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Shifting martingale measures and the birth of a bubble as a submartingale

Francesca Biagini · Hans Föllmer · Sorin Nedelcu

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Abstract In an incomplete financial market model, we study a flow in the space of equivalent martingale measures and the corresponding shifting perception of the fundamental value of a given asset. This allows us to capture the birth of a perceived bubble and to describe it as an initial submartingale which then turns into a supermartingale before it falls back to its initial value zero.

Keywords Bubbles · Strict local martingales · Submartingales · Equivalent martingale measures · Stochastic volatility

Mathematics Subject Classification (2010) 91G99 · 60G07 · 60G44

JEL Classification C60 · C65 · G12

1 Introduction

The notion of an asset price bubble has two ingredients. One is the observed market price of a given financial asset, the other is the asset's intrinsic value, and the bubble is defined as the difference between the two. The intrinsic value, also called

F. Biagini (✉) · S. Nedelcu
Department of Mathematics, Ludwig-Maximilians Universität, Theresienstraße 39, 80333 Munich, Germany
e-mail: biagini@math.lmu.de

S. Nedelcu
e-mail: nedelcu@math.lmu.de

H. Föllmer
Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany
e-mail: foellmer@mathematik.hu-berlin.de

the fundamental value of the asset, is usually defined as the expected sum of future discounted dividends. Since it involves an expectation, this second ingredient of the bubble may involve a considerable amount of model ambiguity: What looks like a bubble to some may be not a bubble for others, if their perception of the fundamental value happens to coincide with the actual price. It has been shown, however, that bubbles arise even in experimental situations where there is no ambiguity about the probabilistic setting, and where market participants are informed of the resulting fundamental value at all times; see Smith et al. [25]. From an economic point of view, the main challenge therefore consists in explaining how such bubbles are generated at the microeconomic level by the interaction of market participants; see for instance Tirole [26], Harrison and Kreps [10], DeLong et al. [7], Scheinkman and Xiong [23], Abreu and Brunnermeier [1], Föllmer et al. [8] and the references therein.

In this paper, however, we make no attempt to contribute to a deeper economic understanding of bubbles on the side of price formation. Instead, we focus on the perception of the fundamental value. More precisely, we consider the following question, which has already been studied by Jarrow et al. [17], and which arises naturally in the standard setting of an incomplete financial market model. Here the discounted price process of a liquid financial asset is given in advance as a semimartingale S on some filtered probability space. If D denotes the associated cumulative discounted dividend process, then absence of arbitrage implies the existence of an equivalent measure which turns the wealth process $W = S + D$ into a local martingale. Following an argument of Harrison and Kreps [10], any such measure can be seen as a prediction scheme that is consistent with the observed price process S if we take a speculative point of view, taking into account not only future dividends, but also the possibility of selling the asset at some future time. However, if we take a fundamental point of view and restrict attention to future dividends, then different martingale measures may give a different assessment. Suppose that at any time, the fundamental value of the asset is computed as the conditional expectation of future discounted dividends under some equivalent local martingale measure. Time consistency would require that all these conditional expectations are computed under the same martingale measure R. Denoting by S^R the resulting fundamental value process, the bubble is now defined as the difference $S - S^R$, and this will be a nonnegative local martingale under R. There is a growing literature about such bubbles and their various effects; see for instance Loewenstein and Willard [20], Cox and Hobson [4], Jarrow and Madan [12], Jarrow/Protter et al. [11, 13, 14, 16, 17]. In Jarrow and Protter [15], the novel concept of a relative asset bubble is introduced, which allows the study of price bubbles for assets with bounded payoffs such as defaultable bonds. The connection between bubbles and the prices of derivatives written on assets whose price process is driven by a strict local martingale has been studied in Pal and Protter [21] and Kardaras et al. [19]. In [19], the authors provide a decomposition of the price of certain classes of path-dependent options into a "non-bubble" term and a default term. In a recent paper which focuses on currency exchange rates, Carr et al. [3] use the Föllmer measure to construct a pricing operator for complete models where the exchange rate is driven by a strict local martingale. This construction allows to preserve put-call parity and also provides the minimal joint replication price for a contingent claim. For a comprehensive survey of the recent mathematical literature on financial bubbles, we refer to Protter [22].

But in such a setting, where the bubble is defined in terms of one fixed martingale measure R, there are only two possibilities: Either the bubble starts at some strictly positive initial value, or it is zero all the time. So how do we capture the birth of a bubble in the standard framework of an incomplete financial market? To this end, we have to give up time consistency and the corresponding choice of one single equivalent martingale measure. While time consistency may be desirable from a normative point of view, there are many factors at work at the microeconomic level that may cause, at the aggregate level, a shift of the martingale measure. In particular, herding behavior of heterogeneous agents with interacting preferences and expectations may have this effect. It is therefore plausible to introduce a dynamics in the space of equivalent local martingale measures, and to look at the corresponding shifting perceptions of the fundamental value. In their paper on "Asset price bubbles in incomplete financial markets" [17], Jarrow, Protter and Shimbo do take that point of view. They consider a dynamics of regime switching, where the martingale measure can only change at certain times. In this picture, a bubble will pop up at some stopping time, and then it will suddenly disappear again at some later stopping time.

In the present paper, we consider a different picture. Our aim is to capture the slow birth of a perceived bubble starting at zero, and to describe it as an initial submartingale. To this end, we fix two martingale measures Q and R. Under the measure Q, the wealth process W is a uniformly integrable martingale, we have $S = S^Q$, and there is no perception of a bubble. Under the measure R, the process W is no longer uniformly integrable, we have $S > S^R$, and so a bubble is perceived under R. A martingale measure is often interpreted as a price equilibrium corresponding to the subjective preferences and expectations of some representative agent; see for example Föllmer and Schied [9, Sect. 3.1]. In the case of the martingale measure Q, this subjective view is "optimistic", or "exuberant": the actual price is seen to be fully justified by the perceived fundamental value. In the case of R, the view is "pessimistic" or "sober", and there is a bubble in the eye of the beholder.

The coexistence of such martingale measures Q and R holds for a wide variety of incomplete financial market models. This is illustrated by a generic example due to Delbaen and Schachermayer [6] and by the stochastic volatility model discussed by Sin [24]. Furthermore, these examples show that typically the following condition is satisfied: The fundamental wealth $W^R = S^R + D$ perceived under the "sober" measure R behaves as a submartingale under the "optimistic" measure Q. In other words, under Q it is expected that the assessment W^R, which seems too pessimistic from that point of view, has a tendency to be adjusted in the upward direction.

In Sect. 3, we study a flow $\mathcal{R} = (R_t)_{t\geq 0}$ in the space of martingale measures that moves from the initial measure Q to the measure R via convex combinations of Q and R, which put an increasing weight on R; for an economic interpretation of such a flow in terms of a microeconomic model of interacting agents in the spirit of [8], see Remark 3.3. The corresponding shifting perception of the fundamental value, computed at time t in terms of the martingale measure R_t, is described by the fundamental value process $S^{\mathcal{R}}$. We denote by $\beta^{\mathcal{R}} = S - S^{\mathcal{R}}$ the resulting $\mathcal{R}$-bubble perceived under the flow $\mathcal{R}$, and we assume that the above condition on the submartingale behavior of W^R under Q is satisfied. In Theorem 3.9, we show that the birth and the subsequent behavior of the $\mathcal{R}$-bubble under the reference measure R can be described as follows: The $\mathcal{R}$-bubble starts from its initial value as a submartingale and then turns into a supermartingale before it finally falls back to zero.

In Sect. 4, we look at the example of Delbaen and Schachermayer [6] where the price process S along with the measures Q and R is defined in terms of two independent continuous martingales, for instance by two independent geometric Brownian motions. Here the processes W^R and $\beta^{\mathcal{R}}$ can be computed explicitly, and we can easily verify our condition on the submartingale behavior of W^R under Q. In Sect. 5, we verify the same condition for a variant of the stochastic volatility model discussed by Sin [24]. But we also show that the model can be modified in such a way that the condition no longer holds. In the final Sect. 6, we change our point of view: Instead of using R as a reference measure, we compute the canonical decomposition of the $\mathcal{R}$-bubble under the measure Q. Here again, the birth of the bubble can be described as an initial submartingale. Its subsequent behavior is now more delicate though, as illustrated in the context of the Delbaen–Schachermayer example.

Our study of a simple flow between two martingale measures of different types complements the study of successive regime switching in [17], and it sheds new light on the birth of a perceived bubble. Both case studies should be seen as first steps towards a systematic investigation of dynamics in the space of martingale measures. Ultimately, any dynamics at that level should be derived from an underlying dynamics at the microeconomic level of interacting market participants and thus be connected with the literature mentioned above, but this is beyond the scope of the present paper.

2 The setting

We consider a market model that contains a risky asset and a money market account. We use the money market account as numéraire, and so we may assume that it is constantly equal to 1. The risky asset generates an uncertain cumulative cash-flow, modeled as a nonnegative increasing and adapted right-continuous process $D = (D_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ that satisfies the usual conditions. In order to simplify the presentation, **we assume that the filtration is such that all martingales have continuous paths.**

Remark 2.1 The process $D = (D_t)_{t\geq 0}$ may be viewed as a cumulative dividend process. There could be some maturity date or default time ζ such that $D_t = D_\zeta$ on $\{\zeta \leq t\}$, and then the value $X := (D_\zeta - D_{\zeta-})1_{\{\zeta<\infty\}}$ can be interpreted as a terminal payoff or liquidation value, as in the setting of [17].

The *market price* of the asset is given by the nonnegative, adapted càdlàg process $S = (S_t)_{t\geq 0}$. We denote by $W = (W_t)_{t\geq 0}$ the corresponding *wealth process* defined by

$$W_t = S_t + D_t, \quad t \geq 0.$$

Our focus will be on the class of globally equivalent local martingale measures for W. More precisely, we denote by $\mathcal{M}_{\mathrm{loc}}(W)$ the class of all probability measures $Q \approx P$ such that W is a local martingale under Q, and we assume that

$$\mathcal{M}_{\mathrm{loc}}(W) \neq \emptyset.$$

This clearly implies that there is no free lunch with vanishing risk (NFLVR); cf. Delbaen and Schachermayer [5]. It is satisfied by our case studies in Sects. 4 and 5.

Remark 2.2 Suppose that in analogy to Bouchard and Nutz [2], we are in a situation of model uncertainty where no probability measure is given ex ante. In this case, we should assume the existence of some local martingale measure Q for W and define $\mathcal{M}_{\mathrm{loc}}(W)$ in terms of the reference measure $P := Q$.

