Cumulativity without closure of the domain under finite unions

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Abstract

For nonmonotonic logics, Cumulativity is an important logical rule. We show here that Cumulativity fans out into an infinity of different conditions, if the domain is not closed under finite unions.

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1 Introduction

1.1 Motivation and history of Cumulativity

Cumulativity is one of the most important properties a nonmonotonic logic can have, and was recognised as such quite long ago, see [Gab85]. It says that adding results proved already to the axioms will not change the set of provable results. More precisely, if $\mid\sim$ is some consequence relation, then cumulativity says:

If $\phi \mid\sim \psi$, then: $\phi \mid\sim \psi'$ iff $\phi \wedge \psi \mid\sim \psi'$.

(The other axiom in Gabbay's system is $\alpha \mid\sim \alpha$. - We modified both slightly here, the precise definition, and other axioms as well as variants are given in Definition 2.3.)

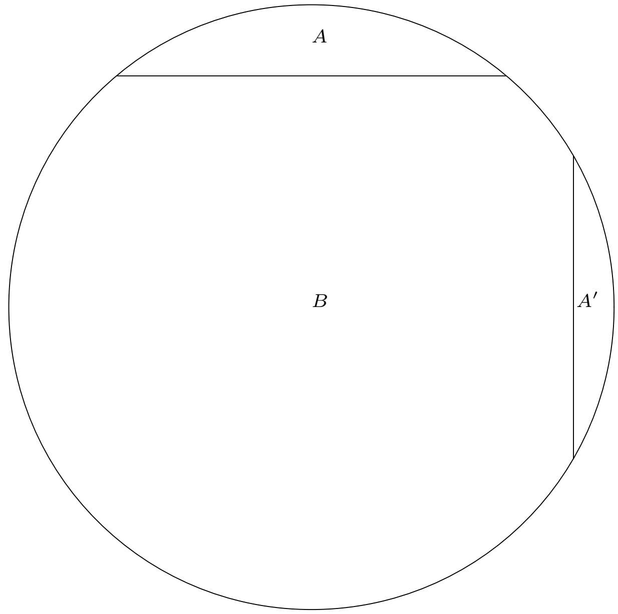
In classical logic, this is trivial, in nonmonotonic logics, this is a non-trivial property.

Whereas the approach of [Gab85] was an abstract consideration of desirable properties, preferential structures were introduced as abstractions of Circumscription independently in [Sho87b] and [BS85]. Roughly, a preferential model is a possible worlds structure with a binary relation $\prec$, which expresses that one world is more "normal" than the other. A precise definition is given below in Definition 2.4.

Both approaches were connected in [KLM90], where a representation theorem was proved, showing that Cumulativity corresponds to the relational property of "smoothness", which says that every possible world is either minimal itself, or there is a smaller one, which is minimal. This property can be violated by non-transitive relations, or infinite descending chains.

Preferential semantics generate, however, a richer logic than the one considered in [Gab85], and, to the authors' knowledge, a precise semantics for this system is given only in [GS08b] by the present authors.

Cumulativity can also be seen as allowing certain manipulations with "small" sets. If $\phi \mid\sim \psi$ expresses that the number of exceptions, i.e. when $\phi \wedge \neg\psi$ holds, is small, then Cumulativity essentiall says that the number of cases where $\phi \wedge \psi \wedge \neg\psi'$ holds is still small in the set of cases where $\phi \wedge \psi$ holds, or, if A and A' are small subsets of B, then A' will still be small in $B - A$ (assume for simplicity that A and A' are disjoint). See Diagram 1.1. A systematic investigation of the connections between "size" and nonmonotonic logic can be found in [GS08c].



Cumulativity

Diagram 1.1

Quite often, the domain on which a logic operates is closed under finite union: if $M(\phi)$ is the set of $\phi-$models, then $M(\phi) \cup M(\phi') = M(\phi \vee \phi')$. This need not always be the case, for instance, not for certain sequent calculi - see Section 3 below - which do not have an "or" in the language.

(Yet, even if the logic allows such closure, i.e. we have "or" in the language, we might simply be unable to observe the result of the closure, we might well know the consequences of ϕ, those of ϕ', but be unable to observe the consequences of $\phi \vee \phi'$. We may think here of an experiment in, e.g., physics, where the input is a formula, and the output is an observation, again a formula. Our experimental setup may just not allow to observe the outcome of $\phi \vee \phi'$ - though the language allows to pose the question $''\phi \vee \psi?''$.)

Quite surprisingly (for the authors at least), absence of this simple closure condition has drastic consequences on cumulativity: the simple short condition breaks up into an infinity of non-equivalent conditions. This is the main result of this paper, and shown by Example 4.1. For illustration, we now give the first three cases of one set of conditions, the full definition is given in Definition 4.1.

We switch to semantics, and give a corresponding condition for model sets. Often, it is good policy to separate the semantical from the proof theoretical conditions, as both have their own problems, and it is easier to treat them separately. More connections are given in Definition 2.3. $\mu(X)$ will now be the set of minimal models of X.

(1) $(\mu Cum0)$ $\mu(X_0) \subseteq U \to X_0 \cap \mu(U) \subseteq \mu(X_0)$,

(2) $(\mu Cum1)$ $\mu(X_0) \subseteq U$, $\mu(X_1) \subseteq U \cup X_0 \to X_0 \cap X_1 \cap \mu(U) \subseteq \mu(X_1)$,

(3) $(\mu Cum2)$ $\mu(X_0) \subseteq U$, $\mu(X_1) \subseteq U \cup X_0$, $\mu(X_2) \subseteq U \cup X_0 \cup X_1 \to X_0 \cap X_1 \cap X_2 \cap \mu(U) \subseteq \mu(X_2)$.

The semantical version of traditional cumulativity is

(μCUM) $\mu(X) \subseteq Y \subseteq X \Rightarrow \mu(X) = \mu(Y)$.

It is easy to show that (μCUM) and closure under finite unions imply $(\mu Cumi)$, but one has to be careful about the prerequisites used, so we refer the reader to Fact 4.1 (5.3).

Let T etc. be sets of formulas, and assume them to be closed under classical deduction, let $M(T)$ be the set of classical models of T. Assume further that the model choice function μ has the property that $\mu(M(T)) = M(T')$ for some T' (i.e. μ is definability preserving in the sense of Definition 2.2), and write $\overline{\overline{T}}$ for $\{\phi : T \mid\sim \phi\}$.

Then these semantical conditions for a model choice function μ translate as follows into logic:

(1) (Cum0) $T \subseteq \overline{\overline{T_0}} \Rightarrow \overline{\overline{T_0}} \subseteq T_0 \cup \overline{\overline{T}}$,

(2) (Cum1) $T \subseteq \overline{\overline{T_0}}$, $T \cap T_0 \subseteq \overline{\overline{T_1}} \Rightarrow \overline{\overline{T_1}} \subseteq T_0 \cup T_1 \cup \overline{\overline{T}}$

(3) (Cum2) $T \subseteq \overline{\overline{T_0}}$, $T \cap T_0 \subseteq \overline{\overline{T_1}}$, $T \cap T_0 \cap T_1 \subseteq \overline{\overline{T_2}} \Rightarrow \overline{\overline{T_2}} \subseteq T_0 \cup T_1 \cup T_2 \cup \overline{\overline{T}}$

Cumulativity for formula sets is now (slightly simplified)

(CUM) $T \subseteq T' \subseteq \overline{\overline{T}} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$.

Seen conversely, closure conditions of the domain - or, better, lack of them - thus reveal themselves as powerful tools to separate logical rules.

1.2 Organisation of the paper

We first give the general framework we work in: We mention the logical properties often considered in the framework of nonmonotonic, and in particular preferential logics, as well as their algebraic counterparts. The reader will find here more than he really needs to understand the main result, but it will help him to put things into perspective. We also show (without proof) connections and differences, which can be quite subtle - see [GS08c] for details and proofs.

We then present two sequent calculi, one due to Lehmann, the other to Arieli and Avron. The first one demonstrated the authors that domain closure questions can be real problems, and were not just due to an incapacity to find a proof with weaker prerequisites. The crucial point in both is that there is no "or" in the language, so the domain is not closed under finite unions.

The main result of the paper, shown in Section 4, Example 4.1 demonstrates that the problem is quite serious, as it splits the otherwise simple condition of Cumulativity into an infinity of conditions, which all collaps to one in the presence of closure of the domain under finite unions. We conclude by giving positive results for the different conditions of cumulativity - see Fact 4.1.

2 Basic definitions for preferential structures and related logics

Definition 2.1

We use $\mathcal{P}$ to denote the power set operator, $\Pi\{X_i : i \in I\} := \{g : g : I \rightarrow \bigcup\{X_i : i \in I\}, \forall i \in I.g(i) \in X_i\}$ is the general cartesian product, $card(X)$ shall denote the cardinality of X, and V the set-theoretic universe we work in - the class of all sets. Given a set of pairs $\mathcal{X}$, and a set X, we denote by $\mathcal{X}\lceil X := \{< x, i > \in \mathcal{X} : x \in X\}$. When the context is clear, we will sometime simply write X for $\mathcal{X}\lceil X$.

$A \subseteq B$ will denote that A is a subset of B or equal to B, and $A \subset B$ that A is a proper subset of B, likewise for $A \supseteq B$ and $A \supset B$.

Given some fixed set U we work in, and $X \subseteq U$, then $\boldsymbol{C}(X) := U - X$.

