12

Reactive Preferential Structures and Non-monotonic Consequence

12.1 Introduction

12.1.1 Overview

Our aim is twofold here. We want to:

1. introduce IBRS (Information Bearing System, defined in Definition 12.1), an abstraction of many semantical structures in nonclassical logics,
2. use IBRS as a generalization of preferential structures to give semantics for logics weaker than preferential logics.

Section 12.1 and Section 12.2 discuss general IBRS, Section 12.3 discusses generalized preferential structures. The main technical developments are in Section 3, the main conceptual discussions are in Sections 1 and 2.

At the beginning of Section 12.3, we briefly argue why we think that the results of this Section are important (they show independence of certain logical rules), and, consequently, why the techniques used to obtain them are important. Section 12.3 is thus a technical justification of the concept of IBRS. (The results are briefly put into perspective at the end of this overview.)

Section 12.1 gives a general framework within which we place ourselves, and then gives the formal definition of IBRS. We then show (not exhaustively!) how our definition covers many different formal logics and their semantics. We also stress the main idea of IBRS, reactivity, whose base is to have not only arrows between points, but also "higher" arrows between points and arrows. This is also basic for the concept of generalized preferential structures, discussed in Section 12.3.

In Section 12.2, we present the outlines of a semantics for general IBRS, together with a circuit semantics for a special case. The main result of Section 12.2 is perhaps a warning: one has to be careful about transmission times in real-life scenarios. Depending on those times, a network may or may not oscillate.

For the reader's convenience, we now describe the main technical results from Section 12.3.

D.M. Gabbay, *Reactive Kripke Semantics*, Cognitive Technologies,
DOI 10.1007/978-3-642-41389-6_12,

The model choice functions μ of usual preferential structures satisfy two basic laws, and if they are smooth, they also satisfy a third law:

1. $(\mu \subseteq)$ $\mu(X) \subseteq X$ - this is trivial,
2. (μPR) $X \subseteq Y \Rightarrow \mu(Y) \cap X \subseteq \mu(X)$ - anything which is not minimal in X will a fortiori not be minimal in Y, if $X \subseteq Y$,
3. (μCUM) $\mu(X) \subseteq Y \subseteq X \Rightarrow \mu(X) = \mu(Y)$ - which holds in smooth structures, but not necessarily in non-smooth structures.

(μPR) is closely linked to the logical rule
(OR) $\alpha \mid\!\sim \beta, \alpha' \mid\!\sim \beta \Rightarrow \alpha \vee \alpha' \mid\!\sim \beta$
(μCUM) to the logical rule
(CUM) $\alpha \mid\!\sim \beta, \alpha \mid\!\sim \beta' \Rightarrow \alpha \wedge \beta \mid\!\sim \beta'$.

Thus, in usual preferential structures (OR) will hold, and if they are smooth, so will (CUM), but we cannot have (CUM) without (OR). The main result of Section 12.3 is now to fully separate them. We can have (OR) without (CUM), but also (CUM) without (OR). The "trick" is that reactive structures allow us to break the coherence condition of (μPR), as a bigger set allows us to "destroy destruction". For all details, the reader is referred to Section 12.3.

We now conclude this overview with an informal motivation, showing that IBRS are a very natural generalisation of existing structures.

Motivation for IBRS and generalized preferential structures

Many semantical structures for nonclassical logics use sets of models, called possible worlds, together with a relation, e.g. accessibility in the case of Kripke models for modal logic, distance in the case of Stalnaker-Lewis semantics for counterfactual conditionals, or comparison of "normality" in the case of preferential structures. IBRS allow not only relations between objects, i.e. possible worlds, but also "higher" relations, where e.g. an object can be in relation with a pair of objects (which are in a "lower" relation), etc.

It is sometimes natural to see the (basic) relations as attacks: if $a \prec b$, then a is considered more "normal" (or less "abnormal") than b, or, we may say that a attacks b's normality, and we may write this as an arrow from a to b. Now when we have an attack from c against this arrow, i.e. the relation $a \prec b$, it is natural to see now the original attack as destroyed, i.e. a does not attack b any more. So $a \prec b$ is still there, but without any effect. This is possible in an IBRS, see Figure 12.1.

The decisive property of preferential structures is the trivial fact that if $a, b \in X \subseteq Y$, and $a \prec b$ "in X", then this will also hold "in Y". This creates strong coherence properties. IBRS can break them: if $c \in Y - X$, and c attacks $a \prec b$, then $a \prec b$ is still there in Y, but not effective any more. Thus, IBRS, or generalized preferential structures, as we will call this special case of IBRS, can give semantics for logics weaker than preferential logics.

Preferential structures themselves were introduced as abstractions of Circumscription independently in [148] and [33]. A precise definition of these structures is given below in Definition 12.12.

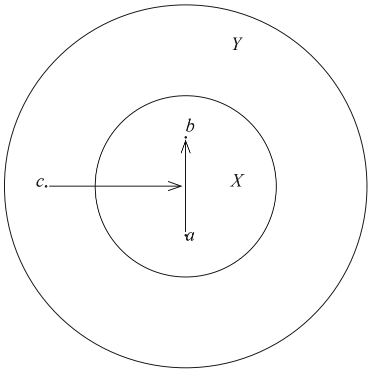

Fig. 12.1: Attacking an arrow

In an abstract consideration of desirable properties a logic might have, [51] examined rules a nonmonotonic consequence relation $\mid\sim$ should satisfy:

1. $\Delta, \alpha \mid\sim \alpha$,
2. $\Delta \mid\sim \alpha \Rightarrow (\Delta \mid\sim \beta \Leftrightarrow \Delta, \alpha \mid\sim \beta)$.

The semantic and the syntactic approaches were connected in [119], where a representation theorem was proved, showing that the system P (which is stronger than Gabbay's system) corresponds to "smooth" preferential structures. System P consists of

1. $\phi \mid\sim \psi, \phi \mid\sim \psi' \Rightarrow \phi \mid\sim \psi \wedge \psi'$,
2. $\phi \mid\sim \psi, \phi' \mid\sim \psi \Rightarrow \phi \vee \phi' \mid\sim \psi$,
3. $\vdash \phi \leftrightarrow \phi' \Rightarrow (\phi \mid\sim \psi \Leftrightarrow \phi' \mid\sim \psi)$,
4. $\phi \mid\sim \psi, \vdash \psi \rightarrow \psi' \Rightarrow \phi \mid\sim \psi'$,
5. $\vdash \phi \rightarrow \phi' \Rightarrow \phi \mid\sim \phi'$,
6. $\phi \mid\sim \psi \Rightarrow (\phi \mid\sim \psi' \Leftrightarrow \phi \wedge \psi \mid\sim \psi')$.

where $\vdash$ is classical provability.

Details can be found in Definition 12.9.

To the authors' knowledge, a precise semantics for Gabbay's system was still lacking. We will give it here, applying the idea of IBRS, which allow us to "break" the relations of preferential structures, and thus their strong coherence conditions, as was already illustrated above.

We will also see that the usual definition of "smoothness" - for every arrow from a to b, there has to be an arrow from a' to b, where a' is a minimal element - is stronger than needed, and a weaker variant, "essential smoothness" (see Definition 12.30 is sufficient to generate the desired property of cumulativity on the logical side.

12.1.2 Introduction to IBRS

The human agent in his daily activity has to deal with many situations involving change. Chief among them are the following:

1. Common sense reasoning from available data. This involves prediction of what unavailable data is supposed to be (non-monotonic deduction) but it is a defeasible prediction, geared towards immediate change. This is formally known as non-monotonic reasoning and is studied by the non-monotonic community.
2. Belief revision, studied by a very large community. The agent is unhappy with the totality of his beliefs which he finds internally unacceptable (usually logically inconsistent but not necessarily so) and needs to change/revise it.
3. Receiving and updating his data, studied by the update community.
4. Making morally correct decisions, studied by the deontic logic community.
5. Dealing with hypothetical and counterfactual situations. This is studied by a large community of philosophers and AI researchers.
6. Considering temporal future possibilities; this is covered by modal and temporal logic.
7. Dealing with properties that persist through time in the near future and with reasoning that is constructive. This is covered by intuitionistic logic.

All the above types of reasoning exist in the human mind and are used continuously and coherently every hour of the day. The formal modelling of these types is done by diverse communities which are largely distinct with no significant communication or cooperation. The formal models they use are very similar and arise from a more general theory, what we might call: "Reasoning with information-bearing binary relations".

Definition 12.1.

1. *An information-bearing binary relation frame IBR, has the form* $(S, \mathfrak{R})$, *where* S *is a nonempty set and* $\mathfrak{R}$ *is a subset of* S_ω, *where* S_ω *is defined by induction as follows:*
   a) $S_0 := S$
   b) $S_{n+1} := S_n \cup (S_n \times S_n)$

c) $S_\omega = \bigcup\{S_n : n \in \omega\}$.

We call elements from S *points or nodes, and elements from* $\mathfrak{R}$ *arrows. Given* $(S, \mathfrak{R})$, *we also set* $\boldsymbol{P}((S, \mathfrak{R})) := S$ *and* $\boldsymbol{A}((S, \mathfrak{R})) := \mathfrak{R}$.

If α *is an arrow, the origin and destination of* α *are defined as usual, and we write* $\alpha : x \to y$ *when* x *is the origin and* y *the destination of the arrow* α. *We also write* $o(\alpha)$ *and* $d(\alpha)$ *for the origin and destination of* α.

2. *Let* Q *be a set of atoms, and* $\boldsymbol{L}$ *be a set of labels (usually* $\{0, 1\}$ *or* $[0, 1]$). *An information assignment* h *on* $(S, \mathfrak{R})$ *is a function* $h : Q \times \mathfrak{R} \to \boldsymbol{L}$.
3. *An information-bearing system IBRS has the form* $(S, \mathfrak{R}, h, Q, \boldsymbol{L})$, *where* S, $\mathfrak{R}$, h, Q, $\boldsymbol{L}$ *are as above.*

See Figure 12.2 for an illustration.

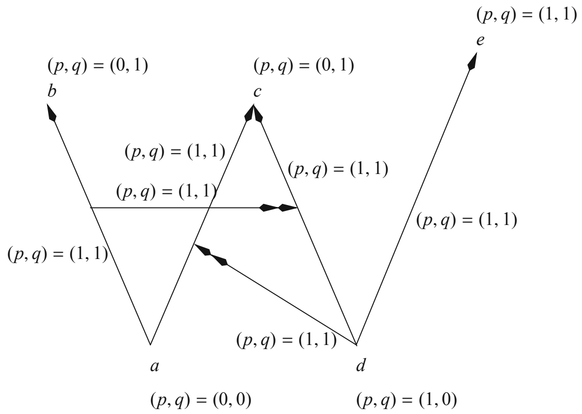

Fig. 12.2: A simple example of an information-bearing system

We have here:

$$S = \{a, b, c, d, e\},$$
$$\mathfrak{R} = S \cup \{(a, b), (a, c), (d, c), (d, e)\} \cup \{((a, b), (d, c)), (d, (a, c))\},$$
$$Q = \{p, q\}.$$

The values of h for p and q are as indicated in the figure. For example $h(p, (d, (a, c))) = 1$.

Thus, IBRS can be used as a source of information for various logics based on the atoms in Q. We now illustrate by listing several such logics.

Modal logic

One can consider the figure as giving rise to two modal logic models, one with actual world a and one with d, these being the two minimal points of the relation. Consider a language with $\Box q$. How do we evaluate $a \models \Box q$?

The modal logic will have to give an algorithm for calculating the values.

Say we choose algorithm $\mathcal{A}_1$ for $a \models \Box q$, namely:

[$\mathcal{A}_1(a, \Box q) = 1$] iff for all $x \in S$ such that $a = x$ or $(a, x) \in \mathfrak{R}$ we have $h(q, x) = 1$.

According to $\mathcal{A}_1$ we get that $\Box q$ is false at a. $\mathcal{A}_1$ gives rise to a T-modal logic. Note that the reflexivity is not anchored in the relation $\mathfrak{R}$ of the network but in the algorithm $\mathcal{A}_1$ in the way we evaluate. We say $(S, \mathfrak{R}, \ldots.) \models \Box\, q$ iff $\Box q$ holds at all minimal points of $(S, \mathfrak{R})$.

For orderings without minimal points we may choose a subset of distinguished points.

Non-monotonic deduction

We can ask whether $p \mid\!\sim q$ according to algorithm $\mathcal{A}_2$ defined below. $\mathcal{A}_2$ says that $p \mid\!\sim q$ holds iff q holds in all minimal models of p. Let us check the value of $\mathcal{A}_2$ in this case:

Let $S_p = \{s \in S \mid h(p, s) = 1\}$. Thus $S_p = \{d, e\}$.

The minimal points of S_p are $\{d\}$. Since $h(q, d) = 0$, we have that $p \not\mid\!\sim q$.

Note that in the cases of modal logic and non-monotonic logic we ignored the arrows $(d, (a, c))$ (i.e. the double arrow from d to the arrow (a, c)) and the h values to arrows. These values do not play a part in the traditional modal or non-monotonic logic. They do play a part in other logics. The attentive reader may already suspect that we have here an opportunity for generalisation of, say, non-monotonic logic, by giving a role to arrow annotations.

Argumentation nets

Here the nodes of S are interpreted as arguments. The atoms $\{p, q\}$ can be interpreted as types of arguments and the arrows, e.g. $(a, b) \in \mathfrak{R}$, as indicating that the argument a is attacking the argument b.

So, for example, let

a = We must win votes.
b = Death sentence for murderers.
c = We must allow abortion for teenagers.
d = Bible forbids taking of life.
q = The argument is a social argument.
p = The argument is a religious argument.
$(d, (a, c))$ = There should be no connection between winning votes and abortion.

$((a,b),(d,c))$ = If we attack the death sentence in order to win votes then we must stress (attack) that there should be no connection between religion (Bible) and social issues.

Thus we have according to this model that supporting abortion can lose votes. The argument for abortion is a social one and the argument from the Bible against it is a religious one.

We can extract information from this IBRS using two algorithms. The modal logic one can check whether for example every social argument is attacked by a religious argument. The answer is no, since the social argument b is attacked only by a which is not a religious argument.

We can also use algorithm $\mathcal{A}_3$ (following Dung) to extract the winning arguments of this system. The arguments a and d are winning since they are not attacked. d attacks the connection between a and c (i.e. stops a attacking c).

The attack of a on b is successful and so b is out. However the arrow (a,b) attacks the arrow (d,c). So c is not attacked at all as both arrrows leading into it are successfully eliminated. So c is in. e is out because it is attacked by d.

So the winning arguments are $\{a,c,d\}$.

In this model we ignore the annotations on arrows. To be consistent in our mathematics we need to say that h is a partial function on $\mathfrak{R}$. The best way is to give a more specific definition of IBRS to make it suitable for each logic.

See also [66] and [14].

Counterfactuals

The traditional semantics for counterfactuals involves closeness of worlds. The clause $y \models p \hookrightarrow q$, where $\hookrightarrow$ is a counterfactual implication is that q holds in all worlds y' "near enough" to y in which p holds. So if we interpret the annotation on arrows as distances then we can define "near" as distance ≤ 2, and we get: $a \models p \hookrightarrow q$ iff in all worlds of p-distance ≤ 2 if p holds so does q. Note that the distance depends on p.

In this case we get that $a \models p \hookrightarrow q$ holds. The distance function can also use the arrows from arrows to arrows, etc. There are many opportunities for generalisation in our IBRS setup.

Intuitionistic persistence

We can get an intuitionistic Kripke model out of this IBRS by letting, for $t,s \in S$, $t\rho_0 s$ iff $t = s$ or $[tRs \wedge \forall q \in Q(h(q,t) \leq h(q,s))]$. We get that

[$r_0 = \{(y,y) \mid y \in S\} \cup \{(a,b),(a,c),(d,e)\}$.]

Let ρ be the transitive closure of ρ_0. Algorithm $\mathcal{A}_4$ evaluates $p \Rightarrow q$ in this model, where $\Rightarrow$ is intuitionistic implication.

$\mathcal{A}_4$: $p \Rightarrow q$ holds at the IBRS iff $p \Rightarrow q$ holds intuitionistically at every ρ-minimal point of (S,ρ).

12.1.3 Purpose of this chapter

In this chapter, we will not cover all these applications of the above abstract definition of IBRS, but will

(1) give an abstract and also a very concrete semantics to IBRS

(2) show that a special case of IBRS generalizes preferential semantics in a very natural way and solves open representation problems for weak logical systems. This is possible, as we can "break" the strong coherence properties of preferential structures by higher arrows, i.e. arrows, which do not go to points, but to arrows themselves.

The notion of IBRS is within the Reactive approach in applied logic. The reader will find more on the Reactive approach in Section 12.1.4.

12.1.4 Concluding discussion - the reactive idea

This section introduces the reactive idea for algorithmic systems. This idea is the intuition behind the general notion of IBRS. We will explain it by looking at a fairly general example involving possible world semantics.

