Complex torus and elliptic curves
Lecture notes

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Abstract

These are notes on the uniformization of complex elliptic curves via Weierstrass functions. Everything is taken out from references [1], [2] and [3], our contribution is just to give a self-contained exposition and more details for certain parts of the original proofs.

Contents

1 Lattices, tori and meromorphic functions

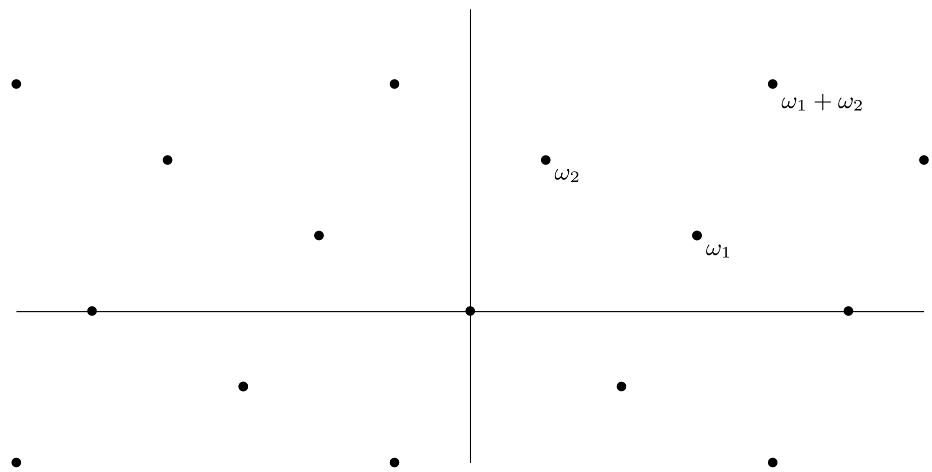
1.1 Change of lattices, complex tori

Let ω_1 and ω_2 be two complex numbers in $\mathbb{C}$ that are free over $\mathbb{R}$ ($\lambda_1\omega_1 + \lambda_2\omega_2 = 0$, λ_1, $\lambda_2 \in \mathbb{R} \Rightarrow \lambda_1 = \lambda_2 = 0$).

They define a 2-dimensional **lattice**

$$\Gamma := \{m\omega_1 + n\omega_2 \mid m,\ n \in \mathbb{Z}\}$$

which is a discrete abelian group in $\mathbb{C}$. For example, when $\omega_1 = 3 + i$ and $\omega_2 = 1 + 2i$ we get



We can define a torus $\mathbb{T}$ by setting

Definition 1.1.1. *The* ***complex torus*** *associated to the lattice* Γ *is the quotient space*

$$\mathbb{T} := \mathbb{C}/_{\Gamma}$$

An element of $\mathbb{T}$ is thus an equivalence class of elements of $\mathbb{C}$ for the equivalence relation

$$z \sim z' \quad \Leftrightarrow \quad \exists n,\, m \in \mathbb{Z} \,/ z = z' + n\omega_1 + m\omega_2$$

The complex structure of $\mathbb{C}$ induces a complex structure on $\mathbb{T}$ such that the projection $\mathbb{C} \to \mathbb{T}$ is holomorphic. This complex structures depends on the lattice Γ. However, if we define Γ' to be the lattice

$$\Gamma' := \{m + n\tau \mid m,\, n \in \mathbb{Z}\}$$

where $\tau := \omega_2/\omega_1$, and $\mathbb{T}'$ to be the torus $\mathbb{C}/\Gamma'$, then the bijective "multiplication by ω_1" map

$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ z & \mapsto & \omega_1 z \end{array}$$

sends Γ' bijectively to Γ.

Proposition 1.1.2. *Multiplication by* ω_1 *induces a biholomorphic isomorphism*

$$\mathbb{T}' \xrightarrow{\cong} \mathbb{T}$$

Thus, we can reduce ourselves to the study of tori given by lattices with $\omega_1 = 1$ and $\omega_2 = \tau$. We can even suppose that $\operatorname{Im} \tau > 0$ since the lattice generated by 1 and τ is the same as the lattice generated by 1 and $-\tau$.

1.2 Meromorphic functions

We now fix a lattice $\Gamma := \{m + n\tau \mid m,\, n \in \mathbb{Z}\}$ and denote by $\mathbb{T} := \mathbb{C}/_{\Gamma}$ the associated torus with projection $\pi : \mathbb{C} \to \mathbb{T}$.

