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A Theory of Hierarchical Consequence and Conditionals

Dov M. Gabbay · Karl Schlechta

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Abstract We introduce $\mathcal{A}$-ranked preferential structures and combine them with an accessibility relation. $\mathcal{A}$-ranked preferential structures are intermediate between simple preferential structures and ranked structures. The additional accessibility relation allows us to consider only parts of the overall $\mathcal{A}$-ranked structure. This framework allows us to formalize contrary to duty obligations, and other pictures where we have a hierarchy of situations, and maybe not all are accessible to all possible worlds. Representation results are proved.

Keywords Deontic logic · Contrary-to-duties · Preferential structures · Ranked structures · Nonmonotonic logic · Representation results

1 Introduction

1.1 The Idea of This Paper

All existing possible world semantics basically share the following semantical recipe:

D. M. Gabbay (✉)
Department of Computer Science, King's College London, Strand, London WC2R 2LS, UK
e-mail: Dov.Gabbay@kcl.ac.uk
URL: www.dcs.kcl.ac.uk/staff/dg

D. M. Gabbay
Department of Computer Science, Bar-Ilan University, 52900 Ramat-Gan, Israel

K. Schlechta
Laboratoire d'Informatique Fondamentale de Marseille, UMR 6166, CNRS and Université de Provence, CMI, 39, rue Joliot-Curie, 13453 Marseille Cedex 13, France
e-mail: ks@cmi.univ-mrs.fr; karl.schlechta@web.de
URL: http//www.cmi.univ-mrs.fr/∼ks

To evaluate a formula at a world/index/context x go to some related world/index/context y and check some other subformulas there. There are variations of this depending whether we are dealing with unary or binary connectives etc.

The forms of words we used here "go to some related worlds" is English style, not actual travel from one world to another. The definition is set theoretical, using relations between points, not spatio temporal. Suppose now that we take the actual travel view, suppose we take the view that to evaluate for example a conditional $A > B$ at world x we actually try to travel to a nearby world y where A holds and check whether B holds? Well, mathematically there is no difference because the properties of the conditional depend on the notion of nearby and on the extensions of A and B. However, when people travel, sometimes they cannot get there, the road is blocked. So what do we do if we cannot get to the world y, how do we check the conditional? Well, we do the best we can, we go to some substitute world y', another world will have to do instead.

Our paper deals with such situations. We have a hierarchy of worlds, a hierarchy of preference relations and a binary relation of where we can get to and where we cannot get to. Of course this can also be made set theoretical, (which is what we do in this paper) but you will be surprised how many applications we have for this point of view.

Section 1.2 (p. 3) gives all the technical definitions. The reader should start reading from Sect. 1.3 (p. 12) where the motivations and explanations begin. The reader can refer back to the precise definitions when needed. It looks complicated mathematically but the ideas are simple, it is just the nature of formalisations that sometimes they become heavy.

1.2 Definitions

1.2.1 Basics

Definition 1.1 We use $\mathcal{P}$ to denote the power set operator, $\Pi\{X_i : i \in I\} := \{g : g : I \to \bigcup\{X_i : i \in I\}, \forall i \in I.g(i) \in X_i\}$ is the general cartesian product, $card(X)$ shall denote the cardinality of X, and V the set-theoretic universe we work in the class of all sets. Given a set of pairs $\mathcal{X}$, and a set X, we denote by $\mathcal{X}\lceil X := \{\langle x, i\rangle \in \mathcal{X} : x \in X\}$. When the context is clear, we will sometime simply write X for $\mathcal{X}\lceil X$. (The intended use is for preferential structures, where x will be a point (intention: a classical propositional model), and i an index, permitting copies of logically identical points.)

$A \subseteq B$ will denote that A is a subset of B or equal to B, and $A \subset B$ that A is a proper subset of B, likewise for $A \supseteq B$ and $A \supset B$.

Given some fixed set U we work in, and $X \subseteq U$, then $\boldsymbol{C}(X) := U - X$.

If $\mathcal{Y} \subseteq \mathcal{P}(X)$ for some X, we say that $\mathcal{Y}$ satisfies

($\cap$) iff it is closed under finite intersections,
($\bigcap$) iff it is closed under arbitrary intersections,
($\cup$) iff it is closed under finite unions,
($\bigcup$) iff it is closed under arbitrary unions,
($\boldsymbol{C}$) iff it is closed under complementation,
($-$) iff it is closed under set difference.

We will sometimes write $A = B \parallel C$ for: $A = B$, or $A = C$, or $A = B \cup C$.
We make ample and tacit use of the Axiom of Choice.

Definition 1.2 $\prec^*$ will denote the transitive closure of the relation $\prec$. If a relation $<$, $\prec$, or similar is given, $a \perp b$ will express that a and b are $< -$ (or $\prec -$) incomparable—context will tell. Given any relation $<$, $\leq$ will stand for $<$ or $=$, conversely, given $\leq$, $<$ will stand for $\leq$, but not $=$, similarly for $\prec$ etc.

Definition 1.3 We work here in a classical propositional language $\mathcal{L}$, a theory T will be an arbitrary set of formulas. Formulas will often be named ϕ, ψ, etc., theories T, S, etc.

$v(\mathcal{L})$ will be the set of propositional variables of $\mathcal{L}$.

$M_{\mathcal{L}}$ will be the set of (classical) models for $\mathcal{L}$, $M(T)$ or M_T is the set of models of T, likewise $M(\phi)$ for a formula ϕ.

$\boldsymbol{D}_{\mathcal{L}} := \{M(T) : T$ a theory in $\mathcal{L}\}$, the set of definable model sets.

Note that, in classical propositional logic, $\emptyset$, $M_{\mathcal{L}} \in \boldsymbol{D}_{\mathcal{L}}$, $\boldsymbol{D}_{\mathcal{L}}$ contains singletons, is closed under arbitrary intersections and finite unions.

An operation $f : \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ for $\mathcal{Y} \subseteq \mathcal{P}(M_{\mathcal{L}})$ is called definability preserving, (dp) or (μdp) in short, iff for all $X \in \boldsymbol{D}_{\mathcal{L}} \cap \mathcal{Y} f(X) \in \boldsymbol{D}_{\mathcal{L}}$.

We will also use (μdp) for binary functions $f : \mathcal{Y} \times \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$—as needed for theory revision—with the obvious meaning.

$\vdash$ will be classical derivability, and

$$\overline{T} := \{\phi : T \vdash \phi\}, \text{ the closure of } T \text{ under } \vdash .$$

$Con(.)$ will stand for classical consistency, so $Con(\phi)$ will mean that ϕ is clasical consistent, likewise for $Con(T)$.$Con(T, T')$ will stand for $Con(T \cup T')$, etc.

Given a consequence relation $\mid\!\sim$, we define

$$\overline{\overline{T}} := \{\phi : T \mid\!\sim \phi\}.$$

(There is no fear of confusion with $\overline{T}$, as it just is not useful to close twice under classical logic.)

$$T \vee T' := \{\phi \vee \phi' : \phi \in T, \phi' \in T'\}.$$

If $X \subseteq M_{\mathcal{L}}$, then $Th(X) := \{\phi : X \models \phi\}$, likewise for $Th(m), m \in M_{\mathcal{L}}$.($\models$ will usually be classical validity.)

1.2.2 Abstract Rules Related to Preferential Structures

We will not need all rules, but most are at least implicitly present, so we give a fuller overall picture by presenting them all.

Note that, in particular, the rules concerning Rationality and Rankedness are important here, as we will have a mixture of plain and of ranked structures.

The precise connections are described—again, in order to give a full picture—in Proposition 1.1 (p. 7), but, for reasons of brevity, not proven, with the exception of those shown in Proposition 3.16 (p. 28). The reader is referred to Gabbay and Schlechta (2009a) for a full proof.

Definition 1.4 We introduce here formally a list of properties of set functions on the algebraic side, and their corresponding logical rules on the other side. Putting them in parallel facilitates orientation, especially when considering representation problems.

We show, wherever adequate, in parallel the formula version in the left column, the theory version in the middle column, and the semantical or algebraic counterpart in the right column. The algebraic counterpart gives conditions for a function $f : \mathcal{Y} \rightarrow \mathcal{P}(U)$, where U is some set, and $\mathcal{Y} \subseteq \mathcal{P}(U)$.

The development in two directions, vertically with often increasing strength, horizontally connecting proof theory with semantics motivates the presentation in a table. The table is split in two, as one table would be too big to print. The first table contains the basic rules, the second one those about cumulativity and rationality.

Basics		
(AND) $\phi \mid\sim \psi, \phi \mid\sim \psi' \Rightarrow$ $\phi \mid\sim \psi \wedge \psi'$	(AND) $T \mid\sim \psi, T \mid\sim \psi' \Rightarrow$ $T \mid\sim \psi \wedge \psi'$	Closure under finite intersection
(OR) $\phi \mid\sim \psi, \phi' \mid\sim \psi \Rightarrow$ $\phi \vee \phi' \mid\sim \psi$	(OR) $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	(μOR) $f(X \cup Y) \subseteq f(X) \cup f(Y)$
(wOR) $\phi \mid\sim \psi,\ \phi' \vdash \psi \Rightarrow$ $\phi \vee \phi' \mid\sim \psi$	(wOR) $\overline{\overline{T}} \cap \overline{T'} \subseteq \overline{\overline{T \vee T'}}$	(μwOR) $f(X \cup Y) \subseteq f(X) \cup Y$
$(disjOR)$ $\phi \vdash \neg\phi',\ \phi \mid\sim \psi,$ $\phi' \mid\sim \psi \Rightarrow \phi \vee \phi' \mid\sim \psi$	$(disjOR)$ $\neg Con(T \cup T') \Rightarrow$ $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$(\mu disjOR)$ $X \cap Y = \emptyset \Rightarrow$ $f(X \cup Y) \subseteq f(X) \cup f(Y)$
(LLE) Left Logical Equivalence $\vdash \phi \leftrightarrow \phi', \phi \mid\sim \psi \Rightarrow$ $\phi' \mid\sim \psi$	(LLE) $\overline{T} = \overline{T'} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$	trivially true
(RW) Right Weakening $\phi \mid\sim \psi, \vdash \psi \rightarrow \psi' \Rightarrow$ $\phi \mid\sim \psi'$	(RW) $T \mid\sim \psi, \vdash \psi \rightarrow \psi' \Rightarrow$ $T \mid\sim \psi'$	upward closure
(CCL) Classical Closure	(CCL) $\overline{\overline{T}}$ is classically closed	trivially true
(SC) Supraclassicality $\phi \vdash \psi \Rightarrow \phi \mid\sim \psi$	(SC) $\overline{T} \subseteq \overline{\overline{T}}$	$(\mu \subseteq)$ $f(X) \subseteq X$
(REF) Reflexivity $T \cup \{\alpha\} \mid\sim \alpha$		
(CP) Consistency Preservation $\phi \mid\sim \bot \Rightarrow \phi \vdash \bot$	(CP) $T \mid\sim \bot \Rightarrow T \vdash \bot$	$(\mu\emptyset)$ $f(X) = \emptyset \Rightarrow X = \emptyset$
		$(\mu\emptyset fin)$ $X \neq \emptyset \Rightarrow f(X) \neq \emptyset$ for finite X
$\overline{\overline{\phi \wedge \phi'}} \subseteq \overline{\overline{\overline{\phi}} \cup \{\phi'\}}$	(PR) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$	(μPR) $X \subseteq Y \Rightarrow$ $f(Y) \cap X \subseteq f(X)$
		$(\mu PR')$ $f(X) \cap Y \subseteq f(X \cap Y)$
(CUT) $T \mid\sim \alpha; T \cup \{\alpha\} \mid\sim \beta \Rightarrow$ $T \mid\sim \beta$	(CUT) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} \subseteq \overline{\overline{T}}$	(μCUT) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(X) \subseteq f(Y)$