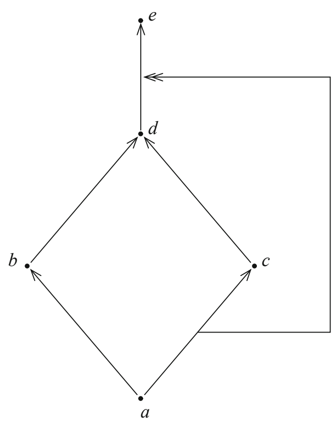

Fig. 12.3

Consider the situation in Figure 12.3. Consider this figure with a starting point at state a. The arrows indicate non-deterministic options for moving along the graph. From a we can reach d in two ways, one through b and one through c. If we move to d through c, passing from a to c (through the arc $a \to c$) sends a signal through the double arrow that disconnects the arc $a \to e$. The idea of these double arrows is the reactive idea.

We use an algorithmic system, and depending on the path the algorithm takes, the system changes.

Many systems can be represented in a graph form and therefore may become reactive.

Here are some examples.

1. Kripke semantics, see Chapter 2. The model can change as we traverse the nodes in the evaluation process.
2. Context free grammars, see Chapter 9. The set of rules available to us can change depending which rules we use (i.e. rules can be active or non-active depending on the use of the other rules).
3. Finite automata, see Chapter 8. The automaton can change its transition table as it changes state.
4. Reactive tableaux or proof theory. The rules can change as some rules are applied.
5. The system of this chapter. With preferential models, a set A has minimal points. When A is enlarged to A', more points and more preferential connections are added but no existing connections are cancelled. Thus minimal points can only be cancelled by addition. The reactive double arrow idea allows us to cancel connections when we enlarge. Figure 12.2 illustrates the situation.

Intuitively, it is easy to see that any clear-cut algorithmic system can be made reactive. The system has to tell us what it means to make a move and so we can change the system as we make such a move. The following question may be asked: can we gain anything by introducing reactivity? Is there any gain in expressive power in addition to a better intuitive representation? The answer is yes.

1. There are logical systems with one modal operator $\Box$ which can be characterized by a class of reactive Kripke frames but cannot be characterized by any class of ordinary Kripke frames.
2. There is no context-free grammar which can recognize the set of words of the form $a^n b^n c^n$, $n \in \omega$. However, there is a reactive context-free grammar which can do the job.
3. Given an ordinary finite automaton with k^n (respectively $k * n$) states, there exists a reactive automaton with $k - n$ (respectively $k + n$) states which recognizes the same inputs.

Let us now gain a bit more insight into what is going on in the Kripke model case, especially with regards to tableaux and axiomatisations. This should be of interest to our readers, since this chapter deals with (preferential) possible worlds.

Consider again Figure 12.3. What we see at world d depends on how we got there. So Figure 12.3 is equivalent to Figure 12.4. The points here are paths. The rectangle around abd and acd indicates these points should have the same assignment (labels) for atoms. We can indicate this by an equivalence relation $\equiv$, $abd \equiv acd$.

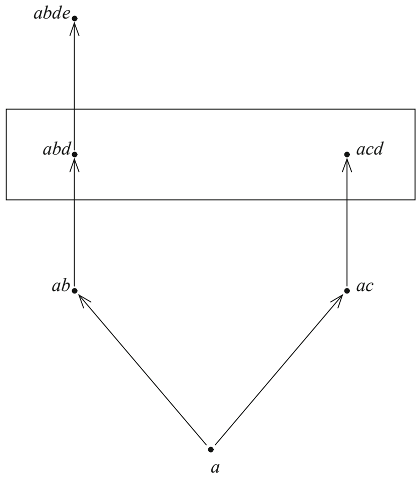

Fig. 12.4

Thus essentially a reactive Kripke model with one-level double arrows for one modality $\Box\phi$ is like an ordinary Kripke model with two modalities $\Box\phi$ and $K\phi$ (necessity and knowledge). This is so because of the unfolding of Figure 12.3 to Figure 12.4. Knowledge is represented by an equivalence relation $\equiv$, $t \models K\phi$ iff for all $s \equiv t$, we have $s \models \phi$. Thus in a reactive Kripke model for $\Box$ with a starting point t_0, we can model knowledge by $K\phi$ holds at t iff ϕ holds at t no matter how we get to t from t_0.

In fact for every t_0 we get a different knowledge operator $K(t_0)$.

How about proof theory? In Figure 12.3 let us assume that q_x holds at x. Thus at a we have

1. $\Diamond\Diamond\Box\bot$ (path through c)
2. $\Diamond\Diamond\Diamond q_e$ (path through b)
3. $\Diamond\Diamond q \to \Box\Box q$ for all atomic q (because there is only one point d).

These hold in Figure 12.3 as a frame. They don't hold in Figure 12.4 as a frame, only if we restrict the assignment for atoms.

12.2 A semantics for IBRS

12.2.1 Introduction

We give here a rough outline of a formal semantics for IBRS. It consists more of some hints where difficulties are, than a finished construction. Still, the authors feel that this is sufficient to complete the work, and, on the other hand, our remarks might be useful to those who intend to finalize the construction.

(1) Nodes and arrows

As we may have counterarguments not only against nodes, but also against arrows, they must be treated basically the same way, i.e. in some way there has to be a positive, but also a negative influence on both. So arrows cannot just be concatenation between the contents of nodes.

We will differentiate between nodes and arrows by labelling arrows in addition with a time delay. We see nodes as situations, where the output is computed instantanously from the input, whereas arrows describe some "force" or "mechanism" which may need some time to "compute" the result from the input.

Consequently, if α is an arrow, and β an arrow pointing to α, then it should point to the input of α, i.e. before the time lapse. Conversely, any arrow originating in α should originate after the time lapse.

Apart from this distinction, we will treat nodes and arrows the same way, so the following discussion will apply to both - which we call just "objects".

(2) Defeasibility

The general idea is to code each object, say X, by $I(X) : U(X) \to C(X)$: If $I(X)$ holds then, unless $U(X)$ holds, consequence $C(X)$ will hold. (We adopted Reiter's notation for defaults, as IBRS have common points with the former.)

The situation is slightly more complicated, as there can be several counterarguments, so $U(X)$ really is an "or". Likewise, there can be several supporting arguments, so $I(X)$ also is an "or".

A counterargument must not always be an argument against a specific supporting argument, but it can be. Thus, we should admit both possibilities. As we can use arrows to arrows, the second case is easy to treat (as is the dual, a supporting argument can be against a specific counterargument). How do we treat the case of unspecific pro- and counterarguments? Probably the easiest way is to adopt Dung's idea: an object is in, if it has at least one support, and no counterargument — see [45]. Of course, other possibilities may be adopted, counting, use of labels, etc., but we just consider the simple case here.

(3) Labels

In the general case, objects stand for some kind of defeasible transmission. We may in some cases see labels as restricting this transmission to certain values. For instance, if the label is $p = 1$ and $q = 0$, then the p-part may be transmitted and the q-part not.

Thus, a transmission with a label can sometimes be considered as a family of transmissions; which ones are active is indicated by the label.

Example 12.2.

In fuzzy Kripke models, labels are elements of [0, 1]. $p = 0.5$ as label for a node m' which stands for a fuzzy model means that the value of p is 0.5. $p = 0.5$ as label for an arrow from m to m' means that p is transmitted with value 0.5. Thus, when we look from m to m', we see p with value $0.5 * 0.5 = 0.25$. So, we have $\Diamond p$ with value 0.25 at m - if, e.g. m, m' are the only models.

(4) Putting things together

If an arrow leaves an object, the object's output will be connected to the (only) positive input of the arrow. (An arrow has no negative inputs from objects it leaves.) If a positive arrow enters an object, it is connected to one of the positive inputs of the object, analogously for negative arrows and inputs.

When labels are present, they are transmitted through some operation.

In slightly more formal terms, we have:

Definition 12.3. *In the most general case, objects of IBRS have the form:* $(\langle I_1, L_1\rangle, \ldots, \langle I_n, L_n\rangle) : (\langle U_1, L'_1\rangle, \ldots, \langle U_n, L'_n\rangle)$, *where the* L_i, L'_i *are labels and the* I_i, U_i *might be just truth values, but can also be more complicated, a (possibly infinite) sequence of some values. Connected objects have, of course, to have corresponding sequences. In addition, the object X has a criterion for each input, whether it is valid or not (in the simple case, this will just be the truth value "TRUE"). If there is at least one positive valid input* I_i, *and no valid negative input* U_i, *then the output* $C(X)$ *and its label are calculated on the basis of the valid inputs and their labels. If the object is an arrow, this will take some time, t, otherwise, this is instantaneous.*

Evaluating a diagram

An evaluation is relative to a fixed input, i.e. some objects will be given certain values, and the diagram is left to calculate the others. It may well be that it oscillates, i.e. shows a cyclic behaviour. This may be true for a subset of the diagram, or the whole diagram. If it is restricted to an unimportant part, we might neglect this. Whether it oscillates or not can also depend on the time delays of the arrows (see Example 12.4.

We therefore define for a diagram Δ

$\alpha \mid\sim_\Delta \beta$ iff

(a) α is a (perhaps partial) input — where the other values are set "not valid"

(b) β is a (perhaps partial) output

(c) after some time, β is stable, i.e. all still possible oscillations do not affect β

(d) the other possible input values do not matter, i.e. whatever the input, the result is the same.

In the cases examined here more closely, all input values will be defined.

12.2.2 A circuit semantics for simple IBRS without labels

It is natural to implement IBRS by (modified) logical circuits. The nodes will be implemented by sub-circuits which can store information, and the arcs by connections between them. As connections can connect connections, connections will not just be simple wires. The objective of the present discussion is to warn the reader that care has to be taken about the temporal behaviour of such circuits, especially when feedback is allowed.

All details are left to those interested in a realization.

Background: It is standard to implement the usual logical connectives by electronic circuits. These components are called gates. Circuits with feedback sometimes show undesirable behaviour when the initial conditions are not specified. (When we switch a circuit on, the outputs of the individual gates can have arbitrary values.) The technical realization of these initial values shows the way to treat defaults. The initial values are set via resistors (on the order of 1 $k\Omega$) between the point in the circuit we want to initialize and the desired tension (say 0 Volt for FALSE, 5 Volt for TRUE). They are called pull-down or pull-up resistors (for default 0 or 5 Volt). When a "real" result comes in, it will override the tension applied via the resistor.

Closer inspection reveals that we have here a three-level default situation: The initial value will be the weakest, which can be overridden by any "real" signal, but a positive argument can be overridden by a negative one. Thus, the biggest resistor will be for the initialization, the smaller ones for the supporting arguments, and the negative arguments have full power. Technical details will be left to the experts.

We give now an example which shows that the delays of the arrows can matter. In one situation, a stable state is reached, in another, the circuit begins to oscillate.

Example 12.4. (In engineering terms, this is a variant of a JK flip-flop with $R * S = 0$, a circuit with feedback.)

We have eight measuring points.

$In1, In2$ are the overall input, $Out1, Out2$ the overall output, $A1, A2, A3, A4$ are auxiliary internal points. All points can be TRUE or FALSE.

The logical structure is as follows:

$A1 = In1 \wedge Out1$, $A2 = In2 \wedge Out2$,

$A3 = A1 \vee Out2$, $A4 = A2 \vee Out1$,

$Out1 = \neg A3$, $Out2 = \neg A4$.

Thus, the circuit is symmetrical, with $In1$ corresponding to $In2$, $A1$ to $A2$, $A3$ to $A4$, $Out1$ to $Out2$.

The input is held constant. See Figure 12.5.

We suppose that the output of the individual gates is present n time slices after the input was present. In the first circuit n will be equal to 1 for all gates, in the second circuit equal to 1 for all but the AND gates, which will take 2 time slices. Thus, in both cases, e.g. $Out1$ at time t will be the negation of $A3$ at time $t-1$. In the first case, $A1$ at time t will be the conjunction of $In1$ and $Out1$ at time $t-1$, and in the second case the conjunction of $In1$ and $Out1$ at time $t-2$.

We initialize $In1$ as TRUE, all others as FALSE. (The initial value of $A3$ and $A4$ does not matter, the behaviour is essentially the same for all such values.)

The first circuit will oscillate with a period of 4, the second circuit will go to a stable state.

We have the following transition tables (time slice shown at left):

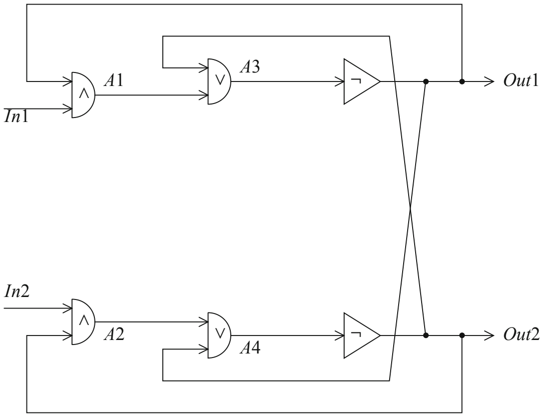

Fig. 12.5: Gate semantics

Table 12.1: Oscillating variant

	*In*1	*In*2	*A*1	*A*2	*A*3	*A*4	*Out*1	*Out*2	
1:	T	F	F	F	F	F	F	F	
2:	T	F	F	F	F	F	T	T	
3:	T	F	T	F	T	T	T	T	
4:	T	F	T	F	T	T	F	F	
5:	T	F	F	F	T	F	F	F	oscillation starts
6:	T	F	F	F	F	F	F	T	
7:	T	F	F	F	T	F	T	T	
8:	T	F	T	F	T	T	F	T	
9:	T	F	F	F	T	F	F	F	back to start of oscillation

Circuit 1, *delay* = 1 everywhere:

Circuit 2, *delay* = 1 everywhere, except for AND with *delay* = 2.

(Thus, *A*1 and *A*2 are held at their intial value up to time 2, then they are calculated using the values of time $t-2$.)

Table 12.2: No oscillation

	*In*1	*In*2	*A*1	*A*2	*A*3	*A*4	*Out*1	*Out*2	
1:	T	F	F	F	F	F	F	F	
2:	T	F	F	F	F	F	T	T	
3:	T	F	F	F	T	T	T	T	
4:	T	F	T	F	T	T	F	F	
5:	T	F	T	F	T	F	F	F	
6:	T	F	F	F	T	F	F	T	stable state reached
7:	T	F	F	F	T	F	F	T	
8:	T	F	F	F	T	F	F	T	

Note that state 6 of circuit 2 is also stable in circuit 1, but it is never reached in that circuit.

12.3 IBRS as generalized preferential structures

12.3.1 Introduction

Overview

The main objective of this Section is to show independence results for certain logical and algebraic rules, which are usually treated together in the system P (see below for P and other concepts mentioned here), and which are all valid in usual preferential structures.

The basic idea is to use generalized preferential structures, i.e. special IBRS, to break the strong coherence conditions of usual preferential structures. This allows us to work without the basic condition (μPR), and we are left with only ($\mu \subseteq$), or can add independently (μCUM), respectively Cumulativity or Cautious Monotony. Thus we can give, for the first time to our knowledge, a structural semantics to certain weak systems of nonmonotonic logics.

We will also modify the usual smoothness condition to "essential smoothness", sufficient for obtaining Cautious Monotony, and thus contribute to a clarification of the concept of smoothness.

In a certain way, our present work goes the opposite way to our [74], where we showed strong conceptual connections between rules like (AND), (OR), Cautious Monotony, etc. There, we worked with an abstract algebraic semantics based on manipulation of the concept of size, here we work with the structural semantics of (generalized) preferential structures. Thus, the present chapter and [74] can be seen as complementary.

To summarize, the main result of this chapter is to show how to work semantically without (μPR), and in particular, how to have (μCUM) and related properties without (μPR). The corresponding logical properties are then easily shown.

Basic definitions, fundamental results

Definition 12.5. *1. We use* $:=$ *and* $:\Leftrightarrow$ *to define the left-hand side by the right-hand side, as in the following two examples:*
$X := \{x\}$ *defines* X *as the singleton with element* x*.*
$X < Y :\Leftrightarrow \forall x \in X \forall y \in Y(x < y)$ *extends the relation* $<$ *from elements to sets.*

2. *We use* $\mathcal{P}$ *to denote the power set operator.*
$\Pi\{X_i : i \in I\} := \{g : g : I \to \bigcup\{X_i : i \in I\}, \forall i \in I.g(i) \in X_i\}$ *is the general Cartesian product,* $X \times X'$ *is the binary Cartesian product.*
$card(X)$ *shall denote the cardinality of* X*, and* V *the set-theoretic universe we work in — the class of all sets.*
Given a set of pairs $\mathcal{X}$*, and a set* X*, we let* $\mathcal{X} \upharpoonright X := \{\langle x, i\rangle \in \mathcal{X} : x \in X\}$*. (When the context is clear, we will sometimes simply write* X *for* $\mathcal{X} \upharpoonright X$*.)*
We will use the same notation $\upharpoonright$ *to denote the restriction of functions and in particular of sequences to a subset of the domain.*
If Σ *is a set of sequences over an index set* X*, and* $X' \subseteq X$*, we will abuse notation and write* $\Sigma \upharpoonright X'$ *for* $\{\sigma \upharpoonright X' : \sigma \in \Sigma\}$*.*
Concatenation of sequences, e.g. of σ *and* σ'*, will be denoted by juxtaposition:* $\sigma\sigma'$*.*

3. $A \subseteq B$ *will denote that* A *is a subset of* B *or equal to* B*, and* $A \subset B$ *that* A *is a proper subset of* B*; likewise for* $A \supseteq B$ *and* $A \supset B$*.*
Given some fixed set U *we work in, and* $X \subseteq U$*,* $\mathbf{C}(X) := U - X$*.*

4. *If* $\mathcal{Y} \subseteq \mathcal{P}(X)$ *for some* X*, we say that* $\mathcal{Y}$ *satisfies*
$(\cap)$ *iff it is closed under finite intersections,*
$(\bigcap)$ *iff it is closed under arbitrary intersections,*
$(\cup)$ *iff it is closed under finite unions,*
$(\bigcup)$ *iff it is closed under arbitrary unions,*
$(\mathbf{C})$ *iff it is closed under complementation,*
$(-)$ *iff it is closed under set difference.*

5. *We will sometimes write* $A = B \parallel C$ *for:* $A = B$*, or* $A = C$*, or* $A = B \cup C$*.*
We make ample and tacit use of the Axiom of Choice.