By definition of the complex structure on $\mathbb{T}$, a function $f : \mathbb{T} \to \mathbb{C}$ is **meromorphic** if and only if the composite $f \circ \pi : \mathbb{C} \to \mathbb{C}$ is. Thus, we are interested in meromorphic functions on $\mathbb{C}$ that are invariant under addition of elements of Γ, i.e. Γ-periodic.

Remark 1.2.1. *Constant functions are holomorphic thus meromorphic. Note that Liouville's theorem implies that bounded holomorphic functions are constant. Since* $\mathbb{T}$ *is compact, any holomorphic function is bounded, thus constant.*

Among meromorphic periodic functions, some are of particular interest for us:

Definition 1.2.2. *The* ***Weierstrass function*** $\wp : \mathbb{C} \to \mathbb{C}$ *is defined by*

$$\wp(z) := \frac{1}{z^2} + \sum_{\substack{\omega \in \Gamma \\ \omega \neq 0}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

for all z *in* $\mathbb{C}$.

Proposition 1.2.3. *The Weierstrass function is*

- *well-defined and holomorphic on* $\mathbb{C} - \Gamma$,
- *meromorphic in each* z *of* Γ *with a pole of order* 2,
- *even and* Γ*-periodic.*

It's derived function is odd, Γ*-periodic, meromorphic with poles of order* 3 *on the lattice and is given by*

$$\wp'(z) = \sum_{\substack{\omega \in \Gamma \\ \omega \neq 0}} \frac{-2}{(z-\omega)^3}$$

Proof. Well defined amounts to prove that the sum involved in the definition of $\wp$ is convergent. For any ω in $\Gamma - \{0\}$

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \frac{\omega^2 - z^2 + 2z\omega - \omega^2}{\omega^2(z-\omega)^2} = \frac{1}{\omega^3}\frac{2z - z^2/\omega}{z/\omega - 1}$$

Since

$$\lim_{\omega \to \infty} \left| \frac{2z - z^2/\omega}{z/\omega - 1} \right| = 2|z|$$

there exists a positive constant $C(z)$ such that

$$\left|\frac{2z - z^2/\omega}{z/\omega - 1}\right| < C(z)$$

for all ω in $\Gamma - \{0\}$. Thus

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| < \frac{C(z)}{|\omega|^3}$$

The fact that $\sum_{\substack{\omega\in\Gamma\\ \omega\neq 0}} \frac{C(z)}{|\omega|^3}$ is convergent implies that

$$\sum_{\substack{\omega\in\Gamma\\ \omega\neq 0}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \tag{1}$$

is absolutely convergent, thus convergent. This proves that $\wp$ is well-defined.

To see that $\wp$ is holomorphic in $z \in \mathbb{C} - \Gamma$, remark that there exists a neighbourhood V of z and a positive number such that the constants $C(z')$ are bounded by C when z' is in V i.e.

$$C(z') < C \quad \forall z' \in V$$

Thus the sum (1) is normaly convergent on V. Since each term $\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$ is holomorphic in z, the limit is also holomorphic in z. Morevover, $\wp'(z)$ can be obtained by deriving term by term so that

$$\wp'(z) = -\sum_{\omega\in\Gamma} \frac{2}{(z-\omega)^3}$$

t is clear that $\wp'$ is meromorphic with poles of order 3 on the lattice and that it is Γ-periodic.

It is clear that each element of the lattice is a pole of order 2 of $\wp$. To see that $\wp$ is even notice that one can reparametrize the sum (1) using $-\omega$ instead of ω.

The periodicity goes as follows: choose a point γ in Γ and set

$$\mathcal{Q}(z) := \wp(z+\gamma) - \wp(z)$$

for all z in $\mathcal{C} - \Gamma$. Then

$$\mathcal{Q}'(z) := \wp'(z+\gamma) - \wp'(z) = 0$$

because $\wp'$ is Γ-periodic. Thus, there exists a constant C (depending on γ *a priori*) such that

$$\mathcal{Q}(z) := \wp(z+\gamma) - \wp(z) = C$$

But for $z = -\gamma/2$ we have that

$$\mathcal{Q}(-\gamma/2) := \wp(\gamma/2) - \wp(-\gamma/2) = 0$$

because $\wp$ is even. Thus, $C = 0$ and $\wp(z+\gamma) = \wp(z)$ for all z in $\mathbb{C} - \Gamma$. This proves that $\wp$ is Γ-periodic.

