i,j}\rho) = \overline{\delta}_{i+m,j+m}(\sigma \otimes \rho)$$

$\square$

Now we can conclude the following theorem which is useful in the characterization in the next section.

Theorem 4.15. *Let $\mathcal{R} \subseteq \mathrm{Rel}(A)$.*
Then $\langle \mathcal{R} \cup \{\delta, \emptyset\} \rangle_\Omega = \langle \langle \langle \mathcal{R} \cup \{A\} \rangle_\otimes \rangle_{\zeta,\tau,\Delta} \rangle_{(\nabla_l)_{l \geq 0}, (\overline{\delta}_{i,j})_{i,j \geq 1}} \cup \{\emptyset\}$.

5. Strong partial clones upside down

In this section, we show that there is a order-reserving bijection between the interval $I(\{\delta,\emptyset\},\mathrm{Rel}(A))$ of domain clones and the lattice of strong partial clones.

We need to recall some things about strong partial clones and the galois connection to relations on A. B. Romov has shown that a strong partial clone is determined by a family of relations of a certain type called irredundant relations. This requires the following:

Let $h \geq 1$ and let ρ be an h-ary relation on A. We say that ρ is *repetition-free* if for all $1 \leq i < j \leq h$, there exists $(a_1,\ldots,a_h) \in \rho$ with $a_i \neq a_j$. Moreover ρ is said to be *irredundant* if it is repetition-free and has no fictitious components, i.e., there is no $i \in [h]$ such that $(a_1,\ldots,a_h) \in \rho$ implies $(a_1,\ldots,\, a_{i-1}, x,\, a_{i+1},\ldots,a_h) \in \rho$ for all $x \in A$.

It can be shown that if μ is a non-empty relation, then one can find an irredundant relation ρ such that $\mathrm{pPol}\,\mu = \mathrm{pPol}\,\rho$ (see [2] and [3]). We have:

Lemma 5.1. *([6]) Let C be a strong partial clone on A. Then there is a non-empty set of irredundant relations $\mathcal{R}$ with $C = \mathrm{pPol}\,\mathcal{R}$.*

The following result, known as the *Definability Lemma*, was first established by B. Romov in [7] (see [3] and Lemma 20.3.4 in [4]).

Lemma 5.2 (Definability Lemma)**.** *Let λ be an irredundant t-ary relation on A, and $\mathcal{R}$ a set of relations on A. Then $\mathrm{pPol}\,\mathcal{R} \subseteq \mathrm{pPol}\,\lambda$ if and only if for each $R \in \mathcal{R}$ there is an $\mathrm{ar}(R)$-ary auxiliary relation γ_R on $[t]$, such that $\{\gamma_R \mid R \in \mathcal{R}\}$ covers $[t]$, and $\lambda = \{\mathbf{x} \in A^t \mid \mathbf{x_i} \in R \text{ for all } R \in \mathcal{R} \text{ and } \mathbf{i} \in \gamma_R\}$.*

We now show that the operations for domains given in the previous section, are equivalent to the one in the previous lemma plus the addition and deletion of duplicate and superficial coordinates.

Lemma 5.3. *Let λ be an irredundant t-ary relation on A, and $\mathcal{R}$ a set of relations on A. Then $\mathrm{pPol}\,\mathcal{R} \subseteq \mathrm{pPol}\,\lambda$ if and only if $\lambda \in \langle\mathcal{R}\rangle_{\zeta,\tau,\Delta,\otimes}$.*

Proof. If $\mathrm{pPol}\,\mathcal{R} \subseteq \mathrm{pPol}\,\lambda$, then by Lemma 5.2 we have for each $R \in \mathcal{R}$ there is an $\mathrm{ar}(R)$-ary auxiliary relation γ_R on $[t]$, such that $\{\gamma_R \mid R \in \mathcal{R}\}$ covers $[t]$, and $\lambda = \{\mathbf{x} \in A^t \mid \mathbf{x_i} \in R$ for all $R \in \mathcal{R}$ and $\mathbf{i} \in \gamma_R\}$.

Since λ is finite, we can assume w.l.o.g., that $\mathcal{R}$ is finite. Then let $X := \{(\mathbf{i}, R) \mid R \in \mathcal{R}, \mathbf{i} \in \gamma_R\}$ be ordered in some (arbitrary) way, i.e., let $\{p_1,\ldots,p_l\} := X$ with $l := |X|$, and $p_j = (\mathbf{i}^j, R^j)$ for all $1 \leq j \leq l$. Let the order be denoted by $\prec$, i.e., $p_i \prec p_j$ iff $i < j$. Then for $p \in X$ let $\mathrm{ar}_\prec(p_v) := \displaystyle\sum_{j=1}^{v-1} \mathrm{ar}(R_j)$.

Furthermore, let $\mathbf{I} = (\mathbf{i}_1,\ldots,\mathbf{i}_l)$, i.e., the concatenation of all $\mathbf{i}_j$.

We can now write $\lambda = \{\mathbf{x} \in A^t \mid \mathbf{x_I} \in \hat{R}\}$, where $\hat{R} := R_1 \otimes R_2 \otimes \cdots \otimes R_l$, and $\otimes$ is the operation on the domains given before. Then every $a \in [t]$ appears at least once in $\mathbf{I}$. With ζ and τ we can reorder the coordinates of $\mathbf{I}$ and $\hat{R}$ such that we can assume $\mathbf{I} = (1,\ldots,1,2,\ldots,2,\ldots,t,\ldots,t)$. Then with Δ (as well as ζ and τ) we can identify the coordinates with the same value in $\mathbf{I}$. Thus we obtain $\lambda = \{\mathbf{x} \in A^t \mid \mathbf{x}_{(1,\dots,t)} \in \phi(\hat{R})\} = \phi(\hat{R})$, where ϕ is some combination of Δ, τ, and ζ.

Thus $\lambda \in \langle \mathcal{R} \rangle_{\zeta,\tau,\Delta,\otimes}$.

Now we let $\lambda \in \langle \mathcal{R} \rangle_{\zeta,\tau,\Delta,\otimes}$, and want to show that $\mathrm{pPol}\,\mathcal{R} \subseteq \mathrm{pPol}\,\lambda$.