Definition 12.6. $<^*$ *will denote the transitive closure of the relation* $<$ *. If* $<$*,* $\prec$*, or a similar relation is given,* $a \perp b$ *will express that* a *and* b *are* $<$*-incomparable (or* $\prec$*-incomparable); context will tell. Given any relation* $<$*,* $\leq$ *will stand for* $<$ *or* $=$*; conversely, given* $\leq$*,* $<$ *will stand for* $\leq$*, but not* $=$*; similarly for* $\prec$*, etc.*

Definition 12.7.

1. V *will be the set of truth values when there are more than the classical ones, TRUE and FALSE.*

We work here in a classical propositional language $\mathcal{L}$; *a theory* T *will be an arbitrary set of formulas. Formulas will often be named* ϕ, ψ, *etc., theories* T, S, *etc.*
$v(\mathcal{L})$ *or simply* L *will be the set of propositional variables of* $\mathcal{L}$.
$F(\mathcal{L})$ *will be the set of formulas of* $\mathcal{L}$.
A propositional model m *will be a function from the set of propositional variables to the set of truth values* V *when we have more than two truth values.*
$M_{\mathcal{L}}$, *or simply* M *when the context is clear, will be the set of (classical) models for* $\mathcal{L}$; $M(T)$ *or* M_T *is the set of models of* T; *likewise for* $M(\phi)$ *for a formula* ϕ.

2. $\boldsymbol{D}_{\mathcal{L}} := \{M(T) : T \text{ a theory in } \mathcal{L}\}$, *the set of definable model sets.*
   Note that, in classical propositional logic, $\emptyset, M_{\mathcal{L}} \in \boldsymbol{D}_{\mathcal{L}}$, $\boldsymbol{D}_{\mathcal{L}}$ *contains singletons, and is closed under arbitrary intersections and finite unions.*
   An operation $f : \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ *for* $\mathcal{Y} \subseteq \mathcal{P}(M_{\mathcal{L}})$ *is called definability preserving,* (dp) *or* (μdp) *for short, iff for all* $X \in \boldsymbol{D}_{\mathcal{L}} \cap \mathcal{Y}$, $f(X) \in \boldsymbol{D}_{\mathcal{L}}$.
   We will also use (μdp) *for binary functions* $f : \mathcal{Y} \times \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ *— as needed for theory revision — with the obvious meaning.*
3. $\vdash$ *will be classical derivability, and*
   $\overline{T} := \{\phi : T \vdash \phi\}$, *the closure of* T *under* $\vdash$.
4. $Con(.)$ *will stand for classical consistency, so* $Con(\phi)$ *will mean that* ϕ *is classically consistent; likewise for* $Con(T)$. $Con(T, T')$ *will stand for* $Con(T \cup T')$, *etc.*
5. *Given a consequence relation* $\mid\!\sim$, *we define*
   $\overline{\overline{T}} := \{\phi : T \mid\!\sim \phi\}$.
   (There is no fear of confusion with $\overline{T}$, *as it just is not useful to close twice under classical logic.)*
6. $T \vee T' := \{\phi \vee \phi' : \phi \in T, \phi' \in T'\}$.
7. *If* $X \subseteq M_{\mathcal{L}}$, *then* $Th(X) := \{\phi : X \models \phi\}$; *likewise for* $Th(m)$, $m \in M_{\mathcal{L}}$. *(*$\models$ *will usually be classical validity.)*

We recollect and note:

Fact 12.8 *Let* $\mathcal{L}$ *be a fixed propositional language,* $\boldsymbol{D}_{\mathcal{L}} \subseteq X$, $\mu : X \to \mathcal{P}(M_{\mathcal{L}})$, *for a* $\mathcal{L}$*-theory* T. *Suppose* $\overline{\overline{T}} = Th(\mu(M_T))$, *let* T, T' *be arbitrary theories, then:*
(1) $\mu(M_T) \subseteq M_{\overline{\overline{T}}}$,
(2) $M_T \cup M_{T'} = M_{T \vee T'}$ *and* $M_{T \cup T'} = M_T \cap M_{T'}$, *moreover, if* T *and* T' *are deductively closed, then also* $M_T \cup M_{T'} = M_{T \cap T'}$, *(3)* $\mu(M_T) = \emptyset \Leftrightarrow \bot \in \overline{\overline{T}}$.
If μ *is definability preserving or* $\mu(M_T)$ *is finite, then the following also hold:*
(4) $\mu(M_T) = M_{\overline{\overline{T}}}$,
(5) $T' \vdash \overline{\overline{T}} \Leftrightarrow M_{T'} \subseteq \mu(M_T)$,
(6) $\mu(M_T) = M_{T'} \Leftrightarrow \overline{T'} = \overline{\overline{T}}$. □

Definition 12.9. *The definitions are given in Tables 12.3, "Logical rules, definitions and connections Part I" and 12.4, "Logical rules, definitions and connections Part*

II", which also show connections between different versions of rules and the semantics. (The tables are split in two, as they would not fit onto one page otherwise.)

Explanation of the tables:

1. *The first table gives the basic properties, the second table those for Cumulativity and Rational Monotony.*
2. *The difference between the first two columns is that the first column treats the formula version of the rule, the second the more general theory (i.e. set of formulas) version.*
3. *"Corr." stands for "Correspondence".*
4. *The third column, "Corr.", is to be understood as follows:*
   Let a logic $\mid\sim$ *satisfy* (LLE) *and* (CCL), *and define a function* $f : \boldsymbol{D}_{\mathcal{L}} \to \boldsymbol{D}_{\mathcal{L}}$ *by* $f(M(T)) := M(\overline{\overline{T}})$. *Then* f *is well defined and satisfies* (μdp), *and* $\overline{\overline{T}} = Th(f(M(T)))$.
   If $\mid\sim$ *satisfies a rule on the left-hand side, then — provided the additional properties noted in the middle for* $\Rightarrow$ *hold, too —* f *will satisfy the property on the right-hand side.*
   Conversely:
   If $f : \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ *is a function, with* $\boldsymbol{D}_{\mathcal{L}} \subseteq \mathcal{Y}$, *and we define a logic* $\mid\sim$ *by* $\overline{\overline{T}} := Th(f(M(T)))$, *then* $\mid\sim$ *satisfies* (LLE) *and* (CCL). *If* f *satisfies* (μdp), *then* $f(M(T)) = M(\overline{\overline{T}})$.
   If f *satisfies a property on the right-hand side, then — provided the additional properties noted in the middle for* $\Leftarrow$ *hold, too —* $\mid\sim$ *will satisfy the property on the left-hand side.*
5. *We use the following abbreviations for those supplementary conditions in the "Correspondence" columns:*
   "$T = \phi$*" means that if one of the theories (the one named the same way in Definition 12.9 is equivalent to a formula, we do not need* (μdp).
   $-(\mu dp)$ *stands for "without* (μdp)*".*
6. $A = B \parallel C$ *will abbreviate* $A = B$, *or* $A = C$, *or* $A = B \cup C$.

Further comments:

1. (PR) *is also called infinite conditionalization We choose this name for its central role for preferential structures* (PR) *or* (μPR).
2. *The system of rules* (AND) (OR) (LLE) (RW) (SC) (CP) (CM) (CUM) *is also called system P (for preferential). Adding* $(RatM)$ *gives the system R (for rationality or rankedness).*
   Roughly, smooth preferential structures generate logics satisfying system P, while ranked structures generate logics satisfying system R.
3. *A logic satisfying* (REF), $(ResM)$, *and* (CUT) *is called a consequence relation.*
4. (LLE) *and* (CCL) *will hold automatically, whenever we work with model sets.*
5. (AND) *is obviously closely related to filters, and corresponds to closure under finite intersections.* (RW) *corresponds to upward closure of filters.*

More precisely, validity of both depend on the definition, and the direction we consider.
Given f and $(\mu \subseteq)$, $f(X) \subseteq X$ generates a principal filter, $\{X' \subseteq X : f(X) \subseteq X'\}$, with the definition, if $X = M(T)$, then $T \mid\sim \phi$ iff $f(X) \subseteq M(\phi)$. Validity of (AND) and (RW) are then trivial.
Conversely, we can define for $X = M(T)$
$\mathcal{X} := \{X' \subseteq X : \exists\phi(X' = X \cap M(\phi)$ *and* $T \mid\sim \phi)\}$.
(AND) then makes $\mathcal{X}$ closed under finite intersections, and (RW) makes $\mathcal{X}$ upward closed. In the infinite case this is usually not yet a filter, as not all subsets of X need to be definable this way. In this case, we complete $\mathcal{X}$ by adding all X'' such that there is $X' \subseteq X'' \subseteq X$, $X' \in \mathcal{X}$.
Alternatively, we can define
$\mathcal{X} := \{X' \subseteq X : \bigcap\{X \cap M(\phi) : T \mid\sim \phi\} \subseteq X'\}$.

6. *(SC) corresponds to the choice of a subset.*
7. *(CP) is somewhat delicate, as it presupposes that the chosen model set is nonempty. This might fail in the presence of ever better choices, without ideal ones; the problem is addressed by the limit versions.*
8. *(PR) is an infinitary version of one half of the deduction theorem: Let T stand for ϕ, T' for ψ, and $\phi \wedge \psi \mid\sim \sigma$, so $\phi \mid\sim \psi \rightarrow \sigma$, but $(\psi \rightarrow \sigma) \wedge \psi \vdash \sigma$.*
9. *(CUM) (whose more interesting half in our context is (CM)) may best be seen as a normal use of lemmas: We have worked hard and found some lemmas. Now we can take a rest, and come back again with our new lemmas. Adding them to the axioms will neither add new theorems, nor prevent old ones from holding. (This is, of course, a meta-level argument concerning an object-level rule. But also object-level rules should — at least generally — have an intuitive justification, which will then come from a meta-level argument.)*

The following results are mentioned here to put our work into perspective. The (somewhat lengthy) proofs can be found in [85].

Fact 12.10 *Table 12.5, "Interdependencies of algebraic rules", is to be read as follows: If the left-hand side holds for some function $f : \mathcal{Y} \rightarrow \mathcal{P}(U)$, and the auxiliary properties noted in the middle also hold for f or $\mathcal{Y}$, then the right-hand side will hold, too - and conversely.*

"sing." will stand for: "$\mathcal{Y}$ contains singletons"

Proposition 12.11. *Table 12.6 "Logical and algebraic rules" is to be read as follows:*

The definitions are given in Definition 12.9.

Let a logic $\mid\sim$ satisfy (LLE) and (CCL), and define a function $f : \boldsymbol{D}_{\mathcal{L}} \rightarrow \boldsymbol{D}_{\mathcal{L}}$ by $f(M(T)) := M(\overline{\overline{T}})$. Then f is well defined, satisfies (μdp), and $\overline{\overline{T}} = Th(f(M(T)))$.

If $\mid\sim$ satisfies a rule in the left-hand side, then — provided the additional properties noted in the middle for $\Rightarrow$ hold, too — f will satisfy the property in the right-hand side.

Table 12.3: Logical rules, definitions and connections Part I

Logical rules, definitions and connections Part I			
Logical rule		Corr.	Model set
Basics			
(SC) Supraclassicality $\alpha \vdash \beta \Rightarrow \alpha \mid\!\sim \beta$	(SC) $\overline{T} \subseteq \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	$(\mu \subseteq)$ $f(X) \subseteq X$
(REF) Reflexivity $T \cup \{\alpha\} \mid\!\sim \alpha$			
(LLE) Left Logical Equivalence $\vdash \alpha \leftrightarrow \alpha', \alpha \mid\!\sim \beta \Rightarrow \alpha' \mid\!\sim \beta$	(LLE) $\overline{T} = \overline{T'} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$		(trivally true)
(RW) Right Weakening $\alpha \mid\!\sim \beta, \vdash \beta \rightarrow \beta' \Rightarrow \alpha \mid\!\sim \beta'$	(RW) $T \mid\!\sim \beta, \vdash \beta \rightarrow \beta' \Rightarrow T \mid\!\sim \beta'$		(upward closure)
(wOR) $\alpha \mid\!\sim \beta, \alpha' \vdash \beta \Rightarrow \alpha \vee \alpha' \mid\!\sim \beta$	(wOR) $\overline{\overline{T}} \cap \overline{T'} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ $\Leftarrow$	(μwOR) $f(X \cup Y) \subseteq f(X) \cup Y$
$(disjOR)$ $\alpha \vdash \neg \alpha', \alpha \mid\!\sim \beta, \alpha' \mid\!\sim \beta \Rightarrow \alpha \vee \alpha' \mid\!\sim \beta$	$(disjOR)$ $\neg Con(T \cup T') \Rightarrow \overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ $\Leftarrow$	$(\mu disjOR)$ $X \cap Y = \emptyset \Rightarrow f(X \cup Y) \subseteq f(X) \cup f(Y)$
(CP) Consistency Preservation $\alpha \mid\!\sim \bot \Rightarrow \alpha \vdash \bot$	(CP) $T \mid\!\sim \bot \Rightarrow T \vdash \bot$	$\Rightarrow$ $\Leftarrow$	$(\mu \emptyset)$ $f(X) = \emptyset \Rightarrow X = \emptyset$
			$(\mu \emptyset fin)$ $X \neq \emptyset \Rightarrow f(X) \neq \emptyset$ for finite X
	(AND_1) $\alpha \mid\!\sim \beta \Rightarrow \alpha \not\mid\!\sim \neg \beta$		
	(AND_n) $\alpha \mid\!\sim \beta_1, \ldots, \alpha \mid\!\sim \beta_{n-1} \Rightarrow \alpha \not\mid\!\sim (\neg \beta_1 \vee \ldots \vee \neg \beta_{n-1})$		
(AND) $\alpha \mid\!\sim \beta, \alpha \mid\!\sim \beta' \Rightarrow \alpha \mid\!\sim \beta \wedge \beta'$	(AND) $T \mid\!\sim \beta, T \mid\!\sim \beta' \Rightarrow T \mid\!\sim \beta \wedge \beta'$		(closure under finite intersection)
(CCL) Classical Closure	(CCL) $\overline{\overline{T}}$ classically closed		(trivally true)
(OR) $\alpha \mid\!\sim \beta, \alpha' \mid\!\sim \beta \Rightarrow \alpha \vee \alpha' \mid\!\sim \beta$	(OR) $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ $\Leftarrow$	(μOR) $f(X \cup Y) \subseteq f(X) \cup f(Y)$
$\overline{\overline{\alpha \wedge \alpha'}} \subseteq \overline{\overline{\overline{\alpha}} \cup \{\alpha'\}}$	(PR) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{\overline{T}} \cup T'}}$	$\Rightarrow$ $\Leftarrow (\mu dp) + (\mu \subseteq)$ $\not\Leftarrow \neg(\mu dp)$ $\Leftarrow (\mu \subseteq)$ $T' = \phi$	(μPR) $X \subseteq Y \Rightarrow f(Y) \cap X \subseteq f(X)$
		$\Leftarrow$ $T' = \phi$	$(\mu PR')$ $f(X) \cap Y \subseteq f(X \cap Y)$
(CUT) $T \mid\!\sim \alpha; T \cup \{\alpha\} \mid\!\sim \beta \Rightarrow T \mid\!\sim \beta$	(CUT) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow \overline{\overline{T'}} \subseteq \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	(μCUT) $f(X) \subseteq Y \subseteq X \Rightarrow f(X) \subseteq f(Y)$

Conversely, if $f : \mathcal{Y} \rightarrow \mathcal{P}(M_{\mathcal{L}})$ is a function, with $\boldsymbol{D}_{\mathcal{L}} \subseteq \mathcal{Y}$, and we define a logic $\mid\!\sim$ by $\overline{\overline{T}} := Th(f(M(T)))$, then $\mid\!\sim$ satisfies (LLE) and (CCL). If f satisfies (μdp), then $f(M(T)) = M(\overline{\overline{T}})$.

If f satisfies a property in the right-hand side, then — provided the additional properties noted in the middle for $\Leftarrow$ hold, too — $\mid\!\sim$ will satisfy the property in the left-hand side.

If "$T = \phi$" is noted in the table, this means that, if one of the theories (the one named the same way in Definition 12.9 is equivalent to a formula, we do not need (μdp).

