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Quantized Anti de Sitter spaces and non-formal deformation quantizations of symplectic symmetric spaces

Pierre Bieliavsky, Laurent Claessens, Daniel Sternheimer, and Yannick Voglaire

Abstract. We realize quantized anti de Sitter space black holes, building Connes spectral triples, similar to those used for quantized spheres but based on Universal Deformation Quantization Formulas (UDF) obtained from an oscillatory integral kernel on an appropriate symplectic symmetric space. More precisely we first obtain a UDF for Lie subgroups acting on a symplectic symmetric space M in a locally simply transitive manner. Then, observing that a curvature contraction canonically relates anti de Sitter geometry to the geometry of symplectic symmetric spaces, we use that UDF to define what we call Dirac-isospectral noncommutative deformations of the spectral triples of locally anti de Sitter black holes. The study is motivated by physical and cosmological considerations.

1. Introduction

1.1. Physical and cosmological motivations. This paper, of independent interest in itself, can also be seen as a small part in a number of long haul programs developed by many in the past decades, with a variety of motivations. The references that follow are minimal and chosen mostly so as to be a convenient starting point for further reading, that includes the original articles quoted therein.

An obvious fact (almost a century old) is that anti de Sitter (AdS) space-time can be obtained from usual Minkowski space-time, deforming it by allowing a (small) non-zero negative curvature. The Poincaré group symmetry of special relativity is then deformed (in the sense of [**Ger64**]) to the AdS group $SO(2,3)$. In $n+1$ space-time dimensions ($n \geq 2$) the corresponding AdS_n groups are $SO(2,n)$. Interestingly these are the conformal groups of flat (or AdS) n space-times. The *deformation philosophy* [**Fla82**] makes it then natural, in the spirit of deformation quantization [**DS02**], to deform these further [**St05, St07**], i.e. quantizing them, which many are doing for Minkowski space-time.

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These deformations have important consequences. Introducing a small negative curvature ρ permits to consider [**AFFS**] massless particles as composites of more fundamental objects, the Dirac *singletons*, so called because they are associated with unitary irreducible representations (UIR) of $SO(2,3)$, discovered by Dirac in 1963, so poor in states that the weight diagram fits on a single line. These have been called Di and Rac and are in fact massless UIR of the Poincaré group in one space dimension less, uniquely extendible to the corresponding conformal group $SO(2,3)$ ($\mathrm{AdS}_4/\mathrm{CFT}_3$ symmetry, a manifestation of 't Hooft's holography). That kinematical fact was made dynamical [**FF88**] in a manner consistent with quantum electrodynamics (QED), the photons being considered as 2-Rac states and the creation and annihilation operators of the naturally confined Rac having unusual commutation relations (a kind of "square root" of the canonical commutation relations for the photon).

Later [**FFS**] this phenomenon has been linked with the (then very recently) observed oscillations of neutrinos (see below, neutrino mixing). Shortly afterwards, making use of flavor symmetry, Frønsdal was able to modify the electroweak model [**Frø00**], obtaining initially massless leptons (see below) that are massified by Yukawa interaction with Higgs particles. (In this model, 5 pairs of Higgs are needed and it predicts the existence of new mesons, parallel to the W and Z of the $U(2)$-invariant electroweak theory, associated with a $U(2)$ flavor symmetry.)

Quantum groups can be viewed [**BGGS**] as an avatar of deformation quantization when dealing with Hopf algebras. Of particular interest here are the quantized AdS groups [**FHT, Sta98**], especially at even root of unity since they have some finite dimensional UIR, a fact generally associated with compact groups and groups of transformations of compact spaces. It is then tempting to consider quantized AdS spaces at even root of unity $q = e^{i\theta}$ as "small black holes" in an ambient Minkowski space that can be obtained as a limit when $\rho q \to -0$. Note that, following e.g. 't Hooft (see e.g. [**Hoo06**] but his approach started around 1980) that some form of communication is possible with quantum black holes by interaction at their surface.

At present, conventional wisdom has it that our universe is made up mostly of "dark energy" (74% according to a recent Wilkinson Microwave Anisotropy Probe, WMAP), then of "dark matter" (22% according to WMAP), and only 4% of "our" ordinary matter, which we can more directly observe. Dark matter is "matter", not directly observed and of unknown composition, that does not emit nor reflect enough electromagnetic radiation to be detected directly, but whose presence can be inferred from gravitational effects on visible matter. According to the Standard Model, dark matter accounts for the vast majority of mass in the observable universe. Dark energy is a hypothetical form of energy that permeates all of space. It is currently the most popular method for explaining recent observations that the universe appears to be expanding at an accelerating rate, as well as accounting for a significant portion of the missing mass in the universe.

The Standard Model of particle physics is a model which incorporates three of the four known fundamental interactions between the elementary particles that make up all matter (the fourth one, weakest but long range, being gravity). It came after the electroweak theory that incorporated QED (electromagnetic interactions) associated with the photon and the so-called weak interactions, associated with the *leptons* that now exist in three generations (flavors): electron, muon and tau, and their neutrinos. The Standard Model, of phenomenological origin, encompasses also the so-called strong interactions, associated with (generally) heavier particles called baryons (the proton and neutron, and many more), now commonly assumed to be bound states of "confined" quarks with gluons, in three "colors". It contains 19 free parameters, plus 10 more in extensions needed to account for the recently observed *neutrino mixing* phenomena, which require nonzero masses for the neutrinos that are traditionally massless in the Standard Model.

Very recently Connes [**CCM**] developed an effective unified theory based on noncommutative geometry (space-time being the product of a Riemannian compact spin 4-manifold and a finite noncommutative geometry) for the Standard Model with neutrino mixing, minimally coupled to gravity. It has 4 parameters less and predicts for the yet elusive Higgs particle (responsible for giving mass to initially massless leptons) a mass that is slightly different (it is at the upper end of the expected mass range) from what is usually predicted. See also [**Co06**], and [**Ba06**] in a Lorentzian framework.

Previously, in part aiming at a possible description of quantum gravity, but mainly in order to study nontrivial examples of noncommutative manifolds, Fröhlich (in a supersymmetric context), then Connes and coworkers had studied quantum spheres in 3 and 4 dimensions [**FGR, CL01, CDV**]. The basic tool there is a spectral triple introduced by Connes [**Co94**]. In the present paper we are developing a similar approach, but for hyperbolic spheres and using integral universal deformation formulas in the deformation quantization approach.

The distant hope is that these quantized AdS spaces (at even root of unity) can be shown to be a kind of "small" black holes at the edge of our Universe in accelerated expansion, from which matter would emerge, possibly created as (quantized) 2-singleton states emerging from them and massified by interaction with ambient dark energy (or dark matter), in a process similar to those of the creation of photons as 2-Rac states and of leptons from 2-singleton states, mentioned above.

As fringe benefits that might explain the acceleration of expansion of our Universe, and the problems of baryogenesis and leptogenesis (see e.g. [**Cl06, SS07**]). Physicists love symmetries and even more to break them (at least at our level). One of the riddles that physics has to face is that, while symmetry considerations suggest that there should be as much matter as antimatter, one observes a huge imbalance in our region of the Universe. In a seminal paper published in 1967 that went largely unnoticed for about 13 years but has now well over a thousand citations (we won't quote it here), Andrei Sakharov addressed that problem, now called baryogenesis. If and when a mechanism along the lines hinted at above can be developed for creating baryons and other particles, it could solve that riddle.

Roughly speaking the idea is that there is no reason, except theological, why everything (whatever that means) would be created "in the beginning", or as conventional wisdom has it now, in a Big Bang. There could very well be "stem cells" of the primordial singularity that would be spread out, like shrapnel, mostly at the edge of the Universe. Our proposal is that these could be described mathematically as quantized AdS black holes. We shall now concentrate our study on them.

1.2. Mathematical introduction. Roughly speaking, a universal deformation formula (briefly UDF) for a given symmetry $\mathcal{G}$ is a procedure that, for every, say, topological algebra $\mathbb{A}$ admitting the symmetry $\mathcal{G}$, produces a deformation $\mathbb{A}_\theta$ of $\mathbb{A}$ within the *same* category of topological algebras. Such a UDF is called *formal* when the category it applies to is that of formal power series in a formal parameter with coefficients in associative algebras.

For instance, Drinfel'd twisting elements in elementary quantum group theory constitute examples of formal UDF's (see e.g. [**CP95**]). Other formal examples in the Hopf algebraic context have been given by Giaquinto and Zhang [**GZ98**]. In [**Zag94**], Zagier produced a formal example from the theory of modular forms. The latter has been used and generalized by Connes and Moscovici in their work on codimension-one foliations [**CM04**].

In [**Rie93**], Rieffel proves that von Neumann's oscillatory integral formula [**vN31**] for the composition of symbols in Weyl's operator calculus actually constitutes an example of a non-formal UDF for the actions of $\mathbb{R}^d$ on associative Fréchet algebras. The latter has been extensively used for constructing large classes of examples of noncommutative manifolds (in the framework of Connes' spectral triples [**Co94**]) via Dirac isospectral deformations[1] of compact spin Riemannian manifolds [**CL01**] (see also [**CDV**]). Some Lorentzian examples have been investigated in [**BDSR**] and [**PS06**]. Other very interesting related approaches can be found in [**HNW, Ga05**] and references quoted therein.