For any probability measure $Q \in \mathcal{M}_{\mathrm{loc}}(W)$ and at any time t, the given price S_t is justified from the point of view of Q if we take into account not only the expectation of the future cumulative cash-flow, but also the option to sell the asset at some future time τ. As in [10], this is made precise by (2.1) below, and in particular by its second part.

Lemma 2.3 *For any $Q \in \mathcal{M}_{\mathrm{loc}}(W)$, the limits $S_\infty := \lim_{t\to\infty} S_t$, $W_\infty := \lim_{t\to\infty} W_t$ and $D_\infty := \lim_{t\to\infty} D_t$ exist a.s. and in $L^1(Q)$, and*

$$S_t = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}_Q[D_\tau - D_t + S_\tau | \mathcal{F}_t] \tag{2.1}$$

$$= \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}_Q[D_\tau - D_t + S_\tau 1_{\{\tau<\infty\}} | \mathcal{F}_t], \tag{2.2}$$

where the essential supremum is taken over all stopping times $\tau \geq t$.

Proof As W is a nonnegative local martingale and hence a supermartingale under Q, the limit $W_\infty := \lim_{t\to\infty} W_t$ exists Q-a.s. and in $L^1(Q)$. So does $S_\infty := \lim_{t\to\infty} S_t$, since the limit $D_\infty := \lim_{t\to\infty} D_t$ exists by monotonicity. Thus the right-hand side of (2.1) is well defined. Moreover,

$$W_t \geq \mathbb{E}_Q[W_\tau | \mathcal{F}_t] \tag{2.3}$$

for any stopping time $\tau \geq t$, and this translates into

$$S_t \geq \mathbb{E}_Q[D_\tau - D_t + S_\tau | \mathcal{F}_t] \geq \mathbb{E}_Q[D_\tau - D_t + S_\tau 1_{\{\tau<\infty\}} | \mathcal{F}_t]. \tag{2.4}$$

On the other hand, we get equality in (2.3), and hence in (2.4), for $n > t$ and $\tau = \sigma \wedge n$ whenever σ is a localizing stopping time for W and Q, and so we have shown (2.1) and (2.2). □

In particular, Lemma 2.3 implies

$$S_t \geq S_t^Q := \mathbb{E}_Q[D_\infty - D_t | \mathcal{F}_t], \tag{2.5}$$

where S^Q is the potential generated by the increasing process D under the measure Q.

Definition 2.4 For $Q \in \mathcal{M}_{\mathrm{loc}}(W)$, the potential S^Q defined in (2.5) is called the *fundamental price* of the asset *perceived under the measure Q*.

Equation (2.1) shows that under any martingale measure $Q \in \mathcal{M}_{\mathrm{loc}}(W)$, the given price of the asset is justified from a speculative point of view, given the possibility of selling the asset at some future time. In this sense different martingale measures agree on the same price S. But they may provide very different assessments S^Q of the asset's fundamental value. Let us discuss this point more precisely.

As in [17], we use the notation

$$\mathcal{M}_{\mathrm{loc}}(W) = \mathcal{M}_{\mathrm{UI}}(W) \cup \mathcal{M}_{\mathrm{NUI}}(W),$$

where $\mathcal{M}_{\mathrm{UI}}(W)$ denotes the class of measures $Q \approx P$ such that W is a uniformly integrable martingale under Q, and where $\mathcal{M}_{\mathrm{NUI}}(W) = \mathcal{M}_{\mathrm{loc}}(W) \setminus \mathcal{M}_{\mathrm{UI}}(W)$. Typically, the classes $\mathcal{M}_{\mathrm{UI}}(W)$ and $\mathcal{M}_{\mathrm{NUI}}(W)$ will both be non-empty, as illustrated in the examples of Sects. 4 and 5. From now on we assume that this is the case:

Standing Assumption 2.5 $\mathcal{M}_{\mathrm{UI}}(W) \neq \emptyset$ and $\mathcal{M}_{\mathrm{NUI}}(W) \neq \emptyset$.

Lemma 2.6 *A measure $Q \in \mathcal{M}_{\mathrm{loc}}(W)$ belongs to $\mathcal{M}_{\mathrm{UI}}(W)$ if and only if*

$$S_t = \mathbb{E}_Q[D_\infty - D_t + S_\infty | \mathcal{F}_t], \quad t \geq 0. \tag{2.6}$$

Proof If $Q \in \mathcal{M}_{\mathrm{UI}}(W)$ then

$$W_t = \mathbb{E}_Q[W_\infty | \mathcal{F}_t], \tag{2.7}$$

and this translates into (2.6). Conversely, condition (2.6) implies (2.7), and so W is a uniformly integrable martingale under Q. □

We are now going to assume that the given market price S is justified not only from a speculative point of view as in (2.1), but also from a fundamental point of view. This means that S should be perceived as the fundamental price for at least one equivalent martingale measure:

Assumption 2.7 There exists $Q \in \mathcal{M}_{\mathrm{loc}}(W)$ such that

$$S = S^Q, \tag{2.8}$$

where S^Q is the fundamental price perceived under Q as defined in (2.5).

Lemma 2.8 *Assumption 2.7 holds if and only if $S_\infty = 0$ a.s., and in this case (2.8) is satisfied if and only if $Q \in \mathcal{M}_{\mathrm{UI}}(W)$.*

Proof In view of (2.1), the condition $S = S^Q$ implies $S_\infty = 0$ a.s. Conversely, if $S_\infty = 0$ a.s., then (2.6) shows that $S = S^Q$ holds iff $Q \in \mathcal{M}_{\mathrm{UI}}(W)$, and by Assumption 2.5 this class is non-empty. □

From now on **we assume that Assumption 2.7 is satisfied**, and so $W_\infty = D_\infty$ a.s.

Definition 2.9 Let $Q \in \mathcal{M}_{\mathrm{UI}}(W)$. The process $W^Q = S^Q + D$ defined by

$$W_t^Q := \mathbb{E}_Q[D_\infty | \mathcal{F}_t], \quad t \geq 0,$$

is called the *fundamental wealth* of the asset *perceived under* Q.

Lemma 2.3 shows that the difference $S - S^Q$, which is nonnegative due to (2.5), does not vanish if $Q \in \mathcal{M}_{\mathrm{NUI}}(W)$, and this can be interpreted as the appearance of a non-trivial "bubble".

Definition 2.10 For any $Q \in \mathcal{M}_{\mathrm{loc}}(W)$, the nonnegative adapted process β^Q defined by

$$\beta^Q := S - S^Q = W - W^Q \geq 0 \tag{2.9}$$

is called the *bubble perceived under* Q or the Q*-bubble*.

Combining the preceding results, we obtain the following description of a Q-bubble.

Corollary 2.11 *A measure* $Q \in \mathcal{M}_{\mathrm{loc}}(W)$ *belongs to* $\mathcal{M}_{\mathrm{UI}}(W)$ *if and only if the* Q*-bubble reduces to the trivial case* $\beta^Q = 0$. *For* $Q \in \mathcal{M}_{\mathrm{NUI}}(W)$, *the* Q*-bubble* β^Q *is a nonnegative local martingale such that* $\beta_0^Q > 0$ *and*

$$\lim_{t\to\infty} \beta_t^Q = 0 \quad \text{a.s. and in } L^1(Q). \tag{2.10}$$

Proof The local martingale property follows from (2.9) since the difference of a local martingale and a uniformly integrable martingale is again a local martingale. Since both S and S^Q converge to 0 almost surely and in $L^1(Q)$, we obtain (2.10). □

For $Q \in \mathcal{M}_{\mathrm{NUI}}(W)$, the Q-bubble β^Q appears immediately at time 0, and then it finally dies out. In order to capture the slow birth of a bubble starting from an initial value 0, we are going to consider a flow in the space $\mathcal{M}_{\mathrm{loc}}(W)$ that begins in $\mathcal{M}_{\mathrm{UI}}(W)$ and then enters the class $\mathcal{M}_{\mathrm{NUI}}(W)$.

3 The birth of a bubble as a submartingale

Consider a flow $\mathcal{R} = (R_t)_{t\geq 0}$ in the space of equivalent local martingale measures, given by a probability measure $R_t \in \mathcal{M}_{\mathrm{loc}}(W)$ for any $t \geq 0$. We assume that $\mathcal{R}$ is càdlàg in the simple sense that the adapted process $W^{\mathcal{R}}$ defined by

$$W_t^{\mathcal{R}} := \mathbb{E}_{R_t}[D_\infty | \mathcal{F}_t], \quad t \geq 0, \tag{3.1}$$

admits a càdlàg version. Then the same is true for the adapted process S^R defined by

$$S_t^{\mathcal{R}} := W_t^{\mathcal{R}} - D_t = \mathbb{E}_{R_t}[D_\infty - D_t | \mathcal{F}_t], \quad t \geq 0.$$

This càdlàg property clearly holds if, as in [17], the flow consists in switching from one martingale measure to another at certain stopping times. It will also be satisfied in the cases studied below.

Definition 3.1 For a càdlàg flow $\mathcal{R} = (R_t)_{t\geq 0}$, we define the $\mathcal{R}$-*bubble* as the non-negative, adapted, càdlàg process

$$\beta^{\mathcal{R}} := W - W^{\mathcal{R}} = S - S^{\mathcal{R}} \geq 0.$$

Clearly, the definition and the analysis of the processes $W^{\mathcal{R}}$, $S^{\mathcal{R}}$ and $\beta^{\mathcal{R}}$ only involve the conditional probability distributions

$$R_t[\,\cdot\,|\mathcal{F}_t], \quad t \geq 0, \tag{3.2}$$

which describe the market's forward-looking view at any time t as described by the local martingale measure $R_t \in \mathcal{M}_{\text{loc}}(W)$. It is thus enough to specify these conditional distributions. Conversely, any such specification that yields the càdlàg property of (3.1) induces a càdlàg flow $\mathcal{R} = (R_t)_{t\geq 0}$ if we fix any measure $Q \in \mathcal{M}_{\text{UI}}(W)$ and define the measure R_t by

$$R_t[A] := \mathbb{E}_Q\big[R_t[A|\mathcal{F}_t]\big] \tag{3.3}$$

for $A \in \mathcal{F}$ and $t \geq 0$.