Definition 12.12. *Fix $U \neq \emptyset$, and consider arbitrary X. Note that this X has not necessarily anything to do with U, or the $\mathcal{U}$ below. Thus, the functions $\mu_{\mathcal{M}}$ below are in principle functions from V to V, where V is the set-theoretical universe we work in.*

Note that we work here often with copies of elements (or models). In other areas of logic, most authors work with valuation functions. The two definitions — copies and valuation functions — are equivalent; a copy $\langle x, i \rangle$ can be seen as a state $\langle x, i \rangle$ with valuation x. In the beginning of research on preferential structures, the notion of copies was widely used, whereas, e.g. [121] used that of valuation functions. There is perhaps a weak justification for the former terminology. In modal logic, even if two states have the same valid classical formulas, they might still be distinguishable by their valid modal formulas. But this depends on the modality being in the object

Table 12.4: Logical rules, definitions and connections Part II

Logical rules, definitions and connections Part II			
Logical rule		Corr.	Model set
Cumulativity			
(wCM) $\alpha\mid\sim\beta, \alpha' \vdash \alpha, \alpha \wedge \beta \vdash \alpha' \Rightarrow$ $\alpha'\mid\sim\beta$			
(CM_2) $\alpha\mid\sim\beta, \alpha\mid\sim\beta' \Rightarrow \alpha \wedge \beta \nvdash \neg\beta'$			
(CM_n) $\alpha\mid\sim\beta_1, \ldots, \alpha\mid\sim\beta_n \Rightarrow$ $\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-1} \nvdash \neg\beta_n$			
(CM) Cautious Monotony $\alpha\mid\sim\beta, \alpha\mid\sim\beta' \Rightarrow$ $\alpha \wedge \beta\mid\sim\beta'$	(CM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} \subseteq \overline{\overline{T'}}$	$\Rightarrow$ $\Leftarrow$	(μCM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) \subseteq f(X)$
or $(ResM)$ Restricted Monotony $T\mid\sim\alpha, \beta \Rightarrow T \cup \{\alpha\}\mid\sim\beta$		$\Rightarrow$ $\Leftarrow$	$(\mu ResM)$ $f(X) \subseteq A \cap B \Rightarrow$ $f(X \cap A) \subseteq B$
(CUM) Cumulativity $\alpha\mid\sim\beta \Rightarrow$ $(\alpha\mid\sim\beta' \Leftrightarrow \alpha \wedge \beta\mid\sim\beta')$	(CUM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}}$	$\Rightarrow$ $\Leftarrow$	(μCUM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) = f(X)$
	$(\subseteq\supseteq)$ $T \subseteq \overline{\overline{T'}}, T' \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} = \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	$(\mu \subseteq\supseteq)$ $f(X) \subseteq Y, f(Y) \subseteq X \Rightarrow$ $f(X) = f(Y)$
Rationality			
$(RatM)$ Rational Monotony $\alpha\mid\sim\beta, \alpha\not\mid\sim\neg\beta' \Rightarrow$ $\alpha \wedge \beta'\mid\sim\beta$	$(RatM)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} \supseteq \overline{\overline{\overline{T'}} \cup T}$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu RatM)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) \subseteq f(Y) \cap X$
	$(RatM=)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{\overline{T'}} \cup T}$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu =)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) = f(Y) \cap X$
	$(Log =')$ $Con(\overline{\overline{T'}} \cup T) \Rightarrow$ $\overline{\overline{T \cup T'}} = \overline{\overline{\overline{T'}} \cup T}$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu =')$ $f(Y) \cap X \neq \emptyset \Rightarrow$ $f(Y \cap X) = f(Y) \cap X$
(DR) $\alpha \vee \beta\mid\sim\gamma \Rightarrow$ $\alpha\mid\sim\gamma$ or $\beta\mid\sim\gamma$	$(Log \parallel)$ $\overline{\overline{T \vee T'}}$ is one of $\overline{\overline{T}}$, or $\overline{\overline{T'}}$, or $\overline{\overline{T}} \cap \overline{\overline{T'}}$ (by (CCL))	$\Rightarrow$ $\Leftarrow$	$(\mu \parallel)$ $f(X \cup Y)$ is one of $f(X)$, $f(Y)$ or $f(X) \cup f(Y)$
	$(Log\cup)$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\neg Con(\overline{\overline{T \vee T'}} \cup T')$	$\Rightarrow (\mu\subseteq) + (\mu =)$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$	$(\mu\cup)$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) \cap Y = \emptyset$
	$(Log\cup')$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\overline{\overline{T \vee T'}} = \overline{\overline{T}}$	$\Rightarrow (\mu\subseteq) + (\mu =)$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$	$(\mu\cup')$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) = f(X)$
			$(\mu\in)$ $a \in X - f(X) \Rightarrow$ $\exists b \in X. a \notin f(\{a,b\})$

language. In most work on preferential structures, the consequence relation is outside the object language, so different states with the same valuation are in a stronger sense copies of each other.

1. *Preferential models* or *structures*.
   a) The version without copies:
      A pair $\mathcal{M} := \langle U, \prec\rangle$ with U an arbitrary set and $\prec$ an arbitrary binary relation on U is called a *preferential model* or *structure*.
   b) The version with copies:
      A pair $\mathcal{M} := \langle \mathcal{U}, \prec\rangle$ with $\mathcal{U}$ an arbitrary set of pairs and $\prec$ an arbitrary binary relation on $\mathcal{U}$ is called a *preferential model* or *structure*.
      If $\langle x, i\rangle \in \mathcal{U}$, then x is intended to be an element of U, and i the index of the copy.
      We sometimes also need copies of the relation $\prec$. We will then replace $\prec$ by one or several arrows α attacking non-minimal elements, e.g. $x \prec y$ will be written $\alpha : x \to y$, $\langle x, i\rangle \prec \langle y, i\rangle$ will be written $\alpha : \langle x, i\rangle \to \langle y, i\rangle$, and finally we might have $\langle \alpha, k\rangle : x \to y$ and $\langle \alpha, k\rangle : \langle x, i\rangle \to \langle y, i\rangle$.
2. *Minimal elements*, the functions $\mu_{\mathcal{M}}$.
   a) The version without copies:
      Let $\mathcal{M} := \langle U, \prec\rangle$, and define

Table 12.5: Interdependencies of algebraic rules

Interdependencies of algebraic rules			
Basics			
(1.1)	(μPR)	$\Rightarrow (\cap) + (\mu \subseteq)$	$(\mu PR')$
(1.2)		$\Leftarrow$	
(2.1)	(μPR)	$\Rightarrow (\mu \subseteq)$	(μOR)
(2.2)		$\Leftarrow (\mu \subseteq) + (-)$	
(2.3)		$\Rightarrow (\mu \subseteq)$	(μwOR)
(2.4)		$\Leftarrow (\mu \subseteq) + (-)$	
(3)	(μPR)	$\Rightarrow$	(μCUT)
(4)	$(\mu \subseteq) + (\mu \subseteq\supseteq) + (\mu CUM)$ $+(\mu RatM) + (\cap)$	$\not\Rightarrow$	(μPR)
Cumulativity			
(5.1)	(μCM)	$\Rightarrow (\cap) + (\mu \subseteq)$	$(\mu ResM)$
(5.2)		$\Leftarrow$ (infin.)	
(6)	$(\mu CM) + (\mu CUT)$	$\Leftrightarrow$	(μCUM)
(7)	$(\mu \subseteq) + (\mu \subseteq\supseteq)$	$\Rightarrow$	(μCUM)
(8)	$(\mu \subseteq) + (\mu CUM) + (\cap)$	$\Rightarrow$	$(\mu \subseteq\supseteq)$
(9)	$(\mu \subseteq) + (\mu CUM)$	$\not\Rightarrow$	$(\mu \subseteq\supseteq)$
Rationality			
(10)	$(\mu RatM) + (\mu PR)$	$\Rightarrow$	$(\mu =)$
(11)	$(\mu =)$	$\Rightarrow$	$(\mu PR) + (\mu RatM)$
(12.1)	$(\mu =)$	$\Rightarrow (\cap) + (\mu \subseteq)$	$(\mu =')$
(12.2)		$\Leftarrow$	
(13)	$(\mu \subseteq) + (\mu =)$	$\Rightarrow (\cup)$	$(\mu \cup)$
(14)	$(\mu \subseteq) + (\mu\emptyset) + (\mu =)$	$\Rightarrow (\cup)$	$(\mu \parallel)$, $(\mu \cup')$, (μCUM)
(15)	$(\mu \subseteq) + (\mu \parallel)$	$\Rightarrow (-)$ of $\mathcal{Y}$	$(\mu =)$
(16)	$(\mu \parallel) + (\mu \in) + (\mu PR)+$ $(\mu \subseteq)$	$\Rightarrow (\cup)$ + sing.	$(\mu =)$
(17)	$(\mu CUM) + (\mu =)$	$\Rightarrow (\cup)$ + sing.	$(\mu \in)$
(18)	$(\mu CUM) + (\mu =) + (\mu \subseteq)$	$\Rightarrow (\cup)$	$(\mu \parallel)$
(19)	$(\mu PR) + (\mu CUM) + (\mu \parallel)$	$\Rightarrow$ sufficient, e.g. true in $\boldsymbol{D}_{\mathcal{L}}$	$(\mu =)$.
(20)	$(\mu \subseteq) + (\mu PR) + (\mu =)$	$\not\Rightarrow$	$(\mu \parallel)$
(21)	$(\mu \subseteq) + (\mu PR) + (\mu \parallel)$	$\not\Rightarrow$ (without $(-)$)	$(\mu =)$
(22)	$(\mu \subseteq) + (\mu PR) + (\mu \parallel)+$ $(\mu =) + (\mu\cup)$	$\not\Rightarrow$	$(\mu \in)$ (thus not representable by ranked structures)

$\mu_{\mathcal{M}}(X) := \{x \in X : x \in U \wedge \neg\exists x' \in X \cap U . x' \prec x\}$.
$\mu_{\mathcal{M}}(X)$ is called the set of *minimal elements* of X (in $\mathcal{M}$).
Thus, $\mu_{\mathcal{M}}(X)$ is the set of elements such that there is no smaller one in X.

b) The version with copies:
Let $\mathcal{M} := \langle \mathcal{U}, \prec \rangle$ be as above. Define
$\mu_{\mathcal{M}}(X) := \{x \in X : \exists \langle x, i \rangle \in \mathcal{U}. \neg\exists \langle x', i' \rangle \in \mathcal{U}(x' \in X \wedge \langle x', i' \rangle' \prec \langle x, i \rangle)\}$.
Thus, $\mu_{\mathcal{M}}(X)$ is the projection on the first coordinate of the set of elements such that there is no smaller one in X.
Again, by abuse of language, we say that $\mu_{\mathcal{M}}(X)$ is the set of *minimal elements* of X in the structure. If the context is clear, we will write just μ.
We sometimes say that $\langle x, i \rangle$ "*kills*" or "*minimizes*" $\langle y, j \rangle$ if $\langle x, i \rangle \prec \langle y, j \rangle$. By abuse of language we also say a set X *kills* or *minimizes* a set Y if for all $\langle y, j \rangle \in \mathcal{U}$, $y \in Y$ there exists $\langle x, i \rangle \in \mathcal{U}$, $x \in X$ s.t. $\langle x, i \rangle \prec \langle y, j \rangle$.

Table 12.6: Logical and algebraic rules

Logical and algebraic rules			
Basics			
(1.1)	(OR)	$\Rightarrow$	(μOR)
(1.2)		$\Leftarrow$	
(2.1)	$(disjOR)$	$\Rightarrow$	$(\mu disjOR)$
(2.2)		$\Leftarrow$	
(3.1)	(wOR)	$\Rightarrow$	(μwOR)
(3.2)		$\Leftarrow$	
(4.1)	(SC)	$\Rightarrow$	$(\mu \subseteq)$
(4.2)		$\Leftarrow$	
(5.1)	(CP)	$\Rightarrow$	$(\mu\emptyset)$
(5.2)		$\Leftarrow$	
(6.1)	(PR)	$\Rightarrow$	(μPR)
(6.2)		$\Leftarrow (\mu dp) + (\mu \subseteq)$	
(6.3)		$\not\Leftarrow -(\mu dp)$	
(6.4)		$\Leftarrow (\mu \subseteq)$ $T' = \phi$	
(6.5)	(PR)	$\Leftarrow$ $T' = \phi$	$(\mu PR')$
(7.1)	(CUT)	$\Rightarrow$	(μCUT)
(7.2)		$\Leftarrow$	
Cumulativity			
(8.1)	(CM)	$\Rightarrow$	(μCM)
(8.2)		$\Leftarrow$	
(9.1)	$(ResM)$	$\Rightarrow$	$(\mu ResM)$
(9.2)		$\Leftarrow$	
(10.1)	$(\subseteq\supseteq)$	$\Rightarrow$	$(\mu \subseteq\supseteq)$
(10.2)		$\Leftarrow$	
(11.1)	(CUM)	$\Rightarrow$	(μCUM)
(11.2)		$\Leftarrow$	
Rationality			
(12.1)	$(RatM)$	$\Rightarrow$	$(\mu RatM)$
(12.2)		$\Leftarrow (\mu dp)$	
(12.3)		$\not\Leftarrow -(\mu dp)$	
(12.4)		$\Leftarrow$ $T = \phi$	
(13.1)	$(RatM =)$	$\Rightarrow$	$(\mu =)$
(13.2)		$\Leftarrow (\mu dp)$	
(13.3)		$\not\Leftarrow -(\mu dp)$	
(13.4)		$\Leftarrow$ $T = \phi$	
(14.1)	$(Log =')$	$\Rightarrow$	$(\mu =')$
(14.2)		$\Leftarrow (\mu dp)$	
(14.3)		$\not\Leftarrow -(\mu dp)$	
(14.4)		$\Leftarrow T = \phi$	
(15.1)	$(Log \parallel)$	$\Rightarrow$	$(\mu \parallel)$
(15.2)		$\Leftarrow$	
(16.1)	$(Log\cup)$	$\Rightarrow (\mu \subseteq) + (\mu =)$	$(\mu\cup)$
(16.2)		$\Leftarrow (\mu dp)$	
(16.3)		$\not\Leftarrow -(\mu dp)$	
(17.1)	$(Log\cup')$	$\Rightarrow (\mu \subseteq) + (\mu =)$	$(\mu\cup')$
(17.2)		$\Leftarrow (\mu dp)$	
(17.3)		$\not\Leftarrow -(\mu dp)$	

$\mathcal{M}$ is also called *injective* or 1-copy iff there is always at most one copy $\langle x, i \rangle$ for each x. Note that the existence of copies corresponds to a non-injective labelling function — as is often used in nonclassical logic, e.g. modal logic.

We say that $\mathcal{M}$ is *transitive*, *irreflexive*, etc., iff $\prec$ is.
Note that $\mu(X)$ might well be empty, even if X is not.

Definition 12.13. *We define the consequence relation of a preferential structure for a given propositional language $\mathcal{L}$.*

1. *a) If m is a classical model of a language $\mathcal{L}$, we say by abuse of language*
   $\langle m, i \rangle \models \phi$ *iff* $m \models \phi$,
   and if X is any set of such pairs, that
   $X \models \phi$ *iff for all* $\langle m, i \rangle \in X$ $m \models \phi$.
   b) If $\mathcal{M}$ is a preferential structure, and X is a set of $\mathcal{L}$-models for a classical propositional language $\mathcal{L}$, or is a set of pairs $\langle m, i \rangle$ where the m are such models, we call $\mathcal{M}$ a classical preferential structure or model.
2. *Validity in a preferential structure, or the semantical consequence relation defined by such a structure:*
   Let $\mathcal{M}$ be as above.
   We define
   $T \models_{\mathcal{M}} \phi$ *iff* $\mu_{\mathcal{M}}(M(T)) \models \phi$, *i.e.* $\mu_{\mathcal{M}}(M(T)) \subseteq M(\phi)$.
3. *$\mathcal{M}$ will be called definability preserving iff for all $X \in \boldsymbol{D}_{\mathcal{L}}$ we have $\mu_{\mathcal{M}}(X) \in \boldsymbol{D}_{\mathcal{L}}$.*

As $\mu_{\mathcal{M}}$ is defined on $\boldsymbol{D}_{\mathcal{L}}$, but need by no means always result in some new definable set, this is (and reveals itself as a quite strong) additional property.

Definition 12.14. *Let $\mathcal{Y} \subseteq \mathcal{P}(U)$. (In applications to logic, $\mathcal{Y}$ will be $\boldsymbol{D}_{\mathcal{L}}$.)*

A preferential structure $\mathcal{M}$ is called $\mathcal{Y}$-smooth iff for every $X \in \mathcal{Y}$ every element $x \in X$ is either minimal in X or above an element which is minimal in X. More precisely:

1. *The version without copies:*
   If $x \in X \in \mathcal{Y}$, then either $x \in \mu(X)$ or there is $x' \in \mu(X).x' \prec x$.
2. *The version with copies:*
   If $x \in X \in \mathcal{Y}$, and $\langle x, i \rangle \in \mathcal{U}$, then either there is no $\langle x', i' \rangle \in \mathcal{U}$, $x' \in X$, $\langle x', i' \rangle \prec \langle x, i \rangle$ or there is a $\langle x', i' \rangle \in \mathcal{U}$, $\langle x', i' \rangle \prec \langle x, i \rangle$, $x' \in X$, s.t. there is no $\langle x'', i'' \rangle \in \mathcal{U}$, $x'' \in X$, with $\langle x'', i'' \rangle \prec \langle x', i' \rangle$.
   (Writing down all details here again might make it easier to read applications of the definition later on.)