□

The Weirestrass function associated to the lattice Γ satisfies a nice differential equation:

Proposition 1.2.4. *The Weierstrass function $\wp$ and its derivative $\wp'$ satisfy*

$$(\wp')^2 = 4\wp^3 + g_2\wp + g_3 \tag{2}$$

*where g_2 and g_3 are the **Eisenstein series** (of the lattice Γ) defined by*

$$g_2 = g_2(\tau) := \sum_{\substack{\omega\in\Gamma\\ \omega\neq 0}} \frac{-60}{\omega^4} = \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ (m,n)\neq(0,0)}} \frac{-60}{(m+n\tau)^4} \quad ; \quad g_3 = g_3(\tau) := \sum_{\substack{\omega\in\Gamma\\ \omega\neq 0}} \frac{-140}{\omega^6} = \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ (m,n)\neq(0,0)}} \frac{-140}{(m+n\tau)^6}$$

Theorem 1.2.5. *The field of meromorphic functions on $\mathbb{T}$, denoted $\mathcal{M}(\mathbb{T})$ is generated by $\wp$ and $\wp'$ i.e.*

$$\mathcal{M}(\mathbb{T}) \cong \mathbb{C}(\wp, \wp')$$

More precisely, relation (2) is the "smallest relation" satisfied by $\wp$ and $\wp'$ i.e.

$$\begin{array}{rcl} \phi : \mathbb{C}(X)[Y]/(Y^2 - 4X^3 - g_2X - g_3) & \longrightarrow & \mathcal{M}(\mathbb{T}) \\ X & \longmapsto & \wp \\ Y & \longmapsto & \wp' \end{array}$$

is an isomorphism of fields.

Proof. We only prove the surjectivity of ϕ. Let $f : \mathbb{T} \to \mathbb{C}$ be a meromorphic function on the torus. If z_0 is a pole of order k of $f \circ \pi$ which is not on the lattice Γ, then setting $f \circ \pi(\wp - \wp(z_0))^k$ gives a holomorphic function in z_0. Indeed, since z_0 doesn't belong to Γ, $\wp$ is homolorphic in z_0 and admits a Taylor expansion near z_0 of the form

$$\wp(z) = \wp(z_0) + \wp'(z_0)(z - z_0) + o(|z - z_0|)$$

This implies that $z \mapsto f(\pi(z))(\wp(z) - \wp(z_0))^k$ admits a finite limit in z_0 ans thus is holomorphic in z_0. Proceeding in the same fashion for all poles that are not on the lattice we get a Γ-periodic function $z \mapsto f(\pi(z))Q(\wp(z))$, where $Q(\wp)$ is a polynomial in $\wp$, which has only possible poles on the lattice (this means that the induced function on the $\mathbb{T}$ can just have a pole in $0 := \pi(\Gamma)$). If we suppose that $z \mapsto f(\pi(z))Q(\wp(z))$ has a pole of even order $2k$ in 0, then there exists a constant c such that

$$z \mapsto f(\pi(z))Q(\wp(z)) - c\wp(z)^k$$

has a pole of order at most $2k - 1$ in 0. Similarly, if the pole as odd order $2k - 1$ we can substract a good multiple of $\wp'$ to lower the order of this pole. We see that repeating this procedure leads to a function that is holomorphic in zero. In other words, there exists a polynomial in two variables $R(X, Y)$ such that the function on the torus $fQ(\wp) - R(\wp, \wp')$ is holomorphic. Since $\mathbb{T}$ is compact, this function is bounded thus constant (by Liouville's theorem). Hence, there exists a constant C in $\mathbb{C}$ such that

$$fQ(\wp) - R(\wp, \wp') = C$$

i.e.

$$f = \frac{C + R(\wp, \wp')}{Q(\wp)}$$

This proves that any meromorphic function f is a rational function in $\wp$ and $\wp'$ which implies that ϕ is surjective. □