First by Lemma 4.12 we have $\lambda \in \langle \hat{R} \rangle_{\zeta,\tau,\Delta}$ for some relation $\hat{R} = R_1 \otimes \cdots \otimes R_l$ with $R_1, \dots, R_l \in \mathcal{R}$. Then $\lambda = \phi(\hat{R})$ where $\phi = \xi_1 \xi_2 \dots \xi_\mu$ for some $\xi_j \in \{\zeta, \tau, \Delta\}$. Then ϕ induces a coordinate mapping $\phi' : [\mathrm{ar}(\hat{R})] \to [t]$. Now we can write

$$\lambda = \{\mathbf{x} \in A^t \mid \mathbf{x}_{\mathbf{i}^j} \in R_j \text{ for all } j \in [l]\}$$

with $\mathbf{i}^j = (\phi'(\mathrm{ar}_{\prec}(R_j) + 1), \dots, \phi'(\mathrm{ar}_{\prec}(R_j) + \mathrm{ar}(R_j)))$.

Furthermore the $\mathbf{i}^j$ do cover $[t]$, since otherwise there would be a fictitious coordinate, i.e., λ would not be irredundant. □

Lemma 5.4. *Let $\lambda \in \langle \mathcal{R} \cup \{\delta\} \rangle_\Omega \setminus \langle \mathcal{R} \rangle_{\zeta,\tau,\Delta,\otimes}$ be non-empty. Then λ is not irredundant, and there is some irredundant relation $\lambda' \in \langle \mathcal{R} \rangle_{\zeta,\tau,\Delta,\otimes}$ with* $\mathrm{pPol}\,\lambda' = \mathrm{pPol}\,\lambda$, *or* $\mathrm{pPol}\,\lambda = \mathrm{Par}(A)$.

Proof. By Theorem 4.15 we have $\langle \mathcal{R} \cup \{\delta\} \rangle_\Omega = \langle \langle \langle \mathcal{R} \cup \{A\} \rangle_\otimes \rangle_{\zeta,\tau,\Delta} \rangle_{(\nabla_l)_{l \geq 0}, (\overline{\delta}_{i,j})_{i,j \geq 1}}$. Thus there is some $\lambda'' \in \langle \langle \mathcal{R} \cup \{A\} \rangle_\otimes \rangle_{\zeta,\tau,\Delta}$ with $\lambda \in \langle \{\lambda''\} \rangle_{(\nabla_l)_{l \geq 0}, (\overline{\delta}_{i,j})_{i,j \geq 1}}$, and $\lambda \neq \lambda''$. Thus ∇_l for some $l \geq 0$, or $\overline{\delta}_{i,j}$ for some $i, j \geq 1$ have been applied at least once to λ'', and therefore λ has a fictious coordinate, or a duplicate coordinate, respectively. That means, λ is not irredundant, and furthermore $\mathrm{pPol}\,\lambda'' = \mathrm{pPol}\,\lambda$.

If λ'' is irredundant, then we can take $\lambda' := \lambda$. Otherwise, λ'' has fictious coordinates, or duplicate coordinates.

If i is a fictious coordinate and λ'' is at least binary, then we can remove it by $\Delta_{i,i+1}\lambda'' \in \langle \langle \mathcal{R} \cup \{A\} \rangle_\otimes \rangle_{\zeta,\tau,\Delta}$. If λ'' is unary, then $\lambda'' = A$, and thus $\mathrm{pPol}\,\lambda = \mathrm{pPol}\,A = \mathrm{Par}(A)$.

If λ'' has a duplicate coordinate there are $i < j$ with $x_i = x_j$ for all $\mathbf{x} \in \lambda''$. Then $\Delta_{i,j}\lambda'' \in \langle \langle \mathcal{R} \cup \{A\} \rangle_\otimes \rangle_{\zeta,\tau,\Delta}$, and has smaller arity.

Repeating this process, until all fictious and duplicate coordinates are removed stops eventually, and we obtain and irredundant $\lambda' \in \langle \langle \mathcal{R} \cup \{A\} \rangle_\otimes \rangle_{\zeta,\tau,\Delta}$, or $\lambda' = A$ and thus $\mathrm{pPol}\,\lambda = \mathrm{pPol}\,A = \mathrm{Par}(A)$. □

From the last two lemmas we can conclude the following nice theorem.

Theorem 5.5. *Let $\mathcal{R} \subseteq \mathrm{Rel}(A)$, and $\lambda \in \mathrm{Rel}(A)$.*
Then $\lambda \in \langle \mathcal{R} \cup \{\delta, \emptyset\} \rangle_\Omega$ if and only if $\mathrm{pPol}\,\mathcal{R} \subseteq \mathrm{pPol}\,\rho$.

For the sublattice of domain clones, which contain the relations δ and $\emptyset$, we use the symbol $\mathcal{L}^\star_{\mathrm{Dom}(A)}$, i.e., $\mathcal{L}^\star_{\mathrm{Dom}(A)} := \mathcal{I}(\langle \delta, \emptyset \rangle_\Omega, \mathrm{Rel}(A))$.

Corollary 5.6. *Let $\mathcal{R}, \mathcal{S} \in \mathcal{L}^\star_{\mathrm{Dom}(A)}$ be domain clones. Then*

$$\mathcal{S} \subseteq \mathcal{R} \iff \mathrm{pPol}\,\mathcal{R} \subseteq \mathrm{pPol}\,\mathcal{S}.$$

Since $(\mathrm{pPol}, \mathrm{pInv})$ is a Galois-connection we have for all $C, D \subseteq \mathrm{Par}(A)$, and $\mathcal{R}, \mathcal{S} \subseteq \mathrm{Rel}(A)$ that

$$\begin{aligned}
&C \subseteq D \Longrightarrow \mathrm{pInv}\, D \subseteq \mathrm{pInv}\, C,\\
&\mathcal{R} \subseteq \mathcal{S} \Longrightarrow \mathrm{pPol}\, \mathcal{S} \subseteq \mathrm{pPol}\, \mathcal{R},\\
&C \subseteq \mathrm{pPol}\,\mathrm{pInv}\, C,\\
&\mathcal{R} \subseteq \mathrm{pInv}\,\mathrm{pPol}\, \mathcal{R}.
\end{aligned}$$

We want to show that the strong partial clones and the domain clones in $\mathcal{L}^{\star}_{\mathrm{Dom}(A)}$ are precisely the Galois-closed sets.

Lemma 5.7. *Let $C \subseteq \mathrm{Par}(A)$. Then $\mathrm{pInv}\, C$ contains $\emptyset$, and δ, and it is closed under ζ, τ, Δ, ∇, and $\otimes$.*

Thus $\mathrm{pInv}\, C$ is a domain clone in $\mathcal{L}^{\star}_{\mathrm{Dom}(A)}$.

Lemma 5.8. *Let $\mathcal{R} \subseteq \mathrm{Rel}(A)$ be a domain clone in $\mathcal{L}^{\star}_{\mathrm{Dom}(A)}$. Then $\mathcal{R} = \mathrm{pInv}\,\mathrm{pPol}\, \mathcal{R}$.*

Proof. We have $\mathcal{R} \subseteq \mathrm{pInv}\,\mathrm{pPol}\, \mathcal{R}$. Assume to the contrary, that there is some $\rho \in (\mathrm{pInv}\,\mathrm{pPol}\, \mathcal{R}) \setminus \mathcal{R}$.