As soon as the flow $\mathcal{R}$ is not constant, it describes a shifting system of predictions $(R_t[\,\cdot\,|\mathcal{F}_t])_{t\geq 0}$ that is not time consistent. Indeed, *time consistency* would amount to the condition that the predictions

$$\pi_t(H) = \int H\, dR_t[\,\cdot\,|\mathcal{F}_t] = \mathbb{E}_{R_t}[H|\mathcal{F}_t], \quad t \geq 0$$

satisfy

$$\pi_s\big(\pi_t(H)\big) = \pi_s(H) \tag{3.4}$$

for any $s \leq t$ and for any bounded measurable contingent claim H. This condition is clearly satisfied if all the conditional distributions in (3.2) belong to the same martingale measure $R_0 \in \mathcal{M}_{\text{loc}}(W)$, and the converse holds as well:

Proposition 3.2 *If* $R_t[\,\cdot\,|\mathcal{F}_t] \neq R_0[\,\cdot\,|\mathcal{F}_t]$ *for some* $t > 0$, *then time consistency fails.*

Proof The assumption implies that for some $A \in \mathcal{F}$ and some $t > 0$, the event

$$B_t = \{R_t[A|\mathcal{F}_t] > R_0[A|\mathcal{F}_t]\}$$

has positive probability $R_0[B_t] > 0$. Then $H := I_{A\cap B_t}$ satisfies

$$\pi_t(H) = \mathbb{E}_{R_t}[H|\mathcal{F}_t] \geq \mathbb{E}_{R_0}[H|\mathcal{F}_t],$$

and the inequality is strict on B_t. Thus we get

$$\pi_0(H) = \mathbb{E}_{R_0}[H] = \mathbb{E}_{R_0}\big[\mathbb{E}_{R_0}[H|\mathcal{F}_t]\big] < \mathbb{E}_{R_0}[\pi_t(H)] = \pi_0\big(\pi_t(H)\big),$$

in contradiction to (3.4). □

In the time consistent case, the conditional probability distributions $R_t[\cdot|\mathcal{F}_t]$ thus all belong to the same local martingale measure $R_0 \in \mathcal{M}_{\text{loc}}(W)$, and so we are in the situation of Corollary 2.11: Either no bubble appears at all, or a bubble already exists at the very beginning.

Let us now look at a time inconsistent situation where the flow $\mathcal{R}$ is not constant. As shown by Lemma 2.8, the $\mathcal{R}$-bubble vanishes at times t when $R_t \in \mathcal{M}_{\text{UI}}(W)$, but it will typically become positive in periods when the flow passes through $\mathcal{M}_{\text{NUI}}(W)$. Let us now focus on the special case where the flow $\mathcal{R}$ consists in moving from some initial measure Q in $\mathcal{M}_{\text{UI}}(W)$ to some measure R in $\mathcal{M}_{\text{NUI}}(W)$ via adapted convex combinations. More precisely, let us fix

$$Q \in \mathcal{M}_{\text{UI}}(W) \quad \text{and} \quad R \in \mathcal{M}_{\text{NUI}}(W) \tag{3.5}$$

and some adapted càdlàg process $\xi = (\xi_t)_{t\geq 0}$ with values in $[0, 1]$ starting in $\xi_0 = 0$. Now suppose that at any time $t \geq 0$, the market's forward-looking view is given by the conditional distribution

$$R_t[\cdot|\mathcal{F}_t] = \xi_t R[\cdot|\mathcal{F}_t] + (1-\xi_t)Q[\cdot|\mathcal{F}_t], \tag{3.6}$$

putting weight ξ_t on the predictions provided by the martingale measure R and the remaining weight on the prediction under Q.

Remark 3.3 The microeconomic model of interacting agents in [8] would suggest the following economic interpretation of such a flow. There are two financial "gurus", one optimistic and one pessimistic, whose subjective views are expressed by the two martingale measures Q and R. Each guru has a group of followers, but the proportion between these two groups is shifting, due to contagion effects. As a result, the temporary price equilibrium at any time t is given by some martingale measure R_t, and in simple cases R_t should be given by a weighted average of Q and R, depending on the present weights of the two groups.

Lemma 3.4 *For the flow $\mathcal{R} = (R_t)_{t\geq 0}$ defined by (3.6) and (3.3), the $\mathcal{R}$-bubble $\beta^{\mathcal{R}} = S - S^{\mathcal{R}}$ is given by*

$$\beta_t^{\mathcal{R}} = \xi_t(S_t - S_t^R) = \xi_t \beta_t^R, \quad t \geq 0. \tag{3.7}$$

The $\mathcal{R}$-bubble starts at $\beta_0^{\mathcal{R}} = 0$, and it dies out in the long run:

$$\lim_{t\to\infty} \beta_t^{\mathcal{R}} = 0 \quad \textit{a.s. and in } L^1(R).$$

Proof Note first that the $\mathcal{R}$-bubble starts at the initial value 0 since $R_0 = Q$ is in $\mathcal{M}_{\text{UI}}(W)$. We have

$$W_t^{\mathcal{R}} = \xi_t \mathbb{E}_R[W_\infty|\mathcal{F}_t] + (1-\xi_t)\mathbb{E}_Q[W_\infty|\mathcal{F}_t] = \xi_t W_t^R + (1-\xi_t)W_t,$$

hence

$$\beta_t^{\mathcal{R}} = W_t - W_t^{\mathcal{R}} = \xi_t(W_t - W_t^R) = \xi_t(S_t - S_t^R) = \xi_t \beta_t^R. \tag{3.8}$$

This implies $\lim_{t\to\infty}\beta_t^{\mathcal{R}}=0$, since β^R converges to 0 by Corollary 2.11 and ξ remains bounded. □

The following proposition shows that the initial behavior of the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ from its starting value 0 is captured by a submartingale property under R, if ξ puts increasing weight on the prediction provided by the measure R.

Proposition 3.5 *If the process ξ is increasing, then the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ is a local submartingale under R. If ξ remains constant after some stopping time τ_1, then $\beta^{\mathcal{R}}$ is a local martingale under R, and hence an R-supermartingale, after time τ_1.*

Proof The R-bubble $\beta^R=W-W^R$ is a local martingale under R as stated in Corollary 2.11. Let σ be a localizing stopping time for β^R under R, that is, the stopped process $(\beta^R)^\sigma_\cdot:=\beta^R_{\cdot\wedge\sigma}$ is an R-martingale. Then the stopped process $(\beta^{\mathcal{R}})^\sigma=(\xi\beta^R)^\sigma$ is an R-submartingale since

$$
\begin{aligned}
(\xi\beta^R)^\sigma_s&=\xi_{s\wedge\sigma}\beta^R_{s\wedge\sigma}=\xi_{s\wedge\sigma}\mathbb{E}_R[\beta^R_{t\wedge\sigma}|\mathcal{F}_s]=\mathbb{E}_R[\xi_{s\wedge\sigma}\beta^R_{t\wedge\sigma}|\mathcal{F}_s]\\
&\le\mathbb{E}_R[\xi_{t\wedge\sigma}\beta^R_{t\wedge\sigma}|\mathcal{F}_s]=\mathbb{E}_R[(\xi\beta^R)^\sigma_t|\mathcal{F}_s]
\end{aligned}
$$

for $s\le t$. To show that $\beta^{\mathcal{R}}$ is a local R-martingale after time τ_1, it is enough to verify that the stopped process $(\beta^{\mathcal{R}})^\sigma$ satisfies

$$
\mathbb{E}_R[(\beta^{\mathcal{R}})^\sigma_\tau]=\mathbb{E}_R[(\beta^{\mathcal{R}})^\sigma_{\tau_1}]
$$

for any stopping time $\tau\ge\tau_1$. Indeed, since $\xi_{\tau\wedge\sigma}=\xi_{\tau_1\wedge\sigma}$, the representation (3.7) of $\beta^{\mathcal{R}}$ allows us to write

$$
\begin{aligned}
\mathbb{E}_R[\beta^{\mathcal{R}}_{\tau\wedge\sigma}]&=\mathbb{E}_R[\xi_{\tau\wedge\sigma}\beta^R_{\tau\wedge\sigma}]=\mathbb{E}_R\big[\xi_{\tau_1\wedge\sigma}\mathbb{E}_R[\beta^R_{\tau\wedge\sigma}|\mathcal{F}_{\tau_1\wedge\sigma}]\big]\\
&=\mathbb{E}_R[\xi_{\tau_1\wedge\sigma}\beta^R_{\tau_1\wedge\sigma}]=\mathbb{E}_R[\beta^{\mathcal{R}}_{\tau_1\wedge\sigma}].
\end{aligned}
$$
□

The situation becomes more delicate if the process ξ is no longer increasing, but only a submartingale under R, as will be the case in the situation considered below in (3.12). Let us first look at the general case where ξ is a special semimartingale with values in [0, 1]. As in (3.8), the bubble $\beta^{\mathcal{R}}$ is given by

$$
\beta^{\mathcal{R}}_t=\xi_t(S_t-S^R_t)=\xi_t\beta^R_t.
$$

Let

$$
\xi=M^\xi+A^\xi \tag{3.9}
$$

denote the canonical decomposition of ξ into a local R-martingale M^ξ and a predictable process A^ξ with paths of bounded variation. Since β^R is a local R-martingale, an application of Itô's integration by parts formula shows that the canonical decomposition of the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}=\xi\beta^R$ takes the form

$$
d\beta^{\mathcal{R}}_t=(\xi_t\,d\beta^R_t+\beta^R_t\,dM^\xi_t)+dA^{\mathcal{R}}_t, \tag{3.10}
$$

where $A^{\mathcal{R}}$ is the predictable process with paths of bounded variation defined by

$$A_t^{\mathcal{R}} = \int_0^t \beta_s^R \, dA_s^\xi + [\xi, \beta^R]_t, \quad t \geq 0. \tag{3.11}$$

Our aim is to clarify the conditions which guarantee that $A^{\mathcal{R}}$ is an increasing process, that is, the bubble $\beta^{\mathcal{R}}$ takes off as a submartingale. In that case, we could say that the "birth" of the bubble takes place while the increase of $A^{\mathcal{R}}$ is strict.

We first state the following criterion for the local submartingale property of $\beta^{\mathcal{R}}$.