When considering the models of a language $\mathcal{L}$, $\mathcal{M}$ will be called smooth iff it is $\boldsymbol{D}_{\mathcal{L}}$-smooth; $\boldsymbol{D}_{\mathcal{L}}$ is the default.

Obviously, the richer the set $\mathcal{Y}$ is, the stronger the condition $\mathcal{Y}$-smoothness will be.

A remark for the intuition: Smoothness is perhaps best motivated through Gabbay's concept of reactive diagrams; see, e.g. [66] and [64], and also [85], [73]. In this concept, smaller, or "better", elements attack bigger, or "less good", elements. But when a attacks b, and b attacks c, then one might consider the attack of b against c weakened by the attack of a against b. In a smooth structure, for every attack against some element x, there is also an uncontested attack against x, as it originates in an element y, which is not attacked itself.

The proofs of the following results can best be found in [147].

Table 12.7, "Preferential representation", summarizes the more difficult half of a full representation result for preferential structures. It shows equivalence between certain abstract conditions for model choice functions and certain preferential structures. They are shown in the respective representation theorems.

"Singletons" means that the domain must contain all singletons, "1 copy" or "≥ 1 copy" means that the structure may contain only one copy for each point, or several, and "$(\mu\emptyset)$" for the preferential structure means that the μ-function of the structure has to satisfy this property.

We call a characterization "normal" iff it is a universally quantified Boolean combination (of any fixed, but perhaps infinite, length) of rules of the usual form. We do not go into details here.

In the second column from the left, "$\Rightarrow$" means, for instance for the smooth case, that for any $\mathcal{Y}$ closed under finite unions, and any choice function f which satisfies the conditions in the left-hand column, there is a (here $\mathcal{Y}$-smooth) preferential structure $\mathcal{X}$ which represents it, i.e. for all $Y \in \mathcal{Y}$, $f(Y) = \mu_{\mathcal{X}}(Y)$, where $\mu_{\mathcal{X}}$ is the model choice function of the structure $\mathcal{X}$. The inverse arrow $\Leftarrow$ means that the model choice function for any smooth $\mathcal{X}$ defined on such a $\mathcal{Y}$ will satisfy the conditions on the left.

12.3.2 Generalized preferential structures

Introduction

Definition 12.15. *An IBRS is called a generalized preferential structure iff the origins of all arrows are points. We will usually write x, y, etc. for points, α, β, etc. for arrows.*

Definition 12.16. *Consider a generalized preferential structure $\mathcal{X}$.*

(1) Level n arrow :

Definition by upward induction.

If $\alpha : x \rightarrow y$, x, y are points, then α is a level 1 arrow.

If $\alpha : x \rightarrow \beta$, x is a point, β a level n arrow, then α is a level $n+1$ arrow. ($o(\alpha)$ is the origin, $d(\alpha)$ is the destination of α.)

$\lambda(\alpha)$ will denote the level of α.

(2) Level n structure :

$\mathcal{X}$ is a level n structure iff all arrows in $\mathcal{X}$ are at most level n arrows.

We consider here only structures of some arbitrary but finite level n.

Table 12.7: Preferential representation

Preferential representation				
μ-function		Pref. Structure		Logic
$(\mu \subseteq)$	$\Leftrightarrow$	reactive	$\Leftrightarrow$	$(LLE) + (CCL) + (SC)$
$(\mu \subseteq) + (\mu CUM)$	$\Rightarrow (\cap)$	reactive + essentially smooth		
$(\mu \subseteq) + (\mu \subseteq\supseteq)$	$\Rightarrow$	reactive + essentially smooth	$\Leftrightarrow$	$(LLE) + (CCL) + (SC) + (\subseteq\supseteq)$
$(\mu \subseteq) + (\mu CUM) + (\mu \subseteq\supseteq)$	$\Leftarrow$	reactive + essentially smooth		
$(\mu \subseteq) + (\mu PR)$	$\Leftarrow$	general	$\Rightarrow (\mu dp)$	$(LLE) + (RW) + (SC) + (PR)$
	$\Rightarrow$		$\Leftarrow$	
			$\not\Rightarrow$ without (μdp)	
			$\not\Leftrightarrow$ without (μdp)	any "normal" characterization of any size
$(\mu \subseteq) + (\mu PR)$	$\Leftarrow$	transitive	$\Rightarrow (\mu dp)$	$(LLE) + (RW) + (SC) + (PR)$
	$\Rightarrow$		$\Leftarrow$	
			$\not\Rightarrow$ without (μdp)	
			$\Leftrightarrow$ without (μdp)	using "small" exception sets
$(\mu \subseteq) + (\mu PR) + (\mu CUM)$	$\Leftarrow$	smooth	$\Rightarrow (\mu dp)$	$(LLE) + (RW) + (SC) + (PR) + (CUM)$
	$\Rightarrow (\cup)$		$\Leftarrow (\cup)$	
	$\not\Rightarrow$ without $(\cup)$		$\not\Rightarrow$ without (μdp)	
$(\mu \subseteq) + (\mu PR) + (\mu CUM)$	$\Leftarrow$	smooth+transitive	$\Rightarrow (\mu dp)$	$(LLE) + (RW) + (SC) + (PR) + (CUM)$
	$\Rightarrow (\cup)$		$\Leftarrow (\cup)$	
			$\not\Rightarrow$ without (μdp)	
			$\Leftrightarrow$ without (μdp)	using "small" exception sets
$(\mu \subseteq) + (\mu =) + (\mu PR) + (\mu =') + (\mu \parallel) + (\mu \cup) + (\mu \cup') + (\mu \in) + (\mu RatM)$	$\Leftarrow$	ranked, ≥ 1 copy		
$(\mu \subseteq) + (\mu =) + (\mu PR) + (\mu \cup) + (\mu \in)$	$\not\Rightarrow$	ranked,		
$(\mu \subseteq) + (\mu =) + (\mu \emptyset)$	$\Leftrightarrow, (\cup)$	ranked, 1 copy + $(\mu \emptyset)$		
$(\mu \subseteq) + (\mu =) + (\mu \emptyset)$	$\Leftrightarrow, (\cup)$	ranked, smooth, 1 copy + $(\mu \emptyset)$		
$(\mu \subseteq) + (\mu =) + (\mu \emptyset fin) + (\mu \in)$	$\Leftrightarrow, (\cup)$, singletons	ranked, smooth, ≥ 1 copy + $(\mu \emptyset fin)$		
$(\mu \subseteq) + (\mu PR) + (\mu \parallel) + (\mu \cup) + (\mu \in)$	$\Leftrightarrow, (\cup)$, singletons	ranked, ≥ 1 copy	$\not\Rightarrow$ without (μdp)	$(RatM), (RatM =), (Log\cup), (Log\cup')$
			$\not\Leftrightarrow$ without (μdp)	any "normal" characterization of any size

(3) We define for an arrow α by induction $O(\alpha)$ and $D(\alpha)$.

If $\lambda(\alpha) = 1$, then $O(\alpha) := \{o(\alpha)\}$, $D(\alpha) := \{d(\alpha)\}$.

If $\alpha : x \to \beta$, then $D(\alpha) := D(\beta)$, and $O(\alpha) := \{x\} \cup O(\beta)$.

Thus, for example, if $\alpha : x \to y$, $\beta : z \to \alpha$, then $O(\beta) := \{x, z\}$, $D(\beta) = \{y\}$. Consider also the arrow $\beta := \langle \beta', l' \rangle$ in Figure 12.7. There, $D(\beta) = \{\langle x, i \rangle\}$, $O(\beta) = \{\langle z', m' \rangle, \langle y, j \rangle\}$.

We will not consider here diagrams with arbitrarily high levels. One reason is that diagrams like the following will have an unclear meaning:

Example 12.17. $\langle \alpha, 1 \rangle : x \to y$,
$\langle \alpha, n + 1 \rangle : x \to \langle \alpha, n \rangle$ $(n \in \omega)$.
Is $y \in \mu(X)$?

Definition 12.18. *Let $\mathcal{X}$ be a generalized preferential structure of (finite) level n.*

We define (by downward induction):

(1) Valid $X \mapsto Y$ arrow :

Let $X, Y \subseteq \boldsymbol{P}(\mathcal{X})$.

$\alpha \in \boldsymbol{A}(\mathcal{X})$ is a valid $X \mapsto Y$ arrow iff

(1.1) $O(\alpha) \subseteq X$, $D(\alpha) \subseteq Y$,

(1.2) $\forall \beta : x' \to \alpha.(x' \in X \Rightarrow \exists \gamma : x'' \to \beta.(\gamma$ is a valid $X \mapsto Y$ arrow)).

We will also say that α is a valid arrow in X, or just valid in X, iff α is a valid $X \mapsto X$ arrow.

(2) Valid $X \Rightarrow Y$ arrow :

Let $X \subseteq Y \subseteq \boldsymbol{P}(\mathcal{X})$.

$\alpha \in \boldsymbol{A}(\mathcal{X})$ is a valid $X \Rightarrow Y$ arrow iff

(2.1) $o(\alpha) \in X$, $O(\alpha) \subseteq Y$, $D(\alpha) \subseteq Y$,

(2.2) $\forall \beta : x' \to \alpha.(x' \in Y \Rightarrow \exists \gamma : x'' \to \beta.(\gamma$ is a valid $X \Rightarrow Y$ arrow)).

Thus, any attack β from Y against α has to be countered by a valid attack on β.

(Note that in particular $o(\gamma) \in X$, and that $o(\beta)$ need not be in X, but can be in the bigger Y.)

Remark 12.19. Note that, in the definition of valid $X \mapsto Y$ arrow, X and Y need not be related, but in the definition of valid $X \Rightarrow Y$ arrow, $X \subseteq Y$.

Let us assume now that $X \subseteq Y$, and look at the remaining differences.

In both cases, $D(\alpha) \subseteq Y$.

In the $X \mapsto Y$ case, $O(\alpha) \subseteq X$, and attacks from X are countered.

In the $X \Rightarrow Y$ case, $o(\alpha) \in X$, $O(\alpha) \subseteq Y$, and attacks from Y are countered.

So the first condition is stronger in the $X \mapsto Y$ case, the second in the $X \Rightarrow Y$ case.

Example 12.20.

(1) Consider the arrow $\beta := \langle \beta', l' \rangle$ in Figure 12.7. $D(\beta) = \{\langle x, i \rangle\}$, $O(\beta) = \{\langle z', m' \rangle, \langle y, j \rangle\}$, and the only arrow attacking β originates outside X, so β is a valid $X \mapsto \mu(X)$ arrow.

(2) Consider the arrows $\langle \alpha', k' \rangle$ and $\langle \gamma', n' \rangle$ in Figure 12.8. Both are valid $\mu(X) \Rightarrow X$ arrows.

Example 12.21. See Figure 12.6.

Let $X \subseteq Y$.

Consider the left-hand side of the diagram.

The fact that δ originates in Y, but not in X, establishes that α is not a valid $X \mapsto Y$ arrow, as the condition $O(\alpha) \subseteq X$ is violated. To be a valid $X \Rightarrow Y$ arrow, we have to show that all attacks on α originating from Y (not only from X) are countered by valid $X \Rightarrow Y$ arrows. This holds, as β_1 is countered by γ_1, β_2 by γ_2. All possible attacks on α, γ_1, γ_2 from outside Y, like ρ, need not be considered.

Consider the right-hand side of the diagram.

The fact that there is no valid counterargument to β_2' establishes that α' is not a valid $X \Rightarrow Y$ arrow. It is a valid $X \mapsto Y$ arrow, as counterarguments to α' like β_2', which do not originate in X, are not considered. The counterargument β_1' is considered, but it is destroyed by the valid $X \mapsto Y$ arrow γ_1'.

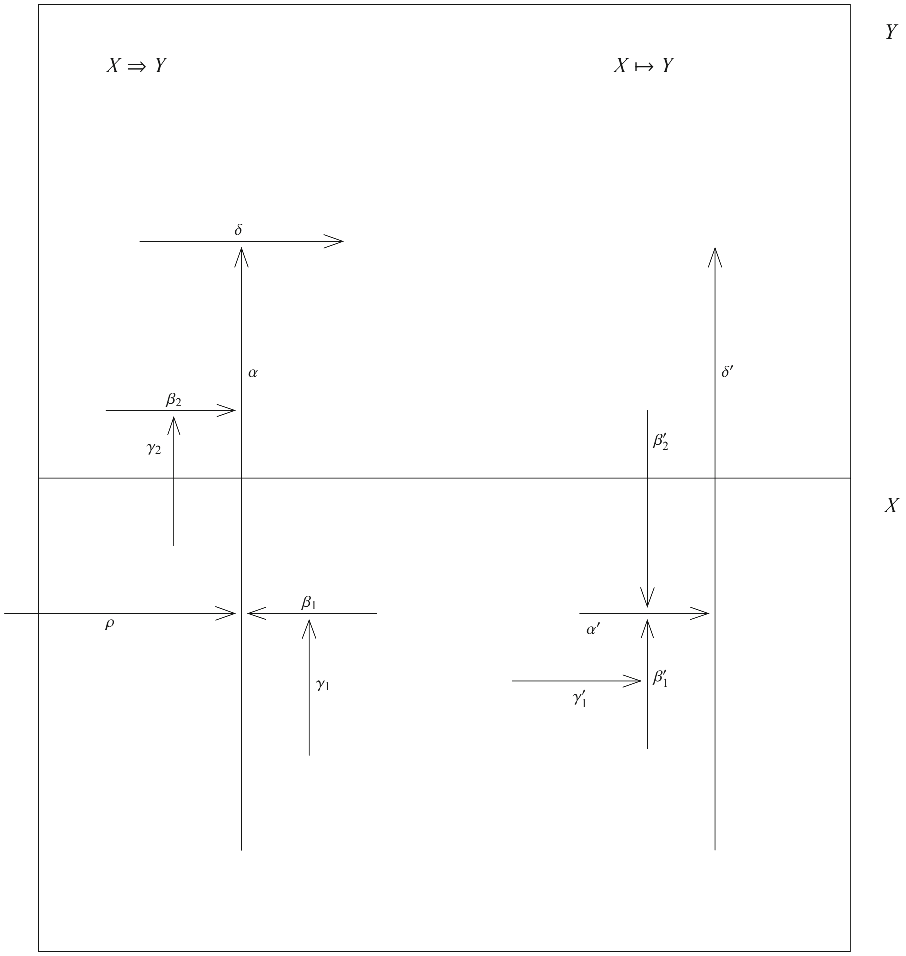

Fig. 12.6

Fact 12.22 *(1) If α is a valid $X \Rightarrow Y$ arrow, then α is a valid $Y \mapsto Y$ arrow.*

(2) If $X \subseteq X' \subseteq Y' \subseteq Y \subseteq \boldsymbol{P}(\mathcal{X})$ and $\alpha \in \boldsymbol{A}(\mathcal{X})$ is a valid $X \Rightarrow Y$ arrow, and $O(\alpha) \subseteq Y'$, $D(\alpha) \subseteq Y'$, then α is a valid $X' \Rightarrow Y'$ arrow.

Proof. Let α be a valid $X \Rightarrow Y$ arrow. We show (1) and (2) together by downward induction (both are trivial).

By the prerequisite, $o(\alpha) \in X \subseteq X'$, $O(\alpha) \subseteq Y' \subseteq Y$, $D(\alpha) \subseteq Y' \subseteq Y$.

Case 1: $\lambda(\alpha) = n$. So α is a valid $X' \Rightarrow Y'$ arrow and a valid $Y \mapsto Y$ arrow.

Case 2: $\lambda(\alpha) = n - 1$. So there is no $\beta : x' \to \alpha$, $y \in Y$, so α is a valid $Y \mapsto Y$ arrow. By $Y' \subseteq Y$, α is a valid $X' \Rightarrow Y'$ arrow.

Case 3: Let the result be shown down to m, $n > m > 1$; let $\lambda(\alpha) = m - 1$. So $\forall \beta : x' \to \alpha(x' \in Y \Rightarrow \exists \gamma : x'' \to \beta$ ($x'' \in X$ and γ is a valid $X \Rightarrow Y$ arrow)). By induction hypothesis γ is a valid $Y \mapsto Y$ arrow, and a valid $X' \Rightarrow Y'$ arrow. So α is a valid $Y \mapsto Y$ arrow, and by $Y' \subseteq Y$, α is a valid $X' \Rightarrow Y'$ arrow.

Definition 12.23. *Let $\mathcal{X}$ be a generalized preferential structure of level n, $X \subseteq \boldsymbol{P}(\mathcal{X})$.*

$\mu(X) := \{x \in X : \exists \langle x, i\rangle . \neg\exists$ *valid* $X \mapsto X$ *arrow* $\alpha : x' \to \langle x, i\rangle\}$.