Since $\mathcal{R}$ is a domain clone, we see that $\mathrm{pPol}\, \mathcal{R} \not\subseteq \mathrm{pPol}\, \rho$, i.e., there is some $f \in (\mathrm{pPol}\, \mathcal{R}) \setminus (\mathrm{pPol}\, \rho)$. Then f does not preserve ρ, and thus $\rho \notin \mathrm{pInv}(\mathrm{pPol}\, \mathcal{R}) \supseteq \mathrm{pInv}\{f\}$. Thus we have a contradiction, and the equation holds. □

As it is known, for each $\mathcal{R} \subseteq \mathrm{Rel}(A)$, the set $\mathrm{pPol}\, \mathcal{R}$ is a strong partial clone.

Lemma 5.9. *Let $C \subseteq \mathrm{Par}(A)$ be a strong partial clone. Then $C = \mathrm{pPol}\,\mathrm{pInv}\, C$.*

Proof. We have $C = \mathrm{pPol}\, \mathcal{R}$ for some $\mathcal{R} \subseteq \mathrm{Rel}(A)$ by Lemma 5.1. Then $\mathcal{R} \subseteq \mathrm{pInv}\,\mathrm{pPol}\, \mathcal{R}$ implies $C = \mathrm{pPol}\, \mathcal{R} \supseteq \mathrm{pPol}\,\mathrm{pInv}\,\mathrm{pPol}\, \mathcal{R} = \mathrm{pPol}\,\mathrm{pInv}\, C$, and together with $C \subseteq \mathrm{pPol}\,\mathrm{pInv}\, C$ we obtain $C = \mathrm{pPol}\,\mathrm{pInv}\, C$. □

Theorem 5.10. *The map pPol is an order-reversing lattice isomorphism from interval $\mathcal{L}^{\star}_{\mathrm{Dom}(A)}$ of domain clones to the lattice of strong partial clones $\mathcal{L}^{\star}_{\mathrm{Par}(A)} := \mathcal{I}(\mathrm{pProj}(A), \mathrm{Par}(A))$, and the map pInv is its inverse lattice isomorphism from $\mathcal{I}(\mathrm{pProj}(A), \mathrm{Par}(A))$ to $\mathcal{L}^{\star}_{\mathrm{Dom}(A)}$.*

We can now combine the two pairs of maps $(\mathrm{dom}, \mathrm{dom}^*)$ and $(\mathrm{pPol}, \mathrm{pInv})$ to obtain the main theorem. Let $\mathcal{L}^{\star}_{\mathrm{pProj}(A)}$ be the lattice of all partial clones in $\mathcal{L}_{\mathrm{pProj}(A)}$ which contain $e_{\emptyset}$ (the unary function with empty domain) and e_{δ} (defined by $e_{\delta} \leq e^2_1$ and $\mathrm{dom}\, e_{\delta} = \delta$).

Theorem 5.11. *The maps $\mathrm{pPol}\,\mathrm{dom}$ and $\mathrm{dom}^*\,\mathrm{pInv}$ are bijective and each is the inverse of the other. They form a pair of order-reversing lattice homomorphisms between the lattices $\mathcal{L}^{\star}_{\mathrm{pProj}(A)}$ and $\mathcal{L}^{\star}_{\mathrm{Par}(A)}$.*

We note that $\mathrm{pProj}(A)$ is the only common point of these two sublattices of the lattice of all partial clones $\mathcal{L}_{\mathrm{Par}(A)}$, and it is the only fix point of $\mathrm{pPol}\,\mathrm{dom}$ and $\mathrm{dom}^*\,\mathrm{pInv}$.

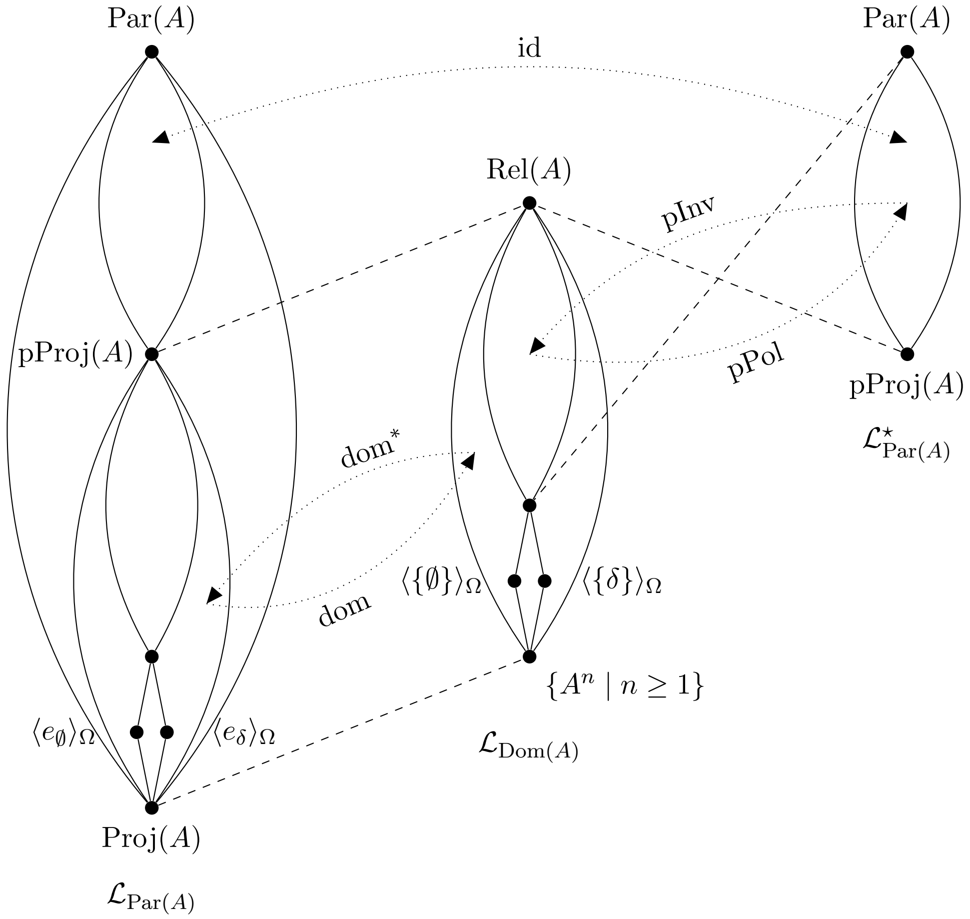

FIGURE 1. The maps between the lattices $\mathcal{L}_{\mathrm{pProj}(A)}$ and $\mathcal{L}_{\mathrm{Dom}(A)}$ on the left, and the maps between $\mathcal{L}^\star_{\mathrm{Dom}(A)}$ and $\mathcal{L}^\star_{\mathrm{Par}(A)}$ on the right. Note that, to improve readability of the figure, the lattice $\mathcal{L}^\star_{\mathrm{Par}(A)}$ appears both in the left lattice as a sublattice, and on the right on its own.