Oscillatory integral UDF's for proper actions of non-Abelian Lie groups have been given in [**Bie02, BiMs, BiMa, BBM**]. Several of them were obtained through geometrical considerations on solvable symplectic symmetric spaces. Nevertheless, the geometry underlying the one in [**BiMs**] remained unclear.

In the present work we build on these works in the AdS context, with when needed reminders of their main features so as to remain largely self-contained. First we show that the latter geometry is that of a solvable symplectic symmetric space which can be viewed as a curvature deformation of the rank one non-compact Hermitian symmetric space. [It can also be viewed as a curvature deformation of the AdS space-time, as we shall see in the last section of the article.]

Next, we develop some generalities on UDF's for groups which act strictly transitively on a symplectic symmetric space. We give some precise criteria. We end the section by providing new examples with exact symplectic forms such as UDF's for solvable one-dimensional extensions of Heisenberg groups, as well as examples with non-exact symplectic forms.

In the last section we apply these developments to noncommutative Lorentzian geometry. In anti de Sitter space $\mathrm{AdS}_{n\geq 3}$, every open orbit $\mathcal{M}_o$ of the Iwasawa component $\mathcal{AN}$ of $SO(2,n)$ is canonically endowed with a causal black hole structure [**CD07**] (generalizing the BTZ-construction in dimension $n=3$). We define the analog of a Dirac-isospectral noncommutative deformation for a triple built on $\mathcal{M}_o$. The deformation is maximal in the sense that its underlying Poisson structure is symplectic on the open $\mathcal{AN}$-orbit $\mathcal{M}_o$. In particular, it does not come from an application of Rieffel's deformation machinery for isometric actions of Abelian Lie groups. Moreover, via the group action, the black hole structure is encoded in the deformed spectral triple, with no other additional geometrical data, in contradistinction with the commutative level[2].

[1]A deformation triple $(\mathcal{A}_\theta, \mathcal{H}_\theta, D_\theta)$ is said **isospectral** when $\mathcal{H}_\theta$ and D_θ are the same for all values of θ.

[2]An interesting challenge would be to analyze which operator algebraic notions attached to the triple are responsible for the singular causality. That is not investigated in the present article.

2. Curvature deformations of rank one Hermitian symmetric spaces and their associated UDF's

2.1. Preliminary set up and reminder. In [**BiMs**], a formal UDF for the actions of the Iwasawa component $\mathcal{R}_0 := \mathcal{AN}$ of $SU(1,n)$ is given in oscillatory integral form. It has been observed in [**BBM**] that this type of UDF is actually non-formal for proper actions on topological spaces. The precise framework and statement are as follows. The group $\mathcal{R}_0$ is a one dimensional extension of the Heisenberg group $\mathcal{N}_0 := H_n$. Through the natural identification $\mathcal{R}_0 = SU(1,n)/U(n)$ induced by the Iwasawa decomposition of $SU(1,n)$, the group $\mathcal{R}_0$ is endowed with a (family of) left-invariant symplectic structure(s) ω. Denoting by $\mathfrak{r}_0 := \mathfrak{a}_0 \times \mathfrak{n}_0$ its Lie algebra, the map

$$\mathfrak{r}_0 \longrightarrow \mathcal{R}_0 : (a,n) \mapsto \exp(a)\exp(n) \tag{2.1}$$

turns out to be a global Darboux chart on $(\mathcal{R}_0,\omega)$. Setting $\mathfrak{n}_0 = V \times \mathbb{R}.Z$ with table $[(x,z)\,,\,(x',z')] = \Omega_V(x,x')\,Z$, and $\mathfrak{r}_0 = \{(a,x,z)\,|\,,a,z\in\mathbb{R}; x\in V\}$, one has

THEOREM 2.1. *For all non-zero $\theta\in\mathbb{R}$, there exists a Fréchet function space $\mathcal{E}_\theta$, $C^\infty_c(\mathcal{R}_0)\subset\mathcal{E}_\theta\subset C^\infty(\mathcal{R}_0)$, such that, defining for all $u,v\in C^\infty_c(\mathcal{R}_0)$*

(2.2)

$$\begin{aligned}
&(u\star_\theta v)(a_0,x_0,z_0)\\
&\quad := \frac{1}{\theta^{\dim\mathcal{R}_0}}\int_{\mathcal{R}_0\times\mathcal{R}_0}\cosh(2(a_1-a_2))\,[\cosh(a_2-a_0)\cosh(a_0-a_1)\,]^{\dim\mathcal{R}_0-2}\\
&\quad \times\exp\Big(\frac{2i}{\theta}\Big\{S_V\big(\cosh(a_1-a_2)x_0,\cosh(a_2-a_0)x_1,\cosh(a_0-a_1)x_2\big)\\
&\quad -\bigoplus_{0,1,2}\sinh(2(a_0-a_1))z_2\Big\}\Big)\\
&\quad \times u(a_1,x_1,z_1)\,v(a_2,x_2,z_2)\,da_1da_2dx_1dx_2dz_1dz_2\ ;
\end{aligned}$$

where $S_V(x_0,x_1,x_2) := \Omega_V(x_0,x_1)+\Omega_V(x_1,x_2)+\Omega_V(x_2,x_0)$ is the phase for the Weyl product on $C^\infty_c(V)$ and $\bigoplus_{0,1,2}$ stands for cyclic summation[3], one has:

(i) *$u\star_\theta v$ is smooth and the map $C^\infty_c(\mathcal{R}_0)\times C^\infty_c(\mathcal{R}_0)\to C^\infty(\mathcal{R}_0)$ extends to an associative product on $\mathcal{E}_\theta$. The pair $(\mathcal{E}_\theta,\star_\theta)$ is a (pre-$C^\star$) Fréchet algebra.*

(ii) *In coordinates (a,x,z) the group multiplication law reads*

$$L_{(a,x,z)}(a',x',z') = \left(a+a', e^{-a'}x+x', e^{-2a'}z+z'+\frac{1}{2}\Omega_V(x,x')e^{-a'}\right).$$

The phase and amplitude occurring in formula (2.2) are both invariant under the left action $L:\mathcal{R}_0\times\mathcal{R}_0\to\mathcal{R}_0$.

(iii) *Formula (2.2) admits a formal asymptotic expansion of the form:*

$$u\star_\theta v\sim\ uv\ +\ \frac{\theta}{2i}\{u,v\}\ +O(\theta^2)\ ;$$

where $\{\,,\,\}$ denotes the symplectic Poisson bracket on $C^\infty(\mathcal{R}_0)$ associated with ω. The full series yields an associative formal star product on $(\mathcal{R}_0,\omega)$ denoted by $\tilde{\star}_\theta$.

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[3]In [**BiMs**], the exponent $\dim\mathcal{R}_0-2$ was forgotten in the expression of the amplitude of the oscillating kernel.

The setting and (i-ii) may be found in [**BiMs**], while (iii) is a straightforward adaptation to $\mathcal{R}_0$ of [**BBM**].

2.2. Geometry underlying the product formula. We start with preliminary material concerning the symmetric spaces.

2.2.1. *Symmetric spaces.* A *symplectic symmetric space* [**Bie95, BCG**] is a triple $(\mathcal{M}, \omega, s)$ where $\mathcal{M}$ is a connected smooth manifold, ω is a non-degenerate two-form on $\mathcal{M}$ and $s : \mathcal{M} \times \mathcal{M} \to \mathcal{M} : (x, y) \mapsto s_x(y)$ is a smooth map such that $\forall x \in \mathcal{M}$ the map $s_x : \mathcal{M} \to \mathcal{M}$ is an involutive diffeomorphism of $\mathcal{M}$ preserving ω and admitting x as an isolated fixed point. Moreover, one requires that the identity $s_x \circ s_y \circ s_x = s_{s_x(y)}$ holds for all $x, y \in \mathcal{M}$ [**Loo69**]. In this situation, if $x \in \mathcal{M}$ and X, Y and Z are smooth tangent vector fields on $\mathcal{M}$,

$$\omega_x(\nabla_X Y, Z) \ := \ \frac{1}{2} X_x.\omega(Y + s_{x_\star} Y \ , \ Z) \tag{2.3}$$

defines an affine connection ∇ on $\mathcal{M}$, the unique affine connection on $\mathcal{M}$ which is invariant under the symmetries $\{s_x\}_{x \in \mathcal{M}}$. It is moreover torsion-free and such that $\nabla\omega = 0$. In particular, the two-form ω is necessarily symplectic.