If $\mathcal{Y} \subseteq \mathcal{P}(X)$ for some X, we say that $\mathcal{Y}$ satisfies

$(\cap)$ iff it is closed under finite intersections,

$(\bigcap)$ iff it is closed under arbitrary intersections,

$(\cup)$ iff it is closed under finite unions,

$(\bigcup)$ iff it is closed under arbitrary unions,

$(\boldsymbol{C})$ iff it is closed under complementation.

We will sometimes write $A = B \parallel C$ for: $A = B$, or $A = C$, or $A = B \cup C$.

We make ample and tacit use of the Axiom of Choice.

Definition 2.2

We work here in a classical propositional language $\mathcal{L}$, a theory T will be an arbitrary set of formulas. Formulas will often be named ϕ, ψ, etc., theories T, S, etc.

$v(\mathcal{L})$ will be the set of propositional variables of $\mathcal{L}$.

$M_{\mathcal{L}}$ will be the set of (classical) models of $\mathcal{L}$, $M(T)$ or M_T is the set of models of T, likewise $M(\phi)$ for a formula ϕ.

$\boldsymbol{D}_{\mathcal{L}} := \{M(T) : T$ a theory in $\mathcal{L}\}$, the set of definable model sets.

Note that, in classical propositional logic, $\emptyset, M_{\mathcal{L}} \in \boldsymbol{D}_{\mathcal{L}}$, $\boldsymbol{D}_{\mathcal{L}}$ contains singletons, is closed under arbitrary intersections and finite unions.

An operation $f : \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ for $\mathcal{Y} \subseteq \mathcal{P}(M_{\mathcal{L}})$ is called definability preserving, (dp) or (μdp) in short, iff for all $X \in \boldsymbol{D}_{\mathcal{L}} \cap \mathcal{Y}$ $f(X) \in \boldsymbol{D}_{\mathcal{L}}$.

We will also use (μdp) for binary functions $f : \mathcal{Y} \times \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ - as needed for theory revision - with the obvious meaning.

$\vdash$ will be classical derivability, and

$\overline{T} := \{\phi : T \vdash \phi\}$, the closure of T under $\vdash$.

$Con(.)$ will stand for classical consistency, so $Con(\phi)$ will mean that ϕ is clasical consistent, likewise for $Con(T)$. $Con(T, T')$ will stand for $Con(T \cup T')$, etc.

Given a consequence relation $\mid\sim$, we define

$\overline{\overline{T}} := \{\phi : T \mid\sim \phi\}$.

(There is no fear of confusion with $\overline{T}$, as it just is not useful to close twice under classical logic.)

$T \vee T' := \{\phi \vee \phi' : \phi \in T, \phi' \in T'\}$.

If $X \subseteq M_{\mathcal{L}}$, then $Th(X) := \{\phi : X \models \phi\}$, likewise for $Th(m)$, $m \in M_{\mathcal{L}}$.

Definition 2.3

We introduce here formally a list of properties of set functions on the algebraic side, and their corresponding logical rules on the other side.

Recall that $\overline{T} := \{\phi : T \vdash \phi\}$, $\overline{\overline{T}} := \{\phi : T \mid\sim \phi\}$, where $\vdash$ is classical consequence, and $\mid\sim$ any other consequence.

We show, wherever adequate, in parallel the formula version in the left column, the theory version in the middle column, and the semantical or algebraic counterpart in the right column. The algebraic counterpart gives conditions for a function $f : \mathcal{Y} \to \mathcal{P}(U)$, where U is some set, and $\mathcal{Y} \subseteq \mathcal{P}(U)$.

When the formula version is not commonly used, we omit it, as we normally work only with the theory version.

Intuitively, A and B in the right hand side column stand for $M(\phi)$ for some formula ϕ, whereas X, Y stand for $M(T)$ for some theory T.

Basics		
(AND) $\phi \mid\sim \psi, \phi \mid\sim \psi' \Rightarrow$ $\phi \mid\sim \psi \wedge \psi'$	(AND) $T \mid\sim \psi, T \mid\sim \psi' \Rightarrow$ $T \mid\sim \psi \wedge \psi'$	Closure under finite intersection
(OR) $\phi \mid\sim \psi, \phi' \mid\sim \psi \Rightarrow$ $\phi \vee \phi' \mid\sim \psi$	(OR) $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	(μOR) $f(X \cup Y) \subseteq f(X) \cup f(Y)$
(wOR) $\phi \mid\sim \psi$, $\phi' \vdash \psi \Rightarrow$ $\phi \vee \phi' \mid\sim \psi$	(wOR) $\overline{\overline{T}} \cap \overline{T'} \subseteq \overline{\overline{T \vee T'}}$	(μwOR) $f(X \cup Y) \subseteq f(X) \cup Y$
$(disjOR)$ $\phi \vdash \neg \phi'$, $\phi \mid\sim \psi$, $\phi' \mid\sim \psi \Rightarrow \phi \vee \phi' \mid\sim \psi$	$(disjOR)$ $\neg Con(T \cup T') \Rightarrow$ $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$(\mu disjOR)$ $X \cap Y = \emptyset \Rightarrow$ $f(X \cup Y) \subseteq f(X) \cup f(Y)$
(LLE) Left Logical Equivalence $\vdash \phi \leftrightarrow \phi', \phi \mid\sim \psi \Rightarrow$ $\phi' \mid\sim \psi$	(LLE) $\overline{T} = \overline{T'} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$	trivially true
(RW) Right Weakening $\phi \mid\sim \psi, \vdash \psi \to \psi' \Rightarrow$ $\phi \mid\sim \psi'$	(RW) $T \mid\sim \psi, \vdash \psi \to \psi' \Rightarrow$ $T \mid\sim \psi'$	upward closure
(CCL) Classical Closure	(CCL) $\overline{\overline{T}}$ is classically closed	trivially true
(SC) Supraclassicality $\phi \vdash \psi \Rightarrow \phi \mid\sim \psi$	(SC) $\overline{T} \subseteq \overline{\overline{T}}$	$(\mu \subseteq)$ $f(X) \subseteq X$
(REF) Reflexivity $\Delta, \alpha \mid\sim \alpha$		
(CP) Consistency Preservation $\phi \mid\sim \bot \Rightarrow \phi \vdash \bot$	(CP) $T \mid\sim \bot \Rightarrow T \vdash \bot$	$(\mu\emptyset)$ $f(X) = \emptyset \Rightarrow X = \emptyset$
		$(\mu\emptyset fin)$ $X \neq \emptyset \Rightarrow f(X) \neq \emptyset$ for finite X
$\overline{\overline{\phi \wedge \phi'}} \subseteq \overline{\overline{\overline{\phi}} \cup \{\phi'\}}$	(PR) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$	(μPR) $X \subseteq Y \Rightarrow$ $f(Y) \cap X \subseteq f(X)$
		$(\mu PR')$ $f(X) \cap Y \subseteq f(X \cap Y)$
(CUT) $\Delta \mid\sim \alpha; \Delta, \alpha \mid\sim \beta \Rightarrow$ $\Delta \mid\sim \beta$	(CUT) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} \subseteq \overline{\overline{T}}$	(μCUT) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(X) \subseteq f(Y)$

Cumulativity		
(CM) Cautious Monotony $\phi \mid\sim \psi, \phi \mid\sim \psi' \Rightarrow$ $\phi \wedge \psi \mid\sim \psi'$	(CM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} \subseteq \overline{\overline{T'}}$	(μCM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) \subseteq f(X)$
or $(ResM)$ Restricted Monotony $\Delta \mid\sim \alpha, \beta \Rightarrow \Delta, \alpha \mid\sim \beta$		$(\mu ResM)$ $f(X) \subseteq A \cap B \Rightarrow f(X \cap A) \subseteq B$
(CUM) Cumulativity $\phi \mid\sim \psi \Rightarrow$ $(\phi \mid\sim \psi' \Leftrightarrow \phi \wedge \psi \mid\sim \psi')$	(CUM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}}$	(μCUM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) = f(X)$
	$(\subseteq\supseteq)$ $T \subseteq \overline{\overline{T'}}, T' \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} = \overline{\overline{T}}$	$(\mu \subseteq\supseteq)$ $f(X) \subseteq Y, f(Y) \subseteq X \Rightarrow$ $f(X) = f(Y)$
Rationality		
$(RatM)$ Rational Monotony $\phi \mid\sim \psi, \phi \not\mid\sim \neg\psi' \Rightarrow$ $\phi \wedge \psi' \mid\sim \psi$	$(RatM)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} \supseteq \overline{\overline{\overline{T'}} \cup T}$	$(\mu RatM)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) \subseteq f(Y) \cap X$
	$(RatM =)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{\overline{T'}} \cup T}$	$(\mu =)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) = f(Y) \cap X$
	$(Log =')$ $Con(\overline{\overline{T'}} \cup T) \Rightarrow$ $\overline{\overline{T \cup T'}} = \overline{\overline{\overline{T'}} \cup T}$	$(\mu =')$ $f(Y) \cap X \neq \emptyset \Rightarrow$ $f(Y \cap X) = f(Y) \cap X$
	$(Log \parallel)$ $\overline{\overline{T \vee T'}}$ is one of $\overline{\overline{T}}$, or $\overline{\overline{T'}}$, or $\overline{\overline{T}} \cap \overline{\overline{T'}}$ (by (CCL))	$(\mu \parallel)$ $f(X \cup Y)$ is one of $f(X)$, $f(Y)$ or $f(X) \cup f(Y)$
	$(Log\cup)$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\neg Con(\overline{\overline{T \vee T'}} \cup T')$	$(\mu\cup)$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) \cap Y = \emptyset$
	$(Log\cup')$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\overline{\overline{T \vee T'}} = \overline{\overline{T}}$	$(\mu\cup')$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) = f(X)$
		$(\mu \in)$ $a \in X - f(X) \Rightarrow$ $\exists b \in X. a \notin f(\{a, b\})$

(PR) is also called infinite conditionalization - we choose the name for its central role for preferential structures (PR) or (μPR).