6. THE LATTICE $\mathcal{L}_{\mathrm{pProj}(A)}$

In the last section we have seen that the lattices $\mathcal{L}^\star_{\mathrm{pProj}(A)}$ and $\mathcal{L}^\star_{\mathrm{Par}(A)}$ are isomorphic with the order by inclusion reversed. Now we want to describe all the other partial clones in the lattice $\mathcal{L}_{\mathrm{pProj}(A)} = \mathcal{I}(\mathrm{Proj}(A), \mathrm{pProj}(A))$. As we have shown this lattice is isomorphic to the lattice of domain clones $\mathcal{L}_{\mathrm{Dom}(A)}$. We know that $\mathcal{L}^\star_{\mathrm{Dom}(A)}$ is isomorphic to $\mathcal{L}^\star_{\mathrm{pProj}(A)}$.

First we consider the problems of minimal and maximal clones in $\mathcal{L}_{\mathrm{Dom}(A)}$.

6.1. **Minimal domain clones.** From the results by Börner, Haddad and Pöschel about minimal partial clones [1] we obtain all minimal clones in $\mathcal{L}_{\mathrm{Dom}(A)}$. But first we need to define some terminology.

Definition 6.1. Let $\rho \in \mathrm{Rel}^{(n)}(A)$ and S_n be the group of permutations on $[n]$. The relation ρ is said to be

(1) *totally symmetric* if for all $\pi \in S_n$ and $(a_1, \dots, a_n) \in A^n$,
$$(a_1, \dots, a_n) \in \rho \iff (a_{\pi(1)}, \dots, a_{\pi(n)}) \in \rho;$$
(2) *totally reflexive* if for every $(a_1, \dots, a_n) \in A^n$ and all $1 \le i < j \le n$, the equality $a_i = a_j$ implies that $(a_1, \dots, a_n) \in \rho$;
(3) *non-trivial* if $\rho \neq A^n$.

Note that any subset of A (including the empty set $\emptyset$) is considered as a totally symmetric and totally reflexive relation.

We get the following theorem.

Theorem 6.2 (Börner, Haddad, Pöschel [1]). *Let $\mathcal{R} \in \mathcal{L}_{\mathrm{Dom}(A)}$ be a minimal domain clone.*

Then $\mathcal{R} = \langle \rho \rangle_\Omega$ for some non-trivial, totally symmetric and totally reflexive relation $\rho \in \mathrm{Rel}^{(n)}(A)$ with $n \le |A|$.

6.2. **Maximal domain clones.** As was shown in [9] there are no minimal strong partial clones. Thus there are no maximal domain clones in the interval $\mathcal{L}^\star_{\mathrm{Dom}(A)}$. The following lemma will let us conclude that there are no maximal domain clones in the lattice $\mathcal{L}_{\mathrm{Dom}(A)}$.

Lemma 6.3. *Let $\mathcal{R} \subset \mathrm{Rel}(A)$ be a domain clone. Then $\langle \mathcal{R} \cup \{\delta, \emptyset\} \rangle_\Omega \neq \mathrm{Rel}(A)$.*

Proof. Let $\mathcal{R}' := \langle \mathcal{R} \cup \{\delta, \emptyset\} \rangle_\Omega$.

Assume to the contrary that $\mathcal{R}' = \mathrm{Rel}(A)$. Then $\{(0,1),(1,0)\}, \{(0,1)\} \cup \delta \in \mathcal{R}'$. Since both of these relations are irredundant we have $\{(0,1),(1,0)\}, \{(0,1)\} \cup \delta \in \langle \mathcal{R} \rangle_{\zeta,\tau,\Delta,\otimes} = \mathcal{R}$. But then $\emptyset = \Delta\{(0,1),(1,0)\} \in \mathcal{R}$, and $\delta = (\{(0,1)\} \cup \delta) \cap (\{(1,0)\} \cup \delta) \in \mathcal{R}$.

Thus $\mathcal{R} = \mathcal{R}' = \mathrm{Rel}(A)$ in contradiction to the assumption. Thus $\mathcal{R}' \neq \mathrm{Rel}(A)$. $\square$

Theorem 6.4. *Let $\mathcal{R} \subset \mathrm{Rel}(A)$ be a domain clone. Then there is some domain clone $\mathcal{R}' \subsetneq \mathrm{Rel}(A)$, with $\mathcal{R} \subset \mathcal{R}' \subset \mathrm{Rel}(A)$, i.e., there are no maximal domain clones in $\mathrm{Rel}(A)$.*

Let us look at domain clones in $\mathcal{L}^\star_{\mathrm{Dom}(A)}$ and what happens if we remove $\langle \delta, \emptyset \rangle_\Omega$ from them. First we look at domain clones generated by just one relation.

7. Intervals $\mathcal{I}(C, \mathrm{Str}(C))$ for total clones C

Lemma 7.1. *Let $C \subsetneq \mathrm{Op}(A)$ be a total clone on A, $D \in \mathcal{I}(C, \mathrm{Str}(C))$, $f \in C^{(n)}$, and $\rho \in \mathrm{dom}^{(n)}(D)$.*

Then $f_\rho \in D$ where $f_\rho \le f$ and $\mathrm{dom}\, f_\rho = \rho$.

Proof. There is some $g \in D$ with $\mathrm{dom}\, g = \rho$. Then $f_\rho = e_1^2(f, g) \in D$. $\square$

Corollary 7.2. *Let $C \subsetneq \mathrm{Op}(A)$ be a total clone on A. Then the map ι_C from $\mathcal{I}(C, \mathrm{Str}(C))$ to $\mathcal{L}_{\mathrm{pProj}(A)} = \mathcal{I}(\mathrm{Proj}(A), \mathrm{pProj}(A))$ defined by $\iota_C(D) = D \cap \mathrm{pProj}(A)$ is injective.*

Proof. By Lemma 7.1 $D \cap \mathrm{pProj}(A) = D' \cap \mathrm{pProj}(A)$ implies $D = D'$, and therefore ι_C is injective. $\square$

Thus we know that the interval $\mathcal{I}(C, \mathrm{Str}(C))$ is at most as complicated as the interval $\mathcal{L}_{\mathrm{pProj}(A)} \cong \mathcal{L}_{\mathrm{Dom}(A)}$. We now want to know how they differ. For that we have to get back to work with functions, and then we will translate this to operations on the domains/relations.