Proposition 3.6 *The $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ is a local R-submartingale if and only if $A^{\mathcal{R}}$ is an increasing process. If ξ is a submartingale, then the local R-submartingale property for $\beta^{\mathcal{R}}$ holds whenever the process $[\xi, \beta^R]$ is increasing.*

Proof The first claim follows immediately from (3.10). If ξ is a submartingale, then A^ξ is an increasing process, and so is the first term on the right-hand side of (3.11) since $\beta^R \geq 0$. Thus $A^{\mathcal{R}}$ increases whenever $[\xi, \beta^R]$ is increasing. □

From now on, we focus on the following special case. Suppose that the flow $\mathcal{R} = (R_t)_{t \geq 0}$ is of the form

$$R_t = (1 - \lambda_t) Q + \lambda_t R, \tag{3.12}$$

where $(\lambda_t)_{t \geq 0}$ is a deterministic càdlàg process of bounded variation that takes values in $[0, 1]$ and starts at $\lambda_0 = 0$. Let us denote by M the uniformly integrable martingale

$$M_t = \mathbb{E}_R\left[\frac{dQ}{dR}\middle|\mathcal{F}_t\right], \quad t \geq 0.$$

Lemma 3.7 *The conditional distributions $R_t[\,\cdot\,|\mathcal{F}_t]$ are of the form (3.6), where the adapted process ξ is given by*

$$\xi_t = \frac{\lambda_t}{\lambda_t + (1 - \lambda_t) M_t}, \quad t \geq 0. \tag{3.13}$$

Proof For any $\mathcal{F}$-measurable $Z \geq 0$ and any $A_t \in \mathcal{F}_t$, we have

$$\begin{aligned}\mathbb{E}_{R_t}[Z; A_t] &= \mathbb{E}_R\big[\big(\lambda_t + (1 - \lambda_t) M_\infty\big) Z; A_t\big] \\ &= \mathbb{E}_R\big[\lambda_t \mathbb{E}_R[Z|\mathcal{F}_t] + (1 - \lambda_t) M_t \mathbb{E}_Q[Z|\mathcal{F}_t]; A_t\big].\end{aligned}$$

Since

$$\left.\frac{dR_t}{dR}\right|_{\mathcal{F}_t} = \lambda_t + (1 - \lambda_t) M_t,$$

we have

$$\lambda_t \left.\frac{dR}{dR_t}\right|_{\mathcal{F}_t} = \xi_t$$

and

$$(1-\lambda_t)M_t \frac{dR}{dR_t}\bigg|_{\mathcal{F}_t} = 1-\xi_t.$$

Thus we can write

$$\mathbb{E}_{R_t}[Z; A_t] = \mathbb{E}_{R_t}\big[\xi_t \mathbb{E}_R[Z|\mathcal{F}_t] + (1-\xi_t)\mathbb{E}_Q[Z|\mathcal{F}_t]; A_t\big],$$

and this amounts to the representation (3.6) of the conditional distribution $R_t[\,\cdot\,|\mathcal{F}_t]$. □

Lemma 3.8 *If* λ *is increasing, the process* $(\xi_t)_{t\geq 0}$ *defined in* (3.13) *is an* R*-submartingale with values in* $[0,1]$*, and its Doob–Meyer decomposition* (3.9) *is given by*

$$M_t^{\xi} = -\int_0^t \frac{\lambda_s(1-\lambda_s)}{(\lambda_s+(1-\lambda_s)M_s)^2}\, dM_s \tag{3.14}$$

and

$$A_t^{\xi} = \int_0^t \frac{M_s}{(\lambda_s+(1-\lambda_s)M_s)^2}\, d\lambda_s + \int_0^t \frac{\lambda_s(1-\lambda_s)^2}{(\lambda_s+(1-\lambda_s)M_s)^3}\, d[M,M]_s. \tag{3.15}$$

Proof Note that $\xi_t = g(M_t, \lambda_t)$, where the function g on $(0,\infty)\times[0,1]$ defined by

$$g(x,y) = \frac{y}{y+(1-y)x} \tag{3.16}$$

is convex in x and increasing in y. Due to Jensen's inequality, this implies

$$\xi_s = g(\mathbb{E}_R[M_t|\mathcal{F}_s], \lambda_s) \leq \mathbb{E}_R[g(M_t,\lambda_s)|\mathcal{F}_s] \leq \mathbb{E}_R[g(M_t,\lambda_t)|\mathcal{F}_s] = \mathbb{E}_R[\xi_t|\mathcal{F}_t]$$

for any $s \leq t$, and so we have shown that ξ is an R-submartingale. Applying Itô's formula to $\xi_t = g(M_t,\lambda_t)$, we obtain the Doob–Meyer decomposition (3.9) with

$$M_t^{\xi} = \int_0^t g_x(M_s,\lambda_s)\, dM_s$$

and

$$A_t^{\xi} = \int_0^t \frac{1}{2} g_{xx}(M_s,\lambda_s)\, d[M,M]_s + \int_0^t g_y(M_s,\lambda_s)\, d\lambda_s,$$

and this yields the explicit expressions (3.14) and (3.15). □

Theorem 3.9 *Consider a flow* $\mathcal{R} = (R_t)_{t\geq 0}$ *of the form* (3.12)*, where* λ *is an increasing, right-continuous function on* $[0,\infty)$ *with values in* $[0,1]$ *and initial value* $\lambda_0 = 0$*. Assume that*

$$W^R \textit{ is a local submartingale under } Q \tag{3.17}$$

or, equivalently, that

$$[W^R, M] \text{ is an increasing process.} \tag{3.18}$$

Then the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ is a local submartingale under R with initial value $\beta_0^{\mathcal{R}} = 0$. After time $t_1 = \inf\{t : \lambda_t = 1\}$, $\beta^{\mathcal{R}}$ is a local martingale under R, and hence an R-supermartingale.

Proof Both W^R and M are martingales under R, and so Itô's product formula

$$d(W^R M) = W^R\, dM + M\, dW^R + d[W^R, M]$$

shows that the quadratic covariation $[W^R, M]$, defined as the predictable process of bounded variation in the canonical decomposition of the semimartingale $W^R M$, is an increasing process if and only if $W^R M$ is a local submartingale under R. But this is equivalent to the condition that W^R is a local submartingale under Q.

Since W is a local martingale under both R and Q, the process WM is a local martingale under R. Thus $[W, M] \equiv 0$, and so we see that

$$[\beta^R, M] = [W - W^R, M] = -[W^R, M] \tag{3.19}$$

is a decreasing process. But this implies that $[\xi, \beta^R]$ is an increasing process. Indeed, since $\xi = g(M, \lambda)$ with g defined by (3.16), we obtain

$$d[\xi, \beta^R] = d[M^\xi, \beta^R] = g_x(M, \lambda)\, d[M, \beta^R],$$

and we have $g_x(M, \lambda) \le 0$ because $g(x, y)$ is decreasing in x. The local submartingale property of $\beta^{\mathcal{R}}$ under R follows from Proposition 3.6. The rest follows as in Proposition 3.5 since $\xi_t = 1$ for $t \ge t_1$. □

Let us now assume that the wealth process W is strictly positive. Then the local R-martingale W admits the representation

$$W = \mathcal{E}(L) = \exp\Big(L - \frac{1}{2}[L, L]\Big),$$

where L is a local martingale under R. The fundamental wealth process W^R perceived under R can now be factorized into the wealth process W and a semimartingale C, i.e.,

$$W_t^R = \mathbb{E}_R[W_\infty^R | \mathcal{F}_t] = W_t C_t, \tag{3.20}$$

where

$$C_t := \mathbb{E}_R\Big[\exp\Big(L_\infty - L_t - \frac{1}{2}([L, L]_\infty - [L, L]_t)\Big)\Big|\mathcal{F}_t\Big]. \tag{3.21}$$

The martingale property of W under Q implies $[W, M] \equiv 0$, and so the factorization (3.20) yields

$$d[W^R, M] = Wd[C, M] + Cd[W, M] = Wd[C, M].$$

Since W is strictly positive, the criterion in Theorem 3.9 now takes the following form.

Corollary 3.10 *The $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ is a local R-submartingale if $[C, M]$ is an increasing process, where C is defined by the factorization $W^R = WC$ in (3.20) and (3.21).*

4 The Delbaen–Schachermayer example

The following situation typically arises in an incomplete financial market model. It was first studied in [6] and then used as a key example in [17].

Let $X^{(1)}$ and $X^{(2)}$ be two independent and strictly positive continuous martingales on our filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ such that $X_0^{(1)} = X_0^{(2)} = 1$ and

$$\lim_{t\uparrow\infty} X_t^{(1)} = \lim_{t\uparrow\infty} X_t^{(2)} = 0 \quad P\text{-a.s.}$$

We fix constants $a \in (0, 1)$ and $b \in (1, \infty)$ and define the stopping times

$$\tau_1 := \inf\{t > 0 : X_t^{(1)} = a\}, \qquad \tau_2 := \inf\{t > 0 : X_t^{(2)} = b\} \tag{4.1}$$

and $\tau := \tau_1 \wedge \tau_2$. Note that $\tau_1 < \infty$ P-a.s., and that an application of the stopping theorem to the martingale $X^{(2)}$ yields

$$P[\tau_2 < \infty | \mathcal{F}_t] = \frac{1}{b} X_{t\wedge\tau_2}^{(2)}. \tag{4.2}$$

Now consider an asset that generates a single payment $X_\tau^{(1)}$ at time τ, and whose price process S is given by $S_t = X_t^1 1_{\{\tau > t\}}$, $t \geq 0$. Thus we have

$$D_t = X_\tau^{(1)} 1_{\{\tau \leq t\}}, \quad t \geq 0,$$

and the wealth process W is given by the process $X^{(1)}$ stopped at τ,

$$W_t = S_t + D_t = X_{\tau\wedge t}^{(1)}, \quad t \geq 0.$$

Clearly, W is a martingale under P and bounded below by a. But it is not uniformly integrable, as shown in [6]. More precisely:

Lemma 4.1 *We have*

$$\mathbb{E}_P[W_\infty | \mathcal{F}_t] = a\left(1 - \frac{1}{b} X_{t\wedge\tau}^{(2)}\right) + \frac{1}{b} X_{t\wedge\tau}^{(1)} X_{t\wedge\tau}^{(2)}, \tag{4.3}$$

and this is strictly smaller than $W_t = X_t^{(1)}$ on the set $\{\tau > t\}$.