It then follows that the group $\mathcal{G} = \mathcal{G}(\mathcal{M}, s)$ generated by the compositions $\{s_x \circ s_y\}_{x,y \in \mathcal{M}}$ is a Lie group of transformations acting transitively on $\mathcal{M}$. The group $\mathcal{G}$ is called the **transvection** group of $\mathcal{M}$. Given a base point o in $\mathcal{M}$, the conjugation by the symmetry s_o defines an involutive automorphism $\tilde{\sigma}$ of $\mathcal{G}$. Its differential at the unit element $\sigma := \tilde{\sigma}_{\star_e}$ induces a decomposition into ± 1-eigenspaces of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The subspace $\mathfrak{k}$ of fixed vectors turns out to be a Lie subalgebra which acts faithfully on the subspace $\mathfrak{p}$ of "anti-fixed" vectors ($\sigma x = -x$ for $x \in \mathfrak{p}$). Moreover $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$. A pair $(\mathfrak{g}, \sigma)$ as above is called a *transvection pair*. The subalgebra $\mathfrak{k}$ corresponds to the Lie algebra of the stabilizer of o in $\mathcal{G}$, while the vector space $\mathfrak{p}$ is naturally identified with the tangent space $T_o(\mathcal{M})$ to $\mathcal{M}$ at point o. In particular the symplectic form at o, ω_o, induces a $\mathfrak{k}$-invariant symplectic bilinear form on $\mathfrak{p}$. Extending the latter by 0 on $\mathfrak{k}$ yields a Chevalley 2-cocycle Ω on $\mathfrak{g}$ with respect to the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$. A triple $(\mathfrak{g}, \sigma, \Omega)$ as above is called a *symplectic transvection triple*. It it said to be *exact* when there exists an element ξ in $\mathfrak{g}^\star$ such that $\delta\xi = \Omega$, where δ denotes the Chevalley coboundary operator. Up to coverings, the correspondence which associates a symplectic transvection triple to a symplectic symmetric space is bijective. More precisely *there is an equivalence of categories between the category of connected simply connected symplectic symmetric spaces and that of symplectic transvection triples* – the notion of morphism being the natural one in both cases. In the above setting, exactness corresponds to the fact that the transvection group acts on $(\mathcal{M}, \omega)$ in a strongly Hamiltonian manner.

The above considerations can be adapted in a natural manner to the Riemannian or pseudo-Riemannian setting[4], essentially by replacing *mutatis-mutandis* the symplectic structure by a metric tensor. The canonical connection (cf. Formula (2.3) above) corresponds in this case to the Levi-Civita connection.

2.2.2. *Symmetric spaces of group type.* We observe that in coordinates (a, x, z) the map $\phi : \mathcal{R}_0 \to \mathcal{R}_0 : (a, x, z) \mapsto (-a, -x, -z)$ preserves the symplectic form ω (because the coordinates are Darboux coordinates), is involutive and admits the unit element $e = (0, 0, 0)$ as an isolated fixed point. It may be therefore called

[4]See [**CP80**] for an excellent reference.

"centered symmetry" of the associative kernel (2.2). Since the kernel is left-invariant as well, such a centered symmetry $s_g : \mathcal{R}_0 \to \mathcal{R}_0 : g' \mapsto s_g(g') := L_g \circ \phi \circ L_{g^{-1}}(g')$ is attached to every point g in $\mathcal{R}_0$. It turns out that endowed with the above family of symmetries, the manifold $\mathcal{R}_0$ becomes a symplectic symmetric space. More precisely:

PROPOSITION 2.2. *Let $s : \mathcal{R}_0 \times \mathcal{R}_0 \to \mathcal{R}_0$ be the map $(g, g') \mapsto s_g(g')$. Then the triple $(\mathcal{R}_0, \omega, s)$ is a symplectic symmetric space whose transvection group is solvable. The underlying affine connection is the unique affine symplectic connection which is invariant under the group of symmetries of the oscillatory kernel (2.2).*

PROOF. In coordinates (a, x, z), the symmetry map is expressed as

$$\begin{aligned} s_{(a,x,z)}(a', x', z') = (2a - a', 2\cosh(a - a')x - x', \\ 2\cosh(2(a - a'))z + \Omega_V(x, x')\sinh(a - a') - z'). \end{aligned} \tag{2.4}$$

One then verifies that it satisfies the defining identity: $s_g \circ s_{g'} \circ s_g = s_{s_g(g')}$. Concerning the solvability of the transvection group, four-dimensional symplectic transvection triples have been classified in [**Bie95, Bie98**]. The one we are concerned with here is given by Table (1) ($\varepsilon = 1$) in Proposition 2.3 of [**Bie98**]; denoting here $\mathfrak{k} = \mathrm{Span}\{u_1, u_2, u_3\}$ and $\mathfrak{p} = \mathrm{Span}\{e_1, e_2, f_1, f_2\}$, it writes:

$$\begin{aligned} [u_2, u_3] = u_1; \quad [u_2, e_2] = e_1; \quad [u_2, f_1] = -f_2; \quad [u_3, f_2] = e_1; \quad [u_1, f_1] = 2e_1; \\ [e_1, f_1] = 2u_1; \quad [e_2, f_1] = u_3; \quad [e_2, f_2] = u_1; \quad [f_1, f_2] = u_2. \end{aligned}$$

In these notations the Lie algebra $\mathfrak{r}_0 = \mathfrak{a}_0 \oplus V \oplus \mathbb{R}Z$ of $\mathcal{R}_0$ (with Darboux chart (2.1)) is generated by $\mathfrak{a}_0 = \mathbb{R}.(-f_1)$, $V = \mathrm{Span}\{u_2 - f_2, u_3 + e_2\}$, $Z = 2(u_1 + e_1)$. We now verify (by a short computation) that for all $a, z \in \mathbb{R}$ and $x \in V$, one has

$$\tilde{\sigma}(\exp(-af_1)\exp(x + 2z(u_1 + e_1))K = (\exp(af_1)\exp(-x - 2z(u_1 + e_1))K,$$

where K denotes the analytic subgroup associated with $\mathfrak{k}$. Thus $\tilde{\sigma}$ gives $s_e = \phi$ on $\mathcal{R}_0$. The triple considered here is therefore precisely the triple that induces on $\mathcal{R}_0$ the present symmetric space structure, hence the transvection group is solvable. The higher dimensional case is similar to the 4-dimensional one.

Note that the above symmetric space structure on $\mathcal{R}_0$ is canonically associated with the data of the oscillatory kernel (2.2). Indeed, coming from the stationary phase expansion of an oscillatory integral, the formal star product $\tilde{\star}_\theta$ mentioned in item (iii) of Theorem 2.1 is *natural* in the sense that for all positive integer r the r-th cochain of the star product is a bidifferential operator of order r. To every such natural star product is uniquely attached a symplectic connection [**Lic82, GR03**]. In our case the latter, being invariant under the symmetries, must coincide with the canonical connection associated with the symmetries $\{s_g\}_{g \in \mathcal{R}_0}$. □

The above considerations lead to the following definitions.

DEFINITION 2.3. Let $\mathcal{R}$ be a Lie group. A **symmetric structure** on $\mathcal{R}$ is a diffeomorphism $\phi : \mathcal{R} \to \mathcal{R}$ such that $\phi^2 = \mathrm{id}_{\mathcal{R}}$; $\phi(e) = e$; $\phi_{\star_e} = -\mathrm{id}_{T_e(\mathcal{R})}$; and setting, for all $g \in \mathcal{R}$, $s_g := L_g \circ \phi \circ L_{g^{-1}}$ then, for all g', we define $s_g \circ s_{g'} \circ s_g = s_{s_g(g')}$.

Or equivalently:

DEFINITION 2.4. A (symplectic) symmetric space, or more generally a homogeneous space, $\mathcal{M}$ of dimension m is **locally of group type** if there exists a m-dimensional (symplectic) Lie subgroup $\mathcal{R}$ of its automorphism group which acts freely on one of its orbits in $\mathcal{M}$. One says that it is **globally** of group type if it is locally and if $\mathcal{R}$ has only one orbit[5].

Lie groups are themselves examples of symmetric spaces (globally) of group type. In the symplectic situation, however, a symplectic symmetric Lie group must be Abelian [**Bie95**]. We will see in what follows other non-Abelian examples.

2.2.3. *Strategy.* Our strategy for constructing UDF's for certain Lie groups may now be easily described: starting with a symplectic symmetric space of group type admitting an invariant deformation quantization — obtained by geometric considerations at the level of the symmetric space structure — one deduces a UDF for every Lie subgroup $\mathcal{R}$ as above by either identifying $\mathcal{M}$ to $\mathcal{R}$, or, in the formal case, by restricting the deformation quantization to an open $\mathcal{R}$-orbit. This type of strategy, in contexts other than symmetric spaces, has already been proposed within a formal framework, see for example [**Xu93, CP95, Huy82, CM04, GZ98**].

3. Construction of UDF's for symmetric spaces of group type

The first example presented in the preceding section as well as the *elementary solvable exact triples* (briefly "ESET") of [**Bie02**] are special cases of the following situation.