Let C be a total clone and let $f \in C^{(n)}$ be an n-ary function. Furthermore, let $\rho \in \mathrm{Rel}^{(m+1)}(A)$ for some $m \geq n$. We define the operator S_f by $S_f(\rho) \in \mathrm{Rel}^{(m)}(A)$ and

$$S_f(\rho) := \{\mathbf{x} \in A^m \mid (f(\mathbf{x}_{(1,\dots,n)}), \mathbf{x}) \in \rho\}.$$

We note that $S_{e_1^1}(\rho) = \Delta\rho$.

Lemma 7.3. *Let $C \subseteq \mathrm{Op}(A)$ be a total clone on A, and $D \in \mathcal{I}(C, \mathrm{Str}(C))$. Then $\mathrm{dom}\, D$ is closed under S_f for all $f \in C$.*

Proof. Let $\rho \in \mathrm{dom}^{(m+1)} D$ and $f \in C^{(n)}$ with $m \geq n$. Then there is some $g \in D$ with $\mathrm{dom}\, g = \rho$.

We consider the partial function $G := g(f(e_1^m, \dots, e_n^m), e_1^m, \dots, e_m^m)$. Then $\mathrm{dom}\, f(e_1^m, \dots, e_n^m) = \{\mathbf{x} \in A^m \mid \mathbf{x}_{(1,\dots,n)} \in \mathrm{dom}\, f\} = A^m$ since $f \in \mathrm{Op}(A)$. Then

$$\mathrm{dom}\, G = \{\mathbf{x} \in A^m \mid (f(\mathbf{x}_{(1,\dots,n)}), \mathbf{x}) \in \mathrm{dom}\, g\}$$

and $G \in D$ imply that $S_f(\rho) = S_f(\mathrm{dom}\, g) = \mathrm{dom}\, G \in \mathrm{dom}\, D$. Thus $\mathrm{dom}\, D$ is closed under S_f for all $f \in C$. □

As a simple example we take $f := c_0$ be the unary constant 0, and $\rho = \{001, 010, 111\}$. Then $S_{c_0}(\rho) = \{01, 10\}$, and $S_{c_0}^2(\rho) = \{1\}$. Thus if C is a total clone and $c_0 \in C$, then $\rho \in \mathrm{dom}\, D$ implies $\{01, 10\}, \{1\} \in \mathrm{dom}\, D$ for each partial clone $D \in \mathcal{I}(C, \mathrm{Str}(C))$.

We have seen that the associated domain clone $\mathrm{dom}\, D$ for a partial clone $D \in \mathcal{I}(C, \mathrm{Str}(C))$ is closed under S_f for all $f \in C$. Now we will show that the converse is also true, i.e., the interval $\{\mathrm{dom}\, D \mid D \in \mathcal{I}(C, \mathrm{Str}(C))\}$ is precisely the set of domain clones closed under S_f for all $f \in C$.

Lemma 7.4. *Let $f \in \mathrm{Op}^{(n)}(A)$ for some $n \geq 1$, and $g_1, \dots, g_n \in \mathrm{Par}^{(m)}(A)$ for some $m \geq 1$. Then*

$$\mathrm{dom}\, f(g_1, \dots, g_n) = \mathrm{dom}\, e_1(g_1, \dots, g_n) = \bigcap_{i=1}^{n} \mathrm{dom}\, g_i.$$

Lemma 7.5. *Let $f, g \in \mathrm{Par}(A)$. Then $\mathrm{dom}(f \star g) \in \langle\{\mathrm{dom}\, f, \mathrm{dom}\, g\}\rangle_{\Omega, S_g}$.*

Proof. Let $f \in \mathrm{Par}^{(n)}(A)$ and $g \in \mathrm{Par}^{(m)}(A)$ for some $n, m \geq 1$.

By the definition of $\star$ we have

$$\begin{aligned}\mathrm{dom}(f \star g) = \{\mathbf{x} \in A^{n+m-1} \mid\ & \mathbf{x}_{(1,\dots,m)} \in \mathrm{dom}(g), \\ & (g(\mathbf{x}_{(1,\dots,m)}), \mathbf{x}_{(m+1,\dots,n+m-1)}) \in \mathrm{dom}(f)\} \\ = (\zeta\nabla)^{n-1}&(\mathrm{dom}\, g) \cap \\ &\{\mathbf{x} \in A^{n+m-1} \mid (g(\mathbf{x}_{(1,\dots,m)}), \mathbf{x}_{(m+1,\dots,n+m-1)}) \in \mathrm{dom}(f)\} \\ = (\zeta\nabla)^{n-1}&(\mathrm{dom}\, g) \cap S_g(\tau\nabla)^n(\mathrm{dom}\, f).\end{aligned}$$

Thus the statement holds. □

Now we can conclude our main theorem for this section.

Theorem 7.6. *Let $C \subseteq \mathrm{Op}(A)$ be a total clone on A, and $\mathcal{R} \in \mathcal{L}_{\mathrm{Dom}(A)}$ a domain clone.*

Then $\mathcal{R}$ is closed under S_f for all $f \in C$ if and only if there is some $D \in \mathcal{I}(C, \mathrm{Str}(C))$ with $\mathcal{R} = \mathrm{dom}\, D$.

If the relation $\delta \in \mathrm{dom}\, D$ we even get another construction S'_f, which is a kind of inverse to S_f. Let $\rho \in \mathrm{Rel}^{(m)}(A)$ and $f \in \mathrm{Op}^{(n)}(A)$ with $m \geq n$. We define S'_f by $S'_f(\rho) \in \mathrm{Rel}^{(m+1)}(A)$ and

$$S'_f(\rho) := \{(f(\mathbf{x}_{(1,\ldots,n)}), \mathbf{x}) \in A^{m+1} \mid \mathbf{x} \in \rho.$$

Then $S_f S'_f(\rho) = \rho$, and $S'_f S_f(\rho) \subseteq \rho$, where the inclusion is normally strict.