Cumulativity		
(CM) Cautious Monotony $\phi \mid\!\sim \psi, \phi \mid\!\sim \psi' \Rightarrow$ $\phi \wedge \psi \mid\!\sim \psi'$	(CM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} \subseteq \overline{\overline{T'}}$	(μCM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) \subseteq f(X)$
or $(ResM)$ Restricted Monotony $T \mid\!\sim \alpha, \beta \Rightarrow T \cup \{\alpha\} \mid\!\sim \beta$		$(\mu ResM)$ $f(X) \subseteq A \cap B \Rightarrow f(X \cap A) \subseteq B$
(CUM) Cumulativity $\phi \mid\!\sim \psi \Rightarrow$ $(\phi \mid\!\sim \psi' \Leftrightarrow \phi \wedge \psi \mid\!\sim \psi')$	(CUM) $T \subseteq \overline{T'} \subseteq \overline{T} \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}}$	(μCUM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) = f(X)$
	$(\subseteq\supseteq)$ $T \subseteq \overline{\overline{T'}}, T' \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} = \overline{\overline{T}}$	$(\mu \subseteq\supseteq)$ $f(X) \subseteq Y, f(Y) \subseteq X \Rightarrow$ $f(X) = f(Y)$
Rationality		
$(RatM)$ Rational Monotony $\phi \mid\!\sim \psi, \phi \not\mid\!\sim \neg\psi' \Rightarrow$ $\phi \wedge \psi' \mid\!\sim \psi$	$(RatM)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} \supseteq \overline{\overline{\overline{T'} \cup T}}$	$(\mu RatM)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) \subseteq f(Y) \cap X$
	$(RatM =)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{\overline{T'} \cup T}}$	$(\mu =)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) = f(Y) \cap X$
	$(Log =')$ $Con(\overline{\overline{T'}} \cup T) \Rightarrow$ $\overline{\overline{T \cup T'}} = \overline{\overline{\overline{T'} \cup T}}$	$(\mu =')$ $f(Y) \cap X \neq \emptyset \Rightarrow$ $f(Y \cap X) = f(Y) \cap X$
	$(Log \parallel)$ $\overline{\overline{T \vee T'}}$ is one of $\overline{\overline{T}}$, or $\overline{\overline{T'}}$, or $\overline{\overline{T}} \cap \overline{\overline{T'}}$ (by (CCL))	$(\mu \parallel)$ $f(X \cup Y)$ is one of $f(X)$, $f(Y)$ or $f(X) \cup f(Y)$
	$(Log\cup)$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\neg Con(\overline{\overline{T \vee T'}} \cup T')$	$(\mu\cup)$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) \cap Y = \emptyset$
	$(Log\cup')$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\overline{\overline{T \vee T'}} = \overline{\overline{T}}$	$(\mu\cup')$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) = f(X)$
		$(\mu \in)$ $a \in X - f(X) \Rightarrow$ $\exists b \in X. a \notin f(\{a, b\})$

Precise connections between the columns are given in Proposition 1.1 (p. 7).

When the formula version is not commonly used, we omit it, as we normally work only with the theory version.

A and B in the right hand side column stand for $M(\phi)$ for some formula ϕ, whereas X, Y stand for $M(T)$ for some theory T.

(PR) is also called infinite conditionalization—we choose the name for its central role for preferential structures (PR) or (μPR).

The system of rules $(AND)(OR)(LLE)(RW)(SC)(CP)(CM)(CUM)$ is also called system P (for preferential), adding $(RatM)$ gives the system R (for rationality or rankedness).

Roughly: Smooth preferential structures generate logics satisfying system P, ranked structures logics satisfying system R.

A logic satisfying (REF), $(ResM)$, and (CUT) is called a consequence relation.

(LLE) and(CCL) will hold automatically, whenever we work with model sets.

(AND) is obviously closely related to filters, and corresponds to closure under finite intersections. (RW) corresponds to upward closure of filters.

More precisely, validity of both depend on the definition, and the direction we consider.

Given f and $(\mu \subseteq)$, $f(X) \subseteq X$ generates a principal filter: $\{X' \subseteq X : f(X) \subseteq X'\}$, with the definition: If $X = M(T)$, then $T \mid\!\sim \phi$ iff $f(X) \subseteq M(\phi)$. Validity of (AND) and (RW) are then trivial.
Conversely, we can define for $X = M(T)$

$$\mathcal{X} := \{X' \subseteq X : \exists\phi(X' = X \cap M(\phi) \text{ and } T \mid\!\sim \phi)\}.$$

(AND) then makes $\mathcal{X}$ closed under finite intersections, (RW) makes $\mathcal{X}$ upward closed. This is in the infinite case usually not yet a filter, as not all subsets of X need to be definable this way. In this case, we complete $\mathcal{X}$ by adding all X'' such that there is $X' \subseteq X'' \subseteq X$, $X' \in \mathcal{X}$.

Alternatively, we can define

$$\mathcal{X} := \{X' \subseteq X : \bigcap\{X \cap M(\phi) : T \mid\!\sim \phi\} \subseteq X'\}.$$

(SC) corresponds to the choice of a subset.
(CP) is somewhat delicate, as it presupposes that the chosen model set is non-empty. This might fail in the presence of ever better choices, without ideal ones; the problem is addressed by the limit versions.
(PR) is an infinitary version of one half of the deduction theorem: Let T stand for ϕ, T' for ψ, and $\phi \wedge \psi \mid\!\sim \sigma$, so $\phi \mid\!\sim \psi \to \sigma$, but $(\psi \to \sigma) \wedge \psi \vdash \sigma$.
(CUM) (whose more interesting half in our context is (CM)) may best be seen as normal use of lemmas: We have worked hard and found some lemmas. Now we can take a rest, and come back again with our new lemmas. Adding them to the axioms will neither add new theorems, nor prevent old ones to hold. (This is, of course, a meta-level argument concerning an object level rule. But also object level rules should—at least generally—have an intuitive justification, which will then come from a meta-level argument.)

Proposition 1.1 *The following table is to be read as follows:*

Let a logic $\mid\!\sim$ satisfy (LLE) and (CCL), and define a function $f : \boldsymbol{D}_{\mathcal{L}} \to \boldsymbol{D}_{\mathcal{L}}$ by $f(M(T)) := M(\overline{\overline{T}})$. Then f is well defined, satisfies (μdp), and $\overline{\overline{T}} = Th(f(M(T)))$.

If $\mid\!\sim$ satisfies a rule in the left hand side, then—provided the additional properties noted in the middle for $\Rightarrow$ hold, too—f will satisfy the property in the right hand side.

Conversely, if $f : \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ is a function, with $\boldsymbol{D}_{\mathcal{L}} \subseteq \mathcal{Y}$, and we define a logic $\mid\!\sim$ by $\overline{\overline{T}} := Th(f(M(T)))$, then $\mid\!\sim$ satisfies (LLE) and (CCL). If f satisfies (μdp), then $f(M(T)) = M(\overline{\overline{T}})$.

If f satisfies a property in the right hand side, then—provided the additional properties noted in the middle for $\Leftarrow$ hold, too—$\mid\!\sim$ will satisfy the property in the left hand side.

If "formula" is noted in the table, this means that, if one of the theories (the one named the same way in Definition 1.4 (p. 4)) is equivalent to a formula, we do not need (μdp).

Basics			
(1.1)	(OR)	$\Rightarrow$	(μOR)
(1.2)		$\Leftarrow$	
(2.1)	$(disjOR)$	$\Rightarrow$	$(\mu disjOR)$
(2.2)		$\Leftarrow$	
(3.1)	(wOR)	$\Rightarrow$	(μwOR)
(3.2)		$\Leftarrow$	
(4.1)	(SC)	$\Rightarrow$	$(\mu \subseteq)$
(4.2)		$\Leftarrow$	
(5.1)	(CP)	$\Rightarrow$	$(\mu\emptyset)$
(5.2)		$\Leftarrow$	
(6.1)	(PR)	$\Rightarrow$	(μPR)
(6.2)		$\Leftarrow (\mu dp) + (\mu \subseteq)$	
(6.3)		$\not\Leftarrow$ without (μdp)	
(6.4)		$\Leftarrow (\mu \subseteq)$ T' a formula	
(6.5)	(PR)	$\Leftarrow$ T' a formula	$(\mu PR')$
(7.1)	(CUT)	$\Rightarrow$	(μCUT)
(7.2)		$\Leftarrow$	
Cumulativity			
(8.1)	(CM)	$\Rightarrow$	(μCM)
(8.2)		$\Leftarrow$	
(9.1)	$(ResM)$	$\Rightarrow$	$(\mu ResM)$
(9.2)		$\Leftarrow$	
(10.1)	$(\subseteq\supseteq)$	$\Rightarrow$	$(\mu \subseteq\supseteq)$
(10.2)		$\Leftarrow$	
(11.1)	(CUM)	$\Rightarrow$	(μCUM)
(11.2)		$\Leftarrow$	
Rationality			
(12.1)	$(RatM)$	$\Rightarrow$	$(\mu RatM)$
(12.2)		$\Leftarrow (\mu dp)$	
(12.3)		$\not\Leftarrow$ without (μdp)	
(12.4)		$\Leftarrow$ T a formula	
(13.1)	$(RatM =)$	$\Rightarrow$	$(\mu =)$
(13.2)		$\Leftarrow (\mu dp)$	
(13.3)		$\not\Leftarrow$ without (μdp)	
(13.4)		$\Leftarrow$ T a formula	
(14.1)	$(Log =')$	$\Rightarrow$	$(\mu =')$
(14.2)		$\Leftarrow (\mu dp)$	
(14.3)		$\not\Leftarrow$ without (μdp)	
(14.4)		$\Leftarrow T$ a formula	
(15.1)	$(Log \parallel)$	$\Rightarrow$	$(\mu \parallel)$
(15.2)		$\Leftarrow$	
(16.1)	$(Log\cup)$	$\Rightarrow (\mu \subseteq) + (\mu =)$	$(\mu\cup)$
(16.2)		$\Leftarrow (\mu dp)$	
(16.3)		$\not\Leftarrow$ without (μdp)	
(17.1)	$(Log\cup')$	$\Rightarrow (\mu \subseteq) + (\mu =)$	$(\mu\cup')$
(17.2)		$\Leftarrow (\mu dp)$	
(17.3)		$\not\Leftarrow$ without (μdp)	

1.2.3 Preferential Structures

Definition 1.5 Fix $U \neq \emptyset$, and consider arbitrary X. Note that this X has not necessarily anything to do with U, or $\mathcal{U}$ below. Thus, the functions $\mu_{\mathcal{M}}$ below are in principle functions from V to V—where V is the set theoretical universe we work in.

Note that we work here often with copies of elements (or models). In other areas of logic, most authors work with valuation functions. Both definitions—copies or valuation functions—are equivalent, a copy $\langle x, i\rangle$ can be seen as a state $\langle x, i\rangle$ with valuation x. In the beginning of research on preferential structures, the notion of copies was widely used, whereas e.g., [KLM90] used that of valuation functions. There is perhaps a weak justification of the former terminology. In modal logic, even if two states have the same valid classical formulas, they might still be distinguishable by their valid modal formulas. But this depends on the fact that modality is in the object language. In most work on preferential structures, the consequence relation is outside the object language, so different states with same valuation are in a stronger sense copies of each other.

(1) Preferential models or structures.

(1.1) The version without copies:
A pair $\mathcal{M} := \langle U, \prec\rangle$ with U an arbitrary set, and $\prec$ an arbitrary binary relation on U is called a preferential model or structure.

(1.2) The version with copies:
A pair $\mathcal{M} := \langle \mathcal{U}, \prec\rangle$ with $\mathcal{U}$ an arbitrary set of pairs, and $\prec$ an arbitrary binary relation on $\mathcal{U}$ is called a preferential model or structure.
If $\langle x, i\rangle \in \mathcal{U}$, then x is intended to be an element of U, and i the index of the copy.
We sometimes also need copies of the relation $\prec$, we will then replace $\prec$ by one or several arrows α attacking non-minimal elements, e.g., $x \prec y$ will be written $\alpha : x \to y$, $\langle x, i\rangle \prec \langle y, i\rangle$ will be written $\alpha : \langle x, i\rangle \to \langle y, i\rangle$, and finally we might have $\langle \alpha, k\rangle : x \to y$ and $\langle \alpha, k\rangle : \langle x, i\rangle \to \langle y, i\rangle$, etc.

(2) Minimal elements, the functions $\mu_{\mathcal{M}}$

(2.1) The version without copies:
Let $\mathcal{M} := \langle U, \prec\rangle$, and define

$$\mu_{\mathcal{M}}(X) := \{x \in X : x \in U \wedge \neg\exists x' \in X \cap U . x' \prec x\}.$$

$\mu_{\mathcal{M}}(X)$ is called the set of minimal elements of X (in $\mathcal{M}$). Thus, $\mu_{\mathcal{M}}(X)$ is the set of elements such that there is no smaller one in X.