Proof Equation (4.3) clearly holds on the set $\{\tau \le t\}$, where both the right-hand side and W_∞ coincide with $X^{(1)}_\tau$. On the set $\{\tau > t\}$, we write

$$\begin{aligned}\mathbb{E}_P[W_\infty|\mathcal{F}_t] &= \mathbb{E}_P[X^{(1)}_\tau|\mathcal{F}_t]\\ &= \mathbb{E}_P\big[X^{(1)}_{\tau_1}1_{\{\tau_2=\infty\}}\big|\mathcal{F}_t\big] + \mathbb{E}_P\big[X^{(1)}_\tau 1_{\{\tau_2<\infty\}}\big|\mathcal{F}_t\big]\\ &= aP[\tau_2=\infty|\mathcal{F}_t] + \mathbb{E}_P\big[\mathbb{E}_P[X^{(1)}_{\tau_1\wedge\tau_2}|\mathcal{F}_t\vee\sigma(\tau_2)]1_{\{\tau_2<\infty\}}\big|\mathcal{F}_t\big].\end{aligned}$$

Since τ_2 is independent of $X^{(1)}$, the last term reduces to

$$X^{(1)}_t P[\tau_2<\infty|\mathcal{F}_t],$$

and in view of (4.2), this implies (4.3). The fact that $\mathbb{E}_R[W_\infty|\mathcal{F}_t] < W_t = X^{(1)}_t$ on $\{\tau > t\}$ follows directly from the definition (4.1) of τ_1 and τ_2. □

Consider the bounded martingale M defined by

$$M_t := X^{(2)}_{t\wedge\tau}, \quad t \ge 0,$$

and denote by Q the probability measure with density

$$\frac{dQ}{dP} = M_\infty = X^{(2)}_\tau > 0.$$

Thus Q is equivalent to P, and it is shown in [6] that W is a uniformly integrable martingale under Q. Indeed, W is a Q-local martingale since $[W,M]\equiv 0$. Moreover we have $\mathbb{E}_P[X^{(1)}_\tau|\tau_2]=1$ on $\{\tau_2<\infty\}$ and $X^{(2)}_\tau = \mathbb{E}_P[X^{(2)}_{\tau_2}1_{\{\tau_2<\infty\}}|\mathcal{F}_\tau]$, hence

$$\begin{aligned}\mathbb{E}_Q[W_\infty] &= \mathbb{E}_P\big[X^{(1)}_\tau X^{(2)}_\tau\big] = \mathbb{E}_P\big[X^{(1)}_\tau X^{(2)}_{\tau_2}1_{\{\tau_2<\infty\}}\big]\\ &= b\mathbb{E}_P\big[\mathbb{E}_P[X^{(1)}_\tau|\tau_2]1_{\{\tau_2<\infty\}}\big] = bP\big[\tau_2<\infty\big] = 1\\ &= W_0,\end{aligned}$$

and this implies uniform integrability of W under Q.

Defining $R := P$, we thus have

$$R \in \mathcal{M}_{\mathrm{NUI}}(W) \quad \text{and} \quad Q \in \mathcal{M}_{\mathrm{UI}}(W).$$

As in Sect. 3, we now consider a flow $\mathcal{R} = (R_t)_{t\ge 0}$ of the form (3.12) and the resulting $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$. In view of (4.3), the fundamental wealth process W^R perceived under R is given by

$$W^R_t = \mathbb{E}_R[W_\infty|\mathcal{F}_t] = a\Big(1-\frac{1}{b}M_t\Big) + \frac{1}{b}W_tM_t, \quad t\ge 0. \tag{4.4}$$

The following proposition shows that condition (3.18) of Theorem 3.9 is satisfied in our present case.

Proposition 4.2 *W^R is a local submartingale under Q.*

Proof Since $[W, M] = 0$, we obtain

$$d[W^R, M] = \frac{1}{b} d[(W - a)M, M] = \frac{1}{b}(W - a)\, d[M, M].$$

Thus $[W^R, M]$ is an increasing process and this amounts to the local submartingale property of W^R under Q. □

In view of (4.4), the R-bubble takes the form

$$\beta^R = W - W^R = (W - a)\Big(1 - \frac{1}{b}M\Big), \tag{4.5}$$

and so the $\mathcal{R}$-bubble is given by

$$\beta^{\mathcal{R}} = \xi\beta^R = \xi(W - W^R) = \xi(W - a)\Big(1 - \frac{1}{b}M\Big).$$

In particular the $\mathcal{R}$-bubble vanishes at time τ, that is, $\beta_t^{\mathcal{R}} = 0$ for $t \geq \tau$. Since we have just verified condition (3.17), the $\mathcal{R}$-bubble takes off from its initial value 0 as a R-submartingale before it finally returns to 0. More precisely:

Corollary 4.3 *The behavior of the $\mathcal{R}$-bubble under the measure R is described by Theorem 3.9.*

5 A stochastic volatility example

In this section, we consider a stochastic volatility model of the form

$$\begin{aligned} dX_t &= \sigma_1 v_t X_t\, dB_t^1 + \sigma_2 v_t X_t\, dB_t^2, \qquad X_0 = x, \\ dv_t &= a_1 v_t\, dB_t^1 + a_2 v_t\, dB_t^2 + a_3 v_t\, dB_t^3, \qquad v_0 = 1, \end{aligned} \tag{5.1}$$

where $B = (B^1, B^2, B^3)$ is a 3-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. We assume that the two vectors $a = (a_1, a_2)$ and $\sigma = (\sigma_1, \sigma_2)$ are not parallel and satisfy $(a \cdot \sigma) > 0$, and that $a_3 \in \{0, 1\}$.

The model (5.1) is a slight modification of the stochastic volatility model studied by C.A. Sin [24]. On the one hand, we drop the drift term in the equation of the process v under the measure P, and this will be convenient for the computation of the fundamental value W^R in Proposition 5.2. On the other hand, our model is driven by a 3-dimensional instead of a 2-dimensional Brownian motion, and this will allow us to construct a counterexample to our condition (3.17).

The following theorem provides the corresponding variant of Theorem 3.9 in [24]; its proof is given in the Appendix.

Theorem 5.1 *There exists a unique solution* (X, v) *of* (5.1).

For any $T > 0$, *the process* $(X_t)_{t\in[0,T]}$ *is a strict local martingale under* P. *Moreover, there exists an equivalent martingale measure* Q *for* X *such that the densities*

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = M_t, \quad 0 \le t \le T,$$

are given by

$$M_t = \mathcal{E}\left(-\int_0^\cdot \frac{v_s(a\cdot\sigma)}{a\cdot\sigma^\perp}\sigma_1^\perp\, dB_s^1 - \int_0^\cdot \frac{v_s(a\cdot\sigma)}{a\cdot\sigma^\perp}\sigma_2^\perp\, dB_s^2 + |\alpha|^2 B_\cdot^3\right)_t, \tag{5.2}$$

where $\mathcal{E}(Z) = \exp(Z - \frac{1}{2}[Z, Z])$ *denotes the stochastic exponential of a continuous semimartingale* Z, *the vector* $\sigma^\perp = (\sigma_1^\perp, \sigma_2^\perp) \neq 0$ *satisfies*

$$\sigma\cdot\sigma^\perp = \sigma_1\sigma_1^\perp + \sigma_2\sigma_2^\perp = 0,$$

and where we put $|\alpha| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. *More precisely, the process* $(X_t)_{t\in[0,T]}$ *is a martingale under* Q *satisfying*

$$\begin{aligned}
dX_t &= \sigma_1 v_t X_t\, dB_t^{Q,1} + \sigma_2 v_t X_t\, dB_t^{Q,2}, \qquad X_0 = x,\\
dv_t &= a_1 v_t\, dB_t^{Q,1} + a_2 v_t\, dB_t^{Q,2} + a_3 v_t\, dB_t^{Q,3} - (a\cdot\sigma)v_t^2\, dt + a_3|\alpha|^2 v_t\, dt,\\
v_0 &= 1,
\end{aligned}$$

where $B^Q = (B^{Q,1}, B^{Q,2}, B^{Q,3})$ *is a* 3*-dimensional Brownian motion under* Q.

In order to return to the setting of Sect. 2, we consider a financial asset that generates a single payment X_T at time T and whose price process S is given by $S_t := X_t$ for $t < T$ and $S_T = 0$. Then the wealth process is given by $W = X$. Theorem 5.1 shows that W is a uniformly integrable martingale under Q, and so we have

$$Q \in \mathcal{M}_{\mathrm{UI}}(W).$$

But Theorem 5.1 also shows that $W = X$ is not uniformly integrable under P, and so we have

$$R := P \in \mathcal{M}_{\mathrm{NUI}}(W).$$

Let us now compute the fundamental value W^R perceived under R, given by

$$W_t^R = \mathbb{E}_R[W_T|\mathcal{F}_t] = \mathbb{E}_R[X_T|\mathcal{F}_t], \quad t\in[0,T].$$

Proposition 5.2 *The process* W^R *admits the factorization* $W^R = WC$, *where the semimartingale* C *is of the form*

$$C_t = 1 + \big(\sigma_1 c_1(t) + \sigma_2 c_2(t)\big)v_t, \quad t\in[0,T].$$

The time-dependent coefficients are given by

$$c_1(t) = \mathbb{E}_R\left[\frac{1}{v_t}\int_0^{T-t} e^{X_u} v_{u+t}\, d\tilde{B}_u^1\right],$$
$$c_2(t) = \mathbb{E}_R\left[\frac{1}{v_t}\int_0^{T-t} e^{X_u} v_{u+t}\, d\tilde{B}_u^2\right], \tag{5.3}$$

and satisfy

$$\sigma_1 c_1(t) + \sigma_2 c_2(t) < 0 \tag{5.4}$$

for any $t \in [0, T)$.

Proof The process X is given by the stochastic exponential

$$X_t = \mathcal{E}\left(\int_0^{\cdot} \sigma_1 v_s\, dB_s^1 + \int_0^{\cdot} \sigma_2 v_s\, dB_s^2\right)_t, \quad t \in [0, T].$$

Thus

$$\frac{X_T}{X_t} = \exp\left(\int_t^T \sigma_1 v_s\, dB_s^1 + \int_t^T \sigma_2 v_s\, dB_s^2 - \frac{1}{2}\int_t^T (\sigma_1^2 + \sigma_2^2) v_s^2\, ds\right)$$
$$= \exp\left(v_t \int_t^T \sigma_1 \frac{v_s}{v_t}\, dB_s^1 + v_t \int_t^T \sigma_2 \frac{v_s}{v_t}\, dB_s^2 - \frac{1}{2} v_t^2 \int_t^T (\sigma_1^2 + \sigma_2^2)(\frac{v_s}{v_t})^2\, ds\right).$$

Clearly, we can write

$$W_t^R = X_t \mathbb{E}_R\left[\frac{X_T}{X_t}\middle|\mathcal{F}_t\right] = W_t C_t,$$

where

$$C_t := \mathbb{E}_R\left[\frac{X_T}{X_t}\middle|\mathcal{F}_t\right] \tag{5.5}$$

for $t \in [0, T]$. Note that

$$\frac{v_u}{v_t} = \exp\left(a_1(B_u^1 - B_t^1) + a_2(B_u^2 - B_t^2) + a_3(B_u^3 - B_t^3) - \frac{1}{2}|\alpha|^2(t-u)\right)$$

is independent of $\mathcal{F}_t$ for $T \geq u \geq t$. Fixing $y := v_t$ and writing

$$Y_u = \sigma_1 y \int_t^{t+u} \frac{v_s}{v_t}\, dB_s^1 + \sigma_2 y \int_t^{t+u} \frac{v_s}{v_t}\, dB_s^2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) y^2 \int_t^{t+u} \left(\frac{v_s}{v_t}\right)^2 ds,$$

for $u \geq 0$, we have $Y_0 = 0$ and

$$Y_{T-t} = \sigma_1 y \int_t^T \frac{v_s}{v_t}\, dB_s^1 + \sigma_2 y \int_t^T \frac{v_s}{v_t}\, dB_s^2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) y^2 \int_t^T \left(\frac{v_s}{v_t}\right)^2 ds.$$