The system of rules (AND) (OR) (LLE) (RW) (SC) (CP) (CM) (CUM) is also called system P (for preferential), adding $(RatM)$ gives the system R (for rationality or rankedness).

Roughly: Smooth preferential structures generate logics satisfying system P, ranked structures logics satisfying system R.

A logic satisfying (REF), $(ResM)$, and (CUT) is called a consequence relation.

(LLE) and(CCL) will hold automatically, whenever we work with model sets.

(AND) is obviously closely related to filters, and corresponds to closure under finite intersections. (RW) corresponds to upward closure of filters.

More precisely, validity of both depend on the definition, and the direction we consider.

Given f and $(\mu \subseteq)$, $f(X) \subseteq X$ generates a pricipal filter: $\{X' \subseteq X : f(X) \subseteq X'\}$, with the definition: If $X = M(T)$, then $T \mid\sim \phi$ iff $f(X) \subseteq M(\phi)$. Validity of (AND) and (RW) are then trivial.

Conversely, we can define for $X = M(T)$

$\mathcal{X} := \{X' \subseteq X : \exists \phi(X' = X \cap M(\phi) \text{ and } T \mid\!\sim \phi)\}.$

(AND) then makes $\mathcal{X}$ closed under finite intersections, (RW) makes $\mathcal{X}$ upward closed. This is in the infinite case usually not yet a filter, as not all subsets of X need to be definable this way. In this case, we complete $\mathcal{X}$ by adding all X'' such that there is $X' \subseteq X'' \subseteq X$, $X' \in \mathcal{X}$.

Alternatively, we can define

$\mathcal{X} := \{X' \subseteq X : \bigcap\{X \cap M(\phi) : T \mid\!\sim \phi\} \subseteq X'\}.$

(SC) corresponds to the choice of a subset.

(CP) is somewhat delicate, as it presupposes that the chosen model set is non-empty. This might fail in the presence of ever better choices, without ideal ones; the problem is addressed by the limit versions.

(PR) is an infinitary version of one half of the deduction theorem: Let T stand for ϕ, T' for ψ, and $\phi \wedge \psi \mid\!\sim \sigma$, so $\phi \mid\!\sim \psi \rightarrow \sigma$, but $(\psi \rightarrow \sigma) \wedge \psi \vdash \sigma$.

(CUM) (whose most interesting half in our context is (CM)) may best be seen as normal use of lemmas: We have worked hard and found some lemmas. Now we can take a rest, and come back again with our new lemmas. Adding them to the axioms will neither add new theorems, nor prevent old ones to hold.

Fact 2.1

This table is to be read as follows: If the left hand side holds for some function $f : \mathcal{Y} \rightarrow \mathcal{P}(U)$, and the auxiliary properties noted in the middle also hold for f or $\mathcal{Y}$, then the right hand side will hold, too - and conversely.

Basics			
(1.1)	(μPR)	$\Rightarrow (\cap) + (\mu \subseteq)$	$(\mu PR')$
(1.2)		$\Leftarrow$	
(2.1)	(μPR)	$\Rightarrow (\mu \subseteq)$	(μOR)
(2.2)		$\Leftarrow (\mu \subseteq)$ + closure under set difference	
(3)	(μPR)	$\Rightarrow$	(μCUT)
(4)	$(\mu \subseteq) + (\mu \subseteq\supseteq) + (\mu CUM) + (\mu RatM) + (\cap)$	$\not\Rightarrow$	(μPR)
Cumulativity			
(5.1)	(μCM)	$\Rightarrow (\cap) + (\mu \subseteq)$	$(\mu ResM)$
(5.2)		$\Leftarrow$ (infin.)	
(6)	$(\mu CM) + (\mu CUT)$	$\Leftrightarrow$	(μCUM)
(7)	$(\mu \subseteq) + (\mu \subseteq\supseteq)$	$\Rightarrow$	(μCUM)
(8)	$(\mu \subseteq) + (\mu CUM) + (\cap)$	$\Rightarrow$	$(\mu \subseteq\supseteq)$
(9)	$(\mu \subseteq) + (\mu CUM)$	$\not\Rightarrow$	$(\mu \subseteq\supseteq)$
Rationality			
(10)	$(\mu RatM) + (\mu PR)$	$\Rightarrow$	$(\mu =)$
(11)	$(\mu =)$	$\Rightarrow$	(μPR),
(12.1)	$(\mu =)$	$\Rightarrow (\cap) + (\mu \subseteq)$	$(\mu =')$,
(12.2)		$\Leftarrow$	
(13)	$(\mu \subseteq)$, $(\mu =)$	$\Rightarrow (\cup)$	$(\mu \cup)$,
(14)	$(\mu \subseteq)$, $(\mu \emptyset)$, $(\mu =)$	$\Rightarrow (\cup)$	$(\mu \parallel)$, $(\mu \cup')$, (μCUM),
(15)	$(\mu \subseteq) + (\mu \parallel)$	$\Rightarrow \mathcal{Y}$ closed under set difference	$(\mu =)$,
(16)	$(\mu \parallel) + (\mu \in) + (\mu PR) + (\mu \subseteq)$	$\Rightarrow (\cup)$ + $\mathcal{Y}$ contains singletons	$(\mu =)$,
(17)	$(\mu CUM) + (\mu =)$	$\Rightarrow (\cup)$ + $\mathcal{Y}$ contains singletons	$(\mu \in)$,
(18)	$(\mu CUM) + (\mu =) + (\mu \subseteq)$	$\Rightarrow (\cup)$	$(\mu \parallel)$,
(19)	$(\mu PR) + (\mu CUM) + (\mu \parallel)$	$\Rightarrow$ sufficient, e.g. true in $\boldsymbol{D}_{\mathcal{L}}$	$(\mu =)$.
(20)	$(\mu \subseteq) + (\mu PR) + (\mu =)$	$\not\Rightarrow$	$(\mu \parallel)$,
(21)	$(\mu \subseteq) + (\mu PR) + (\mu \parallel)$	$\not\Rightarrow$ (without closure under set difference),	$(\mu =)$
(22)	$(\mu \subseteq) + (\mu PR) + (\mu \parallel) + (\mu =) + (\mu \cup)$	$\not\Rightarrow$	$(\mu \in)$ (thus not representability by ranked structures)

Proposition 2.2

The following table is to be read as follows:

Let a logic $\mid\sim$ satisfies (LLE) and (CCL), and define a function $f : \boldsymbol{D}_{\mathcal{L}} \to \boldsymbol{D}_{\mathcal{L}}$ by $f(M(T)) := M(\overline{\overline{T}})$. Then f is well defined, satisfies (μdp), and $\overline{\overline{T}} = Th(f(M(T)))$.

If $\mid\sim$ satisfies a rule in the left hand side, then - provided the additional properties noted in the middle for $\Rightarrow$ hold, too - f will satisfy the property in the right hand side.

Conversely, if $f : \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ is a function, with $\boldsymbol{D}_{\mathcal{L}} \subseteq \mathcal{Y}$, and we define a logic $\mid\sim$ by $\overline{\overline{T}} := Th(f(M(T)))$, then $\mid\sim$ satisfies (LLE) and (CCL). If f satisfies (μdp), then $f(M(T)) = M(\overline{\overline{T}})$.

If f satisfies a property in the right hand side, then - provided the additional properties noted in the middle for $\Leftarrow$ hold, too - $\mid\sim$ will satisfy the property in the left hand side.

If "formula" is noted in the table, this means that, if one of the theories (the one named the same way in Definition 2.3) is equivalent to a formula, we can renounce on (μdp).