(2.2) The version with copies:
Let $\mathcal{M} := \langle \mathcal{U}, \prec\rangle$ be as above. Define

$$\mu_{\mathcal{M}}(X) := \{x \in X : \exists\langle x, i\rangle \in \mathcal{U} . \neg\exists\langle x', i'\rangle \in \mathcal{U}(x' \in X \wedge \langle x', i'\rangle' \prec \langle x, i\rangle)\}.$$

Thus, $\mu_{\mathcal{M}}(X)$ is the projection on the first coordinate of the set of elements such that there is no smaller one in X.

Again, by abuse of language, we say that $\mu_{\mathcal{M}}(X)$ is the set of minimal elements of X in the structure. If the context is clear, we will also write just μ.
We sometimes say that $\langle x, i\rangle$ "kills" or "minimizes" $\langle y, j\rangle$ if $\langle x, i\rangle \prec \langle y, j\rangle$. By abuse of language we also say a set X kills or minimizes a set Y if for all $\langle y, j\rangle \in \mathcal{U}$, $y \in Y$ there is $\langle x, i\rangle \in \mathcal{U}$, $x \in X$ s.t. $\langle x, i\rangle \prec \langle y, j\rangle$.
$\mathcal{M}$ is also called injective or 1-copy, iff there is always at most one copy $\langle x, i\rangle$ for each x. Note that the existence of copies corresponds to a non-injective labelling function—as is often used in nonclassical logic, e.g., modal logic.

We say that $\mathcal{M}$ is transitive, irreflexive, etc., iff $\prec$ is. Note that $\mu(X)$ might well be empty, even if X is not.

Definition 1.6 We define the consequence relation of a preferential structure for a given propositional language $\mathcal{L}$.

(1)

(1.1) If m is a classical model of a language $\mathcal{L}$, we say by abuse of language

$$\langle m, i\rangle \models \phi \text{ iff } m \models \phi,$$

and if X is a set of such pairs, that

$$X \models \phi \text{ iff } \quad \text{forall} \quad \langle m, i\rangle \in X m \models \phi.$$

(1.2) If $\mathcal{M}$ is a preferential structure, and X is a set of $\mathcal{L}$-models for a classical propositional language $\mathcal{L}$, or a set of pairs $\langle m, i\rangle$, where the m are such models, we call $\mathcal{M}$ a classical preferential structure or model.

(2) Validity in a preferential structure, or the semantical consequence relation defined by such a structure:
Let $\mathcal{M}$ be as above.
We define:

$$T \models_{\mathcal{M}} \phi \text{ iff } \mu_{\mathcal{M}}(M(T)) \models \phi, \text{ i.e., } \mu_{\mathcal{M}}(M(T)) \subseteq M(\phi).$$

$\mathcal{M}$ will be called definability preserving iff for all $X \in \boldsymbol{D}_{\mathcal{L}} \mu_{\mathcal{M}}(X) \in \boldsymbol{D}_{\mathcal{L}}$.

As $\mu_{\mathcal{M}}$ is defined on $\boldsymbol{D}_{\mathcal{L}}$, but need by no means always result in some new definable set, this is (and reveals itself as a quite strong) additional property.

Definition 1.7 Let $\mathcal{Y} \subseteq \mathcal{P}(U)$. (In applications to logic, $\mathcal{Y}$ will be $\boldsymbol{D}_{\mathcal{L}}$.)

A preferential structure $\mathcal{M}$ is called $\mathcal{Y}$-smooth iff for every $X \in \mathcal{Y}$ every element $x \in X$ is either minimal in X or above an element, which is minimal in X. More precisely:

(1) The version without copies:
If $x \in X \in \mathcal{Y}$, then either $x \in \mu(X)$ or there is $x' \in \mu(X).x' \prec x$.

(2) The version with copies:
If $x \in X \in \mathcal{Y}$, and $\langle x, i\rangle \in \mathcal{U}$, then either there is no $\langle x', i'\rangle \in \mathcal{U}, x' \in X, \langle x', i'\rangle \prec \langle x, i\rangle$ or there is $\langle x', i'\rangle \in \mathcal{U}, \langle x', i'\rangle \prec \langle x, i\rangle, x' \in X$, s.t. there is no $\langle x'', i''\rangle \in \mathcal{U}, x'' \in X$, with $\langle x'', i''\rangle \prec \langle x', i'\rangle$.

When considering the models of a language $\mathcal{L}$, $\mathcal{M}$ will be called smooth iff it is $\boldsymbol{D}_{\mathcal{L}}$-smooth; $\boldsymbol{D}_{\mathcal{L}}$ is the default.

Obviously, the richer the set $\mathcal{Y}$ is, the stronger the condition $\mathcal{Y}$-smoothness will be.

Fact 1.2 Let $\prec$ be an irreflexive, binary relation on X, then the following two conditions are equivalent:

(1) There is Ω and an irreflexive, total, binary relation $\prec'$ on Ω and a function $f : X \to \Omega$ s.t. $x \prec y \leftrightarrow f(x) \prec' f(y)$ for all $x, y \in X$.
(2) Let $x, y, z \in X$ and $x \perp y$ wrt. $\prec$ (i.e., neither $x \prec y$ nor $y \prec x$), then $z \prec x \to z \prec y$ and $x \prec z \to y \prec z$.

Definition 1.8 We call an irreflexive, binary relation $\prec$ on X, which satisfies (1) (equivalently (2)) of Fact 1.2 (p. 11) , ranked. By abuse of language, we also call a preferential structure $\langle X, \prec\rangle$ ranked, iff $\prec$ is.

1.2.4 $\mathcal{A}$-Ranked Structures

Definition 1.9 We have the usual framework of preferential structures, i.e., either a set with a possibly non-injective labelling function, or, equivalently, a set of possible worlds with copies. The relation of the preferential structure will be fixed, and will not depend on the point m from where we look at it.

Next, we have a set $\boldsymbol{A}$, and a finite, disjoint cover $A_i : i < n$ of $\mathcal{A}$, with a relation "of quality" $<$, $\mathcal{A}$ will denote the A_i (and thus $\boldsymbol{A}$), and $<$, i.e., $\mathcal{A} =< \{A_i : i \in I\}, \langle\rangle$.

By Fact 3.14 (p. 27), we may assume that all A_i are described by a formula.

Finally, we have $\boldsymbol{B} \subseteq \boldsymbol{A}$, the subset of "good" elements of $\boldsymbol{A}$—which we also assume to be described by a formula.

In addition, we have a binary relation of accessibility, R, which we assume transitive—modal operators will be defined relative to $R.R$ determines which part of the preferential structure is visible.

Let $R(s) := \{t : sRt\}$.

Definition 1.10 We repeat here from the introduction, and assume $A_i = M(\alpha_i)$, $B = M(\beta)$, and μ expresses the minimality of the preferential structure.

$$t \models \alpha_i > \beta :\Leftrightarrow \mu(A_i) \cap R(t) \subseteq B,$$

we will also abuse notation and just write

$$t \models A_i > B \text{ in this case.}$$

We then define: $t \models \mathcal{C}$ iff at the smallest i s.t. $\mu(A_i) \cap R(t) \neq \emptyset$, $\mu(A_i) \cap R(t) \subseteq \boldsymbol{B}$ holds.

This motivates the following:

Definition 1.11 Let $\boldsymbol{A}$ be a fixed set, and $\mathcal{A}$ a finite, totally ordered (by $<$) disjoint cover by non-empty subsets of $\boldsymbol{A}$.

For $x \in \boldsymbol{A}$, let $rg(x)$ be the unique $A \in \mathcal{A}$ such that $x \in A$, so $rg(x) < rg(y)$ is defined in the natural way.

A preferential structure $\langle \mathcal{X}, \prec \rangle (\mathcal{X}$ a set of pairs $\langle x, i \rangle)$ is called $\mathcal{A}$-ranked iff for all $x, x' rg(x) < rg(x')$ implies $\langle x, i \rangle \prec \langle x', i' \rangle$ for all $\langle x, i \rangle, \langle x', i' \rangle \in \mathcal{X}$.

Note that automatically for $X \subseteq \boldsymbol{A}, \mu(X) \subseteq A_j$ when j is the smallest i s.t. $X \cap A_i \neq \emptyset$.

The idea is now to make the A_i the layers, and "trigger" the first layer A_j s.t. $\mu(A_j) \cap R(x) \neq \emptyset$, and check whether $\mu(A_j) \cap R(x) \subseteq B_j$. A suitable ranked structure will automatically find this A_j.

More definitions and results for such $\mathcal{A}$ and $\mathcal{C}$ will be found in Sect. 4 (p. 29).

1.3 Description of the Problem

This paper, like all papers about nonmonotonic logics, is about formalization of (an aspect of) common sense reasoning. We often see a hierarchy of situations, e.g.,

(1) it is better to prevent an accident than to help the victims,
(2) it is better to prove a difficult theorem than to prove an easy lemma,
(3) it is best not to steal, but if we have stolen, we should return the stolen object to its legal owner, etc.

On the other hand, it is sometimes impossible to achieve the best objective.

We might have seen the accident happen from far away, so we were unable to interfere in time to prevent it, but we can still run to the scene and help the victims.

We might have seen friends last night and had a drink too many, so today's headaches will not allow us to do serious work, but we can still prove a little lemma.

We might have needed a hammer to smash the windows of a car involved in an accident, so we stole it from a building site, but will return it afterwards.

We see in all cases:

- a hierarchy of situations
- not all situations are possible or accessible for an agent.

In addition, we often have implicitly a "normality" relation:

Normally, we should help the victims, but there might be situations where not: This would expose ourselves to a very big danger, or this would involve neglecting another, even more important task (we are supervisor in a nuclear power plant ....), etc.

Thus, in all "normal" situations where an accident seems imminent, we should try to prevent it. If this is impossible, in all "normal" situations, we should help the victims, etc.

We combine these three ideas

(1) normality,
(2) hierarchy,
(3) accessibility

in the present paper.

Note that it might be well possible to give each situation a numerical value and decide by this value what is right to do—but humans do not seem to think this way, and we want to formalize human common sense reasoning.

Before we begin the formal part, we elaborate above situations with more examples.

- We might have the overall intention to advance computer science.
  So we apply for the job of head of department of computer science at Stanford, and promise every tenured scientist his own laptop.
  Unfortunately, we do not get the job, but become head of computer science department at the local community college. The college does not have research as priority, but we can still do our best to achieve our overall intention, by, say buying good books for the library, or buy computers for those still active in research, etc.
  So, it is reasonable to say that, even if we failed in the best possible situation—it was not accessible to us—we still succeeded in another situation, so we achieved the overall goal.
- The converse is also possible, where better solutions become possible, as is illustrated by the following example.
  The daughter and her husband say to have the overall intention to start a family life with a house of their own, and children.
  Suppose the mother now asks her daughter: You have been married now for two years, how come you are not pregnant?
  Daughter: we cannot afford a baby now, we had to take a huge mortgage to buy our house and we both have to work.
  Mother: *I* shall pay off your mortgage. Get on with it!
  In this case, what was formerly inaccessible, is now accessible, and if the daughter was serious about her intentions—the mother can begin to look for baby carriages.
  Note that we do not distinguish here how the situations change, whether by our own doing, or by someone else's doing, or by some events not controlled by anyone.
- Consider the following hierarchy of obligations making fences as unobtrusive as possible, involving contrary to duty obligations.

  (1) You should have no fence (main duty).
  (2) If this is impossible (e.g., you have a dog which might invade neighbours' property), it should be less than 3 feet high (contrary to duty, but second best choice).
  (3) If this is impossible too (e.g., your dog might jump over it), it should be white (even more contrary to duty, but still better than nothing).
  (4) If all is impossible, you should get the neighbours' consent (etc.).

1.4 The Abstract Problem and Outline of the Solution

The last example can be modelled as follows ($\mu(x)$ is the minimal models of x) :

Layer 1: $\mu(True)$: all best models have no fence.
Layer 2: $\mu(fence)$: all best models with a fence are less than 3 ft. high.
Layer 3: μ(*fence* and more than 3 ft. high): all best models with a tall fence have a white fence.

Layer 4: μ(*fence* and non-white and $\geq$ 3 ft): in all best models with a non-white fence taller than 3 feet, you have permission
Layer 5: all the rest

This will be modelled by a corresponding $\mathcal{A}$-structure.