Applying Itô's formula for the function $f(x) = e^x$, we obtain

$$\begin{aligned} e^{Y_{T-t}} &= e^{Y_0} + \int_0^{T-t} e^{Y_u}\, dY_u + \frac{1}{2}\int_0^{T-t} e^{Y_u}\, d[Y,Y]_u \\ &= e^{Y_0} + \sigma_1 y \int_0^{T-t} \frac{v_{u+t}}{v_t} e^{Y_u}\, d\tilde{B}_u^1 + \sigma_2 y \int_0^{T-t} \frac{v_{u+t}}{v_t} e^{Y_u}\, d\tilde{B}_u^2, \end{aligned}$$

where the Brownian motion $\tilde{B} = (\tilde{B}^1, \tilde{B}^2)$ defined by $\tilde{B}_u^i := B_{t+u}^i - B_t^i$, $i = 1, 2$, is independent of $\mathcal{F}_t$. For fixed $v_t = y$, the conditional expectation (5.5) will thus be equal to the absolute expectation

$$\mathbb{E}_R[e^{Y_{T-t}}] = 1 + \big(\sigma_1 c_1(t) + \sigma_2 c_2(t)\big) y,$$

where $c_1(t)$ and $c_2(t)$ are given by (5.3). It is shown in [24] that an application of Feller's explosion test yields $W_t^R < W_t$ for any $t \in [0, T)$, and this implies (5.4). □

As before, we now consider the flow $\mathcal{R} = (R_t)_{t\geq 0}$ defined by (3.12) and the resulting bubble

$$\beta^{\mathcal{R}} = W - W^{\mathcal{R}} = \xi(W - W^R).$$

Corollary 5.3 *If $a_3 = 0$, the process W^R is a submartingale under the measure Q, and so the behavior of the bubble $\beta^{\mathcal{R}}$ is again described by Theorem 3.9.*

Proof Let us verify the sufficient condition in Corollary 3.10. Since

$$dC_t = \big(\sigma_1 c_1(t) + \sigma_2 c_2(t)\big)\, dv_t + \sigma_1 v_t\, dc_1(t) + \sigma_2 v_t\, dc_2(t),$$

the local martingale part of the semimartingale C is given by

$$M_t^C = \int_0^t a_1\big(\sigma_1 c_1(s) + \sigma_2 c_2(s)\big) v_s\, dB_s^1 + \int_0^t a_2\big(\sigma_1 c_1(s) + \sigma_2 c_2(s)\big) v_s\, dB_s^2.$$

Since (5.2) implies

$$M_t = -\int_0^t \frac{v_s (a\cdot\sigma)}{(a\cdot\sigma^\perp)} \sigma_1^\perp M_s\, dB_s^1 - \int_0^t \frac{v_s (a\cdot\sigma)}{(a\cdot\sigma^\perp)} \sigma_2^\perp M_s\, dB_s^2,$$

we obtain

$$[M, C]_t = [M, M^c]_t = \int_0^t -\big(\sigma_1 c_1(s) + \sigma_2 c_2(s)\big)(a\cdot\sigma) v_s^2 M_s\, ds.$$

This is indeed an increasing process, since the integrand is strictly positive. In view of Corollary 3.10, we have thus shown that $\beta^{\mathcal{R}}$ is a local submartingale under R. □

Let us now modify the model in such a way that condition (3.18) is no longer satisfied. To this end we choose the parameters such that

$$\frac{|\alpha|^2}{(a\cdot\sigma)} > 1,$$

and we introduce the stopping time

$$\tau := \inf\left\{t > 0 : v_t = \frac{|\alpha|^2}{(a\cdot\sigma)}\right\}.$$

Consider a financial asset that generates a single payment X_{τ_0} at time $\tau_0 := T \wedge \tau$ and whose price process S is given again by $S_t := X_t$ for $t < \tau_0$ and $S_t := 0$ for $t \geq \tau_0$. The wealth process is then given again by $W = X$.

Proposition 5.4 *If $a_3 = 1$, the quadratic covariation $[M, C]$ is a decreasing process, and so condition (3.18) is no longer satisfied.*

Proof By using the same computations as in the proof of Proposition 5.2, we obtain

$$\begin{aligned} dC_t &= \big(\sigma_1 c_1(t) + \sigma_2 c_2(t)\big)\, dv_t + \sigma_1 v_t\, dc_1(t) + \sigma_2 v_t\, dc_2(t) \\ &= \big(\sigma_1 c_1(t) + \sigma_2 c_2(t)\big)(a_1 v_t\, dB_t^1 + a_2 v_t\, dB_t^2 + v_t\, dB_t^3) \\ &\quad + \sigma_1 v_t\, dc_1(t) + \sigma_2 v_t\, dc_2(t), \end{aligned}$$

where $c_1(t)$ and $c_2(t)$ are given by (5.3). Hence the local martingale part of C is given by

$$dM_t^C = \big(\sigma_1 c_1(t) + \sigma_2 c_2(t)\big)(a_1 v_t\, dB_t^1 + a_2 v_t\, dB_t^2 + v_t\, dB_t^3).$$

Therefore we obtain

$$\begin{aligned} d[M^C, M]_t &= -\big(\sigma_1 c_1(t) + \sigma_2 c_2(t)\big)(a\cdot\sigma)v_t^2 M_t\, dt \\ &\quad + \big(\sigma_1 c_1(t) + \sigma_2 c_2(t)\big)|a|^2 v_t M_t\, dt \\ &= -\big(\sigma_1 c_1(t) + \sigma_2 c_2(t)\big)\big(-|\alpha|^2 + (a\cdot\sigma)v_t\big)v_t M_t\, dt. \end{aligned}$$

In view of (5.4), the process is decreasing on $[0, \tau_0]$, since $(a\cdot\sigma)v_t - |\alpha|^2 \leq 0$ on $[0, \tau_0]$. □

6 The behavior of the $\mathcal{R}$-bubble under Q

Let us return to the situation of Sect. 3, where the flow $\mathcal{R}$ is given by (3.5) and (3.2), and where the $\mathcal{R}$-bubble is of the form

$$\beta^{\mathcal{R}} = W - W^{\mathcal{R}} = \xi\beta^R;$$

cf. Lemma 3.4. But now we change our point of view: instead of using the reference measure R, we are going the analyze the behavior of the $\mathcal{R}$-bubble under Q.

Let us first focus on the R-bubble $\beta^R = W - W^R = S - S^R$. We retain our condition (3.17) that the fundamental wealth process W^R is a local submartingale under Q, and so its canonical decomposition is of the form

$$W^R = M^Q + A^Q,$$

where M^Q is a Q-local martingale and A^Q is an increasing continuous process of bounded variation.

Proposition 6.1 *Under condition* (3.17), *the R-bubble β^R is a uniformly integrable supermartingale under Q. More precisely, β^R is the Q-potential generated by the increasing process A^Q, that is,*

$$\beta_t^R = \mathbb{E}_R\big[A_\infty^Q - A_t^Q \big| \mathcal{F}_t\big], \quad t \geq 0. \tag{6.1}$$

Proof Since W is uniformly integrable under Q and dominates both M^Q and β^R, the R-bubble

$$\beta^R = W - W^R = (W - M^Q) - A^Q$$

is a uniformly integrable Q-supermartingale. Moreover,

$$\mathbb{E}_Q[M_\infty^Q|\mathcal{F}_t] = \mathbb{E}_Q[W_\infty - A_\infty|\mathcal{F}_t] = W_t - \mathbb{E}_Q[A_\infty^R|\mathcal{F}_t],$$

and this implies (6.1). □

Let us denote by $\tilde{M}$ the Q-martingale

$$\tilde{M}_t := \frac{1}{M_t} = \frac{dR}{dQ}\bigg|_{\mathcal{F}_t}, \quad t \geq 0,$$

and let us represent the $\mathcal{R}$-bubble in the form

$$\beta^{\mathcal{R}} = \tilde{\xi}\tilde{\beta}^R,$$

where $\tilde{\xi} := \xi M$ and $\tilde{\beta}^R := \beta^R \tilde{M}$.

Lemma 6.2 *The process $\tilde{\beta}^R = \beta^R \tilde{M}$ is a local martingale under Q. Under condition* (3.17), *the processes $[\tilde{\beta}^R, \tilde{M}]$ and $[\beta^R, \tilde{M}]$ are both increasing.*

Proof The local martingale property of β^R under R translates into the local martingale property of $\tilde{\beta}^R$ under Q. Under condition (3.17), the process $[\beta^R, M]$ is decreasing; see (3.19). Applying Itô's formula to $\tilde{\beta}^R = \beta^R \tilde{M}$ and $\tilde{M} = M^{-1}$, we obtain

$$d[\tilde{\beta}^R, \tilde{M}] = -\frac{1}{M^3} d[\beta^R, M] + \frac{1}{M^4}\beta^R d[M, M]$$

and so $[\tilde{\beta}^R, \tilde{M}]$ is increasing. Moreover,

$$d[\beta^R, \tilde{M}] = -\frac{1}{M^2} d[\beta^R, M],$$

and so $[\beta^R, \tilde{M}]$ is increasing. □

From now on, we consider the special case where the flow $\mathcal{R} = (R_t)_{t\geq 0}$ is of the form (3.12), i.e.,

$$R_t = (1 - \lambda_t)Q + \lambda_t R,$$

where $(\lambda_t)_{t\geq 0}$ is an increasing càdlàg function that takes values in [0, 1] and starts in $\lambda_0 = 0$. In particular, the process ξ is now given by (3.13).