Basics			
(1.1)	(OR)	$\Rightarrow$	(μOR)
(1.2)		$\Leftarrow$	
(2.1)	$(disjOR)$	$\Rightarrow$	$(\mu disjOR)$
(2.2)		$\Leftarrow$	
(3.1)	(wOR)	$\Rightarrow$	(μwOR)
(3.2)		$\Leftarrow$	
(4.1)	(SC)	$\Rightarrow$	$(\mu \subseteq)$
(4.2)		$\Leftarrow$	
(5.1)	(CP)	$\Rightarrow$	$(\mu\emptyset)$
(5.2)		$\Leftarrow$	
(6.1)	(PR)	$\Rightarrow$	(μPR)
(6.2)		$\Leftarrow (\mu dp) + (\mu \subseteq)$	
(6.3)		$\not\Leftarrow$ without (μdp)	
(6.4)		$\Leftarrow (\mu \subseteq)$ T' a formula	
(6.5)	(PR)	$\Leftarrow$ T' a formula	$(\mu PR')$
(7.1)	(CUT)	$\Rightarrow$	(μCUT)
(7.2)		$\Leftarrow$	
Cumulativity			
(8.1)	(CM)	$\Rightarrow$	(μCM)
(8.2)		$\Leftarrow$	
(9.1)	$(ResM)$	$\Rightarrow$	$(\mu ResM)$
(9.2)		$\Leftarrow$	
(10.1)	$(\subseteq\supseteq)$	$\Rightarrow$	$(\mu \subseteq\supseteq)$
(10.2)		$\Leftarrow$	
(11.1)	(CUM)	$\Rightarrow$	(μCUM)
(11.2)		$\Leftarrow$	
Rationality			
(12.1)	$(RatM)$	$\Rightarrow$	$(\mu RatM)$
(12.2)		$\Leftarrow (\mu dp)$	
(12.3)		$\not\Leftarrow$ without (μdp)	
(12.4)		$\Leftarrow$ T a formula	
(13.1)	$(RatM =)$	$\Rightarrow$	$(\mu =)$
(13.2)		$\Leftarrow (\mu dp)$	
(13.3)		$\not\Leftarrow$ without (μdp)	
(13.4)		$\Leftarrow$ T a formula	
(14.1)	$(Log =')$	$\Rightarrow$	$(\mu =')$
(14.2)		$\Leftarrow (\mu dp)$	
(14.3)		$\not\Leftarrow$ without (μdp)	
(14.4)		$\Leftarrow T$ a formula	
(15.1)	$(Log \parallel)$	$\Rightarrow$	$(\mu \parallel)$
(15.2)		$\Leftarrow$	
(16.1)	$(Log\cup)$	$\Rightarrow (\mu \subseteq) + (\mu =)$	$(\mu\cup)$
(16.2)		$\Leftarrow (\mu dp)$	
(16.3)		$\not\Leftarrow$ without (μdp)	
(17.1)	$(Log\cup')$	$\Rightarrow (\mu \subseteq) + (\mu =)$	$(\mu\cup')$
(17.2)		$\Leftarrow (\mu dp)$	
(17.3)		$\not\Leftarrow$ without (μdp)	

Definition 2.4

Fix $U \neq \emptyset$, and consider arbitrary X. Note that this X has not necessarily anything to do with U, or $\mathcal{U}$ below. Thus, the functions $\mu_{\mathcal{M}}$ below are in principle functions from V to V - where V is the set theoretical universe we work in.

(A) Preferential models or structures.

(1) The version without copies:

A pair $\mathcal{M} :=< U, \prec>$ with U an arbitrary set, and $\prec$ an arbitrary binary relation is called a preferential model or structure.

(2) The version with copies:

A pair $\mathcal{M} :=< \mathcal{U}, \prec>$ with $\mathcal{U}$ an arbitrary set of pairs, and $\prec$ an arbitrary binary relation is called a preferential model or structure.

If $< x, i >\in \mathcal{U}$, then x is intended to be an element of U, and i the index of the copy.

We sometimes also need copies of the relation $\prec$, we will then replace $\prec$ by one or several arrows α attacking non-minimal elements, e.g. $x \prec y$ will be written $\alpha : x \rightarrow y$, $< x, i >\prec< y, i >$ will be written $\alpha :< x, i >\rightarrow< y, i >$, and finally we might have $< \alpha, k >: x \rightarrow y$ and $< \alpha, k >:< x, i >\rightarrow< y, i >$, etc.

(B) Minimal elements, the functions $\mu_{\mathcal{M}}$

(1) The version without copies:

Let $\mathcal{M} :=< U, \prec>$, and define

$\mu_{\mathcal{M}}(X) := \{x \in X : x \in U \wedge \neg\exists x' \in X \cap U.x' \prec x\}.$

$\mu_{\mathcal{M}}(X)$ is called the set of minimal elements of X (in $\mathcal{M}$).

(2) The version with copies:

Let $\mathcal{M} :=< \mathcal{U}, \prec>$ be as above. Define

$\mu_{\mathcal{M}}(X) := \{x \in X : \exists < x, i >\in \mathcal{U}.\neg\exists < x', i' >\in \mathcal{U}(x' \in X \wedge < x', i' >' \prec< x, i >)\}.$

Again, by abuse of language, we say that $\mu_{\mathcal{M}}(X)$ is the set of minimal elements of X in the structure. If the context is clear, we will also write just μ.

We sometimes say that $< x, i >$ "kills" or "minimizes" $< y, j >$ if $< x, i >\prec< y, j >$. By abuse of language we also say a set X kills or minimizes a set Y if for all $< y, j >\in \mathcal{U}$, $y \in Y$ there is $< x, i >\in \mathcal{U}$, $x \in X$ s.t. $< x, i >\prec< y, j >$.

$\mathcal{M}$ is also called injective or 1-copy, iff there is always at most one copy $< x, i >$ for each x. Note that the existence of copies corresponds to a non-injective labelling function - as is often used in nonclassical logic, e.g. modal logic.

We say that $\mathcal{M}$ is transitive, irreflexive, etc., iff $\prec$ is.

Note that $\mu(X)$ might well be empty, even if X is not.

Definition 2.5

We define the consequence relation of a preferential structure for a given propositional language $\mathcal{L}$.

(A)

(1) If m is a classical model of a language $\mathcal{L}$, we say by abuse of language

$< m, i >\models \phi$ iff $m \models \phi$,

and if X is a set of such pairs, that

$X \models \phi$ iff for all $< m,i > \in X$ $m \models \phi$.

(2) If $\mathcal{M}$ is a preferential structure, and X is a set of $\mathcal{L}-$models for a classical propositional language $\mathcal{L}$, or a set of pairs $< m,i >$, where the m are such models, we call $\mathcal{M}$ a classical preferential structure or model.

(B)

Validity in a preferential structure, or the semantical consequence relation defined by such a structure:

Let $\mathcal{M}$ be as above.

We define:

$T \models_{\mathcal{M}} \phi$ iff $\mu_{\mathcal{M}}(M(T)) \models \phi$, i.e. $\mu_{\mathcal{M}}(M(T)) \subseteq M(\phi)$.

$\mathcal{M}$ will be called definability preserving iff for all $X \in \boldsymbol{D}_{\mathcal{L}}$ $\mu_{\mathcal{M}}(X) \in \boldsymbol{D}_{\mathcal{L}}$.

As $\mu_{\mathcal{M}}$ is defined on $\boldsymbol{D}_{\mathcal{L}}$, but need by no means always result in some new definable set, this is (and reveals itself as a quite strong) additional property.

Definition 2.6

Let $\mathcal{Y} \subseteq \mathcal{P}(U)$. (In applications to logic, $\mathcal{Y}$ will be $\boldsymbol{D}_{\mathcal{L}}$.)

A preferential structure $\mathcal{M}$ is called $\mathcal{Y}-$smooth iff in every $X \in \mathcal{Y}$ every element $x \in X$ is either minimal in X or above an element, which is minimal in X. More precisely:

(1) The version without copies:

If $x \in X \in \mathcal{Y}$, then either $x \in \mu(X)$ or there is $x' \in \mu(X).x' \prec x$.

(2) The version with copies:

If $x \in X \in \mathcal{Y}$, and $< x,i > \in \mathcal{U}$, then either there is no $< x',i' > \in \mathcal{U}$, $x' \in X$, $< x',i' > \prec < x,i >$ or there is $< x',i' > \in \mathcal{U}$, $< x',i' > \prec < x,i >$, $x' \in X$, s.t. there is no $< x'',i'' > \in \mathcal{U}$, $x'' \in X$, with $< x'',i'' > \prec < x',i' >$.

When considering the models of a language $\mathcal{L}$, $\mathcal{M}$ will be called smooth iff it is $\boldsymbol{D}_{\mathcal{L}}-$smooth; $\boldsymbol{D}_{\mathcal{L}}$ is the default.

Obviously, the richer the set $\mathcal{Y}$ is, the stronger the condition $\mathcal{Y}-$smoothness will be.

The following table summarizes representation by not necessarily ranked preferential structures. The implications on the right are shown in Proposition 2.2 (going via the $\mu-$functions), those on the left are shown in the respective representation theorems.

$\mu-$ function		Pref.Structure		Logic
$(\mu\subseteq)+(\mu PR)$	$\Leftarrow$	general	$\Rightarrow$ (μdp)	$(LLE)+(RW)+(SC)+(PR)$
	$\Rightarrow$		$\Leftarrow$	
			$\not\Rightarrow$ without (μdp)	
$(\mu\subseteq)+(\mu PR)$	$\Leftarrow$	transitive	$\Rightarrow$ (μdp)	$(LLE)+(RW)+(SC)+(PR)$
	$\Rightarrow$		$\Leftarrow$	
			$\not\Rightarrow$ without (μdp)	
$(\mu\subseteq)+(\mu PR)+(\mu CUM)$	$\Leftarrow$	smooth	$\Rightarrow$ (μdp)	$(LLE)+(RW)+(SC)+(PR)+$ (CUM)
	$\Rightarrow$ $(\cup)$		$\Leftarrow$ $(\cup)$	
			$\not\Rightarrow$ without (μdp)	
$(\mu\subseteq)+(\mu PR)+(\mu CUM)$	$\Leftarrow$	smooth+transitive	$\Rightarrow$ (μdp)	$(LLE)+(RW)+(SC)+(PR)+$ (CUM)
	$\Rightarrow$ $(\cup)$		$\Leftarrow$ $(\cup)$	
			$\not\Rightarrow$ without (μdp)	

3 Motivation - two sequent calculi

3.1 Introduction

This section serves mainly as a posteriori motivation for our examination of weak closure conditions of the domain. The second author realized first when looking at Lehmann's plausibility logic, that absence of $(\cup)$ might be a problem for representation.