Lemma 7.7. *Let $\rho \in \mathrm{Rel}(A)$ and $f \in \mathrm{Op}(A)$. Then $S'_f(\rho) \in \langle\{\rho, \delta\}\rangle_{\zeta,\otimes,S_f}$.*

Proof. Let $\rho \in \mathrm{Rel}^{(m)}(A)$ and $f \in \mathrm{Op}^{(n)}(A)$ with $m \geq n$. Then

$$S'_f(\rho) = \zeta^{-1} S_f \zeta(\delta \otimes \rho).$$

□

7.1. **The domain clone $\langle\{\emptyset, \delta\}\rangle_{\Omega,(S_f)_{f\in C}}$.** Let $Q_C := \langle\{\emptyset, \delta\}\rangle_{\Omega,(S_f)_{f\in C}}$. Then Q_C is a domain clone, and, as we already know, for $C = \mathrm{Proj}(A)$ we have $\mathrm{pPol}\, Q_{\mathrm{Proj}(A)} = \mathrm{Par}(A)$. Now we want to consider in the Boolean case, i.e., $A = \{0, 1\}$, what is $\mathrm{pPol}\, Q_C$ for each total Boolean clone C.

For this let us make several observations concerning the Boolean functions generating the total clones. One step is to determine the total part $P_C := \mathrm{Op}(A) \cap \mathrm{pPol}\, Q_C$ of this clone. The general idea in this search includes two steps:

- First use the functions in a generating set for the C, to obtain some set Q'_C of non-trivial relations in Q_C via the operator S_f from the relations A^n and δ. This gives a suspect $Y_C := \mathrm{Pol}\, Q'_C$ for $\mathrm{Op}(A) \cap \mathrm{pPol}\, Q_C$.
- Then check for each g in the generating set of Y_C, and each f in the generating set of C that $g \in \mathrm{Pol}\, \rho$ implies $g \in \mathrm{Pol}\, S_f(\rho)$.

Lemma 7.8. *Let C, C' be total clones with $C \subseteq C'$. Then $P_{C'} \subseteq P_C$.*

Lemma 7.9. *Let $C \supseteq T_{a,\infty} \cap T_b \cap M$ with $\{a, b\} = \{0, 1\}$. Then $P_C \subseteq \langle c_0, c_1\rangle$.*

Proof. Let w.l.o.g. $a = 0$ and $b = 1$. Then by Theorem 3.2.1.1 [4] we have $m \in C$ where $m(x, y, z) = x \wedge (y \vee z)$. Thus

$$S'_m(A^3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \in Q_C.$$

We define $\rho := \{\mathbf{x} \in A^3 \mid \mathbf{x}_{(1,2,3,1)}, \mathbf{x}_{(1,3,2,1)} \in S'_m(A^3)\}$. Then

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \in Q_C.$$

By Table 10.1 [4] we get $P_C \subseteq \operatorname{Pol}\rho = \langle \wedge, c_0, c_1\rangle$.

We show that $\wedge \notin P_C$. Let $\mathbf{x} = (1,1,0,1)$, $\mathbf{y} = (1,1,1,0)$. Then $\mathbf{x},\mathbf{y} \in S'_m(A^3)$, but $\mathbf{x} \wedge \mathbf{y} = (1,1,0,0) \notin S'_m(A^3)$. Thus $\wedge \notin \operatorname{Pol} S'_m(A^3) \supseteq P_C$. Thus $P_C \subseteq \langle c_0, c_1\rangle$. □

Lemma 7.10. *Let $C \supseteq T_{a,\infty} \cap M$ be a total clone for some $a \in \{0,1\}$. Then $P_C \subseteq \langle c_a\rangle$.*

Proof. Let w.l.o.g. $a = 0$. By Lemma 7.9 we have $P_C \subseteq \langle c_0, c_1\rangle$. By Theorem 3.2.1.1 [4] we have $c_0 \in C$, and thus $\{0\} = \Delta S'_{c_0}(A) \in Q_C$. Since $c_1 \notin \operatorname{Pol}\{0\}$ we obtain $P_C \subseteq \langle c_0\rangle$. □

Lemma 7.11. *Let $a \in \{0,1\}$. Then $P_{T_a} \supseteq \langle c_a\rangle$.*

Proof. Since $c_a \in \operatorname{Pol}\{\emptyset, \delta\} = \operatorname{Pol}\langle\{\emptyset,\delta\}\rangle_\Omega$, we just need to show that $S_f\rho \in \operatorname{Inv} c_a$ for all $\rho \in \operatorname{Inv} c_a$ and $f \in T_a$.

Let $\rho \in \operatorname{Inv} c_a$, and $f \in T_a$. Then $(a,\dots,a) \in \rho$, and $f(a,\dots,a) = a$. Thus $(a,\dots,a) \in S_f\rho$, and consequently $S_f\rho \in \operatorname{Inv} c_a$. This implies $Q_{T_a} \subseteq \operatorname{Inv} c_a$, and therefore $P_{T_a} = \operatorname{Pol} Q_{T_a} \supseteq \operatorname{Pol}\operatorname{Inv} c_a = \langle c_a\rangle$. □

Lemma 7.12. *Let C be a total clone, and $a \in \{0,1\}$. Then $c_a \in P_C$ if and only if $C \subseteq T_a$.*

Proof. First let $C \subseteq T_a$. By Lemmas 7.11 and 7.8 we get $c_a \in P_{T_a} \subseteq P_C$.

Now assume $C \not\subseteq T_a$. Then there is some n-ary function $f \in C$ with $f(a,\dots,a) \neq a$. Then $c_a(\mathbf{x}) = (a,\dots,a) \notin S_f(A^n) \in Q_C$ for all $\mathbf{x} \in S_f(A^n)$. Thus $c_a \notin \operatorname{Pol} S_f(A^n) \supseteq P_C$. □

Corollary 7.13. *Let C be a total clone with $T_{a,\infty} \cap M \subseteq C \subseteq T_a$ for some $a \in \{0,1\}$. Then $P_C = \langle c_a\rangle$.*

Proof. By Lemmas 7.11, 7.8, and 7.10 we have

$$\langle c_a\rangle \subseteq P_{T_a} \subseteq P_C \subseteq \langle c_a\rangle,$$

and thus $P_C = \langle c_a\rangle$. □

Corollary 7.14. $P_{\mathrm{Op}(A)} = P_M = \operatorname{Proj}(A)$.

Proof. By Corollary 7.13 we have $P_{T_a \cap M} = \langle c_a\rangle$ for $a \in \{0,1\}$. Since $M \supseteq T_a \cap M$ for all $a \in \{0,1\}$ we get $P_{\mathrm{Op}(A)} \subseteq P_M \subseteq \langle c_0\rangle \cap \langle c_1\rangle = \operatorname{Proj}(A)$. Since $P_{\mathrm{Op}(A)}$ is a total clone we also have $\operatorname{Proj}(A) \subseteq P_{\mathrm{Op}(A)}$. Thus follows $P_{\mathrm{Op}(A)} = P_M = \operatorname{Proj}(A)$. □

Lemma 7.15. $P_{T_0 \cap T_1} \supseteq \langle c_0, c_1\rangle$.