3.1. Weakly nilpotent solvable symmetric spaces. In this section $(\mathfrak{g},\sigma)$ denotes a complex solvable involutive Lie algebra (in short, iLa) such that if $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is the decomposition into eigenspaces of σ, the action of $\mathfrak{k}$ on $\mathfrak{p}$ is nilpotent[6]. We denote by $\mathrm{pr}_{\mathfrak{p}}:\mathfrak{g}\to\mathfrak{p}$ the projection onto $\mathfrak{p}$ parallel to $\mathfrak{k}$.

Definition 3.1. A **good Abelian subalgebra** (in short, gAs) of $\mathfrak{g}$ is an Abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, contained in $\mathfrak{p}$, supplementary to a σ-stable ideal $\mathfrak{b}$ in $\mathfrak{g}$ and such that the homomorphism $\rho:\mathfrak{a}\to\mathtt{Der}(\mathfrak{b})$ associated with the split extension $0\to\mathfrak{b}\to\mathfrak{g}\to\mathfrak{a}\to 0$ is injective.

Lemma 3.2. *If $(\mathfrak{g},\sigma)$ is not flat (i.e. $[\mathfrak{p},\mathfrak{p}]$ acts nontrivially on $\mathfrak{p}$) then a gAs always exists.*

Proof. Since $\mathfrak{k}$ is nilpotent, $[\mathfrak{k},\mathfrak{p}]\neq\mathfrak{p}$. Moreover by non-flatness there exists $X\in\mathfrak{p}\backslash[\mathfrak{k},\mathfrak{p}]$ not central in $\mathfrak{g}$. Hence for every choice of a subspace V supplementary to $\mathfrak{a}:=\mathbb{R}X$ in $\mathfrak{p}$ and containing $[\mathfrak{k},\mathfrak{p}]$, one has that $\mathfrak{b}:=\mathfrak{k}\oplus V$ is an ideal of $\mathfrak{g}$ supplementary to $\mathfrak{a}$ and on which X acts nontrivially. □

Note that the centralizer $\mathfrak{z}_{\mathfrak{b}}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{b}$ is stable by the involution σ; indeed, for all $a\in\mathfrak{a}$ and $X\in\mathfrak{z}_{\mathfrak{b}}(\mathfrak{a})$, one has $[a,\sigma X]=-\sigma[a,X]=0$. Moreover, the map $\rho:\mathfrak{a}\to\mathtt{End}(\mathfrak{b})$ being injective, we may identify $\mathfrak{a}$ with its image: $\mathfrak{a}=\rho(\mathfrak{a})$. Let $\Sigma:\mathtt{End}(\mathfrak{b})\to\mathtt{End}(\mathfrak{b})$ be the conjugation with respect to the involution $\sigma|_{\mathfrak{b}}\in GL(\mathfrak{b})$, i.e. $\Sigma=Ad(\sigma|_{\mathfrak{b}})$. The automorphism Σ is involutive and preserves the canonical Levi decomposition $\mathtt{End}(\mathfrak{b})=\mathcal{Z}\oplus\mathfrak{sl}(\mathfrak{b})$, where $\mathcal{Z}$ denotes the center of $\mathtt{End}(\mathfrak{b})$. Writing the element $a=\rho(a)\in\mathfrak{a}$ as $a=a_Z+a_0$ within this decomposition, one has: $\Sigma(a)=a_Z+\Sigma(a_0)=-a=-a_Z-a_0$, because the endomorphisms a and $\sigma|_{\mathfrak{b}}$ anticommute. Hence $\Sigma(a_0)=-2a_Z-a_0$ and therefore $a_Z=0$. So, $\mathfrak{a}$

[5]In this case, for every choice of a base point o in $\mathcal{M}$, the map $\mathcal{R}\to\mathcal{M}:g\mapsto g.o$ is a diffeomorphism.

[6]This condition is automatic for a transvection algebra, but not in general. Indeed, consider the 2-dimensional non-Abelian (solvable) algebra $\mathfrak{g}=\mathrm{Span}\{k,p\}$ with table $[k,p]=p$.

actually lies in the semisimple part $\mathfrak{sl}(\mathfrak{b})$. For any $x \in \mathfrak{sl}(\mathfrak{b})$, we denote by $x = x^S + x^N, \quad x^S, x^N \in \mathfrak{sl}(\mathfrak{b})$, its abstract Jordan-Chevalley decomposition. Observe that, denoting by $\mathfrak{sl}(\mathfrak{b}) = \mathfrak{sl}_+ \oplus \mathfrak{sl}_-$ the decomposition in (± 1)-Σ- eigenspaces, one has that $\mathfrak{a} \subset \mathfrak{sl}_-$. Also, the algebra $\mathfrak{a}_N := \{a^N\}_{a\in\mathfrak{a}}$ is an Abelian subalgebra in $\mathfrak{sl}_-$ commuting with $\mathfrak{a}$. Setting $\mathfrak{a}_S := \{a^S\}_{a\in\mathfrak{a}}$, we have

Definition 3.3. A gAs is called **weakly nilpotent** if $\mathfrak{z}_{\mathfrak{b}}(\mathfrak{a}_S) \subset \mathfrak{z}_{\mathfrak{b}}(\mathfrak{a}_N)$ and[7] $\mathfrak{a}_N \subset \mathtt{Der}(\mathfrak{b})$.

Let $\mathfrak{b}^c =: \bigoplus_{\alpha\in\Phi} \mathfrak{b}_\alpha$ be the weight space decomposition with respect to the action of $\mathfrak{a}_S$. Note that for all α, one has $\mathfrak{a}_N.\mathfrak{b}_\alpha \subset \mathfrak{b}_\alpha$. Moreover, for all $X_\alpha \in \mathfrak{b}_\alpha$ and $a^S \in \mathfrak{a}_S$, one has

$$\sigma(a^S.X_\alpha) = \alpha(a^S)\sigma(X_\alpha) = \sigma a^S \sigma^{-1}\sigma X_\alpha = \Sigma(a^S).\sigma(X_\alpha) = -a^S.\sigma(X_\alpha).$$

Therefore, $-\alpha \in \Phi$ and $\sigma\mathfrak{b}_\alpha = \mathfrak{b}_{-\alpha}$. Note in particular that $\sigma\mathfrak{b}_0 = \mathfrak{b}_0$.

Definition 3.4. An involutive Lie algebra $(\mathfrak{g}, \sigma)$ is called **weakly nilpotent** if there exists a sequence of subalgebras $\{\mathfrak{a}_i\}_{0\leq i\leq r}$ of $\mathfrak{g}$ such that

(i) $\mathfrak{a}_0$ is a weakly nilpotent gAs of $\mathfrak{g}$ with associated supplementary ideal $\mathfrak{b}^{(0)}$.

(ii) $\mathfrak{a}_{i+1}$ is a weakly nilpotent gAs of $\mathfrak{z}_{\mathfrak{b}^{(i)}}(\mathfrak{a}_i) \quad (0 \leq i \leq r-1)$ where, for $i \geq 1$, $\mathfrak{b}^{(i)}$ denotes the σ-stable ideal of $\mathfrak{z}_{\mathfrak{b}^{(i-1)}}(\mathfrak{a}_{i-1})$ associated with $\mathfrak{a}_i$.

(iii) $\mathfrak{z}_{\mathfrak{b}^{(r)}}(\mathfrak{a}_r)$ is Abelian.

Proposition 3.5. *Assume the iLa $(\mathfrak{g}, \sigma)$ to be weakly nilpotent. Then there exists a (complex) subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that the restriction $pr_{\mathfrak{p}}|_{\mathfrak{s}} : \mathfrak{s} \to \mathfrak{p}$ is a linear isomorphism.*

Proof. Let $\mathfrak{a}$ be a weakly nilpotent gAs of $\mathfrak{g}$ and set $V_\alpha := \mathfrak{b}_\alpha \oplus \mathfrak{b}_{-\alpha}$ for every $\alpha \in \Phi$. Note that $V_0 = \mathfrak{b}_0$ and that each subspace V_α is then σ-stable and one sets $V_\alpha = \mathfrak{k}_\alpha \oplus \mathfrak{p}_\alpha$ for the corresponding eigenspace decomposition. Choose a partition[8] of $\Phi\backslash\{0\}$ as $\Phi\backslash\{0\} =: \Phi^+ \cup \Phi^-$ with the properties that $-\Phi^+ = \Phi^-$ and that if $\alpha, \beta \in \Phi^+$ with $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$. One has $\mathfrak{b} = \oplus_{\alpha\in\Phi^+} V_\alpha \oplus \mathfrak{b}_0$. We set $\mathfrak{b}^+ := \oplus_{\alpha\in\Phi^+}\mathfrak{b}_\alpha$ and $\mathfrak{p}^+ := \mathrm{pr}_{\mathfrak{p}}(\mathfrak{b}^+)$. It turns out that the restriction map $\mathrm{pr}_{\mathfrak{p}}|_{\mathfrak{b}^+} : \mathfrak{b}^+ \to \mathfrak{p}^+$ is a linear isomorphism. Indeed for $X \in \mathfrak{b}_{\mathfrak{k}} := \mathfrak{b} \cap \mathfrak{k}$ and $a \in \mathfrak{a}_S$ one has $\sigma(a.X) = \sigma a \sigma \sigma X = \Sigma(a).X = -a.X$; hence $\mathfrak{a}_S.\mathfrak{b}_{\mathfrak{k}} \subset \mathfrak{p}$. Therefore, for all $X \in \mathfrak{b}_\alpha \cap \mathfrak{k} \quad \alpha \neq 0$, one can find $a \in \mathfrak{a}_S$ such that $a.X = X \in \mathfrak{p} \cap \mathfrak{k}$; thus $\mathfrak{b}_\alpha \cap \mathfrak{k} = 0$ as soon as $\alpha \neq 0$, yielding $\ker(\mathrm{pr}_{\mathfrak{p}}|_{\mathfrak{b}^+}) = 0$. The last condition of Definition 3.3 implies that $\mathfrak{a}_S$ acts by derivations on $\mathfrak{b}$, hence the usual argument yields that $\mathfrak{b}^+$ is a subalgebra normalized by $\mathfrak{a} \oplus \mathfrak{b}_0$. Moreover the first condition

[7]The last condition is automatic when $\mathfrak{b}$ is Abelian. Observe also that it is satisfied when $\rho(\mathfrak{a})$ is contained in a Levi factor of the derivation algebra $\mathtt{Der}(\mathfrak{b})$.