Definition 12.24. *Let $\mathcal{X}$ be a generalized preferential structure.*

$X \sqsubseteq X'$ *iff*

(1) $X \subseteq X' \subseteq \boldsymbol{P}(\mathcal{X})$,

(2) $\forall x \in X' - X\ \forall \langle x, i\rangle\ \exists \alpha : x' \to \langle x, i\rangle$ (α *is a valid* $X \Rightarrow X'$ *arrow),*

(3) $\forall x \in X\ \exists \langle x, i\rangle$

$(\forall \alpha : x' \to \langle x, i\rangle\ (x' \in X' \Rightarrow \exists \beta : x'' \to \alpha.(\beta$ *is a valid* $X \Rightarrow X'$ *arrow))).*

Note that (3) is not simply the negation of (2):

Consider a level 1 structure. Thus all level 1 arrows are valid, but the source of the arrows must not be neglected.

(2) reads now: $\forall x \in X' - X\ \forall \langle x, i\rangle\ \exists \alpha : x' \to \langle x, i\rangle . x' \in X$

(3) reads: $\forall x \in X\ \exists \langle x, i\rangle\ \neg\exists \alpha : x' \to \langle x, i\rangle . x' \in X'$

This is intended: intuitively, read $X = \mu(X')$, and minimal elements must not be attacked at all, but non-minimals must be attacked from X — which is a modified version of smoothness. More precisely: non-minimal elements (i.e. from $X' - X$) have to be validly attacked from X, minimal elements must not be validly attacked at all from X' (only perhaps from the outside).

Remark 12.25. We note the special case of Definition 12.24 for level 3 structures. We also write it immediately for the intended case $\mu(X) \sqsubseteq X$, and explicitly with copies.

$x \in \mu(X)$ iff

(1) $\exists \langle x, i\rangle \forall \langle \alpha, k\rangle : \langle y, j\rangle \to \langle x, i\rangle$

$(y \in X \Rightarrow \exists \langle \beta', l'\rangle : \langle z', m'\rangle \to \langle \alpha, k\rangle .$

$(z' \in \mu(X) \wedge \neg\exists \langle \gamma', n'\rangle : \langle u', p'\rangle \to \langle \beta', l'\rangle . u' \in X))$.

See Figure 12.7.

$x \in X - \mu(X)$ iff

(2) $\forall \langle x, i\rangle \exists \langle \alpha', k'\rangle : \langle y', j'\rangle \to \langle x, i\rangle$

$(y' \in \mu(X) \wedge$

$(a)\ \neg\exists \langle \beta', l'\rangle : \langle z', m'\rangle \to \langle \alpha', k'\rangle . z' \in X$

or
$(b)\ \forall\langle\beta',l'\rangle : \langle z',m'\rangle \to \langle\alpha',k'\rangle$
$(z' \in X \Rightarrow \exists\langle\gamma',n'\rangle : \langle u',p'\rangle \to \langle\beta',l'\rangle.u' \in \mu(X)))$

See Figure 12.8.

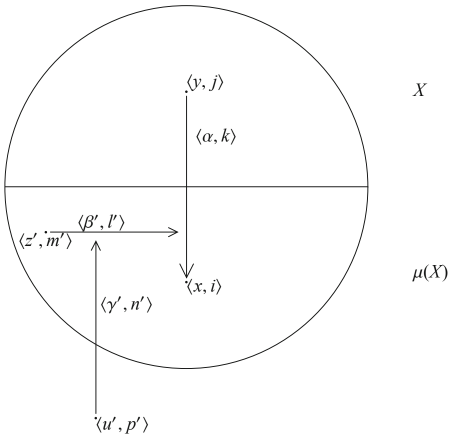

Fig. 12.7: Case 3-1-2

Fact 12.26 *(1) If $X \sqsubseteq X'$, then $X = \mu(X')$,*
(2) $X \sqsubseteq X'$, $X \subseteq X'' \subseteq X' \Rightarrow X \sqsubseteq X''$. (This corresponds to (μCUM).)
(3) $X \sqsubseteq X'$, $X \subseteq Y'$, $Y \sqsubseteq Y'$, $Y \subseteq X' \Rightarrow X = Y$. (This corresponds to $(\mu \subseteq\supseteq)$.)

Proof. (1) Trivial by Fact 12.22 (1).

(2)

We have to show

(a) $\forall x \in X'' - X\ \forall\langle x,i\rangle\ \exists\alpha : x' \to \langle x,i\rangle$ (α is a valid $X \Rightarrow X''$ arrow), and

(b) $\forall x \in X\ \exists\langle x,i\rangle\ (\forall\alpha : x' \to \langle x,i\rangle\ (x' \in X'' \Rightarrow \exists\beta : x'' \to \alpha.(\beta$ is a valid $X \Rightarrow X''$ arrow))).

Both follow from the corresponding condition for $X \Rightarrow X'$, the restriction of the universal quantifier, and Fact 12.22 (2).

(3)

Let $x \in X - Y$.

(a) By $x \in X \sqsubseteq X'$, $\exists\langle x,i\rangle$ s.t. $(\forall\alpha : x' \to \langle x,i\rangle\ (x' \in X' \Rightarrow \exists\beta : x'' \to \alpha.(\beta$ is a valid $X \Rightarrow X'$ arrow))).

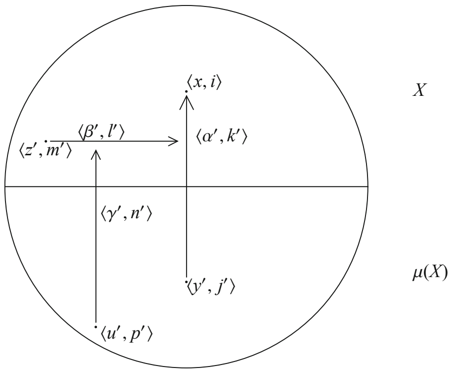

Fig. 12.8: Case 3-2

(b) By $x \notin Y \sqsubseteq \exists \alpha_1 : x' \to \langle x, i \rangle$ α_1 is a valid $Y \Rightarrow Y'$ arrow; in particular $x' \in Y \subseteq X'$. Moreover, $\lambda(\alpha_1) = 1$.

So by (a) $\exists \beta_2 : x'' \to \alpha_1.(\beta_2$ is a valid $X \Rightarrow X'$ arrow), in particular $x'' \in X \subseteq Y'$; moreover $\lambda(\beta_2) = 2$.

It follows by induction from the definition of valid $A \Rightarrow B$ arrows that

$\forall n \exists \alpha_{2m+1}, \lambda(\alpha_{2m+1}) = 2m + 1, \alpha_{2m+1}$ a valid $Y \Rightarrow Y'$ arrow and

$\forall n \exists \beta_{2m+2}, \lambda(\beta_{2m+2}) = 2m + 2, \beta_{2m+2}$ a valid $X \Rightarrow X'$ arrow,

which is impossible, as $\mathcal{X}$ is a structure of finite level.

Definition 12.27. *Let $\mathcal{X}$ be a generalized preferential structure, $X \subseteq \boldsymbol{P}(\mathcal{X})$.*

$\mathcal{X}$ is called totally smooth for X iff

(1) $\forall \alpha : x \to y \in \boldsymbol{A}(\mathcal{X})(O(\alpha) \cup D(\alpha) \subseteq X \Rightarrow \exists \alpha' : x' \to y.x' \in \mu(X))$

(2) if α is valid, then there must also exist such α′ which is valid

(y a point or an arrow).

If $\mathcal{Y} \subseteq \boldsymbol{P}(\mathcal{X})$, then $\mathcal{X}$ is called $\mathcal{Y}$ − totally smooth

iff for all $X \in \mathcal{Y}$ $\mathcal{X}$ is totally smooth for X.

Example 12.28. $X := \{\alpha : a \to b, \alpha' : b \to c, \alpha'' : a \to c, \beta : b \to \alpha'\}$ is not totally smooth,

$X := \{\alpha : a \to b, \alpha' : b \to c, \alpha'' : a \to c, \beta : b \to \alpha', \beta' : a \to \alpha'\}$ is totally smooth.

Example 12.29. Consider $\alpha' : a \to b, \alpha'' : b \to c, \alpha : a \to c, \beta : a \to \alpha$.

Then $\mu(\{a, b, c\}) = \{a\}$, $\mu(\{a, c\}) = \{a, c\}$. Thus, (μCUM) does not hold in this structure. Note that there is no valid arrow from $\mu(\{a, b, c\})$ to c.

Definition 12.30. *Let $\mathcal{X}$ be a generalized preferential structure, $X \subseteq \mathbf{P}(\mathcal{X})$.*

$\mathcal{X}$ is called essentially smooth for X iff $\mu(X) \sqsubseteq X$. If $\mathcal{Y} \subseteq \mathbf{P}(\mathcal{X})$, then $\mathcal{X}$ is called $\mathcal{Y}$ − essentially smooth

iff for all $X \in \mathcal{Y}$ $\mu(X) \sqsubseteq X$.

Example 12.31. It is easy to see that we can distinguish total and essential smoothness in richer structures, as the following Example shows:

We add an accessibility relation R, and consider only those models which are accessible.

Let e.g. $a \to b \to \langle c, 0\rangle, \langle c, 1\rangle$, without transitivity. Thus, only c has two copies. This structure is essentially smooth, but of course not totally so.

Let now mRa, mRb, $mR\langle c, 0\rangle$, $mR\langle c, 1\rangle$, $m'Ra$, $m'Rb$, $m'R\langle c, 0\rangle$.

Thus, seen from m, $\mu(\{a, b, c\}) = \{a, c\}$, but seen from m', $\mu(\{a, b, c\}) = \{a\}$, but $\mu(\{a, c\}) = \{a, c\}$, contradicting (CUM).

Results on not necessarily smooth structures

Example 12.32. We show here $(\mu \subseteq) + (\mu \subseteq\supseteq) + (\mu CUM) + (\mu RatM) + (\cap) \not\Rightarrow (\mu PR)$.

Let $U := \{a, b, c\}$. Let $\mathcal{Y} = \mathcal{P}(U)$. So $(\cap)$ is trivially satisfied. Set $f(X) := X$ for all $X \subseteq U$ except for $f(\{a, b\}) = \{b\}$. Obviously, this cannot be represented by a preferential structure and (μPR) is false for U and $\{a, b\}$. But it satisfies $(\mu \subseteq)$, (μCUM), $(\mu RatM)$. $(\mu \subseteq)$ is trivial. (μCUM) : Let $f(X) \subseteq Y \subseteq X$. If $f(X) = X$, we are done. Consider $f(\{a, b\}) = \{b\}$. If $\{b\} \subseteq Y \subseteq \{a, b\}$, then $f(Y) = \{b\}$, so we are done again. It is shown in Fact 12.10, (8) that $(\mu \subseteq\supseteq)$ follows. $(\mu RatM)$: Suppose $X \subseteq Y$, $X \cap f(Y) \neq \emptyset$, we have to show $f(X) \subseteq f(Y) \cap X$. If $f(Y) = Y$, the result holds by $X \subseteq Y$, so it does if $X = Y$. The only remaining case is $Y = \{a, b\}$, $X = \{b\}$, and the result holds again.

The idea to solve the representation problem illustrated by Example 12.32 is to use the points c and d as bases for counterarguments against $\alpha : b \to a$ - as is possible in IBRS. We do this now. We will obtain a representation for logics weaker than P by generalized preferential structures.

We will now prove a representation theorem, but will make it more general than for preferential structures only. For this purpose, we will introduce some definitions first.

Definition 12.33. *Let $\eta, \rho : \mathcal{Y} \to \mathcal{P}(U)$.*

(1) If $\mathcal{X}$ is a simple structure

$\mathcal{X}$ is called an attacking structure relative to η representing ρ iff

$\rho(X) = \{x \in \eta(X)$: *there is no valid $X - to - \eta(X)$ arrow $\alpha : x' \to x\}$*

for all $X \in \mathcal{Y}$.

(2) If $\mathcal{X}$ is a structure with copies

$\mathcal{X}$ is called an attacking structure relative to η representing ρ iff

$\rho(X) = \{x \in \eta(X)$: *there is $\langle x, i\rangle$ and no valid $X - to - \eta(X)$ arrow $\alpha : \langle x', i'\rangle \to \langle x, i\rangle\}$*

for all $X \in \mathcal{Y}$.

Obviously, in those cases $\rho(X) \subseteq \eta(X)$ *for all* $X \in \mathcal{Y}$.
Thus, $\mathcal{X}$ *is a preferential structure iff* η *is the identity.*
See Figure 12.9 .

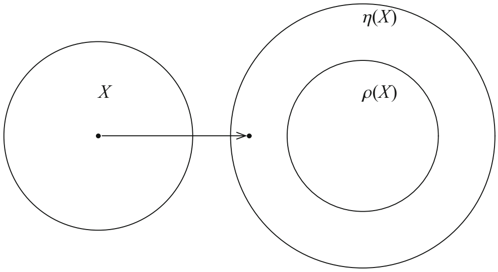

Fig. 12.9: Attacking structure

(Note that it does not seem very useful to generalize the notion of smoothness from preferential structures to general attacking structures, as, in the general case, the minimizing set X and the result $\rho(X)$ may be disjoint.)

The following result is the first positive representation result here, and shows that we can obtain (almost) anything with level 2 structures.

Proposition 12.34. *Let* $\eta, \rho : \mathcal{Y} \to \mathcal{P}(U)$. *Then there is an attacking level 2 structure relative to* η *representing* ρ *iff*
(1) $\rho(X) \subseteq \eta(X)$ *for all* $X \in \mathcal{Y}$,
(2) $\rho(\emptyset) = \eta(\emptyset)$ *if* $\emptyset \in \mathcal{Y}$.

(2) is, of course, void for preferential structures.

Proof. (A) The construction
We perform a two stage construction.
(A.1) Stage 1.
In stage 1, consider (almost as usual)
$\mathcal{U} := \langle \mathcal{X}, \{\alpha_i : i \in I\} \rangle$ where
$\mathcal{X} := \{\langle x, f \rangle : x \in U, f \in \Pi\{X \in \mathcal{Y} : x \in \eta(X) - \rho(X)\}\}$,
$\alpha : x' \to \langle x, f \rangle :\Leftrightarrow x' \in ran(f)$. Attention: $x' \in X$, not $x' \in \rho(X)$!
(A.2) Stage 2.
Let $\mathcal{X}'$ be the set of all $\langle x, f, X \rangle$ s.t. $\langle x, f \rangle \in \mathcal{X}$ and
(a) either X is some dummy value, say $*$
or
(b) all of the following (1) – (4) hold:

(1) $X \in \mathcal{Y}$,
(2) $x \in \rho(X)$,
(3) there is $X' \subseteq X$, $x \in \eta(X') - \rho(X')$, $X' \in \mathcal{Y}$, (thus $ran(f) \cap X \neq \emptyset$ by definition),
(4) $\forall X'' \in \mathcal{Y}.(X \subseteq X'', x \in \eta(X'') - \rho(X'') \Rightarrow (ran(f) \cap X'') - X \neq \emptyset)$.
(Thus, f chooses in (4) X'' also outside X. If there is no such X'', (4) is void, and only (1) – (3) need to hold, i.e. we may take any f with $\langle x, f\rangle \in \mathcal{X}$.)
See Figure 12.10.

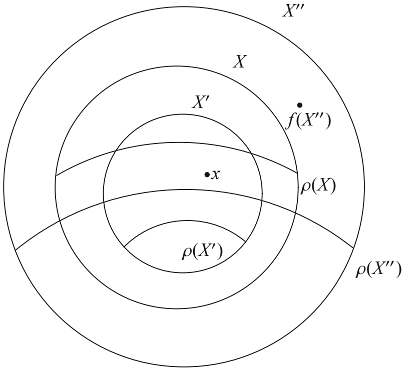

Fig. 12.10: The complicated case

Note: If (1) – (3) are satisfied for x and X, then we will find f s.t. $\langle x, f\rangle \in \mathcal{X}$, and $\langle x, f, X\rangle$ satisfies (1) – (4) : As $X \subset X''$ for X'' as in (4), we find f which chooses such X'' outside of X.

So for any $\langle x, f\rangle \in \mathcal{X}$, there is $\langle x, f, *\rangle$, and maybe also some $\langle x, f, X\rangle$ in $\mathcal{X}'$.

Again for any x', let $\langle x, f, X\rangle \in \mathcal{X}'$

$\alpha : x' \to \langle x, f, X\rangle :\Leftrightarrow x' \in ran(f)$

(A.3) Adding arrows.

Consider x' and $\langle x, f, X\rangle$.

If $X = *$, or $x' \notin X$, we do nothing, i.e. leave a simple arrow $\alpha : x' \to \langle x, f, X\rangle \Leftrightarrow x' \in ran(f)$.

If $X \in \mathcal{Y}$, and $x' \in X$, and $x' \in ran(f)$, we make card(X) many copies of the attacking arrow and have then: $\langle \alpha, x''\rangle : x' \to \langle x, f, X\rangle$ for all $x'' \in X$.

In addition, we add attacks on the $\langle \alpha, x''\rangle : \langle \beta, x''\rangle : x'' \to \langle \alpha, x''\rangle$ for all $x'' \in X$.

The full structure $\mathcal{Z}$ is thus:

$\mathcal{X}'$ is the set of elements.