Proof. By Lemma 7.8 and Corollary 7.13 we have $\langle c_a\rangle = P_{T_a} \subseteq P_{T_0\cap T_1}$ for all $a \in \{0,1\}$, and thus $P_{T_0 \cap T_1} \supseteq \langle c_0, c_1\rangle$. □

Corollary 7.16. *Let C be a total clone with $T_{a,\infty} \cap T_b \cap M \subseteq C \subseteq T_0 \cap T_1$ with $\{a,b\} = \{0,1\}$. Then $P_C = \langle c_0, c_1\rangle$.*

Proof. By Lemmas 7.15, 7.8, and 7.9 we have

$$\langle c_0, c_1\rangle \subseteq P_{T_0\cap T_1} \subseteq P_C \subseteq \langle c_0, c_1\rangle,$$

and thus $P_C = \langle c_0, c_1\rangle$. □

Lemma 7.17. $P_{\langle \wedge, c_0, c_1\rangle} \supseteq \langle\wedge\rangle$.

Proof. Since $\wedge \in \operatorname{Pol}\{\emptyset, \delta\} = \operatorname{Pol}\langle\{\emptyset, \delta\}\rangle_\Omega$, we just need to show that $S_f\rho \in \operatorname{Inv}\wedge$ for all $\rho \in \operatorname{Inv}\wedge$ and $f \in \{\wedge, c_0, c_1\}$.

Let $\rho \in \operatorname{Inv}\wedge$, and $f \in \{\wedge, c_0, c_1\}$. □

Lemma 7.18. $P_{\langle\wedge\rangle} \subseteq \langle\wedge, c_0, c_1\rangle$.

Proof. Since

$$S_\wedge(A^2) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

we get from Table 10.1 [4]

$$P_{\langle\wedge\rangle} = \operatorname{Pol} Q_{\langle\wedge\rangle} \subseteq \operatorname{Pol} S_\wedge(A^2) = \langle\wedge, c_0, c_1\rangle.$$

□

Corollary 7.19.
- $P_{\langle\wedge\rangle} = \langle\wedge, c_0, c_1\rangle$;
- $P_{\langle\wedge, c_a\rangle} = \langle\wedge, c_a\rangle$ *for every* $a \in \{0, 1\}$;
- $P_{\langle\wedge, c_0, c_1\rangle} = \langle\wedge\rangle$.

Proof. Let $C \in \{\langle\wedge\rangle, \langle\wedge, c_0\rangle, \langle\wedge, c_1\rangle, \langle\wedge, c_0, c_1\rangle\}$. By Lemmas 7.17 and 7.18 we have $\langle\wedge\rangle \subseteq P_C \subseteq \langle\wedge, c_0, c_1\rangle$. Then Lemma 7.12 implies the statement of this corollary. □

Lemma 7.20. $P_S = \langle\bar{\cdot}\rangle$.

Proof. By Lemma □

C	$\operatorname{Op}(A) \cap \operatorname{pPol} Q_C$	Elements of Q_C
$\operatorname{Proj}(A)$	$\operatorname{Op}(A)$	
$\langle c_a\rangle$	$\subseteq T_a$	$\{a\}$
$\langle c_0, c_1\rangle$	$\subseteq T_0 \cap T_1$	$\{0\}, \{1\}$
$\langle\bar{\cdot}\rangle$	$\subseteq S$	$\{(0,1), (1,0)\}$
Ω_1	$\subseteq S \cap T_0 \cap T_1$	
$L \cap T_0 \cap T_1$		

7.2. **The interval $\mathcal{I}(C_A, \operatorname{Str}(C_A))$.** Let $C_A := \langle\{c_a \mid a \in A\}\rangle \subseteq \operatorname{Op}(A)$ be the total clone generated by all constant functions in $\operatorname{Op}(A)$.

Frozen partial co-clones, see Nordh and Zanutti [5].

With this information, we might look into the following question. Assuming we know the interval $\mathcal{I}(C, \operatorname{Str}(C))$ for some total clone C. By the lattice isomorphisms given in the previous section, we can associate to each $D \in \mathcal{I}(C, \operatorname{Str}(C))$ containing $e_\emptyset$ and e_δ the strong partial clone $\operatorname{pPol}\operatorname{dom}\iota_C D$. Which strong partial clones do we get? What does it tell us about the lattice of strong partial clones?

The restriction, that D should contain $e_\emptyset$ and e_δ could be abandoned. This might give a more precise picture, but we have to take care that the map $\operatorname{pPol}\operatorname{dom}\iota_C D$ is not injective in this case.

We will look at a few examples.

8. Intervals of domain clones for strong partial clones

As stated in Theorem 5.10 there is a bijection between the interval $\mathcal{L}^{\star}_{\mathrm{Dom}(A)}$ of domain clones and the lattice $\mathcal{L}^{\star}_{\mathrm{Par}(A)}$ of all strong partial clones. So we know the structure of the interval of domain clones if and only if we know the structure of the lattice of strong partial clones. But what about the rest of the lattice $\mathcal{L}_{\mathrm{Dom}(A)}$ of all domain clones.

Let $D \in \mathcal{L}_{\mathrm{Dom}(A)}$ be a domain clone. We define $D^{\uparrow}$ and $D^{\downarrow}$ by

$$D^{\downarrow} := \langle \{R \in D \mid R \neq \emptyset, R \text{ is irredundant}\}\rangle_{\mathrm{Dom}(A)},$$
$$D^{\uparrow} := \langle D \cup \{\emptyset, \delta\}\rangle_{\mathrm{Dom}(A)}.$$

Since $\emptyset$ and δ are preserved by every partial function, and they are not needed in the construction of irredundant relations, we have

$$\mathrm{pPol}\, D^{\uparrow} = \mathrm{pPol}\, D = \mathrm{pPol}\, D^{\downarrow}.$$

Furthermore, $D^{\uparrow} = \mathrm{pInv}\,\mathrm{pPol}\, D$, and thus the biggest domain clone D' with $\mathrm{pPol}\, D' = \mathrm{pPol}\, D$. Similarly, $D^{\downarrow}$ is the smallest domain clone D' with $\mathrm{pPol}\, D' = \mathrm{pPol}\, D$. The only relations possibly missing from $D^{\downarrow}$, are the ones with duplicate coordinates.