In summary:

(1) We have a hierarchy of situations, where one group (e.g., preventing accidents) is strictly better than another group (e.g., helping victims).
(2) Within each group, preferences are not so clear (first help person A, or person B, first call ambulance, etc.?).
(3) We have a subset of situations which are attainable, this can be modelled by an accessibility relation which tells us which situations are possible or can be reached.

The problem is then to find a structure which is sufficiently rich to express our examples, and their abstract description in above three points (and, we may add, still relatively close to known structures). In addition, we should be able to characterize such structures by representation results. This is what we will do here.

Traditionally, deontic logic uses preferential structures as semantics, see Hansson (1971), so looking for some such structure was natural. The idea of combining simple with ranked preferential structures came quite naturally, and the representation proofs turned out to be relatively straightforward, using techniques developed before by the authors. Putting a Kripke structure on top, was again a straightforward move. So we can say that the present work is well in the general tradition of philosophical logic. The authors worked quite extensively on representation results for preferential structures, coming from a proof theoretical point of view, see Gabbay (1985), or from a more semantical point of view, see e.g., Schlechta (2004). They investigated systems of different strength, among them general and also ranked structures, so the present work is also well in the personal tradition of both authors. as it is in the personal research tradition of the authors.

More precisely, we combine all three ideas, consider what we call $\mathcal{A}$-ranked structures, structures which are organized in levels A_1, A_2, A_3, etc., where all elements of A_1 are better than any element of A_2—this is basically rankedness, and where inside each A_i we have an arbitrary relation of preference. Thus, an $\mathcal{A}$-ranked structure is between a simple preferential structure and a fully ranked structure. See Fig. 1 (p. 14).

Remark It is not at all necessary that the rankedness relation between the different layers and the relation inside the layers express the same concept. For instance, rankedness may express deontic preference, whereas the inside relation expresses normality or some usualness.

In addition, we have an accessibility relation R, which tells us which situations are reachable.

It is perhaps easiest to motivate the precise choice of modelling by layered (or contrary to duty) obligations.

For any point t, let $R(t) := \{s : tRs\}$, the set of R-reachable points from t. Given a preferential structure $\mathcal{X} := \langle X, \prec \rangle$, we can relativize $\mathcal{X}$ by considering only those points in X, which are reachable from t.

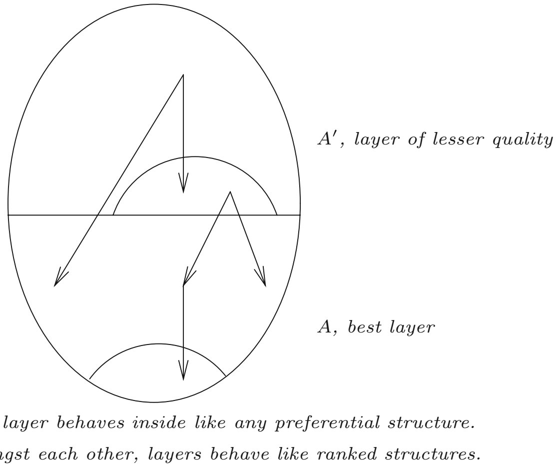

Fig. 1 A- ranked structure

Let $X' \subseteq X$, and $\mu(X')$ the minimal points of X, we will now consider $\mu(X') \cap R(t)$-attention, not: $\mu(X' \cap R(t))$! This choice is motivated by the following: norms are universal, and do not depend on one's situation t. Universality is expressed by $\mu(X')$, and we choose among the universally best those which are reachable from t. By "norms are universal" we mean here something like the universality of "you should not steal". If circumstances justify the violation, then the norm is still violated, but this might be seen as a lesser evil.

If $\mathcal{X}$ describes a simple obligation, then we are obliged to Y iff $\mu(X') \cap R(t) \neq \emptyset$, and $\mu(X') \cap R(t) \subseteq Y$. The first clause excludes obligations to the unattainable.

If an $\mathcal{A}$-ranked structure has two or more layers, then we are, if possible, obliged to fulfill the lower obligation, e.g., prevent an accident, but if this is impossible, we are obliged to fulfill the upper obligation, e.g., help the victims, etc.

See Fig. 2 (p. 15).

Let now $\boldsymbol{B}$ be a subset of the union of all layers A. Then we say that m satisfies $\langle \mathcal{A}, \boldsymbol{B} \rangle$ iff in the lowest layer A where $\mu(A) \cap R(m) \neq \emptyset \mu(A) \cap R(m) \subseteq \boldsymbol{B}$. See Fig. 3 (p. 30).

1.5 Historical Remarks

(1) In an abstract consideration of desirable properties a logic might have, Gabbay (1985) examined rules a nonmonotonic consequence relation $\mid\sim$ should satisfy:

(1.1) (REF) $\Delta, \alpha \mid\sim \alpha$,
(1.2) (CUM) $\Delta \mid\sim \alpha \Rightarrow (\Delta \mid\sim \beta \Leftrightarrow \Delta, \alpha \mid\sim \beta)$.

Preferential structures themselves were introduced as abstractions of Circumscription independently in Schlechta (2004) and Bossu and Siegel (1985). A precise definition of these structures is given below in Definition 1.5 (p. 9).

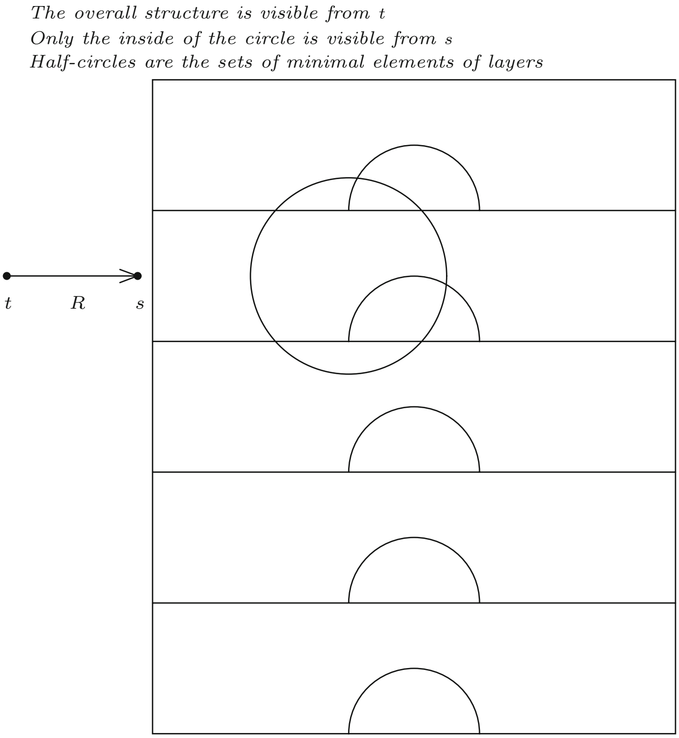

Fig. 2 A- ranked structure and accessibility

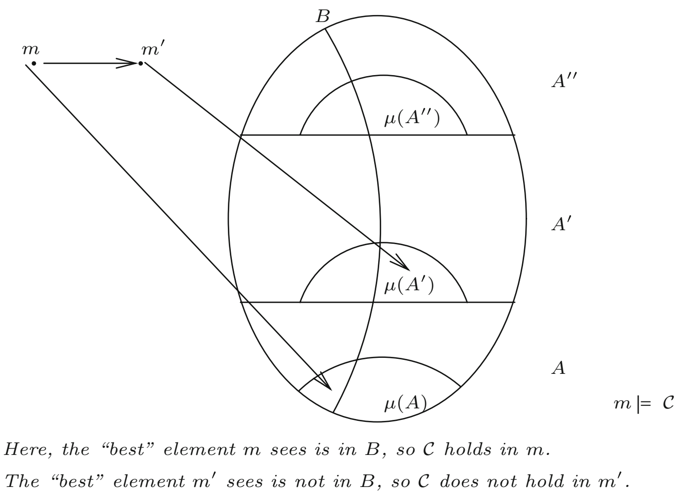

Fig. 3 Validity of $\mathcal{C}$ from m and m'

Both, the semantic and the syntactic, approaches were connected in Kraus et al. (1990), where a representation theorem was proved, showing that the (stronger than $Gabbay'$s) system P corresponds to "smooth" preferential structures. System P consists of

(1.1) (AND) $\phi \mid\sim \psi, \phi \mid\sim \psi' \Rightarrow \phi \mid\sim \psi \wedge \psi'$,
(1.2) (OR) $\phi \mid\sim \psi, \phi' \mid\sim \psi \Rightarrow \phi \vee \phi' \mid\sim \psi$,
(1.3) (LLE) $\vdash \phi \leftrightarrow \phi' \Rightarrow (\phi \mid\sim \psi \Leftrightarrow \phi' \mid\sim \psi)$,
(1.4) (RW) $\phi \mid\sim \psi, \vdash \psi \rightarrow \psi' \Rightarrow \phi \mid\sim \psi'$,
(1.5) (SC) $\vdash \phi \rightarrow \phi' \Rightarrow \phi \mid\sim \phi'$,
(1.6) (CUM) $\phi \mid\sim \psi \Rightarrow (\phi \mid\sim \psi' \Leftrightarrow \phi \wedge \psi \mid\sim \psi')$.

where $\vdash$ is classical provability.
Details can be found in Definition 1.4 (p. 4).

(2) Ranked preferential structures were introduced in Lehmann and Magidor (1992), see Definition 1.8 (p. 11). On the logical side, they correspond to above system P, plus the additional axiom:
(RatM) $\phi \mid\sim \psi, \phi \not\mid\sim \neg\psi' \Rightarrow \phi \wedge \psi' \mid\sim \psi$.
(3) Accessibility relations in possible worlds semantics go back (at least) to Kripke's semantics for modal logics.

1.6 Formal Modelling and Summary of Results

We started with an investigation of "best fulfillment" of abstract requirements, and contrary to duty obligations.—See also Gabbay (2008a,b).

It soon became evident that semi-ranked preferential structures give a natural semantics to contrary to duty obligations, just as simple preferential structures give a natural semantics to simple obligations—the latter goes back to Hansson (1971).

A semi-ranked—or $\mathcal{A}$-ranked preferential structure, as we will call them later, as they are based on a system of sets $\mathcal{A}$- has a finite number of layers, which amongst them are totally ordered by a ranking, but the internal ordering is just any (binary) relation. It thus has stronger properties than a simple preferential structure, but not as strong ones as a (totally) ranked structure.

The idea is to put the (cases of the) strongest obligation at the bottom, and the weaker ones more towards the top. Then, fulfillment of a strong obligation makes the whole obligation automatically satisfied, and the weaker ones are forgotten.

Beyond giving a natural semantics to contrary to duty obligations, semi-ranked structures seem very useful for other questions of knowledge representation. For instance, any blackbird might seem a more normal bird than any penguin, but we might not be so sure within each set of birds.

Thus, this generalization of preferential semantics seems very natural and welcome.

The second point of this paper is to make some, but not necessarily all, situations accessible to each point of departure. Thus, if we imagine agent a to be at point p, some fulfillments of the obligation, which are reachable to agent a' from point p' might just be impossible to reach for him. Thus, we introduce a second relation, of accessibility in the intuitive sense, denoting situations which can be reached. If this relation is transitive, then we have restrictions on the set of reachable situations: if p

is accessible from p', and p can access situation s, then so can p', but not necessarily the other way round.

For the rest of this section, the reader will need to leaf back to the list of definitions.

On the formal side, we characterize:

(1) $\mathcal{A}$-ranked structures,
(2) satisfaction of an $\mathcal{A}$-ranked conditional once an accessibility relation between the points p, p', etc. is given.

(1) will give a complete correspondence between proof theory and semantics of $\mathcal{A}$-ranked structures.
(2) will show how things change when we move along paths of developments, where R codes the steps of possible development.

For the convience of the reader, we now state the main formal results of this paper—together with the more unusual definitions.

On (1):

Let $\boldsymbol{A}$ be a fixed set, and $\mathcal{A}$ a finite, totally ordered (by $<$) disjoint cover by non-empty subsets of $\boldsymbol{A}$.

For $x \in \boldsymbol{A}$, let $rg(x)$ be the unique $A \in \mathcal{A}$ such that $x \in A$, so $rg(x) < rg(y)$ is defined in the natural way.