[8]Such a partition can be defined as follows. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}(\mathfrak{b})$ containing $\mathfrak{a}_S$ and let $\Lambda \in \mathfrak{h}^\star$ denote the set of weights of the representation of $\mathfrak{sl}(\mathfrak{b})$ on $\mathfrak{b}$. Note that the restriction map $\lambda \mapsto \lambda|_{\mathfrak{a}_S}$ from Λ to $\mathfrak{a}_S^\star$ is surjective onto Φ. Let $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$ be a real decomposition such that the restriction of the Killing form to $\mathfrak{h}_{\mathbb{R}}$ is positive definite. Every weight in Λ is then real valued when restricted to $\mathfrak{h}_{\mathbb{R}}$ [**Kna01**]. Now, any choice of a basis of $\mathfrak{h}_{\mathbb{R}}$ defines a partial ordering on Λ with the desired properties. To pass to the set of weights Φ, consider the $\mathfrak{h}_{\mathbb{R}}$-components, $\mathfrak{a}_S^{\mathbb{R}}$, of $\mathfrak{a}_S$ viewed as a vector subspace of $\mathfrak{h}$. The restriction map $\rho : \Phi \to \Phi|_{\mathfrak{a}_S^{\mathbb{R}}}$ is then a bijection. Indeed, for $\lambda \in \ker(\rho)$, one has, by $\mathbb{C}$-linearity, $\lambda(a+ia') = \lambda(a)+i\lambda(a') = 0\,\forall a, a' \in \mathfrak{a}_S^{\mathbb{R}}$. Hence $\lambda = 0$ as an element of Φ. Therefore, an order on Λ induces on Φ an order having the same properties.

of the same definition implies $[\mathfrak{a}, \mathfrak{b}_0] = 0$. The proposition follows by induction. One starts applying the above considerations to $\mathfrak{a} = \mathfrak{a}_0$ and $\mathfrak{b} = \mathfrak{b}^{(0)}$. This yields a subalgebra $\mathfrak{s}_0$ supplementary to $\mathfrak{z}_{\mathfrak{b}^{(0)}}(\mathfrak{a}_0)$ and normalized by it. One then sets $\mathfrak{g}_1 := \mathfrak{z}_{\mathfrak{b}^{(0)}}(\mathfrak{a}_0)$, considers a weakly nilpotent $\mathfrak{a}_1$ in $\mathfrak{g}_1$ and gets a subalgebra $\mathfrak{s}_1$ that is now supplementary to and normalized by $\mathfrak{z}_{\mathfrak{b}^{(1)}}(\mathfrak{a}_1)$. Applying this procedure inductively, one gets a sequence a subalgebras $\mathfrak{s}_0, ..., \mathfrak{s}_{r-1}, \mathfrak{s}_r := \mathfrak{z}_{\mathfrak{b}_r}(\mathfrak{a}_r) \cap \mathfrak{p}$ such that $\mathfrak{s}_{i+1}$ normalizes $\mathfrak{s}_i$ and $\mathfrak{s}_i \cap \mathfrak{s}_j = 0$, and defines $\mathfrak{s} := \oplus_i \mathfrak{s}_i$. □

REMARK 3.6. In the exact symplectic case, one has $\mathfrak{p}_0 \perp \mathfrak{a}$ since $\xi[\mathfrak{a}, \mathfrak{b}_0] = 0$.

3.2. Darboux charts and kernels: an extension lemma. Let $(\mathfrak{s}_i, \Omega_i)$ with $i = 1, 2$ be symplectic Lie algebras, and denote by $(\mathbf{S}_i, \omega_i)$ the corresponding simply connected symplectic Lie groups (ω_i being left-invariant). Given a homomorphism $\rho : \mathfrak{s}_1 \to \mathtt{Der}(\mathfrak{s}_2) \cap \mathfrak{sp}(\Omega_2)$, form the corresponding semi-direct product $\mathfrak{s} := \mathfrak{s}_1 \times_\rho \mathfrak{s}_2$ and consider the associated simply connected Lie group $\mathbf{S}$ endowed with the left-invariant symplectic structure ω defined by $\omega_e := \Omega_1 \oplus \Omega_2$ (e denotes the unit element in $\mathbf{S}$).

LEMMA 3.7. *If* $\phi_i : (\mathfrak{s}_i, \Omega_i) \to (\mathbf{S_i}, \omega_i) \quad (i = 1, 2)$ *are Darboux charts, the map* $\phi : (\mathfrak{s}, \Omega := \Omega_1 \oplus \Omega_2) \longrightarrow (\mathbf{S}, \omega) : (X_1, X_2) \mapsto \phi_2(X_2).\phi_1(X_1)$ *is Darboux.*

PROOF. For $X \in \mathfrak{s}_1$ and $Y \in \mathfrak{s}_2$ one has:

$$\begin{aligned} L_{\phi^{-1}_{\star_\phi}}\phi_\star(X) = L_{\phi^{-1}_{\star_\phi}}\Big(R_{\phi_1{}_{\star_{\phi_2}}}(\phi_{2\star}X)\Big) &= L_{\phi_1^{-1}{}_\star} L_{\phi_2^{-1}{}_\star} R_{\phi_1{}_\star}\phi_{2\star}X \\ &= \mathtt{Ad}\phi_1^{-1}\left(L_{\phi_2^{-1}{}_{\star_{\phi_2}}}\phi_{2\star}X\right) \end{aligned}$$

while for $Y \in \mathfrak{s}_1$ one has

$$L_{\phi^{-1}{}_{\star_\phi}}\phi_\star(Y) = L_{\phi_1^{-1}{}_\star} L_{\phi_2^{-1}{}_\star} L_{\phi_2{}_\star}\phi_{1\star}Y = L_{\phi_1^{-1}{}_\star}\phi_{1\star}Y.$$

Hence

$$\omega_\phi(X, Y) = \Omega\Big(\mathtt{Ad}\phi_1^{-1}\big(L_{\phi_2^{-1}{}_{\star_{\phi_2}}}\phi_{2\star}X\big)\,,\, L_{\phi_1^{-1}{}_\star}\phi_{1\star}Y\Big) = 0,$$

because the first (resp. the second) argument belongs to $\mathfrak{s}_2$ (resp. to $\mathfrak{s}_1$), and for $X, X' \in \mathfrak{s}_2$,

$$\begin{aligned} \omega_\phi(X, X') &= \Omega\Big(\mathtt{Ad}\phi_1^{-1}\big(L_{\phi_2^{-1}{}_{\star_{\phi_2}}}\phi_{2\star}X\big)\,,\,\mathtt{Ad}\phi_1^{-1}\big(L_{\phi_2^{-1}{}_{\star_{\phi_2}}}\phi_{2\star}X'\big)\Big) \\ &= \mathtt{Ad}\phi_1^{-1\star}\Omega_2\Big(L_{\phi_2^{-1}{}_{\star_{\phi_2}}}\phi_{2\star}X\,,\,L_{\phi_2^{-1}{}_{\star_{\phi_2}}}\phi_{2\star}X'\Big) \\ &= \Omega_2\Big(L_{\phi_2^{-1}{}_{\star_{\phi_2}}}\phi_{2\star}X\,,\,L_{\phi_2^{-1}{}_{\star_{\phi_2}}}\phi_{2\star}X'\Big) \\ &= \omega_{2\star_{\phi_2}}(X, X') \ = \Omega_2(X, X'). \end{aligned}$$

A similar (and simpler) computation applies for two elements of $\mathfrak{s}_1$. □

A direct computation shows

LEMMA 3.8. *Let* $K_i \in Fun((\mathbf{S}_i)^3)$ *be a left-invariant three point kernel on* $\mathbf{S}_i$ *($i = 1, 2$). Assume* $K_2 \otimes 1 \in Fun((\mathbf{S})^3)$ *is invariant under conjugation by elements of* $\mathbf{S}_1$*. Then* $K := K_1 \otimes K_2 \in Fun((\mathbf{S})^3)$ *is left-invariant (under* $\mathbf{S}$*).*

In particular, given associative kernels satisfying the above hypotheses, their tensor product defines an associative invariant kernel on the semidirect product $\mathbf{S}$. We will call it **extension product** of K_2 by K_1.