Since $(D^{\uparrow})^{\downarrow} = D^{\downarrow}$, we need only to consider domain clones the D with $D = D^{\uparrow}$. These are exactly the domain clones in the interval $\mathcal{L}^{\star}_{\mathrm{Dom}(A)}$, i.e., $D = \mathrm{pInv}\, C$ for some strong partial clone C. Thus the intervals $\mathcal{I}(D^{\downarrow}, D^{\uparrow})$ can be indexed by the strong partial clones, and to keep the notation simpler we define $\mathcal{I}_{\mathrm{Dom}}(C)$ for a strong partial clone C by $\mathcal{I}_{\mathrm{Dom}}(C) := \mathcal{I}((\mathrm{pInv}\, C)^{\downarrow}, \mathrm{pInv}\, C)$.

A natural question concerns the size of the interval $\mathcal{I}_{\mathrm{Dom}}(C)$ for any strong partial clone C. We will see that there are strong partial clones where the size is equal to the continuum.

We use the definitions from [8] to give continuum many domain clones in $\mathcal{I}_{\mathrm{Dom}}(C)$ for a single strong partial clone C. Let $R^{0,2}_{C,n}$ and $R^{0,2}_{K,n}$ be two n-ary relations defined by

$$R^{0,2}_{C,n}(x_1, \dots, x_n) := \bigwedge_{i \in [n]} \rho_{0,2}(x_i, x_{i+1 \bmod n}),$$
$$R^{0,2}_{K,n}(x_1, \dots, x_n) := \bigwedge_{\substack{i,j \in [n] \\ i \neq j}} \rho_{0,2}(x_i, x_j).$$

Furthermore, let

$$R^{0,2}_{n} := R^{0,2}_{C,n} \times R^{0,2}_{K,n}.$$

Let $\hat{R}^{0,2}_n$ be the $(2n+1)$-ary relation obtained from $R^{0,2}_n$ by duplicating the first coordinate, i.e., $\hat{R}^{0,2}_n := \{\mathbf{x}_{(1,1,2,\dots,2n)} \mid \mathbf{x} \in R^{0,2}_n\}$. Clearly, by identifying the first two coordinates of $\hat{R}^{0,2}_n$ we obtain $R^{0,2}_n$ and thus $\langle R^{0,2}_n\rangle_{\mathrm{Dom}(A)} \subseteq \langle \hat{R}^{0,2}_n\rangle_{\mathrm{Dom}(A)}$.

Let $\hat{\mathbb{N}} := \{n \in \mathbb{N} \mid n \text{ odd}, n \geq 3\}$, and $\mathcal{R} := \{R^{0,2}_n, \hat{R}^{0,2}_n \mid n \in \hat{\mathbb{N}}\}$.

Lemma 8.1. *Let $n \in \hat{\mathbb{N}}$. Then $\hat{R}^{0,2}_n \notin \langle \mathcal{R} \setminus \{\hat{R}^{0,2}_n\}\rangle_{\mathrm{Dom}(A)}$.*

Proof. Let $\mathcal{R}' := \langle \mathcal{R} \setminus \{\hat{R}_n^{0,2}\}\rangle_{\mathrm{Dom}(A)}$ and $R := \hat{R}_n^{0,2}$.

Since $(1,\dots,1) \notin R_m^{0,2}, \hat{R}_m^{0,2}$ for all $m \geq 3$, and $(1,1) \in \delta$, we see that $\delta \notin \mathcal{R}'$.

Assume to the contrary that $R \in \mathcal{R}'$. Then we can write

$$R = \{\mathbf{x} \in \mathbf{2}^{2n+1} \mid \mathbf{x}_\mathbf{i} \in S \text{ for all } \mathbf{i} \in \gamma_S \text{ and } S \in \mathcal{R}'\} \tag{1}$$

for some auxiliary relations γ_S for all $S \in \mathcal{R}'$.

We can apply Lemma 5.2 [8] and see that $\gamma_S = \emptyset$ for every S with bigger arity than R. Similarly, for every S with arity smaller than $2n$ we see that it embeds into the second part of R.

Thus we obtain $\gamma_S \neq \emptyset$ iff $S = R_n^{0,2}$. Basically, the only possible construction is $R = \{\mathbf{x} \in \mathbf{2}^{2n+1} \mid \mathbf{x}_{(1,3,4,\dots,2n+1)}, \mathbf{x}_{(2,3,4,\dots,2n+1)} \in R_n^{0,2}\}$. But then $(1,0,\dots,0) \in R$ in contradiction to the fact, that the first two coordinates of R are equal. $\square$

Now we can give the continuum many domain clones to some strong partial clone C. Let $N \subseteq \hat{\mathbb{N}}$. For a given function $\phi : N \to \{0,1\}$ let $\mathcal{R}_\phi := \langle \{R_n^{0,2} \mid n \in N, \phi(n) = 0\} \cup \{\hat{R}_n^{0,2} \mid n \in N, \phi(n) = 1\}\rangle_{\mathrm{Dom}(A)}$. Let $\mathcal{R}_N := \langle \{\emptyset, \delta\} \cup \{R_n^{0,2} \mid n \in N\}\rangle_{\mathrm{Dom}(A)}$. Clearly, $\mathcal{R}_\phi^{\uparrow} = \mathcal{R}_N$.

Lemma 8.2. *Let $\phi, \psi : N \to \{0,1\}$. Then $\mathcal{R}_\phi = \mathcal{R}_\psi \iff \phi = \psi$.*

Proof. W.l.o.g. $\phi(n) = 1 \neq \psi(n)$ for some $n \in N$. Then $\hat{R}_n^{0,2} \in \mathcal{R}_\phi$, but $\hat{R}_n^{0,2} \notin \mathcal{R}_\psi$ by Lemma 8.1. $\square$

Theorem 8.3. *Let $N \subseteq \hat{\mathbb{N}}$. Then $|\mathcal{I}_{\mathrm{Dom}}(\mathrm{pPol}\,\mathcal{R}_N)| \geq 2^{|N|}$.*

Proof. Let $\Phi := \{\phi \mid \phi : N \to \{0,1\}\}$. Clearly, $|\Phi| \geq 2^{|N|}$. By Lemma 8.2 we have that $|Q| = |\Phi|$ for $Q := \{\mathcal{R}_\phi \mid \phi \in \Phi\}$. Furthermore, $Q \subseteq \mathcal{I}_{\mathrm{Dom}}(\mathrm{pPol}\,\mathcal{R}_N)$, and thus $|\mathcal{I}_{\mathrm{Dom}}(\mathrm{pPol}\,\mathcal{R}_N)| \geq 2^{|N|}$. $\square$

Corollary 8.4. *Let $N \subseteq \hat{\mathbb{N}}$ be infinite. Then $\mathcal{I}_{\mathrm{Dom}}(\mathrm{pPol}\,\mathcal{R}_N)$ contains continuum many domain clones.*

This shows that the lattice of partial clones is "much more" complicated than the lattice of strong partial clones, even if we only restrict to the partial projections.

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