A preferential structure $\langle \mathcal{X}, \prec \rangle (\mathcal{X}$ a set of pairs $\langle x, i \rangle)$ is called $\mathcal{A}$-ranked iff for all $x, x' rg(x) < rg(x')$ implies $\langle x, i \rangle \prec \langle x', i' \rangle$ for all $\langle x, i \rangle, \langle x', i' \rangle \in \mathcal{X}$. See Definition 1.5 (p. 9) for the definition of preferential structures, and Fig. 1 (p. 14) for an illustration.

We then have:

Let $\mid\sim$ be a logic for $\mathcal{L}$. Set $T^{\mathcal{M}} := Th(\mu_{\mathcal{M}}(M(T)))$, and $\overline{\overline{T}} := \{\phi : T \mid\sim \phi\}$. where $\mathcal{M}$ is a preferential structure.

(1) Then there is a (transitive) definability preserving classical preferential model $\mathcal{M}$ s.t. $\overline{\overline{T}} = T^{\mathcal{M}}$ iff (LLE), (CCL), (SC), (PR) hold for all $T, T' \subseteq \mathcal{L}$.
(2) The structure can be chosen smooth, iff, in addition (CUM) holds.
(3) The structure can be chosen $\mathcal{A}$-ranked, iff, in addition ($\mathcal{A}$-min) $T \not\vdash \neg\alpha_i$ and $T \not\vdash \neg\alpha_j, i < j$ implies $\overline{\overline{T}} \vdash \neg\alpha_j$ holds.

See Definition 1.6 (p. 10) for the logic defined by a preferential structure, Definition 1.4 (p. 4) for the logical conditions, Definition 1.7 (p. 10) for smoothness.

On (2)

Given a transitive accessibility relation R, $R(m) := \{x : mRx\}$.

Given $\mathcal{A}$ as above, let $\boldsymbol{B} \subseteq \boldsymbol{A}$ be the set of "good" points in $\boldsymbol{A}$, and set $\mathcal{C} := \langle \mathcal{A}, \boldsymbol{B} \rangle$.

We define:

(1) $\mu(\mathcal{A}) := \bigcup\{\mu(A_i) : i \in I\}$
(warning: this is NOT $\mu(\boldsymbol{A})$)
(2) $\mathcal{A}_m := R(m) \cap \boldsymbol{A}$,
(3) $\mu(\mathcal{A}_m) := \bigcup\{\mu(A_i) \cap R(m) : i \in I\}$
(3a) $\nu(\mathcal{A}_m) := \mu(\mu(\mathcal{A}_m))$
(thus $\nu(\mathcal{A}_m) = \{a \in \boldsymbol{A} : \exists A \in \mathcal{A}(a \in \mu(A), a \in R(m)$, and $\neg\exists a'(\exists A' \in \mathcal{A}(a' \in \mu(A'), a' \in R(m), a' \prec a)))\}$.

(4) $m \models \mathcal{C} :\leftrightarrow \nu(\mathcal{A}_m) \subseteq \boldsymbol{B}$.

See Fig. 3 (p. 30)

Then the following hold:

Let $m, m' \in M$, A, $A' \in \mathcal{A}$, $\boldsymbol{A}$ be the set of models of α.

(1) $m \models \Box\neg\alpha$, mRm$'$ $\Rightarrow m' \models \Box\neg\alpha$
(2) mRm', $\nu(\mathcal{A}_m) \cap A \neq \emptyset$, $\nu(\mathcal{A}_{m'}) \cap A' \neq \emptyset$, $\Rightarrow A \leq A'$ (in the ranking)
(3) mRm', $\nu(\mathcal{A}_m) \cap A \neq \emptyset$, $\nu(\mathcal{A}_{m'}) \cap A' \neq \emptyset$, $m \models \mathcal{C}$, $m' \not\models \mathcal{C}$, $\Rightarrow A < A'$

Conversely, these conditions suffice to construct an accessibility relation between M and $\boldsymbol{A}$ satisfying them, so they are sound and complete.

1.7 Overview

We next point out some connections with other domains of artificial intelligence and computer science.

We then put our work in perspective with a summary of logical and semantical conditions for nonmonotonic and related logics, and present basic definitions for preferential structures.

Next, we will give special definitions for our framework.

We then start the main formal part, and prove representation results for $\mathcal{A}$-ranked structures, first for the general case, then for the smooth case. The general case needs more work, as we have to do a (minor) modification of the not $\mathcal{A}$-ranked case. The smooth case is easy, we simply have to append a small construction. Both proofs are given in full detail, in order to make the text self-contained.

Finally, we characterize changes due to restricted accessibility.

2 Connections with Other Concepts

2.1 Hierarchical Conditionals and Programs

Our situation is now very similar to a sequence of computer program instructions:

if A_1 then do B_1;
else if A_2 then do B_2;
else if A_3 then do B_3;

where we can see the B_i as subroutines.

We can deepen this analogy in two directions:

(1) connect it to Update
(2) put an imperative touch to it.

In both cases, we differentiate between different degrees of fulfillment of $\mathcal{C}$: the lower the level is which is fulfilled, the better.

(1) We can consider all threads of developments which lead to a model m where $m \models \mathcal{C}, \mathcal{C}$ a desirable result. Then we take as best threads those which lead to the

best fulfillment of $\mathcal{C}$. So the degree of fulfillment gives the order by which we should do the update. (This is then not update in the sense that we choose the most normal developments, but rather we choose the most desirable ones.) We will not pursue this line any further here, but leave it for future research.

(2) We introduce an imperative operator, say!.! means that one should fulfill $\mathcal{C}$ as best as possible by suitable choices. We will elaborate this now.

First, we can easily compare the degree of satisfaction of $\mathcal{C}$ of two models:

Definition 2.1 Let $m, m' \models \mathcal{C}$, and define $m < m' :\Rightarrow \mu(\mu(\mathcal{A}_m) \cup \mu(\mathcal{A}_{m'})) \cap \mu(\mathcal{A}_{m'}) = \emptyset$.($\mu$ is, as usual, relative to some fixed $\leq_t$.)

For two sets of models, X, X', the situation does not seem so easy. So suppose that $X, X' \models \mathcal{C}$. First, we have to decide how to compare this, we do by the maximum: $X < X'$ iff the worst satisfaction of all $x \in X$ is better than the worst satisfaction in X'. More precisely, we look at all $\gamma(\mathcal{C})$ for all $x \in X$, take the maximum (which exists, as $\mathcal{A}$ is finite), and then compare the maxima for X and for X'.

2.2 Connection with Theory Revision

In particular, the situation of contrary to duty obligations (see Sect. 1 (p. 2)) shows an intuitive similarity to theory revision, see Alchourron et al. (1985). You have the duty not to have a fence. If this is impossible (read: inconsistent), then it should be white. So the duty is revised.

But there is also a formal analogy: As is well known, AGM revision (with fixed left hand side theory, K, the theory which is to be revised, "knowledge base", therefore "K" in AGM notation) corresponds to a ranked order of models, where models of K have lowest rank (or: distance 0 from K-models). The structures we consider ($\mathcal{A}$-rankings) are partially ranked, i.e., there is only a partial ranked preference, inside the layers, nothing is said about the ordering. This partial ranking is natural, as we have only a limited number of cases to consider.

But we use the revision order (based on K, so it really is a $\leq_K$ relation) differently: We do not revise K, but use only the order to choose the first layer which has non-empty intersection with the set of possible cases. Still, the spirit (and formal apparatus) of revision is there, just used somewhat differently. The K-relation expresses here deontic quality, and if the best situation is impossible, we choose the second best, etc.

Full theory revision with variable K can be expressed by a distance d between models (see Lehmann et al. (2001)), where $K * \phi$, the result of revising K by the formula ϕ, is defined by the set of ϕ models which have minimal distance from the set of K models.

We can now generalize our idea of layered structure to a partial distance as follows: For instance, $d(K, \phi)$, the distance between the models of K and the models of ϕ is defined, $d(K, \phi')$ too, and we know that all ϕ-models with minimal distance to $M(K)$ have smaller distance than the ϕ'-models with minimal distance to $M(K)$. But we do NOT know a precise distance for other ϕ-models, we can sometimes compare, but not always. We may also know that all ϕ-models are closer to $M(K)$ than any ϕ'-model is, but for a and a', both ϕ-models, we might not know if one or the other is closer to $M(K)$, or if they have the same distance.

3 Representation Results for $\mathcal{A}$-Ranked Structures

3.1 Discussion

The not necessarily smooth and the smooth case will be treated differently.

Strangely, the smooth case is simpler, as an added new layer in the proof settles it. Yet, this is not surprising when looking closer, as minimal elements never have higher rank, and we know from (μCUM) that minimizing by minimal elements suffices. All we have to add that any element in the minimal layer minimizes any element higher up. For this reason, we mostly only quote earlier results.

In the simple, not necessarily smooth, case, we have to go deeper into the original proof to obtain the result. Therefore, we will give the full proof here.

The reader, after having gone through the proof, might wonder why the following idea, inspired by the treatment of the smooth case, will not work: Instead of minimizing by arbitrary elements, minimize only by elements of minimal rank, as the following example shows. If it worked, we might add just another layer to the original proof without $(\mu\mathcal{A})$, (see Definition 3.1 (p. 22)), as in the smooth case.

Example 3.1 Consider the base set $\{a, b, c\}$, $\mu(\{a, b, c\}) = \{b\}$, $\mu(\{a, b\}) = \{a, b\}$, $\mu(\{a, c\}) = \emptyset$, $\mu(\{b, c\}) = \{b\}$, $\mathcal{A}$ defined by $\{a, b\} < \{c\}$. Obviously, $(\mu\mathcal{A})$ is satisfied. μ can be represented by the (not transitive!) relation $a \prec c \prec a, b \prec c$, which is $\mathcal{A}$-ranked. But trying to minimize a in $\{a, b, c\}$ in the minimal layer will lead to $b \prec a$, and thus $a \notin \mu(\{a, b\})$, which is wrong. □

The proofs of the general and transitive general case are adaptations of earlier proofs by the second author, but the basic ideas are not new, and were published before—see e.g., Schlechta (2004), or Schlechta (1992). The quoted results for the smooth case are slightly stronger than those published in Schlechta (2004), as we work without closure under finite intersection, and the reader is referred to Gabbay and Schlechta (2009b) for the new proofs. The rest is almost verbatim the same, we only add a supplementary layer in the end (Fact 3.9 (p. 25)), which will make the construction $\mathcal{A}$-ranked. Thus, we only give the small new part, and not the quite long proofs of the basic results.

In the following, we will assume the partition $\mathcal{A}$ to be given. We could also construct it from the properties of μ, but this would need stronger closure properties of the domain. The construction of $\mathcal{A}$ is more difficult than the construction of the ranking in fully ranked structures, as $x \in \mu(X)$, $y \in X - \mu(X)$ will guarantee only $rg(x) \leq rg(y)$, and not $rg(x) < rg(y)$, as is the case in the latter situation. This corresponds to the separate treatment of the α and other formulas in the logical version, discussed in Sect. 3.4 (p. 27).

3.2 $\mathcal{A}$-Ranked General and Transitive Structures

We will show here the following representation results:

Let $\mathcal{A}$ be given.

An operation $\mu : \mathcal{Y} \to \mathcal{P}(Z)$ is representable by an $\mathcal{A}$-ranked preferential structure iff μ satisfies $(\mu \subseteq)$, (μPR), $(\mu \mathcal{A})$ (Proposition 3.3 (p. 23)), and, moreover, the structure can be chosen transitive (Proposition 3.5 (p. 23)).

Note that we carefully avoid any unnecessary assumptions about the domain $\mathcal{Y} \subseteq \mathcal{P}(Z)$ of the function μ.

Definition 3.1 We define a new condition:

Let $\mathcal{A}$ be given as defined in Definition 1.11 (p. 11).

$(\mu \mathcal{A})$ If $X \in \mathcal{Y}$, $A, A' \in \mathcal{A}$, $A < A'$, $X \cap A \neq \emptyset$, $X \cap A' \neq \emptyset$ then $\mu(X) \cap A' = \emptyset$.

This new condition will be central for the modified representation.

3.2.1 The Basic, not Necessarily Transitive, Case

Definition 3.2 For $x \in Z$, let $\mathcal{Y}_x := \{Y \in \mathcal{Y}: x \in Y - \mu(Y)\}$, $\Pi_x := \Pi \mathcal{Y}_x$.