2 The link with elliptic curves

2.1 The projective plane, complex elliptic curves

Definition 2.1.1. *The* ***complex projective plane****, denoted $\mathbb{C}P^2$ is the set of complex lines in $\mathbb{C}^3$. Any non-zero vector (z_1, z_2, z_3) in $\mathbb{C}^3$ spans a complex line i.e. an element of $\mathbb{C}P^2$ that we denote by $[z_1 : z_2 : z_3]$ (homogenous coordinates). $\mathbb{C}P^2$ is an affine complex variety with open affine cover given by U_1, U_2, U_3 where*

$$U_i := \{L = [z_1, z_2, z_3] \in \mathbb{C}P^2 \mid z_i \neq 0\}.$$

A ***projective curve*** *is a closed subset of $\mathbb{C}P^2$ which is the zero-locus of a homogenous polynomial P in $\mathbb{C}[X, Y, Z]$. It is said to be* ***non-singular*** *when the partial derivatives of P never vanish simultaneously on its zero locus.*

An ***elliptic curve*** *is a non-singular projective curve which is the zero locus of a polynomial P such that there exists an affine chart in which P takes the form*

$$Y^2 - 4X^3 - aX - b$$

Remark 2.1.2. *To any polynomial $P := \sum_{i,j} a_{i,i} X^i Y^j$ of degree n in $\mathbb{C}[X, Y]$, we can associate a homogenous polynomial $\tilde{P}$ of degree n in $\mathbb{C}[X, Y, Z]$ by setting*

$$\tilde{P} := \sum_{i,j} a_{i,i} X^i Y^j Z^{n-i-j}$$

*The projective curve corresponding to the zero locus of $\tilde{P}$ will be referred to as the projective curve a**ssociated to** P in the sequel.*

Proposition 2.1.3. *The projective curve associated to a polynomial*

$$P := Y^2 - X^3 - aX - b$$

is non singular (i.e. elliptic) if and only if the discriminant

$$\Delta(a, b) := 4a^3 + 27b^2$$

is not zero.

Proof. We have that $\frac{\partial P}{\partial Y}(x,y)=0$ if and only if $y=0$. Thus, the only points (x,y) of the zero locus of P (on the curve) where $\frac{\partial P}{\partial Y}$ vanishes are of the form $(x,0)$. But such a point is on the curve if and only if

$$P(x,0) = -x^3 - ax - b = 0$$

If x_1, x_2, x_3 are the roots of X^3+aX+b, this condition is equivalent to the fact that $x=x_i$ for some $i\in\{1,2,3\}$. But

$$\frac{\partial P}{\partial X} = -\frac{\partial}{\partial X}\big((X-x_1)(X-x_2)(X-x_3)\big) = -((X-x_2)(X-x_3)+(X-x_1)(X-x_3)+(X-x_1)(X-x_2))$$

thus we see that $\frac{\partial P}{\partial X}(x_i,0)=0$ if and only if there exists $j\neq i$ such that $x_i=x_j$ i.e. if and only if P has a root of multiplicity strictly greater than one (meaning that $\operatorname{card}\{x_1,x_2,x_3\}\leqslant 2$).

Now recall that the coefficients a and b of the polynomial X^3+aX+b can be expressed in terms of its roots x_i in the following manner

$$a = x_1x_2+x_1x_3+x_2x_3 \qquad \text{and} \qquad b=-x_1x_2x_3$$

so that

$$\Delta(a,b) := 4a^3+27b^2 = 4(x_1x_2+x_1x_3+x_2x_3)^3+27(x_1x_2x_3)^2$$

Since there is no term of degree 2 in x^3+aX+b the sum of the roots is zero thus $x_3=-x_2-x_1$. Hence

$$\begin{aligned}\Delta(a,b) =& -4(x_1^2+x_2x_1+x_2^2)^3+27(x_1^2x_2+x_1x_2^2)^2\\ =& -4(x_1^6+x_1^3x_2^3+x_2^6+3(x_1^5x_2+2x_1^4x_2^2+2x_1^2x_2^4+x_1x_2^5)+6x_1^3x_2^3)+27(x_1^4x_2^2+2x_1^3x_2^3+x_1^2x_2^4)\\ =& -4x_1^6-12x_1^5x_2+3x_1^4x_2^2+26x_1^3x_2^3+3x_1^2x_2^4-12x_1x_2^5-4x_2^6\end{aligned}$$