Beyond motivation, the reader will see here two "real life" examples where closure under $(\cup)$ is not given, and thus problems arise. So this is also a warning against a too naive treatment of representation problems, neglecting domain closure issues.

3.2 Plausibility Logic

Discussion of plausibility logic

Plausibility logic was introduced by D. Lehmann [Leh92a], [Leh92b] as a sequent calculus in a propositional language without connectives. Thus, a plausibility logic language $\mathcal{L}$ is just a set, whose elements correspond to propositional variables, and a sequent has the form $X \mid\!\sim Y$, where X, Y are $\underline{finite}$ subsets of $\mathcal{L}$, thus, in the intuitive reading, $\bigwedge X \mid\!\sim \bigvee Y$. (We use $\mid\!\sim$ instead of the $\vdash$ used in [Leh92a], [Leh92b] and continue to reserve $\vdash$ for classical logic.)

The details:

Notation 3.1

We abuse notation, and write $X \mid\!\sim a$ for $X \mid\!\sim \{a\}$, $X, a \mid\!\sim Y$ for $X \cup \{a\} \mid\!\sim Y$, $ab \mid\!\sim Y$ for $\{a,b\} \mid\!\sim Y$, etc. When discussing plausibility logic, X, Y, etc. will denote finite subsets of $\mathcal{L}$, a, b, etc. elements of $\mathcal{L}$.

We first define the logical properties we will examine.

Definition 3.1

X and Y will be finite subsets of $\mathcal{L}$, a, etc. elements of $\mathcal{L}$. The base axiom and rules of plausibility logic are (we use the prefix "Pl" to differentiate them from the usual ones):

(PlI) (Inclusion): $X \mid\!\sim a$ for all $a \in X$,

(PlRM) (Right Monotony): $X \mid\!\sim Y \Rightarrow X \mid\!\sim a, Y$,

(PlCLM) (Cautious Left Monotony): $X \mid\!\sim a$, $X \mid\!\sim Y \Rightarrow X, a \mid\!\sim Y$,

(PlCC) (Cautious Cut): $X, a_1 \ldots a_n \mid\!\sim Y$, and for all $1 \leq i \leq n$ $X \mid\!\sim a_i, Y \Rightarrow X \mid\!\sim Y$,

and as a special case of (PlCC):

(PlUCC) (Unit Cautious Cut): $X, a \mid\!\sim Y$, $X \mid\!\sim a, Y \Rightarrow X \mid\!\sim Y$.

and we denote by PL, for plausibility logic, the full system, i.e. $(PlI)+(PlRM)+(PlCLM)+(PlCC)$. $\square$

We now adapt the definition of a preferential model to plausibility logic. This is the central definition on the semantic side.

Definition 3.2

Fix a plausibility logic language $\mathcal{L}$. A model for $\mathcal{L}$ is then just an arbitrary subset of $\mathcal{L}$.

If $\mathcal{M} :=< M, \prec>$ is a preferential model s.t. M is a set of (indexed) $\mathcal{L}-$models, then for a finite set $X \subseteq \mathcal{L}$ (to be imagined on the left hand side of $\mid\sim$!), we define

(a) $m \models X$ iff $X \subseteq m$

(b) $M(X) := \{m$: $< m, i >\in M$ for some i and $m \models X\}$

(c) $\mu(X) := \{m \in M(X)$: $\exists < m, i >\in M.\neg\exists < m', i' >\in M$ $(m' \in M(X) \wedge < m', i' >\prec< m, i >)\}$

(d) $X \models_{\mathcal{M}} Y$ iff $\forall m \in \mu(X).m \cap Y \neq \emptyset$. $\square$

(a) reflects the intuitive reading of X as $\bigwedge X$, and (d) that of Y as $\bigvee Y$ in $X \mid\sim Y$. Note that X is a set of "formulas", and $\mu(X) = \mu_{\mathcal{M}}(M(X))$.

We note as trivial consequences of the definition.

Fact 3.1

(a) $a \models_{\mathcal{M}} b$ iff for all $m \in \mu(a).b \in m$

(b) $X \models_{\mathcal{M}} Y$ iff $\mu(X) \subseteq \bigcup\{M(b) : b \in Y\}$

(c) $m \in \mu(X) \wedge X \subseteq X' \wedge m \in M(X') \rightarrow m \in \mu(X')$. $\square$

We note without proof: $(PlI) + (PlRM) + (PlCC)$ is complete (and sound) for preferential models

We note the following fact for smooth preferential models:

Fact 3.2

Let U, X, Y be any sets, $\mathcal{M}$ be smooth for at least $\{Y, X\}$ and let $\mu(Y) \subseteq U \cup X$, $\mu(X) \subseteq U$, then $X \cap Y \cap \mu(U) \subseteq \mu(Y)$. (This is, of course, a special case of $(\mu Cum1)$, see Definition 4.1.

Example 3.1

Let $\mathcal{L} := \{a, b, c, d, e, f\}$, and $\mathcal{X} := \{a \mid\sim b,\ b \mid\sim a,\ a \mid\sim c,\ a \mid\sim fd,\ dc \mid\sim ba,\ dc \mid\sim e,\ fcba \mid\sim e\}$. We show that $\mathcal{X}$ does not have a smooth representation.

Fact 3.3

$\mathcal{X}$ does not entail $a \mid\sim e$.

See [Sch96-3] for a proof.

Suppose now that there is a smooth preferential model $\mathcal{M} =< M, \prec>$ for plausibility logic which represents $\mid\sim$, i.e. for all X, Y finite subsets of $\mathcal{L}$ $X \mid\sim Y$ iff $X \models_{\mathcal{M}} Y$. (See Definition 3.2 and Fact 3.1.)

$a \mid\sim a$, $a \mid\sim b$, $a \mid\sim c$ implies for $m \in \mu(a)$ $a, b, c \in m$. Moreover, as $a \mid\sim df$, then also $d \in m$ or $f \in m$. As $a \not\mid\sim e$, there must be $m \in \mu(a)$ s.t. $e \notin m$. Suppose now $m \in \mu(a)$ with $f \in m$. So $a, b, c, f \in m$, thus by $m \in \mu(a)$ and Fact 3.1, $m \in \mu(a, b, c, f)$. But $fcba \mid\sim e$, so $e \in m$. We thus have shown that $m \in \mu(a)$ and $f \in m$ implies $e \in m$. Consequently, there must be $m \in \mu(a)$ s.t. $d \in m$, $e \notin m$. Thus, in particular, as $cd \mid\sim e$, there is $m \in \mu(a)$, $a, b, c, d \in m$, $m \notin \mu(cd)$. But by $cd \mid\sim ab$, and $b \mid\sim a$, $\mu(cd) \subseteq M(a) \cup M(b)$

and $\mu(b) \subseteq M(a)$ by Fact 3.1. Let now $T := M(cd)$, $R := M(a)$, $S := M(b)$, and $\mu_{\mathcal{M}}$ be the choice function of the minimal elements in the structure $\mathcal{M}$, we then have by $\mu(S) = \mu_{\mathcal{M}}(M(S))$:

1. $\mu_{\mathcal{M}}(T) \subseteq R \cup S$,

2. $\mu_{\mathcal{M}}(S) \subseteq R$,

3. there is $m \in S \cap T \cap \mu_{\mathcal{M}}(R)$, but $m \notin \mu_{\mathcal{M}}(T)$,

but this contradicts above Fact 3.2. □ (Example Plausi-1)

3.3 A comment on the work by Arieli and Avron

We turn to a similar case, published in [AA00]. Definitions are due to [AA00], for motivation the reader is referred there.

Definition 3.3

(1) A Scott consequence relation, abbreviated scr, is a binary relation $\vdash$ between sets of formulae, that satisfies the following conditions:

(s-R) if $\Gamma \cap \Delta \neq \emptyset$, the $\Gamma \vdash \Delta$ (M) if $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, then $\Gamma' \vdash \Delta'$ (C) if $\Gamma \vdash \psi, \Delta$ and $\Gamma', \psi \vdash \Delta'$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$

(2) A Scott cautious consequence relation, abbreviated sccr, is a binary relation $\mid\!\sim$ between nonempty sets of formulae, that satisfies the following conditions:

(s-R) if $\Gamma \cap \Delta \neq \emptyset$, the $\Gamma \mid\!\sim \Delta$ (CM) if $\Gamma \mid\!\sim \Delta$ and $\Gamma \mid\!\sim \psi$, then $\Gamma, \psi \mid\!\sim \Delta$ (CC) if $\Gamma \mid\!\sim \psi$ and $\Gamma, \psi \mid\!\sim \Delta$, then $\Gamma \mid\!\sim \Delta$.

Example 3.2

We have two consequence relations, $\vdash$ and $\mid\!\sim$.