Note that $\emptyset \notin \mathcal{Y}_x$, $\Pi_x \neq \emptyset$, and that $\Pi_x = \{\emptyset\}$ iff $\mathcal{Y}_x = \emptyset$.

Claim 3.1 Let $\mu : \mathcal{Y} \to \mathcal{P}(Z)$ satisfy $(\mu \subseteq)$ and (μPR), and let $U \in \mathcal{Y}$. Then $x \in \mu(U) \leftrightarrow x \in U \wedge \exists f \in \Pi_x . ran(f) \cap U = \emptyset$.

Proof Case 1: $\mathcal{Y}_x = \emptyset$, thus $\Pi_x = \{\emptyset\}$. "$\to$": Take $f := \emptyset$. "$\leftarrow$": $x \in U \in \mathcal{Y}$, $\mathcal{Y}_x = \emptyset \to x \in \mu(U)$ by definition of $\mathcal{Y}_x$.

Case 2: $\mathcal{Y}_x \neq \emptyset$. "$\to$": Let $x \in \mu(U) \subseteq U$. It suffices to show $Y \in \mathcal{Y}_x \to Y - U \neq \emptyset$. But if $Y \subseteq U$ and $Y \in \mathcal{Y}_x$, then $x \in Y - \mu(Y)$, contradicting (μPR). "$\leftarrow$": If $x \in U - \mu(U)$, then $U \in \mathcal{Y}_x$, so $\forall f \in \Pi_x . ran(f) \cap U \neq \emptyset$. □

Construction 3.1 Let $\mathcal{X} := \{\langle x, f \rangle : x \in Z \wedge f \in \Pi_x\}$, and $\langle x', f' \rangle \prec \langle x, f \rangle :\leftrightarrow x' \in ran(f)$. Let $\mathcal{Z} := \langle \mathcal{X}, \prec \rangle$.

Corollary 3.2 *Let* $\mu : \mathcal{Y} \to \mathcal{P}(Z)$ *satisfy* $(\mu \subseteq)$, (μPR), $(\mu \mathcal{A})$, *and let* $U \in \mathcal{Y}$.

If $x \in U$ *and* $\exists x' \in U . rg(x') < rg(x)$, *then* $\forall f \in \Pi_x . ran(f) \cap U \neq \emptyset$.

Proof By $(\mu \mathcal{A}) x \notin \mu(U)$, thus by Claim 3.1 (p. 22) $\forall f \in \Pi_x . ran(f) \cap U \neq \emptyset$. □

Proposition 3.3 *Let* $\mathcal{A}$ *be given.*

An operation $\mu : \mathcal{Y} \to \mathcal{P}(Z)$ *is representable by an* $\mathcal{A}$*-ranked preferential structure iff* μ *satisfies* $(\mu \subseteq)$, (μPR), $(\mu \mathcal{A})$.

Proof One direction is trivial. The central argument is: If $a \prec b$ in X, and $X \subseteq Y$, then $a \prec b$ in Y, too.

We turn to the other direction. The preferential structure is defined in Construction 3.2 (p. 23), Claim 3.4 (p. 23) shows representation.

Construction 3.2 Let $\mathcal{X} := \{\langle x, f \rangle : x \in Z \wedge f \in \Pi_x\}$, and $\langle x', f' \rangle \prec \langle x, f \rangle :\leftrightarrow x' \in ran(f)$ or $rg(x') < rg(x)$.

Note that, as $\mathcal{A}$ is given, we also know $rg(x)$.

Let $\mathcal{Z} := \langle \mathcal{X}, \prec \rangle$.

Obviously, $\mathcal{Z}$ is $\mathcal{A}$-ranked.

Claim 3.4 For $U \in \mathcal{Y}$, $\mu(U) = \mu_{\mathcal{Z}}(U)$.

Proof By Claim 3.1 (p. 22), it suffices to show that for all $U \in \mathcal{Y} x \in \mu_{\mathcal{Z}}(U) \leftrightarrow x \in U$ and $\exists f \in \Pi_x . ran(f) \cap U = \emptyset$. So let $U \in \mathcal{Y}$.

" $\rightarrow$ ": If $x \in \mu_{\mathcal{Z}}(U)$, then there is $\langle x, f \rangle$ minimal in $\mathcal{X}\lceil U$—where $\mathcal{X}\lceil U := \{\langle x, i \rangle \in \mathcal{X} : x \in U\}$), so $x \in U$, and there is no $\langle x', f' \rangle \prec \langle x, f \rangle, x' \in U$, so by $\Pi_{x'} \neq \emptyset$ there is no $x' \in ran(f), x' \in U$, but then $ran(f) \cap U = \emptyset$.

" $\leftarrow$ ": If $x \in U$, and there is $f \in \Pi_x, ran(f) \cap U = \emptyset$, then by Corollary 3.2 (p. 22), there is no $x' \in U, rg(x') < rg(x)$, so $\langle x, f \rangle$ is minimal in $\mathcal{X}\lceil U$. □

(Claim 3.4 (p. 23) and Proposition 3.3 (p. 23))

3.2.2 The Transitive Case

Proposition 3.5 *Let $\mathcal{A}$ be given.*

An operation $\mu : \mathcal{Y} \rightarrow \mathcal{P}(Z)$ is representable by an $\mathcal{A}$-ranked transitive preferential structure iff μ satisfies $(\mu \subseteq)$, (μPR), $(\mu \mathcal{A})$.

Construction 3.3

(1) For $x \in Z$, let T_x be the set of trees t_x s.t.
   - (a) all nodes are elements of Z,
   - (b) the root of t_x is x,
   - (c) $height(t_x) \leq \omega$,
   - (d) if y is an element in t_x, then there is $f \in \Pi_y := \Pi\{Y \in \mathcal{Y}\colon y \in Y - \mu(Y)\}$ s.t. the set of children of y is $ran(f) \cup \{y' \in Z : rg(y') < rg(y)\}$.

(2) For $x, y \in Z, t_x \in T_x, t_y \in T_y$, set $t_x \rhd t_y$ iff y is a (direct) child of the root x in t_x, and t_y is the subtree of t_x beginning at y.

(3) Let $\mathcal{Z} := \langle \{\langle x, t_x \rangle : x \in Z, t_x \in T_x\}, \langle x, t_x \rangle \succ \langle y, t_y \rangle$ iff $t_x \rhd t_y \rangle$.

Fact 3.6

(1) The construction ends at some y iff $\mathcal{Y}_y = \emptyset$ and there is no y' s.t. $rg(y') < rg(y)$, consequently $T_x = \{x\}$ iff $\mathcal{Y}_x = \emptyset$ and there are no x' with lesser rang. (We identify the tree of height 1 with its root.)

(2) We define a special tree tc_x for all x : For all nodes y in tc_x, the successors are as follows: if $\mathcal{Y}_y \neq \emptyset$, then z is an successor iff $z = y$ or $rg(z) < rg(y)$; if $\mathcal{Y}_y = \emptyset$, then z is an successor iff $rg(z) < rg(y)$. (In the first case, we make $f \in \mathcal{Y}_y$ always choose y itself.) tc_x is an element of T_x. Thus, with (1), $T_x \neq \emptyset$ for any x. Note: $tc_x = x$ iff $\mathcal{Y}_x = \emptyset$ and x has minimal rang.

(3) If $f \in \Pi_x$, then the tree tf_x with root x and otherwise composed of the subtrees tc_y for $y \in ran(f) \cup \{y' : rg(y') < rg(y)\}$ is an element of T_x. (Level 0 of tf_x has x as element, the $t_y's$ begin at level 1.)

(4) If y is an element in t_x and t_y the subtree of t_x starting at y, then $t_y \in T_y$.

(5) $\langle x, t_x \rangle \succ \langle y, t_y \rangle$ implies $y \in ran(f) \cup \{x' : rg(x') < rg(x)\}$ for some $f \in \Pi_x$. □

Claim 3.7 (p. 24) shows basic representation.

Claim 3.7 $\forall U \in \mathcal{Y}.\mu(U) = \mu_{\mathcal{Z}}(U)$

Proof By Claim 3.1 (p. 22), it suffices to show that for all $U \in \mathcal{Y} x \in \mu_{\mathcal{Z}}(U) \leftrightarrow x \in U \wedge \exists f \in \Pi_x.ran(f) \cap U = \emptyset$.

Fix $U \in \mathcal{Y}$.

" $\rightarrow$ ": $x \in \mu_{\mathcal{Z}}(U) \rightarrow$ ex. $\langle x, t_x \rangle$ minimal in $\mathcal{Z}\lceil U$, thus $x \in U$ and there is no $\langle y, t_y \rangle \in \mathcal{Z}$, $\langle y, t_y \rangle \prec \langle x, t_x \rangle$, $y \in U$. Let f define the first part of the set of children of the root x in t_x. If $ran(f) \cap U \neq \emptyset$, if $y \in U$ is a child of x in t_x, and if t_y is the subtree of t_x starting at y, then $t_y \in T_y$ and $\langle y, t_y \rangle \prec \langle x, t_x \rangle$, contradicting minimality of $\langle x, t_x \rangle$ in $\mathcal{Z}\lceil U$. So $ran(f) \cap U = \emptyset$.

" $\leftarrow$ ": Let $x \in U$, and $\exists f \in \Pi_x.ran(f) \cap U = \emptyset$. By Corollary 3.2 (p. 22), there is no $x' \in U, rg(x') < rg(x)$. If $\mathcal{Y}_x = \emptyset$, then the tree tc_x has no $\triangleright$-successors in U, and $\langle x, tc_x \rangle$ is $\succ$-minimal in $\mathcal{Z}\lceil U$. If $\mathcal{Y}_x \neq \emptyset$ and $f \in \Pi_x$ s.t. $ran(f) \cap U = \emptyset$, then $\langle x, tf_x \rangle$ is again $\succ$-minimal in $\mathcal{Z}\lceil U$. □

We consider now the transitive closure of $\mathcal{Z}$. (Recall that $\prec^*$ denotes the transitive closure of $\prec$.) Claim 3.8 (p. 25) shows that transitivity does not destroy what we have achieved. The trees tf_x play a crucial role in the demonstration.

Claim 3.8 Let $\mathcal{Z}' := \langle \{\langle x, t_x \rangle : x \in Z, t_x \in T_x\}, \langle x, t_x \rangle \succ \langle y, t_y \rangle$ iff $t_x \triangleright^* t_y \rangle$. Then $\mu_{\mathcal{Z}} = \mu_{\mathcal{Z}'}$.

Proof Suppose there is $U \in \mathcal{Y}$, $x \in U$, $x \in \mu_{\mathcal{Z}}(U)$, $x \notin \mu_{\mathcal{Z}'}(U)$. Then there must be an element $\langle x, t_x \rangle \in \mathcal{Z}$ with no $\langle x, t_x \rangle \succ \langle y, t_y \rangle$ for any $y \in U$. Let $f \in \Pi_x$ determine the first part of the set of children of x in t_x, then $ran(f) \cap U = \emptyset$, consider tf_x. All elements $w \neq x$ of tf_x are already in $ran(f)$, or $rg(w) < rg(x)$ holds. (Note that the elements chosen by rang in tf_x continue by themselves or by another element of even smaller rang, but the rang order is transitive.) But all w s.t. $rg(w) < rg(x)$ were already successors at level 1 of x in tf_x. By Corollary 3.2 (p. 22), there is no $w \in U, rg(w) < rg(x)$. Thus, no element $\neq x$ of tf_x is in U. Thus there is no $\langle z, t_z \rangle \prec^* \langle x, tf_x \rangle$ in $\mathcal{Z}$ with $z \in U$, so $\langle x, tf_x \rangle$ is minimal in $\mathcal{Z}'\lceil U$, contradiction. □

(Claim 3.8 (p. 25) and Proposition 3.5 (p. 23))

3.3 $\mathcal{A}$-Ranked Smooth Structures

All smooth cases have a simple solution. We use one of our existing proofs for the not necessarily $\mathcal{A}$-ranked case, and add one little result, Fact 3.9 (p. 25). The main results we use are Proposition 3.11 (p. 26) for the simple case, and Proposition 3.13 (p. 27) for the transitive case. We will not give the proofs for both results, as they are quite long, and already published in Schlechta (2004), Chapt. 3.3. What we just said is not quite true. Both proofs there use a stronger prerequisite for the domain (closure under finite intersection, too) than we do, the new, stronger, results are semi-published in Gabbay and Schlechta (2009b). The reader can also check there that the prerequisite $\langle x, i \rangle \prec \langle y, j \rangle$ implies $rg(x) \leq rg(y)$ of the following Fact is satisfied.