Proposition 6.3 *The process $\tilde{\xi} = \xi M$ is a submartingale under Q. More precisely, the Doob–Meyer decomposition of $\tilde{\xi}$ under Q is given by*

$$\tilde{\xi}_t = \tilde{M}^{\xi} + \tilde{A}^{\xi} \tag{6.2}$$

with

$$d\tilde{M}^{\xi} = -\frac{\lambda^2}{(\lambda \tilde{M} + (1 - \lambda))^2} d\tilde{M}$$

and

$$d\tilde{A}^{\xi} = \frac{1}{(\lambda \tilde{M} + (1 - \lambda))^2} d\lambda + \frac{\lambda^3}{(\lambda \tilde{M} + (1 - \lambda))^3} d[\tilde{M}, \tilde{M}]. \tag{6.3}$$

Proof Note that

$$\tilde{\xi}_t = \tilde{g}(\tilde{M}_t, \lambda_t),$$

where

$$\tilde{g}(x, y) = \frac{y}{xy + (1 - y)}$$

is convex in $x \in (0, \infty)$ and increasing in $y \in [0, 1]$. As in the proof of Lemma 3.8, it follows that $\tilde{\xi}$ is a Q-submartingale. The explicit form of its Doob–Meyer decomposition is obtained by applying Itô's formula, using

$$\tilde{g}_x(x, y) = -\frac{y^2}{(xy + (1 - y))^2}, \qquad \tilde{g}_y(x, y) = \frac{1}{(xy + (1 - y))^2} \tag{6.4}$$

and

$$\tilde{g}_{xx}(x, y) = \frac{2y^3}{(xy + (1 - y))^3}. \tag{6.5}$$

□

Let us now describe the behavior of the $\mathcal{R}$-bubble $\beta^{\mathcal{R}} = \xi\beta^R = \tilde{\xi}\tilde{\beta}^R$ under the measure Q.

Proposition 6.4 *Under Q the $\mathcal{R}$-bubble has the canonical decomposition*

$$\beta^{\mathcal{R}} = \tilde{M}^{\mathcal{R}} + \tilde{A}^{\mathcal{R}},$$

where the local martingale $\tilde{M}^{\mathcal{R}}$ is given by

$$d\tilde{M}^{\mathcal{R}} = \tilde{\xi}\, d\tilde{\beta}^R + \tilde{\beta}^R\, dM^{\tilde{\xi}}.$$

The process $\tilde{A}^{\mathcal{R}}$ takes the form

$$d\tilde{A}^{\mathcal{R}} = \frac{\tilde{M}}{\lambda\tilde{M} + (1-\lambda)}(\beta^R\, d\lambda - dD), \tag{6.6}$$

where D denotes the increasing process given by

$$dD = \frac{\lambda^2(1-\lambda)\beta^R}{\tilde{M}(\lambda\tilde{M} + (1-\lambda))}\, d[\tilde{M},\tilde{M}] + \lambda^2\, d[\beta^R,\tilde{M}].$$

Proof Applying integration by parts to $\beta^{\mathcal{R}} = \tilde{\xi}\tilde{\beta}^R$ and using the Doob–Meyer decomposition (6.2) of $\tilde{\xi}$, we obtain

$$\begin{aligned} d\beta^{\mathcal{R}} &= \tilde{\xi}\, d\tilde{\beta}^R + \tilde{\beta}^R\, d\tilde{\xi} + d[\tilde{\beta}^R,\tilde{\xi}] \\ &= (\tilde{\xi}\, d\tilde{\beta}^R + \tilde{\beta}^R\, d\tilde{M}^{\xi}) + (\tilde{\beta}^R\, d\tilde{A}^{\xi} + d[\tilde{\beta}^R,\tilde{\xi}]) \\ &=: d\tilde{M}^{\mathcal{R}} + d\tilde{A}^{\mathcal{R}}. \end{aligned}$$

In view of Lemma 6.2, $\tilde{M}^{\mathcal{R}}$ is a local martingale under Q, and so the finite variation part is given by $\tilde{A}^{\mathcal{R}}$. Since $\tilde{\xi} = \tilde{g}(\tilde{M},\lambda)$ and $\tilde{\beta}^{\mathcal{R}} = \beta^R\tilde{M}$, we obtain

$$d[\tilde{\beta}^R,\tilde{\xi}] = \tilde{g}(\tilde{M},\lambda)\, d[\tilde{\beta}^R,\tilde{M}] = \tilde{g}_x(\tilde{M},\lambda)(\beta^R\, d[\tilde{M},\tilde{M}] + \tilde{M}[\beta^R,\tilde{M}]).$$

Combined with (6.4) and (6.3), this yields

$$\begin{aligned} d\tilde{A}^{\mathcal{R}} &= \frac{\beta^R\tilde{M}}{(\lambda\tilde{M} + (1-\lambda))^2}\, d\lambda + \frac{\beta^R\tilde{M}\lambda^3}{(\lambda\tilde{M} + (1-\lambda))^3}\, d[\tilde{M},\tilde{M}] \\ &\quad - \frac{\beta^R\lambda^2}{(\lambda\tilde{M} + (1-\lambda))^2}\, d[\tilde{M},\tilde{M}] - \frac{\lambda^2\tilde{M}}{(\lambda\tilde{M} + (1-\lambda))^2}\, d[\beta^R,\tilde{M}] \\ &= \frac{\beta^R\tilde{M}}{(\lambda\tilde{M} + (1-\lambda))^2}\, d\lambda - \frac{\beta^R\lambda^2(1-\lambda)}{(\lambda\tilde{M} + (1-\lambda))^3}\, d[\tilde{M},\tilde{M}] \\ &\quad - \frac{\lambda^2\tilde{M}}{(\lambda\tilde{M} + (1-\lambda))^2}\, d[\beta^R,\tilde{M}] \\ &= \frac{\tilde{M}}{(\lambda\tilde{M} + (1-\lambda))^2}(\beta^R\, d\lambda - dD). \end{aligned}$$

Note that the process D is indeed increasing due to Lemma 6.2. □

Definition 6.5 We say that the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ behaves locally as a strict Q-submartingale in a given random period if $\tilde{A}^{\mathcal{R}}$ is strictly increasing in that period.

The preceding proposition shows that the $\mathcal{R}$-bubble behaves like a Q-supermartingale in periods where λ stays constant. In order to induce a strict submartingale behavior under Q, the increase in λ must be strong enough to compensate for the increase in D. Typically this will be the case during the initial period when the $\mathcal{R}$-bubble is born, as long as λ and hence D still remain small enough to be compensated by the initial increase of λ.

Let us illustrate the qualitative behavior of the $\mathcal{R}$-bubble under Q in the specific situation of the Delbaen–Schachermayer example in Sect. 4. According to (4.5), the R-bubble now takes the form

$$\beta^R = (W-a)\left(1-\frac{1}{b}M\right). \tag{6.7}$$

Since $[W,\tilde{M}]=0$ and $d[M,\tilde{M}] = -\tilde{M}^{-2}\,d[\tilde{M},\tilde{M}]$, the increasing process $[\beta^R,\tilde{M}]$ is given by

$$d[\beta^R,\tilde{M}] = \frac{1}{b}(W-a)\tilde{M}^{-2}\,d[\tilde{M},\tilde{M}]. \tag{6.8}$$

Let us denote by

$$\phi = \frac{d\lambda}{d[\tilde{M},\tilde{M}]}$$

the density of the absolutely continuous part of λ with respect to $[\tilde{M},\tilde{M}]$.

Corollary 6.6 *The $\mathcal{R}$-bubble behaves locally as a strict Q-submartingale in periods where*

$$\phi_t > \lambda_t^2\left(1-\lambda_t\left(1-\frac{1}{b}\right)\right)\left(\tilde{M}_t-\frac{1}{b}\right)^{-1}\left(\lambda_t\tilde{M}_t+(1-\lambda_t)\right)^{-1}. \tag{6.9}$$

Proof In view of (6.6)–(6.8) and after cancellation of the common term $W-a$, the condition $d\tilde{A}^{\mathcal{R}}>0$ takes the form

$$\left(1-\frac{1}{b}M_t\right)\phi_t \ge \lambda_t^2(1-\lambda_t)\left(1-\frac{1}{b}M_t\right)\tilde{M}_t^{-1}\left(\lambda_t\tilde{M}_t+(1-\lambda_t)\right)^{-1}+\frac{\lambda_t^2}{b}\tilde{M}_t^{-2}.$$

Multiplying by $\tilde{M}_t(\lambda_t\tilde{M}_t+(1-\lambda_t))$, we obtain

$$\left(\tilde{M}_t-\frac{1}{b}\right)\left(\lambda_t\tilde{M}_t+(1-\lambda_t)\right)\phi_t \ge \lambda_t^2\left(1-\lambda_t\left(1-\frac{1}{b}\right)\right).$$

□

Let us now consider the special case where the martingale $X^{(2)}$ in Sect. 4 is of the form $dX^{(2)} = X^{(2)}\,dB$ for some Brownian motion B. Then $d[\tilde{M},\tilde{M}] = \tilde{M}^2\,dt$ up to the stopping time τ introduced in Sect. 4.

Let λ be continuous and piecewise differentiable with right-continuous derivative λ'. Then the density ϕ is given by $\phi = \tilde{M}^{-2}\lambda'$. Introducing the functions

$$f(x,t) := \left(1-\frac{1}{b}x\right)\left(\lambda(t)+\left(1-\lambda(t)\right)x\right)\lambda'(t)$$

and

$$h(t) := \lambda^2(t)\big(1-\lambda(t)\big)\Big(1-\frac{1}{b}\Big),$$

we can now describe the behavior of the $\mathcal{R}$-bubble under Q as follows.

Corollary 6.7 *Up to time τ, the $\mathcal{R}$-bubble $\beta^{\mathcal{R}}$ behaves locally as a strict Q-submartingale as long as the process (M_t, t) stays in the domain*

$$D_+ := \{(x,t) : f(x,t) > h(t)\},$$

and as a strict supermartingale under Q as long as it stays in

$$D_- := \{(x,t) : f(x,t) < h(t)\}.$$

In particular, if $\lambda'(0) > 0$, then $\beta^{\mathcal{R}}$ behaves as a strict Q-submartingale up to the exit time

$$\sigma := \inf\{t > 0 : (M_t, t) \notin D_+\} > 0$$

from D_+.