If $x' \in ran(f)$, and $X = *$ or $x' \notin X$ then $\alpha : x' \to \langle x, f, X\rangle$.

If $x' \in ran(f)$, and $X \neq *$ and $x' \in X$ then

(a) $\langle \alpha, x''\rangle : x' \to \langle x, f, X\rangle$ for all $x'' \in X$,

(b) $\langle \beta, x'' \rangle : x'' \rightarrow \langle \alpha, x'' \rangle$ for all $x'' \in X$.
See Figure 12.11.

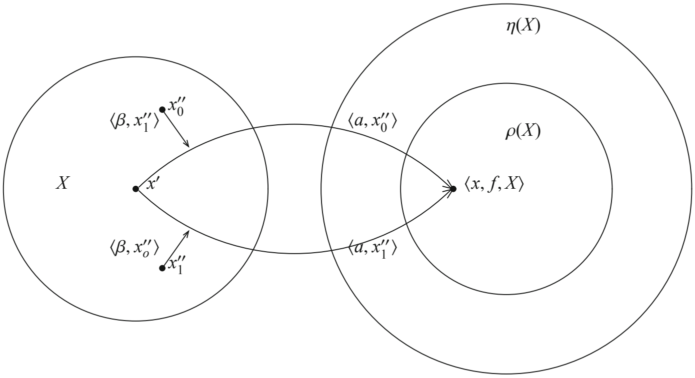

Fig. 12.11: Attacking structure

(B) Representation

We have to show that this structure represents ρ relative to η.

Let $y \in \eta(Y)$, $Y \in \mathcal{Y}$.

Case 1. $y \in \rho(Y)$.

We have to show that there is $\langle y, g, Y'' \rangle$ s.t. there is no valid $\alpha : y' \rightarrow \langle y, g, Y'' \rangle$, $y' \in Y$. In Case 1.1 below, Y'' will be $*$, in Case 1.2, Y'' will be Y, g will be chosen suitably.

Case 1.1. There is no $Y' \subseteq Y, y \in \eta(Y') - \rho(Y')$, $Y' \in \mathcal{Y}$.

So for all Y' with $y \in \eta(Y') - \rho(Y')$ $Y' - Y \neq \emptyset$. Let $g \in \Pi\{Y' - Y : y \in \eta(Y') - \rho(Y')\}$. Then $ran(g) \cap Y = \emptyset$, and $\langle y, g, * \rangle$ is not attacked from Y. ($\langle y, g \rangle$ was already not attacked in $\mathcal{X}$.)

Case 1.2. There is $Y' \subseteq Y, y \in \eta(Y') - \rho(Y')$, $Y' \in \mathcal{Y}$.

Now let $\langle y, g, Y \rangle \in \mathcal{X}'$, s.t. $g(Y'') \notin Y$ if $Y \subseteq Y''$, $y \in \eta(Y'') - \rho(Y'')$, $Y'' \in \mathcal{Y}$. As noted above, such g and thus $\langle y, g, Y \rangle$ exist. Fix $\langle y, g, Y \rangle$.

Consider any $y' \in ran(g)$. If $y' \notin Y$, y' does not attack $\langle y, g, Y \rangle$ in Y. Suppose $y' \in Y$. We had made card(Y) many copies $\langle \alpha, y'' \rangle$, $y'' \in Y$ with $\langle \alpha, y'' \rangle : y' \rightarrow \langle y, g, Y \rangle$ and had added the level 2 arrows $\langle \beta, y'' \rangle : y'' \rightarrow \langle \alpha, y'' \rangle$ for $y'' \in Y$. So all copies $\langle \alpha, y'' \rangle$ are destroyed in Y. This was done for all $y' \in Y$, $y' \in ran(g)$, so $\langle y, g, Y \rangle$ is now not (validly) attacked in Y any more.

Case 2. $y \in \eta(Y) - \rho(Y)$.

Let $\langle y, g, Y' \rangle$ (where Y' can be $*$) be any copy of y, we have to show that there is $z \in Y$, $\alpha : z \to \langle y, g, Y' \rangle$, or some $\langle \alpha, z' \rangle : z \to \langle y, g, Y' \rangle$, $z' \in Y'$, which is not destroyed by some level 2 arrow $\langle \beta, z' \rangle : z' \to \langle \alpha, z' \rangle$, $z' \in Y$.

As $y \in \eta(Y) - \rho(Y)$, $ran(g) \cap Y \neq \emptyset$, so there is $z \in ran(g) \cap Y$. Fix such z. (We will modify the choice of z only in Case 2.2.2 below.)

Case 2.1. $Y' = *$.

As $z \in ran(g)$, $\alpha : z \to \langle y, g, * \rangle$. (There were no level 2 arrows introduced for this copy.)

Case 2.2. $Y' \neq *$.

So $\langle y, g, Y' \rangle$ satisfies the conditions (1) – (4) of (b) at the beginning of the proof.

If $z \notin Y'$, we are done, as $\alpha : z \to \langle y, g, Y' \rangle$, and there were no level 2 arrows introduced in this case. If $z \in Y'$, we had made card(Y') many copies $\langle \alpha, z' \rangle$, $\langle \alpha, z' \rangle$: $z \to \langle y, g, Y' \rangle$, one for each $z' \in Y'$. Each $\langle \alpha, z' \rangle$ was destroyed by $\langle \beta, z' \rangle : z' \to \langle \alpha, z' \rangle$, $z' \in Y'$.

Case 2.2.1. $Y' \not\subseteq Y$.

Let $z'' \in Y'$-Y, then $\langle \alpha, z'' \rangle : z \to \langle y, g, Y' \rangle$ is destroyed only by $\langle \beta, z'' \rangle : z'' \to \langle \alpha, z'' \rangle$ in Y', but not in Y, as $z'' \notin Y$, so $\langle y, g, Y' \rangle$ is attacked by $\langle \alpha, z'' \rangle : z \to \langle y, g, Y' \rangle$, valid in Y.

Case 2.2.2. $Y' \subset Y$ ($Y = Y'$ is impossible, as $y \in \rho(Y')$, $y \notin \rho(Y)$).

Then there was by definition (condition (b) (4)) some $z' \in (ran(g) \cap Y) - Y'$ and $\alpha : z' \to \langle y, g, Y' \rangle$ is valid, as $z' \notin Y'$. (In this case, there are no copies of α and no level 2 arrows.)

Corollary 12.35. *(1) We cannot distinguish general structures of level 2 from those of higher levels by their ρ-functions relative to η.*

(2) Let U be the universe, $\mathcal{Y} \subseteq \mathcal{P}(U)$, $\mu : \mathcal{Y} \to \mathcal{P}(U)$. Then any μ satisfying ($\mu \subseteq$) can be represented by a level 2 preferential structure. (Choose η = identity.)

Again, we cannot distinguish general structures of level 2 from those of higher levels by their μ-functions.

A remark on the function η :

We can also obtain the function η via arrows. Of course, then we need positive arrows (not only negative arrows against negative arrows, as we first need to have something positive).

If η is the identity, we can make a positive arrow from each point to itself. Otherwise, we can connect every point to every point by a positive arrow, and then choose those we really want in η by a choice function obtained from arrows just as we obtained ρ from arrows.

Results on total smoothness

Fact 12.36 *Let $X, Y \in \mathcal{Y}$, $\mathcal{X}$ a level n structure. Let $\langle \alpha, k \rangle : \langle x, i \rangle \to \langle y, j \rangle$, where $\langle y, j \rangle$ may itself be (a copy of) an arrow.*

(1) Let $n > 1$, $X \subseteq Y$, $\langle \alpha, k \rangle \in X$ a level $n - 1$ arrow in $\mathcal{X} \upharpoonright X$. If $\langle \alpha, k \rangle$ is valid in $\mathcal{X} \upharpoonright Y$, then it is valid in $\mathcal{X} \upharpoonright X$.

(2) Let $\mathcal{X}$ be totally smooth, $\mu(X) \subseteq Y$, $\mu(Y) \subseteq X$, $\langle \alpha, k \rangle \in \mathcal{X} \upharpoonright X \cap Y$, then $\langle \alpha, k \rangle$ is valid in $\mathcal{X} \upharpoonright X$ iff it is valid in $\mathcal{X} \upharpoonright Y$.

Note that we will also sometimes write X for $\mathcal{X} \upharpoonright X$, when the context is clear.

Proof. (1) If $\langle \alpha, k \rangle$ is not valid in $\mathcal{X} \upharpoonright X$, then there must be a level n arrow $\langle \beta, r \rangle : \langle z, s \rangle \to \langle \alpha, k \rangle$ in $\mathcal{X} \upharpoonright X \subseteq \mathcal{X} \upharpoonright Y$. $\langle \beta, r \rangle$ must be valid in $\mathcal{X} \upharpoonright X$ and $\mathcal{X} \upharpoonright Y$, as there are no level $n + 1$ arrows. So $\langle \alpha, k \rangle$ is not valid in $\mathcal{X} \upharpoonright Y$, *contradiction*.

(2) By downward induction. Case $n : \langle \alpha, k \rangle \in \mathcal{X} \upharpoonright X \cap Y$, so it is valid in both as there are no level $n + 1$ arrows. Case $m \to m - 1$: Let $\langle \alpha, k \rangle \in \mathcal{X} \upharpoonright X \cap Y$ be a level $m - 1$ arrow valid in $\mathcal{X} \upharpoonright X$, but not in $\mathcal{X} \upharpoonright Y$. So there must be a level m arrow $\langle \beta, r \rangle : \langle z, s \rangle \to \langle \alpha, k \rangle$ valid in $\mathcal{X} \upharpoonright Y$. By total smoothness, we may assume $z \in \mu(Y) \subseteq X$, so $\langle \beta, r \rangle \in \mathcal{X} \upharpoonright X$ is valid by induction hypothesis. So $\langle \alpha, k \rangle$ is not valid in $\mathcal{X} \upharpoonright X$, *contradiction*.

Corollary 12.37. *Let $X, Y \in \mathcal{Y}$, $\mathcal{X}$ a totally smooth level n structure, $\mu(X) \subseteq Y$, $\mu(Y) \subseteq X$. Then $\mu(X) = \mu(Y)$.*

Proof. Let $x \in \mu(X) - \mu(Y)$. Then by $\mu(X) \subseteq Y$, $x \in Y$, so there must be for all $\langle x, i \rangle \in \mathcal{X}$ an arrow $\langle \alpha, k \rangle : \langle y, j \rangle \to \langle x, i \rangle$ valid in $\mathcal{X} \upharpoonright Y$, without loss of generality $y \in \mu(Y) \subseteq X$ by total smoothness. So by Fact 12.36, (2), $\langle \alpha, k \rangle$ is valid in $\mathcal{X} \upharpoonright X$. This holds for all $\langle x, i \rangle$, so $x \notin \mu(X)$, *contradiction*.

Fact 12.38 *There are situations satisfying* $(\mu \subseteq) + (\mu CUM) + (\cap)$ *which cannot be represented by level 2 totally smooth preferential structures.*

The proof is given in the following example.

Example 12.39. Let $Y := \{x, y, y'\}$, $X := \{x, y\}$, $X' := \{x, y'\}$. Let $\mathcal{Y} := \mathcal{P}(Y)$. Let $\mu(Y) := \{y, y'\}$, $\mu(X) := \mu(X') := \{x\}$, and $\mu(Z) := Z$ for all other sets.

Obviously, this satisfies $(\cap)$, $(\mu \subseteq)$, and (μCUM).

Suppose $\mathcal{X}$ is a totally smooth level 2 structure representing μ.

So $\mu(X) = \mu(X') \subseteq Y - \mu(Y)$, $\mu(Y) \subseteq X \cup X'$. Let $\langle x, i \rangle$ be minimal in $\mathcal{X} \upharpoonright X$. As $\langle x, i \rangle$ cannot be minimal in $\mathcal{X} \upharpoonright Y$, there must be $\alpha : \langle z, j \rangle \to \langle x, i \rangle$, valid in $\mathcal{X} \upharpoonright Y$.

Case 1: $z \in X'$.

So $\alpha \in \mathcal{X} \upharpoonright X'$. If α is valid in $\mathcal{X} \upharpoonright X'$, there must be $\alpha' : \langle x', i' \rangle \to \langle x, i \rangle$, $x' \in \mu(X')$, valid in $\mathcal{X} \upharpoonright X'$, and thus in $\mathcal{X} \upharpoonright X$, by $\mu(X) = \mu(X')$ and Fact 12.36 (2). This is impossible, so there must be $\beta : \langle x', i' \rangle \to \alpha$, $x' \in \mu(X')$, valid in $\mathcal{X} \upharpoonright X'$. As β is in $\mathcal{X} \upharpoonright Y$ and $\mathcal{X}$ is a level ≤ 2 structure, β is valid in $\mathcal{X} \upharpoonright Y$, so α is not valid in $\mathcal{X} \upharpoonright Y$, *contradiction*.

Case 2: $z \in X$.

α cannot be valid in $\mathcal{X} \upharpoonright X$, so there must be $\beta : \langle x', i' \rangle \to \alpha$, $x' \in \mu(X)$, valid in $\mathcal{X} \upharpoonright X$. Again, as β is in $\mathcal{X} \upharpoonright Y$ and $\mathcal{X}$ is a level ≤ 2 structure, β is valid in $\mathcal{X} \upharpoonright Y$, so α is not valid in $\mathcal{X} \upharpoonright Y$, *contradiction*.

It is unknown to the authors whether an analogue is true for essential smoothness, i.e. whether there are examples of such a function μ which need at least level 3 essentially smooth structures for representation. Proposition 12.45 below shows that such structures suffice, but we do not know whether level 3 is necessary.

Fact 12.40 *The above Example 12.39 can be solved by a totally smooth level 3 structure:*

Let $\alpha_1 : x \to y$, $\alpha_2 : x \to y'$, $\alpha_3 : y \to x$, $\beta_1 : y \to \alpha_2$, $\beta_2 : y' \to \alpha_1$, $\beta_3 : y \to \alpha_3$, $\beta_4 : x \to \alpha_3$, $\gamma_1 : y' \to \beta_3$, $\gamma_2 : y' \to \beta_4$.

See Figure 12.12.

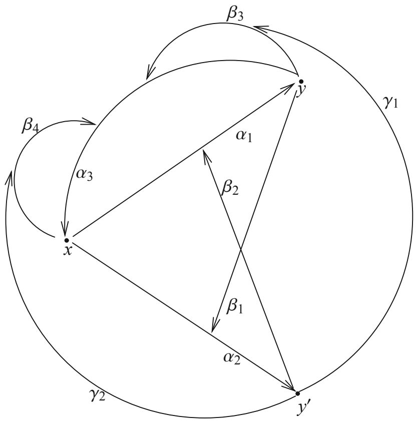

Fig. 12.12: Solution by smooth level 3 structure

The subdiagram generated by X contains $\alpha_1, \alpha_3, \beta_3$, and β_4. $\alpha_1, \beta_3, \beta_4$ are valid, so $\mu(X) = \{x\}$.

The subdiagram generated by X' contains α_2. α_2 is valid, so $\mu(X') = \{x\}$.

In the full diagram, $\alpha_3, \beta_1, \beta_2, \gamma_1$, and γ_2 are valid, so $\mu(Y) = \{y, y'\}$.

Remark 12.41. Example 12.42 together with Corollary 12.37 show that $(\mu \subseteq)$ and (μCUM) without $(\cap)$ do not guarantee representability by a level n totally smooth structure.

Example 12.42. We show here $(\mu \subseteq) + (\mu CUM) \not\Rightarrow (\mu \subseteq\supseteq)$.

Consider $X := \{a, b, c\}$, $Y := \{a, b, d\}$, $f(X) := \{a\}$, $f(Y) := \{a, b\}$, $\mathcal{Y} := \{X, Y\}$. (If $f(\{a, b\})$ were defined, we would have $f(X) = f(\{a, b\}) = f(Y)$, *contradiction*.)

Obviously, $(\mu \subseteq)$ and (μCUM) hold, but not $(\mu \subseteq\supseteq)$.

Results on essential smoothness

Definition 12.43. *Let* $\mu : \mathcal{Y} \to \mathcal{P}(U)$ *and* $\mathcal{X}$ *be given, let* $\alpha : \langle y, j\rangle \to \langle x, i\rangle \in \mathcal{X}$.

Define

$\boldsymbol{O}(\alpha) := \{Y \in \mathcal{Y} : x \in Y - \mu(Y), y \in \mu(Y)\}$,

$\boldsymbol{D}(\alpha) := \{X \in \mathcal{Y} : x \in \mu(X), y \in X\}$,

$\Pi(\boldsymbol{O}, \alpha) := \Pi\{\mu(Y) : Y \in \boldsymbol{O}(\alpha)\}$,

$\Pi(\boldsymbol{D}, \alpha) := \Pi\{\mu(X) : X \in \boldsymbol{D}(\alpha)\}$.

The following lemma is probably the main technical result of the chapter.

Lemma 12.44. *Let U be the universe, $\mu : \mathcal{Y} \to \mathcal{P}(U)$. Let μ satisfy* $(\mu \subseteq) + (\mu \subseteq\supseteq)$.