3.3. Examples.

3.3.1. *One dimensional split extensions of Heisenberg algebras.* The strategy is as follows. First we observe that (the connected simply connected group associated to) every such extension acts simply transitively on the symmetric space $\mathcal{R}_0$, and is therefore diffeomorphic to it. Then, we remark that this diffeomorphism can be chosen to be symplectic. The latter is almost a homomorphism, up to an action of a subgroup of the automorphism group of the kernel on $\mathcal{R}_0$ (this is done by embedding the extension in a subgroup of the automorphism group of the kernel containing $\mathcal{R}_0$). Finally we check that the pullback of that kernel by the diffeomorphism gives an invariant kernel on the extension.

So let us start with a Heisenberg Lie algebra $\mathfrak{h}_n = V \oplus \mathbb{R}Z$, where (V, Ω_V) is a $2n$-dimensional symplectic vector space, Z is central, and $[v, v'] = \Omega_V(v, v')Z$. One easily sees that a split extension (with $i \circ p = \mathrm{Id}$) $\mathfrak{r}_n$ of $\mathfrak{h}_n$ by $\mathfrak{a} = \mathbb{R}A$:

$$0 \to \mathfrak{h}_n \to \mathfrak{r}_n \underset{i}{\overset{p}{\rightleftarrows}} \mathfrak{a} \to 0,$$

must be such that, for all $v \in V$ and $z \in \mathbb{R}$, $[A, v + zZ] = X.v + \mu(v)Z + 2dzZ$ with $X \in \mathrm{End}(V)$, $\mu \in V^*$ and $d \in \mathbb{R}$ such that $X - d\mathrm{Id} \in \mathfrak{sp}(V, \Omega_V)$. As symplectic form on the associated group, we choose the left-invariant 2-form whose value at the identity is the Chevalley 2-coboundary $\Omega = -\delta Z^* = Z^*[\cdot, \cdot]$, which is a natural generalization of that on $\mathcal{R}_0$ in [**BiMs**]. This 2-coboundary is nondegenerate if and only if $d \neq 0$, which we will assume from now on[9], and consider extensions with parameters dX, $d\mu$ and $2d$, which we will denote by $(dX, d\mu, 2d)$.

We have the following symplectic Lie algebra isomorphisms:

(i) $(dX, d\mu, 2d) \cong (X, \mu, 2)$ for all $d \in \mathbb{R}_0$, through the map $L(a, v, z) = (da, v, z)$.

(ii) $(X, \mu, d) \cong (X, 0, 2)$ for all $\mu \in V^*$, through the map $L(a, v, z) = (a, v + au, z)$ with $i_u\Omega_V = \mu$.

(iii) $(X, 0, 2) \cong (X', 0, 2)$ through the map $L(a, v, z) = (a, M.v, z)$, if and only if $M \in Sp(V, \Omega_V)$ is such that $MXM^{-1} = X'$, i.e. if $X - \mathrm{Id}$ and $X' - \mathrm{Id}$ belong to the same adjoint orbit of $Sp(V, \Omega_V)$.

Thus we concentrate on algebras of type $(X, 0, 2)$ from which we can recover the quantization of the others.

Now let $\mathfrak{r}_0 = (I, 0, 2)$, $\mathfrak{r}' = (X, 0, 2)$, and $\mathcal{R}_0$, $\mathcal{R}'$ the associated groups. Elements of these algebras will be denoted respectively by $aA + v + zZ$ and $aA' + v + zZ$. The difference between the actions of A' and A on V is $\bar{X} := X - \mathrm{Id}$ which lies in $\mathfrak{sp}(V, \Omega_V)$. Extending the action $\bar{X}$ on $\mathfrak{r}_0$ by $[\bar{X}, aA + v + zZ] := \bar{X}.v$, we can therefore view $\mathfrak{r}_0$ and $\mathfrak{r}'$ as subalgebras of the semidirect product $\mathfrak{g} = \mathfrak{r}_0 \times \mathfrak{s}$ of $\mathfrak{r}_0$ by $\mathfrak{s} = \mathrm{span}(\bar{X}) \subset \mathfrak{sp}(V, \Omega_V)$. At the level of the groups ($\mathcal{S}$ corresponding to $\mathfrak{s}$), on the one hand we can identify $\mathcal{R}_0$ with $\mathcal{G}/\mathcal{S}$ as manifolds, and on the other hand as a subgroup of $\mathcal{G}$, $\mathcal{R}'$ acts on the quotient. That action is simply transitive. Indeed, on $\mathcal{R}_0$ and $\mathcal{R}'$, we have global coordinate maps $I(aA + v + zZ) = \exp(aA)\exp(v + zZ)$ and $I'(aA' + v + zZ) = \exp(aA')\exp(v + zZ)$ such that $I'(aA' + v + zZ) = \exp(a\bar{X}).I(aA + v + zZ)$, giving a decomposition $g = sr \in \mathcal{R}'$ with $s \in \mathcal{S}$ and $r \in \mathcal{R}_0$. Thus, acting on $e\mathcal{S} \in \mathcal{G}/\mathcal{S}$, such an element g gives

[9] Quantizing extensions with $d = 0$ requires a non exact 2-form. Since our method needs an exact one, we shall have to apply it on central extensions of our algebras. An example of that procedure is given in the next section.

$g \cdot e\mathcal{S} = C_s(r)\mathcal{S} = I(aA + e^{a\bar{X}}v + zZ)\mathcal{S}$, where C_s denotes the action by conjugation, $C_s(g) = sgs^{-1}$. The map $\phi : aA' + v + zZ \mapsto aA + e^{a\bar{X}}v + zZ$ is a diffeomorphism from $\mathfrak{r}'$ to $\mathfrak{r}_0$, and the corresponding map $\Phi = I \circ \phi \circ I'^{-1}$, or $sr \mapsto C_s(r)$, is a diffeomorphism from $\mathcal{R}'$ to $\mathcal{R}_0$. We now observe

PROPOSITION 3.9. (i) *For $g = sr \in \mathcal{R}'$, we have $\Phi \circ L_g = L_{\Phi(g)} \circ C_s \circ \Phi$.*
(ii) *The kernel K on $\mathcal{R}_0$ is invariant under the conjugations by $\mathcal{S}$.*
(iii) *Denoting by ω and ω' the left-invariant 2-forms with value $-\delta Z^*$ at the identity on $\mathcal{R}_0$ and $\mathcal{R}'$ respectively, we have $\Phi^*\omega = \omega'$.*

PROOF. (i) For all $g = sr$, $g' = s'r' \in \mathcal{R}'$, we have

$$\Phi \circ L_g(g') = \Phi(ss'C_{s'^{-1}}(r)r') = C_{ss'}(C_{s'^{-1}}(r)r') = L_{\Phi(g)} \circ C_s \circ \Phi(g').$$

(ii) Recall that for $s = e^{a'\bar{X}}$, $C_s(I(aA + v + zZ)) = I(aA + e^{a'\bar{X}}v + zZ)$ and that $\bar{X} \in Sp(V, \Omega_V)$. Therefore in the kernel (2.2) the amplitude is independent of v, and in its phase, the function S_V is invariant under the symplectomorphisms of V. The kernel as a whole is thus also invariant under $Sp(V, \Omega_V)$.
(iii) By left-invariance, the condition is $\omega_e \circ \left(L_{\Phi(g)^{-1}} \circ \Phi \circ L_g\right)_{*e} = \omega'_e$. Using the first property of Φ above and the invariance of ω_e under $Sp(V, \Omega_V)$, we get $\omega_e \circ \left(L_{\Phi(g)^{-1}} \circ \Phi \circ L_g\right)_{*e} = \omega_e \circ (C_s \circ \Phi)_{*e} = \omega_e \circ \phi_{*0}$, which is readily seen to be equal to ω'_e.

□

Defining a kernel $K' : \mathcal{R}' \times \mathcal{R}' \times \mathcal{R}' \to \mathbb{C}$ on $\mathcal{R}'$ by $K' = \Phi^*K$, we now have

PROPOSITION 3.10. *The kernel K' is*
(i) *invariant under the diagonal left action of $\mathcal{R}'$,*
(ii) *associative.*

Together with the functional space $\Phi^*\mathcal{E}_\theta$, it thus defines a WKB quantization of $\mathcal{R}'$.