On the other hand,

$$\begin{aligned}(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2 =&(x_1-x_2)^2(2x_1+x_2)^2(x_1+2x_2)^2\\ =&(2x_1^3+3x_1^2x_2-3x_1x_2^2-2x_2^3)^2\\ =&4x_1^6+12x_1^5x_2-3x_1^4x_2^2-26x_1^3x_2^3-3x_1^2x_2^4+12x_1x_2^5+4x_2^6\\ =&-\Delta(a,b)\end{aligned}$$

Hence

$$\left\{\begin{array}{l}P(x,y)=0\\ \frac{\partial P}{\partial Y}(x,y)=0\\ \frac{\partial P}{\partial X}(x,y)=0\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}y=0\\ x\in\{x_1,x_2,x_3\}\\ \operatorname{card}\{x_1,x_2,x_3\}\leqslant 2\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}y=0\\ x\in\{x_1,x_2,x_3\}\\ (x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2=-\Delta(a,b)=0\end{array}\right.$$

Thus the curve is singular if and only if $\Delta(a,b)=0$.

□

2.2 The uniformization isomorphism

In the first section, we have seen that given τ in the upper half plane, the torus $\mathbb{T}:=\mathbb{C}/\Gamma$ (where Γ is the lattice generated by 1 and τ) is endowed with a particular meromorphic function: the Weierstrass function $\wp:\mathbb{T}\to\mathbb{C}$ which satisfies the differential equation:

$$(\wp')^2 = 4\wp^3+g_2\wp+g_3$$

or equivalently

$$(\frac{\wp'}{2})^2 = \wp^3+\frac{g_2}{4}\wp+\frac{g_3}{4}$$

This means that for any z in $\mathbb{C}-\Gamma$, the element $[\wp(z):\frac{\wp'(z)}{2}:1]$ of $\mathbb{C}P^2$ belongs to the projective curve associated to the polynomial $Y^2-X^3-\frac{g_2}{4}X-\frac{g_3}{4}$. Recall that g_2 and g_3 depend on the choice of τ.

Proposition 2.2.1. *For any τ with strictly positive imaginary part,*

$$\Delta(\frac{g_2}{4},\frac{g_3}{4})\neq 0.$$

Thus, the projective curve associated to $Y^2-X^3-\frac{g_2}{4}X-\frac{g_3}{4}$ is elliptic.

Let $\mathcal{C}(\tau)$ be the elliptic curve associated to $Y^2-X^3-\frac{g_2}{4}X-\frac{g_3}{4}$.

Theorem 2.2.2. [Uniformization isomorphism] *The map*

$$\begin{array}{ccl} \mathbb{T} & \to & \mathcal{C}(\tau) \subset \mathbb{C}P^2 \\ [z] & \mapsto & \begin{cases} [\wp(z) : \wp'(z)/2 : 1] & \textit{if } z \notin \Gamma, \\ [1 : 1 : 0] & \textit{if } z \in \Gamma. \end{cases} \end{array}$$

is a biholomorphism.

The reciproque of the preceeding theorem is true:

Theorem 2.2.3. *Any complex elliptic curve is "isomorphic" to an elliptic curve of the form $\mathcal{C}(\tau)$ for a certain τ, thus analytically isomorphic to a complex torus.*

Thus, we see that, thanks to the uniformization isomorphism theorem and its reciproque, any complex elliptic curve $\mathcal{C}$ can be identified with the complex torus $\mathbb{T}$ associated to a certain lattice Γ (associated to a certan choice of τ). But since $\mathbb{T}$ is a quotient of abelian groups, it is itself an abelian group (with neutral element $[0]$). This implies that any elliptic $\mathcal{C}$ can be endowed with an abelian group multiplication. Is there a way to describe this multiplication geometrically ? This might be the subject of next lectures...

References

[1] Martin Schlichenmaier, *An introduction to Riemann Surfaces, Algebraic Curves and Moduli Spaces.* Lecture Notes in Physics 322, Springer-Verlag, 1989.

[2] Christophe Delaunay, *Cryptographie sur les courbes elliptiques.* `http://math.univ-lyon1.fr/~wagner/coursDelaunay.pdf`

[3] Richard Schwartz, *Notes on Weiertrass Uniformization.* `http://www.math.brown.edu/~res/M154/notes9.pdf`