Let $\mathcal{X}$ be a level 1 preferential structure, $\alpha : \langle y, j\rangle \to \langle x, i\rangle$, $\boldsymbol{O}(\alpha) \neq \emptyset$, $\boldsymbol{D}(\alpha) \neq \emptyset$.

We can modify $\mathcal{X}$ to a level 3 structure $\mathcal{X}'$ by introducing level 2 and level 3 arrows s.t. no copy of α is valid in any $X \in \boldsymbol{D}(\alpha)$, and in every $Y \in \boldsymbol{O}(\alpha)$ at least one copy of α is valid. (More precisely, we should write $\mathcal{X}' \restriction X$, etc.)

Thus, in $\mathcal{X}'$,

(1) $\langle x, i\rangle$ will not be minimal in any $Y \in \boldsymbol{O}(\alpha)$,

(2) if α is the only arrow minimizing $\langle x, i\rangle$ in $X \in \boldsymbol{D}(\alpha)$, $\langle x, i\rangle$ will now be minimal in X.

The construction is made independently for all such arrows $\alpha \in \mathcal{X}$.

Proof. (1) The construction

Make $\Pi(\boldsymbol{D}, \alpha)$ many copies of α : $\{\langle \alpha, f\rangle : f \in \Pi(\boldsymbol{D}, \alpha)\}$, all $\langle \alpha, f\rangle : \langle y, j\rangle \to \langle x, i\rangle$. Note that $\langle \alpha, f\rangle \in X$ for all $X \in \boldsymbol{D}(\alpha)$ and $\langle \alpha, f\rangle \in Y$ for all $Y \in \boldsymbol{O}(\alpha)$.

Add to the structure $\langle \beta, f, X_r, g\rangle : \langle f(X_r), i_r\rangle \to \langle \alpha, f\rangle$, for any $X_r \in \boldsymbol{D}(\alpha)$, and $g \in \Pi(\boldsymbol{O}, \alpha)$ (and some or all i_r - this does not matter).

For all $Y_s \in \boldsymbol{O}(\alpha)$:

if $\mu(Y_s) \not\subseteq X_r$ and $f(X_r) \in Y_s$, then add to the structure $\langle \gamma, f, X_r, g, Y_s\rangle$: $\langle g(Y_s), j_s\rangle \to \langle \beta, f, X_r, g\rangle$ (again for all or some j_s),

if $\mu(Y_s) \subseteq X_r$ or $f(X_r) \notin Y_s$, $\langle \gamma, f, X_r, g, Y_s\rangle$ is not added.

See Figure 12.13.

(2) Let $X_r \in \boldsymbol{D}(\alpha)$. We have to show that no $\langle \alpha, f\rangle$ is valid in X_r. Fix f.

$\langle \alpha, f\rangle$ is in X_r, so we have to show that for at least one $g \in \Pi(\boldsymbol{O}, \alpha)$ $\langle \beta, f, X_r, g\rangle$ is valid in X_r, i.e. that for this g, no $\langle \gamma, f, X_r, g, Y_s\rangle : \langle g(Y_s), j_s\rangle \to \langle \beta, f, X_r, g\rangle$, $Y_s \in \boldsymbol{O}(\alpha)$ attacks $\langle \beta, f, X_r, g\rangle$ in X_r.

We define g. Take $Y_s \in \boldsymbol{O}(\alpha)$.

Case 1: $\mu(Y_s) \subseteq X_r$ or $f(X_r) \notin Y_s$: choose arbitrary $g(Y_s) \in \mu(Y_s)$.

Case 2: $\mu(Y_s) \not\subseteq X_r$ and $f(X_r) \in Y_s$: Choose $g(Y_s) \in \mu(Y_s) - X_r$.

In Case 1, $\langle \gamma, f, X_r, g, Y_s\rangle$ does not exist, so it cannot attack $\langle \beta, f, X_r, g\rangle$.

In Case 2, $\langle \gamma, f, X_r, g, Y_s\rangle : \langle g(Y_s), j_s\rangle \to \langle \beta, f, X_r, g\rangle$ is not in X_r, as $g(Y_s) \notin X_r$.

Thus, no $\langle \gamma, f, X_r, g, Y_s\rangle : \langle g(Y_s), j_s\rangle \to \langle \beta, f, X_r, g\rangle$, $Y_s \in \boldsymbol{O}(\alpha)$ attacks $\langle \beta, f, X_r, g\rangle$ in X_r.

So $\forall \langle \alpha, f\rangle : \langle y, j\rangle \to \langle x, i\rangle$

$$
\begin{aligned}
y \in X_r \Rightarrow \exists \langle \beta, f, X_r, g\rangle &: \langle f(X_r), i_r\rangle \to \langle \alpha, f\rangle \\
&(f(X_r) \in \mu(X_r) \wedge \neg\exists \langle \gamma, f, X_r, g, Y_s\rangle : \\
&\quad \langle g(Y_s), j_s\rangle \to \langle \beta, f, X_r, g\rangle . g(Y_s) \in X_r)).
\end{aligned}
$$

But $\langle \beta, f, X_r, g\rangle$ was constructed only for $\langle \alpha, f\rangle$, so was $\langle \gamma, f, X_r, g, Y_s\rangle$, and there was no other $\langle \gamma, i\rangle$ attacking $\langle \beta, f, X_r, g\rangle$, so we are done.

(3) Let $Y_s \in \boldsymbol{O}(\alpha)$. We have to show that at least one $\langle \alpha, f\rangle$ is valid in Y_s.

We define $f \in \Pi(\boldsymbol{D}, \alpha)$. Take X_r.

If $\mu(X_r) \not\subseteq Y_s$, choose $f(X_r) \in \mu(X_r) - Y_s$. If $\mu(X_r) \subseteq Y_s$, choose arbitrary $f(X_r) \in \mu(X_r)$.

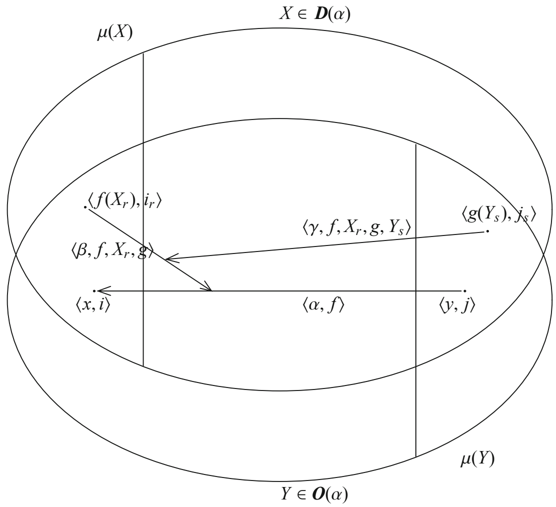

Fig. 12.13: The construction

All attacks on $\langle x, f \rangle$ have the form $\langle \beta, f, X_r, g \rangle : \langle f(X_r), i_r \rangle \to \langle \alpha, f \rangle$, $X_r \in \boldsymbol{D}(\alpha)$, $g \in \Pi(\boldsymbol{O}, \alpha)$. We have to show that they are either not in Y_s, or that they are themselves attacked in Y_s.

Case 1: $\mu(X_r) \not\subseteq Y_s$. Then $f(X_r) \notin Y_s$, so $\langle \beta, f, X_r, g \rangle : \langle f(X_r), i_r \rangle \to \langle \alpha, f \rangle$ is not in Y_s (for no g).

Case 2: $\mu(X_r) \subseteq Y_s$. Then $\mu(Y_s) \not\subseteq X_r$ by ($\mu \subseteq\supseteq$) and $f(X_r) \in Y_s$, so $\langle \beta, f, X_r, g \rangle : \langle f(X_r), i_r \rangle \to \langle \alpha, f \rangle$ is in Y_s (for all g). Take any $g \in \Pi(\boldsymbol{O}, \alpha)$. As $\mu(Y_s) \not\subseteq X_r$ and $f(X_r) \in Y_s$, $\langle \gamma, f, X_r, g, Y_s \rangle : \langle g(Y_s), j_s \rangle \to \langle \beta, f, X_r, g \rangle$ is defined, and $g(Y_s) \in \mu(Y_s)$, so it is in Y_s (for all g). Thus, $\langle \beta, f, X_r, g \rangle$ is attacked in Y_s.

Thus, for this f, all $\langle \beta, f, X_r, g \rangle$ are either not in Y_s, or attacked in Y_s, thus for this f, $\langle \alpha, f \rangle$ is valid in Y_s.

So for this $\langle x, i \rangle$

$\exists \langle \alpha, f \rangle : \langle y, j \rangle \to \langle x, i \rangle . y \in \mu(Y_s) \wedge$

(a) $\neg \exists \langle \beta, f, X_r, g \rangle : \langle f(X_r), i \rangle \to \langle \alpha, f \rangle . f(X_r) \in Y_s$

or

(b) $\forall \langle \beta, f, X_r, g \rangle : \langle f(X_r), i \rangle \to \langle \alpha, f \rangle$

$(f(X_r) \in Y_s \Rightarrow$

$\exists \langle \gamma, f, X_r, g, Y_s \rangle : \langle g(Y_s), j_s \rangle \to \langle \beta, f, X_r, g \rangle . g(Y_s) \in \mu(Y_s))$.

As we made copies of α only, introduced only βs attacking the α-copies, and γs attacking the βs, the construction is independent for different αs.

Proposition 12.45. *Let U be the universe, $\mu : \mathcal{Y} \to \mathcal{P}(U)$.*

Then any μ satisfying $(\mu \subseteq)$, $(\cap)$, (μCUM) (or, alternatively, $(\mu \subseteq)$ and $(\mu \subseteq\supseteq)$) can be represented by a level 3 essentially smooth structure.

Proof. In stage 1, consider as usual $\mathcal{U} := \langle \mathcal{X}, \{\alpha_i : i \in I\}\rangle$ where $\mathcal{X} := \{\langle x, f\rangle : x \in U, f \in \Pi\{\mu(X) : X \in \mathcal{Y}, x \in X - \mu(X)\}\}$, and set $\alpha : \langle x', f'\rangle \to \langle x, f\rangle :\Leftrightarrow x' \in ran(f)$.

For stage 2:

Any level 1 arrow $\alpha : \langle y, j\rangle \to \langle x, i\rangle$ was introduced in stage 1 by some $Y \in \mathcal{Y}$ s.t. $y \in \mu(Y)$, $x \in Y - \mu(Y)$. Do the construction of Lemma 12.44 for all level 1 arrows of $\mathcal{X}$ in parallel or successively.

We have to show that the resulting structure represents μ and is essentially smooth. (Level 3 is obvious.)

(1) Representation

Suppose $x \in Y - \mu(Y)$. Then there was in stage 1 for all $\langle x, i\rangle$ some $\alpha : \langle y, j\rangle \to \langle x, i\rangle$, $y \in \mu(Y)$. We examine the y.

If there is no X s.t. $x \in \mu(X)$, $y \in X$, then there were no βs and γs introduced for this $\alpha : \langle y, j\rangle \to \langle x, i\rangle$, so α is valid.

If there is X s.t. $x \in \mu(X)$, $y \in X$, consider $\alpha : \langle y, j\rangle \to \langle x, i\rangle$. So $X \in \boldsymbol{D}(\alpha)$, $Y \in \boldsymbol{O}(\alpha)$, so we did the construction of Lemma 12.44, and by its result, $\langle x, i\rangle$ is not minimal in Y.

Thus, in both cases, $\langle x, i\rangle$ is successfully attacked in Y, and no $\langle x, i\rangle$ is a minimal element in Y.

Suppose $x \in \mu(X)$ (we change notation to conform to Lemma 12.44. Fix $\langle x, i\rangle$.

If there is no $\alpha : \langle y, j\rangle \to \langle x, i\rangle$, $y \in X$, then $\langle x, i\rangle$ is minimal in X, and we are done.

If there is α or $\langle \alpha, k\rangle : \langle y, j\rangle \to \langle x, i\rangle$, $y \in X$, then α originated from stage 1 through some Y s.t. $x \in Y - \mu(Y)$, and $y \in \mu(Y)$. (Note that stage 2 of the construction did not introduce any new level 1 arrows - only copies of existing level 1 arrows.) So $X \in \boldsymbol{D}(\alpha)$, $Y \in \boldsymbol{O}(\alpha)$, so we did the construction of Lemma 12.44, and by its result, $\langle x, i\rangle$ is minimal in X, and we are done again.

In both cases, all $\langle x, i\rangle$ are minimal elements in X.

(2) Essential smoothness. We have to show the conditions of Definition 12.24. We will, however, work with the reformulation given in Remark 12.25.

Case (1), $x \in \mu(X)$.

Case (1.1), there is $\langle x, i\rangle$ with no $\langle \alpha, f\rangle : \langle y, j\rangle \to \langle x, i\rangle$, $y \in X$. There is nothing to show.

Case (1.2), for all $\langle x, i\rangle$ there is $\langle \alpha, f\rangle : \langle y, j\rangle \to \langle x, i\rangle$, $y \in X$.

α was introduced in stage 1 by some Y s.t. $x \in Y - \mu(Y)$, $y \in X \cap \mu(Y)$, so $X \in \boldsymbol{D}(\alpha)$, $Y \in \boldsymbol{O}(\alpha)$. In the proof of Lemma 12.44, at the end of (2), it was shown that

$$\begin{aligned}\exists\langle \beta, f, X_r, g\rangle : \langle f(X_r), i_r\rangle \to \langle \alpha, f\rangle \\ (f(X_r) \in \mu(X_r) \wedge \\ \neg\exists\langle \gamma, f, X_r, g, Y_s\rangle : \langle g(Y_s), j_s\rangle \to \langle \beta, f, X_r, g\rangle . g(Y_s) \in X_r).\end{aligned}$$

By $f(X_r) \in \mu(X_r)$, condition (1) of Remark 12.25 is true.

Case (2), $x \notin \mu(Y)$. Fix $\langle x, i\rangle$. (We change notation back to Y.)

In stage 1, we constructed $\alpha : \langle y, j\rangle \to \langle x, i\rangle, y \in \mu(Y)$, so $Y \in \boldsymbol{O}(\alpha)$.

If $\boldsymbol{D}(\alpha) = \emptyset$, then there is no attack on α, and the condition (2) of Remark 12.25 is trivially true.

If $\boldsymbol{D}(\alpha) \neq \emptyset$, we did the construction of Lemma 12.44, so

$\exists\langle\alpha, f\rangle : \langle y, j\rangle \to \langle x, i\rangle . y \in \mu(Y_s) \wedge$

(a) $\neg\exists\langle\beta, f, X_r, g\rangle : \langle f(X_r), i\rangle \to \langle\alpha, f\rangle . f(X_r) \in Y_s$

or

(b) $\forall\langle\beta, f, X_r, g\rangle : \langle f(X_r), i\rangle \to \langle\alpha, f\rangle$

$(f(X_r) \in Y_s \Rightarrow$

$\exists\langle\gamma, f, X_r, g, Y_s\rangle : \langle g(Y_s), j_s\rangle \to \langle\beta, f, X_r, g\rangle . g(Y_s) \in \mu(Y_s))$.

As the only attacks on $\langle\alpha, f\rangle$ had the form $\langle\beta, f, X_r, g\rangle$, and $g(Y_s) \in \mu(Y_s)$, condition (2) of Remark 12.25 is satisfied.

As we said after Example 12.39, we do not know if level 3 is necessary for representation. We also do not know whether the same can be achieved with level 3, or higher, totally smooth structures.

Translation to logics

We turn to the translation to logics.

Proposition 12.46. *Let* $\mid\!\sim$ *be a logic for* $\mathcal{L}$. *Set* $T^{\mathcal{M}} := Th(\mu_{\mathcal{M}}(M(T)))$, *where* $\mathcal{M}$ *is a generalized preferential structure, and* $\mu_{\mathcal{M}}$ *its choice function. Then*

(1) there is a level 2 preferential structure $\mathcal{M}$ *s.t.* $\overline{\overline{T}} = T^{\mathcal{M}}$ *iff* (LLE), (CCL), (SC) *hold for all* $T, T' \subseteq \mathcal{L}$.

(2) there is a level 3 essentially smooth preferential structure $\mathcal{M}$ *s.t.* $\overline{\overline{T}} = T^{\mathcal{M}}$ *iff* (LLE), (CCL), (SC), $(\subseteq\supseteq)$ *hold for all* $T, T' \subseteq \mathcal{L}$.

Proof. The proof is an immediate consequence of Corollary 12.35 (2), Fact 12.26, and Propositions 12.45 and 12.11 (10) and (11).

(More precisely, for (2): Let $\mathcal{M}$ be an essentially smooth structure, then by Definition 12.30 for all X $\mu(X) \sqsubseteq X$. Consider (μCUM). So by Fact 12.26 (2) $\mu(X') \subseteq X'' \subseteq X' \Rightarrow \mu(X') \sqsubseteq X''$, so by Fact 12.26 (1) $\mu(X') = \mu(X'')$. $(\mu \subseteq\supseteq)$ is analogous, using Fact 12.26 (3).

We leave aside the generalization of preferential structures to attacking structures relative to η, as this can cause problems, without giving real insight: It might well be that $\rho(X) \not\subseteq \eta(X)$, but, still, $\rho(X)$ and $\eta(X)$ might define the same theory - due to definability problems.