A quantization of these groups can be obtained by the same method as in [**BiMs**]. So let us first quickly review that method. On a connected simply connected symplectic solvable Lie group $(\mathcal{R}, \omega)$, with ω a left-invariant exact symplectic form (so that the action by left translations is strongly Hamiltonian), one chooses a global Darboux chart $I : \mathfrak{r} \to \mathcal{R}$ for which the Moyal star product is *covariant*, i.e. if for $X \in \nabla$, $\lambda_X \in C^\infty(\mathfrak{r})$ denotes the (dual) moment map in these coordinates, and $\star^M_\theta$ the Moyal product on $\mathfrak{r}$, one has $[\lambda_X, \lambda_Y]_{\star^M_\theta} = 2\theta\{\lambda_X, \lambda_Y\}$. Such charts always exist on these groups (see [**Puk90**], [**AC90**]).

Covariance of the Moyal product implies that $\rho_\theta : \mathfrak{r} \to \mathrm{End}(C^\infty(\mathfrak{r})[[\theta]]) : X \mapsto [\lambda_X, \cdot]_{\star^M_\theta}$ is a representation of $\mathfrak{r}$ by derivations of the algebra $(C^\infty(\mathfrak{r})[[\theta]], \star^M_\theta)$. In order to find an invariant product on $\mathfrak{r}$, one then tries to find an invertible operator T_θ which intertwines this action and that by fundamental vector fields, i.e. such that $T_\theta^{-1} \circ \rho_\theta(X) \circ T_\theta = X^*$. Those found up to now were all integral operators of the type $\mathcal{F}^{-1} \circ \phi^*_\theta \circ \mathcal{F}$, where $\mathcal{F}$ is a partial Fourier transform and ϕ_θ a diffeomorphism (see the proof of Theorem 4.6 for the precise form of the one in [**BiMs**]). An invariant product $\star_\theta$ on $\mathfrak{r}$ is then defined by $u \star_\theta v = T_\theta^{-1}(T_\theta u \star^W_\theta T_\theta v)$, where $\star^W_\theta$ is the Weyl product on $\mathfrak{r}$. Modulo some work on the function spaces, this gives a WKB invariant quantization of $\mathcal{R}$. In our case, choosing as Darboux chart the map $I(aA + v + zZ) = \exp(aA)\exp(e^{-a\bar{X}}v + zZ)$, one checks that the same integral operator T_θ as in [**BiMs**] works here, giving thus rise to the same kernel (2.2). This reflects again the fact that our groups are all subgroups of the automorphism group of a symplectic symmetric space on which they act simply transitively.

3.3.2. *Non exact example.* As mentioned before, all the examples of symplectic symmetric Lie groups shown up to now were endowed with an exact symplectic form, as our method requires exactness in order for the left translations to be strongly Hamiltonian. We present here an example with a non exact symplectic form showing that, as expected, considering a central extension allows to apply the same method.

Let $\mathfrak{r} = \mathrm{span}\,\langle A, V_1, V_2, W_1, W_2, Z\rangle$ be the Lie algebra defined by $[A, V_i] = V_i$, $[A, W_i] = -W_i$, $[A, Z] = 2Z$, $[V_1, V_2] = Z$, and choose the non exact[10] Chevalley 2-cocycle Ω: $\Omega(A, Z) = 1$, $\Omega(V_i, W_i) = 1$, $\Omega(V_1, V_2) = 1/2$. We define $\mathfrak{g}$ as the central extension of $\mathfrak{r}$ by the element E with commutators $[X, Y]_{\mathfrak{g}} = [X, Y]_{\mathfrak{r}} + \Omega(X, Y).E$ for all $X, Y \in \mathfrak{g}$, where we extended $[\cdot, \cdot]_{\mathfrak{r}}$ and Ω by zero on E. Then $\Omega = -\delta E^*$ is a 2-coboundary.

Now the connected simply connected Lie group $\mathcal{R}$ whose Lie algebra is $\mathfrak{r}$ can be realized as the coadjoint orbit $\mathcal{O}$ of E^* in $\mathfrak{g}^*$, and a global Darboux chart J from $\mathbb{R}^6$ to $\mathcal{O}$ is given by:

$$\begin{aligned} J(q_1, p_1, q_2, p_2, q_3, p_3) \quad &= \quad \exp(q_3 H) \exp((p_3 + q_1 p_1 + q_2 p_2 - p_1 p_2/2)Z) \\ &\quad\quad \exp((q_2 + p_1/4)W_1) \exp(p_2 V_1) \\ &\quad\quad \exp((q1 - p_2/4)W_2) \exp(p_1 V_2) \cdot E^*. \end{aligned}$$

In this chart, the (dual) moment maps are linear in the p_i, so that the Moyal product is covariant. From now on, the method outlined above can be carried on the same way as in [**BiMs**], with the same integral operator T_θ as before.

3.3.3. *The Iwasawa factor of* $\mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{so}(2, 3)$. The Lie algebra $\mathfrak{g}_0 := \mathfrak{sp}(n, \mathbb{R})$ of the group $\mathcal{G}_0 := Sp(n, \mathbb{R})$ is defined as the set of $2n \times 2n$ real matrices X such that ${}^\tau X F + F X = 0$ where $F := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. One has

$$\mathfrak{g}_0 = \{\begin{pmatrix} A & S_1 \\ S_2 & -{}^\tau A \end{pmatrix} \text{ where } A \in \mathrm{Mat}(n \times n, \mathbb{R}) \text{ and } S_i = {}^\tau S_i\,; i = 1, 2\}. \tag{3.1}$$

In particular, $F \in \mathcal{G}_0$ and a Cartan involution of $\mathfrak{g}_0$ is given by $\theta := \mathtt{Ad}(F)$. The corresponding Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is then given by

$$\mathfrak{k}_0 \simeq \mathfrak{u}(n) \text{ and } \mathfrak{p}_0 = \{\begin{pmatrix} S & S' \\ S' & -S \end{pmatrix}\}, \tag{3.2}$$

where the matrices S and S' are symmetric. For $n = 2$, a maximal Abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}_0$ is generated by $H_1 = E_{11} - E_{33}$ and $H_2 = E_{22} - E_{44}$ where as usual E_{ij} denotes the matrix whose component are zero except the element ij which is one. The restricted roots Φ w.r.t $\mathfrak{a}$ are then given by

$$\Phi = \{\alpha_0,\ \alpha_1,\ \alpha_2 := \alpha_0 + \alpha_1,\ \alpha_3 := \alpha_0 + 2\alpha_1\} \tag{3.3}$$

with $\alpha_0 := 2H_2^*$, $\alpha_1 := H_1^* - H_2^*$, hence $\alpha_2 := H_1^* + H_2^*$ and $\alpha_3 := 2H_1^*$, where $H_i^*(H_j) := \delta_{ij}$. The corresponding root spaces $\mathfrak{g}_{\alpha_i}$ $(i = 0, ..., 3)$ are one-dimensional, generated respectively by $N_0 = E_{24}$, $N_1 = E_{12} - E_{43}$, $N_2 = E_{14} + E_{23}$, $N_3 = E_{13}$.

[10] One can actually show that every symplectic 2-cocycle on $\mathfrak{r}$ is non exact and that what follows can be applied to any one of them.

With this choice of generators, the minimal parabolic subalgebra $\mathfrak{s} := \mathfrak{a} \oplus \mathfrak{n}$ with $\mathfrak{n} := \bigoplus_{i=0}^{3} \mathfrak{g}_{\alpha_i}$ has the following multiplication table:

$$[H_1, N_1] = N_1, \qquad [H_1, N_2] = N_2, \tag{3.4a}$$

$$[H_2, N_1] = -N_1, \qquad [H_2, N_2] = N_2, \tag{3.4b}$$

$$[H_1, N_3] = 2N_3, \qquad [H_2, N_0] = 2N_0, \tag{3.4c}$$

$$[N_0, N_1] = -N_2, \qquad [N_1, N_2] = 2N_3; \tag{3.4d}$$

the other brackets being zero. Setting

$$\mathfrak{s}_1 := \mathrm{Span}\{H_2, N_0\} \text{ and } \mathfrak{s}_2 := \mathrm{Span}\{H_1, N_1, N_2, N_3\}, \tag{3.5}$$

one observes that $\mathfrak{s}$ is a split extension of $\mathfrak{s}_2$ by $\mathfrak{s}_1$:

$$0 \longrightarrow \mathfrak{s}_2 \longrightarrow \mathfrak{s} \longrightarrow \mathfrak{s}_1 \longrightarrow 0. \tag{3.6}$$

Note that $\mathfrak{s}_1$ is a minimal parabolic subalgebra of $\mathfrak{su}(1,1)$ while $\mathfrak{s}_2$ is a minimal parabolic subalgebra of $\mathfrak{su}(1,n)$. In particular, the Lie algebra $\mathfrak{s}$ is exact symplectic w.r.t. the element $\xi := \xi_1 \oplus \xi_2$ of $\mathfrak{s}^\star$ with $\xi_i \in \mathfrak{s}_i^\star \quad (i = 1, 2)$ defined as $\xi_1 := N_0^*$ and $\xi_2 := N_3^*$.

One therefore obtains a UDF for proper actions of $\mathcal{AN}$ by direct application of the above extension lemma 3.8.