The rules to consider are

LCC^n $\frac{\Gamma|\sim\psi_1,\Delta\ldots\Gamma|\sim\psi_n,\Delta\Gamma,\psi_1,\ldots,\psi_n|\sim\Delta}{\Gamma|\sim\Delta}$

RW^n $\frac{\Gamma|\sim\psi_i,\Delta i=1\ldots n\Gamma,\psi_1,\ldots,\psi_n\vdash\phi}{\Gamma|\sim\phi,\Delta}$

Cum $\Gamma, \Delta \neq \emptyset$, $\Gamma \vdash \Delta \rightarrow \Gamma \mid\!\sim \Delta$

RM $\Gamma \mid\!\sim \Delta \rightarrow \Gamma \mid\!\sim \psi, \Delta$

CM $\frac{\Gamma|\sim\psi\Gamma|\sim\Delta}{\Gamma,\psi|\sim\Delta}$

$s - R$ $\Gamma \cap \Delta \neq \emptyset \rightarrow \Gamma \mid\!\sim \Delta$

M $\Gamma \vdash \Delta$, $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta' \rightarrow \Gamma' \vdash \Delta'$

C $\frac{\Gamma_1\vdash\psi,\Delta_1\Gamma_2,\psi\vdash\Delta_2}{\Gamma_1,\Gamma_2\vdash\Delta_1,\Delta_2}$

Let $\mathcal{L}$ be any set. Define now $\Gamma \vdash \Delta$ iff $\Gamma \cap \Delta \neq \emptyset$. Then $s - R$ and M for $\vdash$ are trivial. For C : If $\Gamma_1 \cap \Delta_1 \neq \emptyset$ or $\Gamma_1 \cap \Delta_1 \neq \emptyset$, the result is trivial. If not, $\psi \in \Gamma_1$ and $\psi \in \Delta_2$, which implies the result. So $\vdash$ is a scr.

Consider now the rules for a sccr which is $\vdash$ $-$plausible for this $\vdash$. Cum is equivalent to $s-$R, which is essentially (PlI) of Plausibility Logic. Consider RW^n. If ϕ is one of the ψ_i, then the consequence

$\Gamma \mid\!\sim \phi, \Delta$ is a case of one of the other hypotheses. If not, $\phi \in \Gamma$, so $\Gamma \mid\!\sim \phi$ by $s-$R, so $\Gamma \mid\!\sim \phi, \Delta$ by RM (if Δ is finite). So, for this $\vdash$, RW^n is a consequence of $s - R +$ RM.

We are left with LCC^n, RM, CM, $s-$R, it was shown in [Sch04] and [Sch96-3] that this does not suffice to guarantee smooth representability, by failure of $(\mu Cum1)$ - see Definition 4.1.

4 Cumulativity without $(\cup)$

4.1 Introduction

Comment 4.1

We show here that, without sufficient closure properties, there is an infinity of versions of cumulativity, which collaps to usual cumulativity when the domain is closed under finite unions. Closure properties thus reveal themselves as a powerful tool to show independence of properties.

We work in some fixed arbitrary set Z, all sets considered will be subsets of Z.

Unless said otherwise, we use without further mentioning (μPR) and $(\mu \subseteq)$.

We first give the definition of the new conditions in this introduction, and then state and prove the main result (Example 4.1), and finally show some properties of the new conditions (Fact 4.1).

Definition 4.1

For any ordinal α, we define

$(\mu Cum\alpha)$:

If for all $\beta \leq \alpha$ $\mu(X_\beta) \subseteq U \cup \bigcup\{X_\gamma : \gamma < \beta\}$ hold, then so does $\bigcap\{X_\gamma : \gamma \leq \alpha\} \cap \mu(U) \subseteq \mu(X_\alpha)$.

$(\mu Cumt\alpha)$:

If for all $\beta \leq \alpha$ $\mu(X_\beta) \subseteq U \cup \bigcup\{X_\gamma : \gamma < \beta\}$ hold, then so does $X_\alpha \cap \mu(U) \subseteq \mu(X_\alpha)$.

(“ t ” stands for transitive, see Fact 4.1, (2.2) below.)

$(\mu Cum\infty)$ and $(\mu Cumt\infty)$ will be the class of all $(\mu Cum\alpha)$ or $(\mu Cumt\alpha)$ - read their “conjunction”, i.e. if we say that $(\mu Cum\infty)$ holds, we mean that all $(\mu Cum\alpha)$ hold.

Note Cum-Alpha

The first conditions thus have the form:

$(\mu Cum0)$ $\mu(X_0) \subseteq U \rightarrow X_0 \cap \mu(U) \subseteq \mu(X_0)$,

$(\mu Cum1)$ $\mu(X_0) \subseteq U$, $\mu(X_1) \subseteq U \cup X_0 \rightarrow X_0 \cap X_1 \cap \mu(U) \subseteq \mu(X_1)$,

$(\mu Cum2)$ $\mu(X_0) \subseteq U$, $\mu(X_1) \subseteq U \cup X_0$, $\mu(X_2) \subseteq U \cup X_0 \cup X_1 \rightarrow X_0 \cap X_1 \cap X_2 \cap \mu(U) \subseteq \mu(X_2)$.

$(\mu Cumt\alpha)$ differs from $(\mu Cum\alpha)$ only in the consequence, the intersection contains only the last X_α - in particular, $(\mu Cum0)$ and $(\mu Cumt0)$ coincide.

Recall that condition $(\mu Cum1)$ is the crucial condition in [Leh92a], which failed, despite (μCUM), but which has to hold in all smooth models. This condition $(\mu Cum1)$ was the starting point of the investigation.

4.2 The results

Example 4.1

This important example shows that the conditions $(\mu Cum\alpha)$ and $(\mu Cumt\alpha)$ defined in Definition 4.1 are all different in the absence of $(\cup)$, in its presence they all collaps (see Fact 4.1 below). More precisely, the following (class of) $example(s)$ shows that the $(\mu Cum\alpha)$ increase in strength. For any finite or infinite ordinal $\kappa > 0$ we construct an example s.t.

(a) (μPR) and $(\mu \subseteq)$ hold

(b) (μCUM) holds

(c) $(\bigcap)$ holds

(d) $(\mu Cumt\alpha)$ holds for $\alpha < \kappa$

(e) $(\mu Cum\kappa)$ fails.

Proof:

We define a suitable base set and a non-transitive binary relation $\prec$ on this set, as well as a suitable set $\mathcal{X}$ of subsets, closed under arbitrary intersections, but not under finite unions, and define μ on these subsets as usual in preferential structures by $\prec$. Thus, (μPR) and $(\mu \subseteq)$ will hold. It will be immediate that $(\mu Cum\kappa)$ fails, and we will show that (μCUM) and $(\mu Cumt\alpha)$ for $\alpha < \kappa$ hold by examining the cases.

For simplicity, we first define a set of generators for $\mathcal{X}$, and close under $(\bigcap)$ afterwards. The set U will have a special position, it is the "useful" starting point to construct chains corresponding to above definitions of $(\mu Cum\alpha)$ and $(\mu Cumt\alpha)$.

In the sequel, i, j will be successor ordinals, λ etc. limit ordinals, α, β, κ any ordinals, thus e.g. $\lambda \leq \kappa$ will imply that λ is a limit ordinal $\leq \kappa$, etc.

The base set and the relation $\prec$:

$\kappa > 0$ is fixed, but arbitrary. We go up to $\kappa > 0$.

The base set is $\{a, b, c\} \cup \{d_\lambda : \lambda \leq \kappa\} \cup \{x_\alpha : \alpha \leq \kappa + 1\} \cup \{x'_\alpha : \alpha \leq \kappa\}$. $a \prec b \prec c$, $x_\alpha \prec x_{\alpha+1}$, $x_\alpha \prec x'_\alpha$, $x'_0 \prec x_\lambda$ (for any λ) - $\prec$ is NOT transitive.

The generators:

$U := \{a, c, x_0\} \cup \{d_\lambda : \lambda \leq \kappa\}$ - i.e. $\ldots .\{d_\lambda : lim(\lambda) \wedge \lambda \leq \kappa\}$,

$X_i := \{c, x_i, x'_i, x_{i+1}\}$ $(i < \kappa)$,

$X_\lambda := \{c, d_\lambda, x_\lambda, x'_\lambda, x_{\lambda+1}\} \cup \{x'_\alpha : \alpha < \lambda\}$ $(\lambda < \kappa)$,

$X'_\kappa := \{a, b, c, x_\kappa, x'_\kappa, x_{\kappa+1}\}$ if κ is a successor,

$X'_\kappa := \{a, b, c, d_\kappa, x_\kappa, x'_\kappa, x_{\kappa+1}\} \cup \{x'_\alpha : \alpha < \kappa\}$ if κ is a limit.

Thus, $X'_\kappa = X_\kappa \cup \{a, b\}$ if X_κ were defined.

Note that there is only one X'_κ, and X_α is defined only for $\alpha < \kappa$, so we will not have X_α and X'_α at the same time.

Thus, the values of the generators under μ are:

$\mu(U) = U$,

$\mu(X_i) = \{c, x_i\}$,
$\mu(X_\lambda) = \{c, d_\lambda\} \cup \{x'_\alpha : \alpha < \lambda\}$,
$\mu(X'_i) = \{a, x_i\}$ $(i > 0!)$,
$\mu(X'_\lambda) = \{a, d_\lambda\} \cup \{x'_\alpha : \alpha < \lambda\}$.
(We do not assume that the domain is closed under μ.)