Fact 3.9 Let $(\mu\mathcal{A})$ hold, and let $\mathcal{Z} = \langle \mathcal{X}, \prec \rangle$ be a smooth preferential structure representing μ, i.e., $\mu = \mu_{\mathcal{Z}}$.

Suppose that $\langle x, i\rangle \prec \langle y, j\rangle$ implies $rg(x) \leq rg(y)$.
Define $\mathcal{Z}' := \langle \mathcal{X}, \sqsubset\rangle$ where $\langle x, i\rangle \sqsubset \langle y, j\rangle$ iff $\langle x, i\rangle \prec \langle y, j\rangle$ or $rg(x) < rg(y)$.
Then $\mathcal{Z}'$ is $\mathcal{A}$-ranked.
$\mathcal{Z}'$ is smooth, too, and $\mu_{\mathcal{Z}} = \mu_{\mathcal{Z}'} =: \mu'$.
In addition, if $\prec$ is free from cycles, so is $\sqsubset$, if $\prec$ is transitive, so is $\sqsubset$.

Proof $\mathcal{A}$-rankedness is trivial.

Suppose $\langle x, i\rangle$ is $\prec$-minimal, but not $\sqsubset$-minimal. Then there must be $\langle y, j\rangle \sqsubset \langle x, i\rangle, \langle y, j\rangle \not\prec \langle x, i\rangle, y \in X$, so $rg(y) < rg(x)$. By $(\mu\mathcal{A})$, all $x \in \mu(X)$ have minimal $\mathcal{A}$-rang among the elements of X, so this is impossible. Thus, μ-minimal elements stay μ'-minimal, so smoothness will also be preserved—remember that we increased the relation.

By prerequisite, there cannot be any cycle involving only $\prec$, but the rang order is free from cycles, too, and $\prec$ respects the rang order, so $\sqsubset$ is free from cycles.

Let $\prec$ be transitive, so is the rang order. But if $\langle x, i\rangle \prec \langle y, j\rangle$ and $rg(y) < rg(z)$ for some $\langle z, k\rangle$, then by prerequisite $rg(x) \leq rg(y)$, so $rg(x) < rg(z)$, so $\langle x, i\rangle \sqsubset \langle z, k\rangle$ by definition. Likewise for $rg(x) < rg(y)$ and $\langle y, j\rangle \prec \langle z, k\rangle$. □

3.3.1 The Basic Smooth, not Necessarily Transitive Case

We will show the following representation result:

Proposition 3.10 *Let $\mathcal{A}$ be given.*

Let $\mathcal{Y}$ be closed under finite unions, and $\mu : \mathcal{Y} \to \mathcal{P}(Z)$. Then there is a $\mathcal{Y}$-smooth $\mathcal{A}$-ranked preferential structure $\mathcal{Z}$, s.t. for all $X \in \mathcal{Y}\mu(X) = \mu_{\mathcal{Z}}(X)$ iff μ satisfies $(\mu \subseteq), (\mu PR), (\mu CUM), (\mu\mathcal{A})$.

To prove Proposition 3.10 (p. 26), we use

Proposition 3.11 *Let $\mu : \mathcal{Y} \to \mathcal{P}(U)$ satisfy $(\mu \subseteq)$, (μPR), and (μCUM), and the domain $\mathcal{Y}(\cup)$.*

Then there is a $\mathcal{Y}$-smooth preferential structure $\mathcal{X}$ s.t. $\mu = \mu_{\mathcal{X}}$. See e.g., Schlechta (2004).

Proof of Proposition 3.10 (p. 26) Consider the construction in the proof of Proposition 3.11 (p. 26) in Schlechta (2004) or Gabbay and Schlechta (2009b). It respects the rang order with respect to $\mathcal{A}$, i.e., that $\langle x', \sigma'\rangle \prec' \langle x, \sigma\rangle$ implies $rg(x') \leq rg(x)$. By definition, $x' \in \bigcup\{ran(\sigma_i) : i \in \omega\}$. If $x' \in ran(\sigma_0)$, then for some $Yx' \in \mu(Y), x \in Y - \mu(Y)$, so $rg(x') \leq rg(x)$ by $(\mu\mathcal{A})$. If $x' \in ran(\sigma_i), i > 0$, then for some $Xx', x \in \mu(X)$, so $rg(x) = rg(x')$ by $(\mu\mathcal{A})$. □

(Proposition 3.10 (p. 26))

3.3.2 The Transitive Smooth Case

We will show here the transitive analogon of Proposition 3.10 (p. 26):

Proposition 3.12 *Let $\mathcal{A}$ be given.*

Let $\mathcal{Y}$ be closed under finite unions, and $\mu : \mathcal{Y} \to \mathcal{P}(Z)$. Then there is a $\mathcal{Y}$-smooth $\mathcal{A}$-ranked transitive preferential structure $\mathcal{Z}$, s.t. for all $X \in \mathcal{Y} \mu(X) = \mu_{\mathcal{Z}}(X)$ iff μ satisfies $(\mu \subseteq)$, (μPR), (μCUM), $(\mu\mathcal{A})$.

To prove Proposition 3.12 (p. 27), we use:

Proposition 3.13 *Let $\mu : \mathcal{Y} \to \mathcal{P}(U)$ satisfy $(\mu \subseteq)$, (μPR), and (μCUM), and the domain $\mathcal{Y}(\cup)$.*

Then there is a transitive $\mathcal{Y}$-smooth preferential structure $\mathcal{X}$ s.t. $\mu = \mu_{\mathcal{X}}$. See e.g., Schlechta (2004).

Proof of Proposition 3.12 (p. 27) Consider the construction in the proof of Proposition 3.13 (p. 27) in Schlechta (2004) or Gabbay and Schlechta (2009b).

Note that in $\mathcal{Z}$ defined by

$\mathcal{Z} :=< \{\langle x, t_x\rangle : x \in Z, t_x \in T_x\}, \langle x, t_x\rangle \succ \langle y, t_y\rangle$ iff $t_x \rhd^* t_y >, t_x \rhd t_y$ implies $rg(y) \leq rg(x)$.

But by construction of the trees, $x_n \in Y_{n+1}$, and $x_{n+1} \in \mu(U_n \cup Y_{n+1})$, so $rg(x_{n+1}) \leq rg(x_n)$. □

(Proposition 3.12 (p. 27))

3.4 The Logical Properties with Definability Preservation

First, a small fact about the $\mathcal{A}$.

Fact 3.14 Let $\mathcal{A}$ be as above (and thus finite). Then each A_i is equivalent to a formula α_i.

Proof We use the standard topology and its compactness. By definition, each $M(A_i)$ is closed, by finiteness all unions of such $M(A_i)$ are closed, too, so $\boldsymbol{C}(M(A_i))$ is closed. By compactness, each open cover $X_j : j \in J$ of the clopen $M(A_i)$ contains a finite subcover, so also $\bigcup\{M(A_j) : j \neq i\}$ has a finite open cover. But the $M(\phi)$, ϕ a formula form a basis of the closed sets, so we are done. □

Most (with the exception of the material on $\mathcal{A}$-rankedness) of the following was already published in various articles and books by the second author, see e.g., Schlechta (2004), but is repeated here for completeness' sake.

Proposition 3.15 *Let $\mid\!\sim$ be a logic for $\mathcal{L}$. Set $T^{\mathcal{M}} := Th(\mu_{\mathcal{M}}(M(T)))$, where $\mathcal{M}$ is a preferential structure.*

(1) Then there is a (transitive) definability preserving classical preferential model $\mathcal{M}$ s.t. $\overline{\overline{T}} = T^{\mathcal{M}}$ iff (LLE), (CCL), (SC), (PR) hold for all $T, T' \subseteq \mathcal{L}$.
(2) The structure can be chosen smooth, iff, in addition (CUM) holds.
(3) The structure can be chosen $\mathcal{A}$-ranked, iff, in addition ($\mathcal{A}$-min) $T \not\vdash \neg\alpha_i$ and $T \not\vdash \neg\alpha_j, i < j$ implies $\overline{\overline{T}} \vdash \neg\alpha_j$ holds.

The proof is an immediate consequence of Proposition 3.16 (p. 28) and the respective above results. This proposition (or its analogue) was mostly already shown in Schlechta (1992) and Schlechta (1996) and is repeated here for completeness' sake, but with a new and partly stronger proof.

Proposition 3.16 *Consider for a logic $\mid\sim$ on $\mathcal{L}$ the properties (LLE), (CCL), (SC), (PR), (CUM), ($\mathcal{A}$-min) hold for all $T, T' \subseteq \mathcal{L}$. and for a function $\mu : \boldsymbol{D}_{\mathcal{L}} \to \mathcal{P}(M_{\mathcal{L}})$ the properties $(\mu dp)\mu$ is definability preserving, i.e., $\mu(M(T)) = M(T')$ for some T' $(\mu \subseteq)$, (μPR), (μCUM), $(\mu\mathcal{A})$ for all $X, Y \in \boldsymbol{D}_{\mathcal{L}}$. It then holds:*

(a) If μ satisfies (μdp), $(\mu \subseteq)$, (μPR), then $\mid\sim$ defined by $\overline{\overline{T}} := T^{\mu} := Th(\mu(M(T)))$ satisfies (LLE), (CCL), (SC), (PR). If μ satisfies in addition (μCUM), then (CUM) will hold, too. If μ satisfies in addition $(\mu\mathcal{A})$, then ($\mathcal{A}$-min) will hold, too.
(b) If $\mid\sim$ satisfies (LLE), (CCL), (SC), (PR), then there is $\mu : \boldsymbol{D}_{\mathcal{L}} \to \mathcal{P}(M_{\mathcal{L}})$ s.t. $\overline{\overline{T}} = T^{\mu}$ for all $T \subseteq \mathcal{L}$ and μ satisfies (μdp), $(\mu \subseteq)$, (μPR). If, in addition, (CUM) holds, then (μCUM) will hold, too. If, in addition, ($\mathcal{A}$-min) holds, then $(\mu\mathcal{A})$ will hold, too.

Proof Set $\mu(T) := \mu(M(T))$, note that $\mu(T \cup T') := \mu(M(T \cup T')) = \mu(M(T) \cap M(T'))$.

(a) Suppose $\overline{\overline{T}} = T^{\mu}$ *for* some such μ, and all T.
(LLE): If $\overline{T} = \overline{T'}$, then $M(T) = M(T')$, so $\mu(T) = \mu(T')$, and $T^{\mu} = T'^{\mu}$.
(CCL) and (SC) are trivial.
We show (PR): $M(\overline{\overline{T}} \cup T') = M(\overline{\overline{T}}) \cap M(T') =_{(\mu dp)} \mu(T) \cap M(T') =_{(\mu\subseteq)} \mu(T) \cap M(T) \cap M(T') = \mu(T) \cap M(T \cup T') \subseteq_{(\mu PR)} \mu(T \cup T') =_{(\mu dp)} M(\overline{\overline{T \cup T'}})$. Let now $\phi \in \overline{\overline{T \cup T'}}$, so ϕ holds in all $m \in M(\overline{\overline{T \cup T'}})$, so ϕ holds in all $m \in M(\overline{\overline{T}} \cup T')$, so $\overline{\overline{T}} \cup T' \vdash \phi$, so $\phi \in \overline{\overline{\overline{T}} \cup T'}$.
We turn to (CUM):
Let $T \subseteq \overline{T'} \subseteq \overline{\overline{T}}$. Thus by (μCum) and $\mu(T) \subseteq M(\overline{\overline{T}}) \subseteq M(T') \subseteq M(T)$, so $\mu(T) = \mu(T')$, so $\overline{\overline{T}} = Th(\mu(T)) = Th(\mu(T')) = \overline{\overline{T'}}$.
($\mathcal{A}$-min) is trivial.
(b) Let $\mid\sim$ satisfy $(LLE) - (CUM)$ for all T. We define μ and show $\overline{\overline{T}} = T^{\mu}$. (CUM) will be needed only to show (μCUM).
If $X = M(T)$ for some $T \subseteq \mathcal{L}$, set $\mu(X) := M(\overline{\overline{T}})$.
If $X = M(T) = M(T')$, then $\overline{T} = \overline{T'}$, thus $\overline{\overline{T}} = \overline{\overline{T'}}$ by (LLE), so $M(\overline{\overline{T}}) = M(\overline{\overline{T'}})$, and μ is well-defined. Moreover, μ satisfies (μdp), and by (SC), $\mu(X) \subseteq X$. We show $\overline{\overline{T}} = T^{\mu}$: Let now $T \subseteq \mathcal{L}$ be given. Then $\phi \in T^{\mu} :\leftrightarrow \forall m \in \mu(T).m \models \phi \leftrightarrow \forall m \in M(\overline{\overline{T}}).m \models \phi \leftrightarrow \overline{\overline{T}} \vdash \phi \leftrightarrow \phi \in \overline{\overline{T}}$ (as $\overline{\overline{T}}$ is classically closed).
Next, we show that the above defined μ satisfies (μPR). Suppose $X := M(T)$, $Y := M(T')$, $X \subseteq Y$, we have to show $\mu(Y) \cap X \subseteq \mu(X)$. By prerequisite, $\overline{T'} \subseteq \overline{T}$, so $\overline{T \cup T'} = \overline{T}$, so $\overline{\overline{T \cup T'}} = \overline{\overline{T}}$ by (LLE). By (PR) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{\overline{T'}} \cup T}}$, so $\mu(Y) \cap X = \mu(T') \cap M(T) = M(\overline{\overline{T'}} \cup T) \subseteq M(\overline{\overline{T \cup T'}}) = M(\overline{\overline{T}}) = \mu(X)$, using (μdp).