Proof In our special situation, (6.9) amounts to the condition $f(M_t, t) > h(t)$, and the condition $f(M_t, t) < h(t)$ is equivalent to $d\tilde{A}^{\mathcal{R}} < 0$. Note that $\lambda'(0) > 0$ implies $(1, 0) \in D_+$, hence $(M_t, t) \in D_+$ for small enough t, and so the exit time from D_+ is strictly positive. □

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Appendix

This section contains the proof of Theorem 5.1. We proceed as in the proof of Theorem 3.3 in [24]. Note first that there exists a unique solution (X, v) of (5.1). Indeed, the process v satisfies the 1-dimensional stochastic differential equation

$$dv_t = |\alpha| v_t \, dW_t, \quad 0 \le t \le T, \tag{A.1}$$

with respect to the Brownian motion

$$W_t = |\alpha|^{-1}(\alpha_1 B_t^1 + \alpha_2 B_t^2 + \alpha_3 B_t^3).$$

It follows that (A.1) admits a unique solution $v = \mathcal{E}(|\alpha| W)$. Therefore X is uniquely determined as the stochastic exponential of the square integrable process

$$\int_0^{\cdot} \sigma_1 v_s \, dB_s^1 + \int_0^{\cdot} \sigma_2 v_s \, dB_s^2.$$

Let us now show that $(X_t)_{t\in[0,T]}$ is a strict local martingale under P. It follows from Lemma 4.2 of [24] that the expectation of the local martingale X under P can be computed as

$$\mathbb{E}_P[X_T] = X_0 P\big[w_t \text{ does not explode on } [0,T]\big],$$

where $(w_t)_{t\in[0,T]}$ is given by

$$dw_t = a_1 w_t\, dB_t^1 + a_2 w_t\, dB_t^2 + a_3 w_t\, dB_t^3 + (a\cdot\sigma) w_t^2\, dt, \qquad w_0 = 1.$$

Then we have

$$dw_t = |\alpha| w_t\, dW_t + (a\cdot\sigma) w_t^2\, dt. \tag{A.2}$$

It follows from Lemma 4.3 of [24] that the unique solution of (A.2) explodes to $+\infty$ in finite time with positive probability. This implies that $\mathbb{E}_P[X_T] < X_0$, and therefore X is a strict local martingale under P.

Now we have to prove that the process $(M_t)_{t\in[0,T]}$ is indeed a Radon–Nikodým density process, i.e., that it is a true martingale under the measure P. It follows from Lemma 4.2 of [24] that the expectation under P of M_T is given by

$$\mathbb{E}_P[M_T] = M_0 P\big[\hat{v}_t \text{ does not explode on } [0,T]\big], \tag{A.3}$$

where $(\hat{v}_t)_{t\in[0,T]}$ satisfies

$$\begin{aligned} d\hat{v}_t &= a_1 \hat{v}_t\, dB_t^1 + a_2 \hat{v}_t\, dB_t^2 + a_3 \hat{v}_t\, dB_t^3 - (a\cdot\sigma)(\hat{v}_t)^2\, dt + a_3|\alpha|^2 \hat{v}_t\, dt \\ &= |\alpha| \hat{v}_t\, dW_t - (a\cdot\sigma)(\hat{v}_t)^2\, dt + a_3|\alpha|^2 \hat{v}_t\, dt. \end{aligned}$$

The explosion time of $(\hat{v}_t)_{t\in[0,T]}$ is given by

$$\tau_\infty = \inf\{t \geq 0 : \hat{v}_t \notin (0,\infty)\}.$$

We apply Feller's test to $\hat{v}$ (see Sect. 5.5 of Karatzas and Shreve [18]) in order to prove that

$$P[\tau_\infty = +\infty] = P\big[\hat{v}_t \text{ does not explode on } [0,T]\big] = 1.$$

To this end we compute the scale function

$$p(x) = \int_1^x \exp\left(-2\int_1^y \frac{-(a\cdot\sigma)z^2 + a_3|\alpha|^2 z}{|\alpha|^2 z^2}\, dz\right) dy,$$

and examine the limits $\lim_{x\downarrow 0} p(x)$ and $\lim_{x\uparrow\infty} p(x)$.

We distinguish two cases:

Case 1: $a_3 = 0$. We have

$$\begin{aligned} p(x) &= \int_1^x \exp\left(\frac{2(a\cdot\sigma)}{|\alpha|^2}\int_1^y dz\right) dy \\ &= k\int_1^x \exp\left(\frac{2(a\cdot\sigma)y}{|\alpha|^2}\right) dy = k_1 \frac{|\alpha|^2}{2(a\cdot\sigma)} \exp\left(\frac{2(a\cdot\sigma)x}{|\alpha|^2}\right) - k_2 \end{aligned}$$

with $k, k_1, k_2 \in \mathbb{R}_+$. Clearly

$$\lim_{x\uparrow\infty} p(x) = +\infty,$$

since $a \cdot \sigma > 0$. Therefore it follows from Problem 5.27 of [18] that

$$u(\infty) = +\infty,$$

where

$$u(x) = \int_1^x p'(y) \int_1^y \frac{2}{p'(z)|\alpha|^2 z^2}\, dz\, dy.$$

Furthermore

$$\lim_{x\to 0+} p(x) = k_1 \frac{|\alpha|^2}{2(a\cdot\sigma)} - k_2 > -\infty.$$

As required by Feller's test, we now compute

$$\begin{aligned}
\lim_{x\to 0+} u(x) &= \lim_{x\to 0+} \int_1^x p'(y) \int_1^y \frac{2}{|\alpha|^2 z^2 p'(z)}\, dz\, dy \\
&= \lim_{x\to 0+} \int_1^x \frac{2}{|\alpha|^2 z^2 p'(z)} \int_z^x p'(y)\, dy\, dz \\
&= \lim_{x\to 0+} \int_1^x \frac{2}{|\alpha|^2 z^2} \exp\left(-\frac{2(a\cdot\sigma)z}{|\alpha|^2}\right) \int_z^x \exp\left(\frac{2(a\cdot\sigma)y}{|\alpha|^2}\right) dy\, dz \\
&\geq \lim_{x\to 0+} e^{-\frac{2(a\cdot\sigma)}{|\alpha|^2}} \int_1^x \frac{2}{|\alpha|^2 z^2} \int_z^x dy\, dz \\
&= \lim_{x\to 0+} \left(e^{-\frac{2(a\cdot\sigma)}{|\alpha|^2}} \frac{2}{|\alpha|^2} \int_1^x \frac{1}{z^2}(x-z)\, dz \right) \\
&= e^{-\frac{2(a\cdot\sigma)}{|\alpha|^2}} \frac{2}{|\alpha|^2} \lim_{x\to 0+} (-\log x - x + 1) = +\infty.
\end{aligned}$$

Applying Theorem 5.29 of [18], we obtain

$$P[\tau_\infty = +\infty] = 1.$$

This shows that $\hat{v}$ does not explode on $[0, T]$. In view of (A.3), we have proved $\mathbb{E}_P[M_T] = M_0$. Thus the process $(M_t)_{t\in[0,T]}$ is a true martingale, and we denote by $Q \approx P$ the probability measure with the density process given by M.

Applying Girsanov's theorem, we see that under the measure Q, the bivariate process (X, v) satisfies

$$\begin{aligned}
dX_t &= \sigma_1 v_t X_t\, dB_t^{Q,1} + \sigma_2 v_t X_t\, dB_t^{Q,2}, \qquad X_0 = x, \\
dv_t &= a_1 v_t\, dB_t^{Q,1} + a_2 v_t\, dB_t^{Q,2} - (a\cdot\sigma) v_t^2\, dt, \qquad v_0 = 1.
\end{aligned}$$

Thus X is a positive local Q-martingale. To show that is a true martingale, it is enough to show that it has constant expectation. By applying Lemma 4.2 from [24], we obtain

$$\mathbb{E}_Q[X_T] = X_0 Q\big[\bar{v}_t \text{ does not explode on } [0,T]\big],$$

where

$$d\bar{v}_t = a_1 \bar{v}_t\, dB_t^1 + a_2 \bar{v}_t\, dB_t^2. \tag{A.4}$$

Since (A.4) has linear coefficients, it follows from Remark 5.19 in [18] that it has a non-exploding solution. Therefore $(X_t)_{t\in[0,T]}$ is a Q-martingale.

Case 2: $a_3 = 1$. The scale function is in this case equal to

$$\begin{aligned} p(x) &= \int_0^x \exp\left(-2\int_1^y \frac{-(a\cdot\sigma)z^2 + |\alpha|_3^2 z}{|\alpha|^2 z^2}\, dz\right) dy \\ &= k\int_1^x \exp\left(2\frac{(a\cdot\sigma)y}{|\alpha|^2}\right) y^{-2}\, dy, \end{aligned}$$

where $k \in \mathbb{R}_+$. We examine the limits $\lim_{x\downarrow 0} p(x)$ and $\lim_{x\uparrow\infty} p(x)$.

We have

$$\lim_{x\downarrow 0} p(x) = \lim_{x\downarrow 0} k\int_1^x \exp\left(2\frac{(a\cdot\sigma)y}{|\alpha|^2}\right) y^{-2}\, dy = -\infty.$$

Then it follows from Problem 5.27 of [18] that

$$u(0+) = +\infty,$$

where

$$u(x) = \int_1^x p'(y)\int_1^y \frac{2}{p'(z)|\alpha|^2 z^2}\, dz\, dy.$$

Furthermore, we have

$$\lim_{x\uparrow\infty} p(x) = \lim_{x\uparrow\infty} k\int_1^x \exp\left(2\frac{(a\cdot\sigma)y}{|\alpha|^2}\right) y^{-2}\, dy = +\infty.$$

Then it follows from Problem 5.27 of [18] that

$$u(\infty) = +\infty.$$

Applying Theorem 5.29 of [18], we obtain

$$P[\tau_\infty = +\infty] = 1.$$

Therefore $\hat{v}$ does not explode on $[0,T]$. Thus the process M is a true martingale.

Applying Girsanov's theorem, we see that under the measure Q, the bivariate process (X, v) satisfies

$$\begin{aligned} dX_t &= \sigma_1 v_t X_t\, dB_t^{1,Q} + \sigma_2 v_t X_t\, dB_t^{2,Q}, \quad t\in[0,T], \\ dv_t &= a_1 v_t\, dB_t^{1,Q} + a_2 v_t\, dB_t^{2,Q} + v_t\, dB_t^{3,Q} - (a\cdot\sigma)v_t^2\, dt + |\alpha|^2 v_t\, dt. \end{aligned}$$

Thus X is a positive local Q-martingale. As in the previous case, in order to show that is a true martingale, it is enough to show that it has constant expectation. By applying Lemma 4.2 from [24], we obtain

$$\mathbb{E}_Q[X_T] = X_0 Q\big[\hat{w}_t \text{ does not explode on } [0,T]\big],$$

where

$$d\hat{w}_t = a_1 \hat{w}_t \, dB_t^{1,Q} + a_2 \hat{w}_t \, dB_t^{2,Q} + \hat{w}_t \, dB_t^{3,Q} + |\alpha|^2 \hat{w}_t \, dt. \tag{A.5}$$

Due to the linearity of the coefficients, it follows from Remark 5.19 in [18] that (A.5) has a non-exploding solution. Therefore $(X_t)_{t\in[0,T]}$ is a Q-martingale.

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