Intersections:

We consider first pairwise intersections:

(1) $U \cap X_0 = \{c, x_0\}$,

(2) $U \cap X_i = \{c\}$, $i > 0$,

(3) $U \cap X_\lambda = \{c, d_\lambda\}$,

(4) $U \cap X'_i = \{a, c\}$ $(i > 0!)$,

(5) $U \cap X'_\lambda = \{a, c, d_\lambda\}$,

(6) $X_i \cap X_j$:

(6.1) $j = i + 1$ $\{c, x_{i+1}\}$,

(6.2) else $\{c\}$,

(7) $X_i \cap X_\lambda$:

(7.1) $i < \lambda$ $\{c, x'_i\}$,

(7.2) $i = \lambda + 1$ $\{c, x_{\lambda+1}\}$,

(7.3) $i > \lambda + 1$ $\{c\}$,

(8) $X_\lambda \cap X_{\lambda'}$: $\{c\} \cup \{x'_\alpha : \alpha \leq min(\lambda, \lambda')\}$.

As X'_κ occurs only once, $X_\alpha \cap X'_\kappa$ etc. give no new results.

Note that μ is constant on all these pairwise intersections.

Iterated intersections:

As c is an element of all sets, sets of the type $\{c, z\}$ do not give any new results. The possible subsets of $\{a, c, d_\lambda\}$: $\{c\}$, $\{a, c\}$, $\{c, d_\lambda\}$ exist already. Thus, the only source of new sets via iterated intersections is $X_\lambda \cap X_{\lambda'} = \{c\} \cup \{x'_\alpha : \alpha \leq min(\lambda, \lambda')\}$. But, to intersect them, or with some old sets, will not generate any new sets either. Consequently, the example satisfies $(\bigcap)$ for $\mathcal{X}$ defined by U, X_i $(i < \kappa)$, X_λ $(\lambda < \kappa)$, X'_κ, and above paiwise intersections.

We will now verify the positive properties. This is tedious, but straightforward, we have to check the different cases.

Validity of (μCUM)**:**

Consider the prerequisite $\mu(X) \subseteq Y \subseteq X$. If $\mu(X) = X$ or if $X - \mu(X)$ is a singleton, X cannot give a violation of (μCUM). So we are left with the following candidates for X :

(1) $X_i := \{c, x_i, x'_i, x_{i+1}\}$, $\mu(X_i) = \{c, x_i\}$

Interesting candidates for Y will have 3 elements, but they will all contain a. (If $\kappa < \omega : U = \{a, c, x_0\}$.)

(2) $X_\lambda := \{c, d_\lambda, x_\lambda, x'_\lambda, x_{\lambda+1}\} \cup \{x'_\alpha : \alpha < \lambda\}$, $\mu(X_\lambda) = \{c, d_\lambda\} \cup \{x'_\alpha : \alpha < \lambda\}$

The only sets to contain d_λ are X_λ, U, $U \cap X_\lambda$. But $a \in U$, and $U \cap X_\lambda$ ist finite. (X_λ and X'_λ cannot be present at the same time.)

(3) $X'_i := \{a, b, c, x_i, x'_i, x_{i+1}\}$, $\mu(X'_i) = \{a, x_i\}$

a is only in U, X'_i, $U \cap X'_i = \{a, c\}$, but $x_i \notin U$, as $i > 0$.

(4) $X'_\lambda := \{a, b, c, d_\lambda, x_\lambda, x'_\lambda, x_{\lambda+1}\} \cup \{x'_\alpha : \alpha < \lambda\}$, $\mu(X'_\lambda) = \{a, d_\lambda\} \cup \{x'_\alpha : \alpha < \lambda\}$

d_λ is only in X'_λ and U, but U contains no x'_α.

Thus, (μCUM) holds trivially.

$(\mu Cumt\alpha)$ **hold for** $\alpha < \kappa$**:**

To simplify language, we say that we reach Y from X iff $X \neq Y$ and there is a sequence X_β, $\beta \leq \alpha$ and $\mu(X_\beta) \subseteq X \cup \bigcup\{X_\gamma : \gamma < \beta\}$, and $X_\alpha = Y$, $X_0 = X$. Failure of $(\mu Cumt\alpha)$ would then mean that there are X and Y, we can reach Y from X, and $x \in (\mu(X) \cap Y) - \mu(Y)$. Thus, in a counterexample, $Y = \mu(Y)$ is impossible, so none of the intersections can be such Y.

To reach Y from X, we have to get started from X, i.e. there must be Z s.t. $\mu(Z) \subseteq X$, $Z \not\subseteq X$ (so $\mu(Z) \neq Z$). Inspection of the different cases shows that we cannot reach any set Y from any case of the intersections, except from (1), (6.1), (7.2).

If Y contains a globally minimal element (i.e. there is no smaller element in any set), it can only be reached from any X which already contains this element. The globally minimal elements are a, x_0, and the d_λ, $\lambda \leq \kappa$.

By these observations, we see that X_λ and X'_κ can only be reached from U. From no X_α U can be reached, as the globally minimal a is missing. But U cannot be reached from X'_κ either, as the globally minimal x_0 is missing.

When we look at the relation $\prec$ defining μ, we see that we can reach Y from X only by going upwards, adding bigger elements. Thus, from X_α, we cannot reach any X_β, $\beta < \alpha$, the same holds for X'_κ and X_β, $\beta < \kappa$. Thus, from X'_κ, we cannot go anywhere interesting (recall that the intersections are not candidates for a Y giving a contradiction).

Consider now X_α. We can go up to any $X_{\alpha+n}$, but not to any X_λ, $\alpha < \lambda$, as d_λ is missing, neither to X'_κ, as a is missing. And we will be stopped by the first $\lambda > \alpha$, as x_λ will be missing to go beyond X_λ. Analogous observations hold for the remaining intersections (1), (6.1), (7.2). But in all these sets we can reach, we will not destroy minimality of any element of X_α (or of the intersections).

Consequently, the only candidates for failure will all start with U. As the only element of U not globally minimal is c, such failure has to have $c \in Y - \mu(Y)$, so Y has to be X'_κ. Suppose we omit one of the X_α in the sequence going up to X'_κ. If $\kappa \geq \lambda > \alpha$, we cannot reach X_λ and beyond, as x'_α will be missing. But we cannot go to $X_{\alpha+n}$ either, as $x_{\alpha+1}$ is missing. So we will be stopped at X_α. Thus, to see failure, we need the full sequence $U = X_0$, $X'_\kappa = Y_\kappa$, $Y_\alpha = X_\alpha$ for $0 < \alpha < \kappa$.

$(\mu Cum\kappa)$ **fails:**

The full sequence $U = X_0$, $X'_\kappa = Y_\kappa$, $Y_\alpha = X_\alpha$ for $0 < \alpha < \kappa$ shows this, as $c \in \mu(U) \cap X'_\kappa$, but $c \notin \mu(X'_\kappa)$.

Consequently, the example satisfies $(\bigcap)$, (μCUM), $(\mu Cumt\alpha)$ for $\alpha < \kappa$, and $(\mu Cum\kappa)$ fails.

□

To put our work more into perspective, we mention and prove now some positive results about the $(\mu Cum\alpha)$ and $(\mu Cumt\alpha)$.

Fact 4.1

We summarize some properties of $(\mu Cum\alpha)$ and $(\mu Cumt\alpha)$ - sometimes with some redundancy. Unless said otherwise, α, β etc. will be arbitrary ordinals.

For (1) to (6) (μPR) and $(\mu \subseteq)$ are assumed to hold, for (7) only $(\mu \subseteq)$.

(1) Downward:

(1.1) $(\mu Cum\alpha) \rightarrow (\mu Cum\beta)$ for all $\beta \leq \alpha$

(1.2) $(\mu Cumt\alpha) \rightarrow (\mu Cumt\beta)$ for all $\beta \leq \alpha$

(2) Validity of $(\mu Cum\alpha)$ and $(\mu Cumt\alpha)$:

(2.1) All $(\mu Cum\alpha)$ hold in smooth preferential structures

(2.2) All $(\mu Cumt\alpha)$ hold in transitive smooth preferential structures

(2.3) $(\mu Cumt\alpha)$ for $0 < \alpha$ do not necessarily hold in smooth structures without transitivity, even in the presence of $(\bigcap)$

(3) Upward:

(3.1) $(\mu Cum\beta) + (\cup) \rightarrow (\mu Cum\alpha)$ for all $\beta \leq \alpha$

(3.2) $(\mu Cumt\beta) + (\cup) \rightarrow (\mu Cumt\alpha)$ for all $\beta \leq \alpha$

(3.3) $\{(\mu Cumt\beta) : \beta < \alpha\} + (\mu CUM) + (\bigcap) \not\rightarrow (\mu Cum\alpha)$ for $\alpha > 0$.

(4) Connection $(\mu Cum\alpha)/(\mu Cumt\alpha)$:

(4.1) $(\mu Cumt\alpha) \rightarrow (\mu Cum\alpha)$

(4.2) $(\mu Cum\alpha) + (\bigcap) \not\rightarrow (\mu Cumt\alpha)$

(4.3) $(\mu Cum\alpha) + (\cup) \rightarrow (\mu Cumt\alpha)$

(5) (μCUM) and $(\mu Cumi)$:

(5.1) $(\mu CUM) + (\cup)$ entail:

(5.1.1) $\mu(A) \subseteq B \rightarrow \mu(A \cup B) = \mu(B)$

(5.1.2) $\mu(X) \subseteq U$, $U \subseteq Y \rightarrow \mu(Y \cup X) = \mu(Y)$

(5.1.3) $\mu(X) \subseteq U$, $U \subseteq Y \rightarrow \mu(Y) \cap X \subseteq \mu(U)$

(5.2) $(\mu Cum\alpha) \rightarrow (\mu CUM)$ for all α

(5.3) $(\mu CUM) + (\cup) \rightarrow (\mu Cum\alpha)$ for all α

(5.4) $(\mu CUM) + (\cap) \rightarrow (\mu Cum0)$

(6) (μCUM) and $(\mu Cumt\alpha)$:

(6.1) $(\mu Cumt\alpha) \rightarrow (\mu CUM)$ for all α

(6.2) $(\mu CUM) + (\cup) \rightarrow (\mu Cumt\alpha)$ for all α

(6.3) $(\mu CUM) \not\rightarrow (\mu Cumt\alpha)$ for all $\alpha > 0$

(7) $(\mu Cum0) \rightarrow (\mu PR)$

Proof of Fact Cum-Alpha

We prove these facts in a different order: (1), (2), (5.1), (5.2), (4.1), (6.1), (6.2), (5.3), (3.1), (3.2), (4.2), (4.3), (5.4), (3.3), (6.3), (7).