($\mu\mathcal{A}$) is trivial.
It remains to show (μCUM). So let $X = M(T), Y = M(T')$, and $\mu(T) := M(\overline{\overline{T}}) \subseteq M(T') \subseteq M(T) \rightarrow \overline{T} \subseteq \overline{T'} \subseteq \overline{\overline{T}} = \overline{\overline{(\overline{T})}} \rightarrow \overline{\overline{T}} = \overline{\overline{(\overline{T})}} = \overline{\overline{(\overline{T'})}} = \overline{\overline{T'}} \rightarrow \mu(T) = M(\overline{\overline{T}}) = M(\overline{\overline{T'}}) = \mu(T')$, thus $\mu(X) = \mu(Y)$. □
(Proposition 3.16 (p. 28))

4 Formal Results and Representation for Hierarchical Conditionals

We look here at the following problem:
Given

(1.1) a finite, ordered partition $\mathcal{A}$ of $\boldsymbol{A}$, $\mathcal{A} = \langle\{A_i : i \in I\}, <\rangle$
(1.2) a normality relation $\prec$, which is an $\mathcal{A}$-ranking, defining a choice function μ on subsets of $\boldsymbol{A}$, (so, obviously, $A < A'$ iff $\mu(A \cup A') \cap A' = \emptyset$),
(1.3) a subset $\boldsymbol{B} \subseteq \boldsymbol{A}$, and we set $\mathcal{C} := \langle\mathcal{A}, \boldsymbol{B}\rangle$ (thus, the B_i are just $A_i \cap \boldsymbol{B}$, this way of writing saves a little notation),
(2.1) a set of models M,
(2.2) an accessibility relation R on M, with some finite upper bound on R-chains,
(2.3) an unknown extension of R to pairs (m, a), $m \in M$, $a \in \boldsymbol{A}$,
(3.1) a notion of validity $m \models \mathcal{C}$, for $m \in M$, defined by $m \models \mathcal{C}$ iff $\{a \in \boldsymbol{A} : \exists A \in \mathcal{A}(a \in \mu(A), a \in R(m)$, and
$$\neg\exists a'(\exists A' \in \mathcal{A}(a' \in \mu(A'), a' \in R(m), a' \prec a\} \subseteq \boldsymbol{B},$$
(3.2) a subset M' of M give a criterion which decides whether it is possible to construct the extension of R to pairs (m, a) s.t. $\forall m \in M.(m \in M' \Leftrightarrow m \models \mathcal{C})$.
We first show some elementary facts on the situation, and give the criterion in Proposition 4.4 (p. 31), together with the proof that it does what is wanted.

Fact 4.1 Reachability for a transitive relation is characterized by

$$y \in R(x) \rightarrow R(y) \subseteq R(x)$$

Proof Define directly xRz iff $z \in R(x)$. This does it. □

Let now S be a set with an accessibility relation R', generated by transitive closure from the intransitive subrelation R. All modal notation will be relative to this R.
Let $\boldsymbol{A} = M(\alpha)$, $A_i = M(\alpha_i)$, the latter is justified by Fact 3.14 (p. 27).

Definition 4.1 (1) $\mu(\mathcal{A}) := \bigcup\{\mu(A_i) : i \in I\}$
(warning: this is NOT $\mu(\boldsymbol{A})$)
(2) $\mathcal{A}_m := R(m) \cap \boldsymbol{A}$,
(3) $\mu(\mathcal{A}_m) := \bigcup\{\mu(A_i) \cap R(m) : i \in I\}$
(3a) $\nu(\mathcal{A}_m) := \mu(\mu(\mathcal{A}_m))$
(thus $\nu(\mathcal{A}_m) = \{a \in \boldsymbol{A} : \exists A \in \mathcal{A}(a \in \mu(A), a \in R(m)$, and
$$\neg\exists a'(\exists A' \in \mathcal{A}(a' \in \mu(A'), a' \in R(m), a' \prec a)))\}.$$
(4) $m \models \mathcal{C} :\leftrightarrow \nu(\mathcal{A}_m) \subseteq \boldsymbol{B}$.
See Fig. 3 (p. 30)

We have the following Fact for $m \models \mathcal{C}$:

Fact 4.2 Let $m, m' \in M$, $A, A' \in \mathcal{A}$.

(1) $m \models \Box\neg\alpha$, mRm$' \Rightarrow m' \models \Box\neg\alpha$
(2) mRm', $\nu(\mathcal{A}_m) \cap A \neq \emptyset$, $\nu(\mathcal{A}_{m'}) \cap A' \neq \emptyset$, $\Rightarrow A \leq A'$
(3) mRm', $\nu(\mathcal{A}_m) \cap A \neq \emptyset$, $\nu(\mathcal{A}_{m'}) \cap A' \neq \emptyset$, $m \models \mathcal{C}$, $m' \not\models \mathcal{C}$, $\Rightarrow A < A'$

Proof Trivial. □

Fact 4.3 We can conclude from above properties that there are no arbitrarily long R-chains of models m, changing from $m \models \mathcal{C}$ to $m \not\models \mathcal{C}$ and back.

Proof Trivial: By Fact 4.2 (p. 31), (3), any change from $\models \mathcal{C}$ to $\not\models \mathcal{C}$ results in a strict increase in rank. □

We solve now the representation task described at the beginning of Sect. 4 (p. 29), all we need are the properties shown in Fact 4.2 (p. 31).

(Note that constructing R between the different m, m' is trivial: we could just choose the empty relation.)

Proposition 4.4 *If the properties of Fact 4.2 (p. 31) hold, we can extend R to solve the representation problem described at the beginning of this Sect. 4 (p. 29).*

Proof By induction on R. This is possible, as the depth of R on M was assumed to be finite.

Construction 4.1 We choose now elements as possible, which ones are chosen exactly does not matter.

$X_i := \{b_i, c_i\}$ iff $\mu(A_i) \cap \boldsymbol{B} \neq \emptyset$ and $\mu(A_i) - \boldsymbol{B} \neq \emptyset$, $b_i \in \mu(A_i) \cap \boldsymbol{B}$, $c_i \in \mu(A_i) - \boldsymbol{B}$.
$X_i := \{c_i\}$ iff $\mu(A_i) \cap \boldsymbol{B} = \emptyset$ and $\mu(A_i) - \boldsymbol{B} \neq \emptyset$, $c_i \in \mu(A_i) - \boldsymbol{B}$
$X_i := \{b_i\}$ iff $\mu(A_i) \cap \boldsymbol{B} \neq \emptyset$ and $\mu(A_i) - \boldsymbol{B} = \emptyset$, $b_i \in \mu(A_i) \cap \boldsymbol{B}$,
$X_i := \emptyset$ iff $\mu(A_i) = \emptyset$.
Case 1:
Let m be R-minimal and $m \models \mathcal{C}$. Let i_0 be the first i s.t. $b_i \in X_i$, make $\gamma(m) := i_0$, and make $R(m) := \{b_{i_0}\} \cup \bigcup\{X_i : i > i_0\}$. This makes $\mathcal{C}$ hold. (This leaves us as many possibilities open as possible—remember we have to decrease the set of reachable elements now.)
Case 2:
Let m be R-minimal and $m \not\models \mathcal{C}$. Let i_0 be the first i s.t. $c_i \in X_i$, make $\gamma(m) := i_0$, and make $R(m) := \bigcup\{X_i : i \geq i_0\}$. This makes $\mathcal{C}$ false.
Let all R-predecessors of m be determined, and $i := max\{\gamma(m') : m'Rm\}$.
Case 3: $m \models \mathcal{C}$. Let j be the smallest $i' \geq i$ with $\mu(A_{i'}) \cap \boldsymbol{B} \neq \emptyset$. Let $R(m) := \{b_j\} \cup \bigcup\{X_k : k > j\}$, and $\gamma(m) := j$.
Case 4: $m \not\models \mathcal{C}$.
Case 4.1: For all $m'Rm$ with $i = \gamma(m')m' \not\models \mathcal{C}$.
Take one such m' and set $R(m) := R(m')$, $\gamma(m) := i$.

Case 4.2: There is $m'Rm$ with $i = \beta(m')m' \models \mathcal{C}$.
Let j be the smallest $i' > i$ with $\mu(A_{i'}) - \boldsymbol{B} \neq \emptyset$. Let $R(m) := \bigcup\{X_k : k \geq j\}$.
(Remark: To go from $\models$ to $\not\models$, we have to go higher in the hierarchy.)
Obviously, validity is done as it should be. It remains to show that the sets of reachable elements decrease with R.

Fact 4.5 In above construction, if mRm′, then $R(m') \subseteq R(m)$.

Proof By induction, considering R.□ (Fact 4.5 (p. 32) and Proposition 4.4 (p. 31))

We consider an example for illustration.

Example 4.1 Let $a_1 R a_2 R c R c_1, b_1 R b_2 R b_3 R c R d_1 R d_2$.
Let $\mathcal{C} = (A_1 > B_1, \ldots, A_n > B_n)$ with the C_i consistency with $\mu(A_i)$.
Let $\mu(A_2) \cap B_2 = \emptyset$, $\mu(A_3) \subseteq B_3$, and for the other i hold neither of these two.
Let $a_1, a_2, b_2, c_1, d_2 \models \mathcal{C}$, the others $\not\models \mathcal{C}$.
Let $\mu(A_1) = \{a_{1,1}, a_{1,2}\}$, with $a_{1,1} \in B_1, a_{1,2} \notin B_1$,
$\mu(A_2) = \{a_{2,1}\}$, $\mu(A_3) = \{a_{3,1}\}$ (there is no reason to differentiate),
and the others like $\mu(A_1)$. Let $\mu A := \bigcup\{\mu(A_i) : i \leq n\}$.
We have to start at a_1 and b_1, and make $R(x)$ progressively smaller.
Let $R(a_1) := \mu A - \{a_{1,2}\}$, so $a_1 \models \mathcal{C}$. Let $R(a_2) = R(a_1)$, so again $a_2 \models \mathcal{C}$.
Let $R(b_1) := \mu A - \{a_{1,1}\}$, so $b_1 \not\models \mathcal{C}$. We now have to take $a_{1,2}$ away, but $a_{2,1}$ too to be able to change. So let $R(b_2) := R(b_1) - \{a_{1,2}, a_{2,1}\}$, so we begin at $\mu(A_3)$, which is a (positive) singleton. Then let $R(b_3) := R(b_2) - \{a_{3,1}\}$.
We can choose $R(c) := R(b_3)$, as $R(b_3) \subseteq R(a_2)$.
Let $R(c_1) := R(c) - \{a_{4,2}\}$ to make $\mathcal{C}$ hold again. Let $R(d_1) := R(c)$, and $R(d_2) := R(c_1)$. □

5 Conclusion

We introduced a new type of semantic structure, intermediate between general and ranked preferential structures, and gave a representation theorem for those structures. These structures are well adapted to represent hierarchical conditionals, and thus contrary-to-duty conditionals, with different degrees of fulfillment. We also pointed out connections to distance based Theory Revision and Update, as well as to programming, but they are not pursued.

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