(1.1)

For $\beta < \gamma \leq \alpha$ set $X_\gamma := X_\beta$. Let the prerequisites of $(\mu Cum\beta)$ hold. Then for γ with $\beta < \gamma \leq \alpha$ $\mu(X_\gamma) \subseteq X_\beta$ by $(\mu \subseteq)$, so the prerequisites of $(\mu Cum\alpha)$ hold, too, so by $(\mu Cum\alpha)$ $\bigcap\{X_\delta : \delta \leq \beta\} \cap \mu(U)$ $= \bigcap\{X_\delta : \delta \leq \alpha\} \cap \mu(U) \subseteq \mu(X_\alpha) = \mu(X_\beta)$.

(1.2)

Analogous.

(2.1)

Proof by induction.

$(\mu Cum0)$ Let $\mu(X_0) \subseteq U$, suppose there is $x \in \mu(U) \cap (X_0 - \mu(X_0))$. By smoothness, there is $y \prec x$, $y \in \mu(X_0) \subseteq U$, *contradiction* (The same arguments works for copies: all copies of x must be minimized by some $y \in \mu(X_0)$, but at least one copy of x has to be minimal in U.)

Suppose $(\mu Cum\beta)$ hold for all $\beta < \alpha$. We show $(\mu Cum\alpha)$. Let the prerequisites of $(\mu Cum\alpha)$ hold, then those for $(\mu Cum\beta)$, $\beta < \alpha$ hold, too. Suppose there is $x \in \mu(U) \cap \bigcap\{X_\gamma : \gamma \leq \alpha\} - \mu(X_\alpha)$. So by $(\mu Cum\beta)$ for $\beta < \alpha$ $x \in \mu(X_\beta)$ moreover $x \in \mu(U)$. By smoothness, there is $y \in \mu(X_\alpha) \subseteq U \cup \bigcup\{X_\beta : \beta < \alpha\}$, $y \prec x$, but this is a contradiction. The same argument works again for copies.

(2.2)

We use the following Fact: Let, in a smooth transitive structure, $\mu(X_\beta) \subseteq U \cup \bigcup\{X_\gamma : \gamma < \beta\}$ for all $\beta \leq \alpha$, and let $x \in \mu(U)$. Then there is no $y \prec x$, $y \in U \cup \bigcup\{X_\gamma : \gamma \leq \alpha\}$.

Proof of the Fact by induction: $\alpha = 0 : y \in U$ is impossible: if $y \in X_0$, then if $y \in \mu(X_0) \subseteq U$, which is impossible, or there is $z \in \mu(X_0)$, $z \prec y$, so $z \prec x$ by transitivity, but $\mu(X_0) \subseteq U$. Let the result hold for all $\beta < \alpha$, but fail for α, so $\neg\exists y \prec x.y \in U \cup \bigcup\{X_\gamma : \gamma < \alpha\}$, but $\exists y \prec x.y \in U \cup \bigcup\{X_\gamma : \gamma \leq \alpha\}$, so $y \in X_\alpha$. If $y \in \mu(X_\alpha)$, then $y \in U \cup \bigcup\{X_\gamma : \gamma < \alpha\}$, but this is impossible, so $y \in X_\alpha - \mu(X_\alpha)$, let by smoothness $z \prec y$, $z \in \mu(X_\alpha)$, so by transitivity $z \prec x$, *contradiction*. The result is easily modified for the case with copies.

Let the prerequisites of $(\mu Cumt\alpha)$ hold, then those of the Fact will hold, too. Let now $x \in \mu(U) \cap (X_\alpha - \mu(X_\alpha))$, by smoothness, there must be $y \prec x$, $y \in \mu(X_\alpha) \subseteq U \cup \bigcup\{X_\gamma : \gamma < \alpha\}$, contradicting the Fact.

(2.3)

Let $\alpha > 0$, and consider the following structure over $\{a, b, c\} : U := \{a, c\}$, $X_0 := \{b, c\}$, $X_\alpha := \ldots :=$ $X_1 := \{a, b\}$, and their intersections, $\{a\}$, $\{b\}$, $\{c\}$, $\emptyset$ with the order $c \prec b \prec a$ (without transitivity). This is preferential, so (μPR) and $(\mu \subseteq)$ hold. The structure is smooth for U, all X_β, and their intersections. We have $\mu(X_0) \subseteq U$, $\mu(X_\beta) \subseteq U \cup X_0$ for all $\beta \leq \alpha$, so $\mu(X_\beta) \subseteq U \cup \bigcup\{X_\gamma : \gamma < \beta\}$ for all $\beta \leq \alpha$ but $X_\alpha \cap \mu(U) = \{a\} \not\subseteq \{b\} = \mu(X_\alpha)$ for $\alpha > 0$.

(5.1)

(5.1.1) $\mu(A) \subseteq B \rightarrow \mu(A \cup B) \subseteq \mu(A) \cup \mu(B) \subseteq B \rightarrow_{(\mu CUM)} \mu(B) = \mu(A \cup B)$.

(5.1.2) $\mu(X) \subseteq U \subseteq Y \rightarrow$ (by (1)) $\mu(Y \cup X) = \mu(Y)$.

(5.1.3) $\mu(Y) \cap X =$ (by (2)) $\mu(Y \cup X) \cap X \subseteq \mu(Y \cup X) \cap (X \cup U) \subseteq$ (by (μPR)) $\mu(X \cup U) =$ (by (1)) $\mu(U)$.

(5.2)

Using (1.1), it suffices to show $(\mu Cum0) \rightarrow (\mu CUM)$. Let $\mu(X) \subseteq U \subseteq X$. By $(\mu Cum0)$ $X \cap \mu(U) \subseteq \mu(X)$, so by $\mu(U) \subseteq U \subseteq X \rightarrow \mu(U) \subseteq \mu(X)$. $U \subseteq X \rightarrow \mu(X) \cap U \subseteq \mu(U)$, but also $\mu(X) \subseteq U$, so $\mu(X) \subseteq \mu(U)$.

(4.1)

Trivial.

(6.1)

Follows from (4.1) and (5.2).

(6.2)

Let the prerequisites of $(\mu Cumt\alpha)$ hold.

We first show by induction $\mu(X_\alpha \cup U) \subseteq \mu(U)$.

Proof:

$\alpha = 0 : \mu(X_0) \subseteq U \to \mu(X_0 \cup U) = \mu(U)$ by (5.1.1). Let for all $\beta < \alpha$ $\mu(X_\beta \cup U) \subseteq \mu(U) \subseteq U$. By prerequisite, $\mu(X_\alpha) \subseteq U \cup \bigcup\{X_\beta : \beta < \alpha\}$, thus $\mu(X_\alpha \cup U) \subseteq \mu(X_\alpha) \cup \mu(U) \subseteq \bigcup\{U \cup X_\beta : \beta < \alpha\}$,

so $\forall \beta < \alpha$ $\mu(X_\alpha \cup U) \cap (U \cup X_\beta) \subseteq \mu(U)$ by (5.1.3), thus $\mu(X_\alpha \cup U) \subseteq \mu(U)$.

Consequently, under the above prerequisites, we have $\mu(X_\alpha \cup U) \subseteq \mu(U) \subseteq U \subseteq U \cup X_\alpha$, so by (μCUM) $\mu(U) = \mu(X_\alpha \cup U)$, and, finally, $\mu(U) \cap X_\alpha = \mu(X_\alpha \cup U) \cap X_\alpha \subseteq \mu(X_\alpha)$ by (μPR).

Note that finite unions take us over the limit step, essentially, as all steps collaps, and $\mu(X_\alpha \cup U)$ will always be $\mu(U)$, so there are no real changes.

(5.3)

Follows from (6.2) and (4.1).

(3.1)

Follows from (5.2) and (5.3).

(3.2)

Follows from (6.1) and (6.2).

(4.2)

Follows from (2.3) and (2.1).

(4.3)

Follows from (5.2) and (6.2).

(5.4)

$\mu(X) \subseteq U \to \mu(X) \subseteq U \cap X \subseteq X \to \mu(X \cap U) = \mu(X) \to X \cap \mu(U) = (X \cap U) \cap \mu(U) \subseteq \mu(X \cap U) = \mu(X)$

(3.3)

See Example 4.1.

(6.3)

This is a consequence of (3.3).

(7)

Trivial. Let $X \subseteq Y$, so by $(\mu \subseteq)$ $\mu(X) \subseteq X \subseteq Y$, so by $(\mu Cum0)$ $X \cap \mu(Y) \subseteq \mu(X)$.

□

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