4. Isospectral deformations of anti de Sitter black holes

4.1. Anti de Sitter black holes. Anti de Sitter (AdS) black holes have been introduced by Bañados, Teitelbaum, Zannelli and Henneaux [**BTZ, BHTZ**] as connected locally AdS space-times $\mathcal{M}$ (possibly with boundary and corners) admitting a *singular* causal structure in the following sense:

Condition BH. There exists a closed subset $\mathcal{S}$ in $\mathcal{M}$ called the *singularity* such that the subset $\mathcal{M}_{\mathrm{bh}}$ constituted by all the points x such that every light like geodesic issued from x ends in $\mathcal{S}$ within a finite time is a proper open subset of $\mathcal{M}_{\mathrm{phys}} := \mathcal{M} \backslash \mathcal{S}$.

Originally such solutions were constructed in space-time dimension 3, but they exist in arbitrary dimension $n \geq 3$ (see [**BDSR, CD07**]). More precisely the structure may be described as follows. Take $\mathcal{G} := SO(2, n-1)$ (the AdS group), fix a Cartan involution θ and a θ-commuting involutive automorphism σ of $\mathcal{G}$ such that the subgroup $\mathcal{H}$ of $\mathcal{G}$ of the elements fixed by σ is locally isomorphic to $SO(1, n-1)$. The quotient space $\mathcal{M} := \mathcal{G}/\mathcal{H}$ is an n-dimensional Lorentzian symmetric space, the *anti de Sitter space-time*. It is a solution of the Einstein equations without source. Let $\mathfrak{g}$ denote the Lie algebra of $\mathcal{G}$ and denote by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ the ± 1-eigenspace decomposition with respect to the differential at e of σ that we denote again by σ. Denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition induced by θ, consider a σ-stable maximally Abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ and choose accordingly a positive system of roots. Denote by $\mathfrak{n}$ the corresponding nilpotent subalgebra. Set $\overline{\mathfrak{n}} := \theta(\mathfrak{n})$, $\mathfrak{r} := \mathfrak{a} \oplus \mathfrak{n}$ and $\overline{\mathfrak{r}} := \mathfrak{a} \oplus \overline{\mathfrak{n}}$. Finally denote by $\mathcal{R} := \mathcal{AN}$ and $\overline{\mathcal{R}} := A\overline{N}$ the corresponding analytic subgroups of $\mathcal{G}$. One then has

Proposition 4.1. [**BDSR, CD07**] *The groups $\mathcal{R}$ and $\overline{\mathcal{R}}$ admit open orbits and finitely many closed orbits in the AdS space $\mathcal{M}$. Prescribing as singular the union of all closed orbits (of $\mathcal{R}$ and $\overline{\mathcal{R}}$) defines a structure of causal black hole on an open subset $\mathcal{M}_{phys}$ in $\mathcal{M}$ (in the sense of the above condition* **(BH)***). In* *particular, every open orbit $\mathcal{M}_o$ of $\mathcal{R}$ in $\mathcal{M}$ containing $\mathcal{M}_{phys}$ is itself endowed with a black hole structure.*

Recall that if J denotes an element of $Z(\mathcal{K})$ whose associated conjugation coincides with the Cartan involution θ then the $\mathcal{R}$-orbit $\mathcal{M}_o$ in $\mathcal{G}/\mathcal{H}$ of an element $u\mathcal{H}$ with $u^2 = J$ is open and contains $\mathcal{M}_{phys}$, see [**CD07**]. Remark that the extension lemma 3.8 yields an oscillatory integral UDF for proper actions of $\mathcal{R}$. But here the situation is simplified by the following observation — for convenience of the presentation we write it below for $n = 4$ but the results are valid for any $n \geq 3$.

PROPOSITION 4.2. *The $\mathcal{R}$-homogeneous space $\mathcal{M}_o$ admits a unique structure of globally group type symplectic symmetric space. The latter is isomorphic to $(\mathcal{R}_0, \omega, s)$ described in section 2.*

For the proof, recall first (see [**CD07**]) that the solvable part of the Iwasawa decomposition of $\mathfrak{so}(2,3)$ may be realized with as nilpotent part $\mathfrak{n} = \{W, V, M, L\}$ and Abelian $\mathfrak{a} = \{J_1, J_2\}$ with the commutator table $[V,W] = M$, $[V,L] = 2W$, $[J_1,W] = W$, $[J_2,V] = V$, $[J_1,L] = L$, $[J_2,L] = -L$, $[J_1,M] = M$, $[J_2,M] = M$. Notice that $W, J_1 \in \mathfrak{h}$, and $J_2 \in \mathfrak{q}$. This decomposition is related to the one given in (3.4) by

$$\begin{aligned} N_0 &= L & N_2 &= 2W & H_1 &= J_1 + J_2 \\ N_1 &= V & N_3 &= M & H_2 &= J_1 - J_2. \end{aligned}$$

We choose to study the orbit of the element $\vartheta = u\mathcal{H}$ with

$$u = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & I_{3\times 3} \end{pmatrix}.$$

We denote by $\mathcal{R}_\theta$ its stabilizer group in $\mathcal{R}$, by $\mathfrak{r}_\theta$ the Lie algebra of $\mathcal{R}_\theta$; $\mathfrak{r}'$ is the subalgebra of $\mathfrak{r}$ generated by elements of $\mathfrak{r}$ minus the generator of $\mathfrak{r}_\theta$ and $\mathcal{R}'$ is the analytic subgroup of $\mathcal{R}$ whose algebra is $\mathfrak{r}'$.

LEMMA 4.3. *The action of $\mathcal{R}'$ on $\mathcal{U}$ is simply transitive, i.e. $\mathcal{R}'u\mathcal{H} = \mathcal{R}u\mathcal{H}$.*

PROOF. The first step is to prove that $\mathcal{R}_\theta$ is connected and $\mathcal{U}$ simply connected in order to prevent any double covering problem. The stabilizer of $u\mathcal{H}$ is

$$\mathcal{R}_\theta = \{r \in \mathcal{R} \mid r \cdot u\mathcal{H} = u\mathcal{H}\} = \{r \in \mathcal{R} \mid C_{u^{-1}}(r) \in \mathcal{H}\}. \tag{4.1}$$

Since $\mathcal{R}$ is an exponential group, we have $\mathfrak{r}_\theta = \{X \in \mathcal{R} \mid \mathtt{Ad}(u^{-1})X \in \mathfrak{h}\}$ with $\mathcal{R}_\theta = \exp \mathfrak{r}_\theta$. The set $\mathfrak{r}_\theta$ being connected, $\mathcal{R}_\theta$ is connected too. A long exact sequence argument using the fibration $\mathcal{R}_\theta \to \mathcal{R} \to \mathcal{U}$ shows that $H^0(\mathcal{R}_\theta) \simeq H^1(\mathcal{U})$, which proves that $\mathcal{U}$ is simply connected.

As an algebra, $\mathfrak{r}$ is a split extension $\mathfrak{r} = \mathfrak{r}_\theta \oplus_{\mathtt{ad}} \mathfrak{r}'$. Hence, as group, $\mathcal{R} = \mathcal{R}_\theta\mathcal{R}'$, or equivalently $\mathcal{R} = \mathcal{R}'\mathcal{R}_\theta$. This proves that the action is transitive. The action is even simply transitive because $\mathcal{U}$ is simply connected. □

Let us now find the algebra $\mathfrak{r}_\theta$. The Cartan involution $X \mapsto -X^t$ is implemented as C_J with

$$J = \begin{pmatrix} -I_{2\times 2} & \\ & I_{3\times 3} \end{pmatrix}.$$

Using the relations $u^2 = J$ and $\sigma(u) = u^{-1}$, one sees that $C_{u^{-1}}(r) \in \mathcal{H}$ if and only if $\sigma\Big(C_{u^{-1}}r\Big) = C_{u^{-1}}r$. This condition is equivalent to $\theta\sigma(r) = r$. The involution σ

splits $\mathfrak{a}$ into two parts: $\mathfrak{a} = \mathfrak{a}^+ \oplus \mathfrak{a}^-$ with $J_1 \in \mathfrak{a}^+ = \mathfrak{a} \cap \mathfrak{h}$ and $J_2 \in \mathfrak{a}^- \cap \mathfrak{q}$. Let $\beta_1, \beta_2 \in \mathfrak{a}^*$ be the dual basis; we have $W \in \mathfrak{g}_{\beta_1}$, $V \in \mathfrak{g}_{\beta_2}$, $L \in \mathfrak{g}_{\beta_1-\beta_2}$, $M \in \mathfrak{g}_{\beta_1+\beta_2}$, and, in terms of positive roots, the space $\mathfrak{n}$ is given by $W \in \mathfrak{g}_{\alpha+\beta}$, $V \in \mathfrak{g}_\beta$, $L \in \mathfrak{g}_\alpha$, $M \in \mathfrak{g}_{\alpha+2\beta}$. We are now able to compute the vectors $X \in \mathfrak{r}$ such that $\sigma\theta(X) = X$. Let us take $X \in \mathfrak{r} = \mathfrak{a} \oplus \mathfrak{n}$ and apply $\sigma\theta$:

$$\begin{aligned} X &= X_{J_1} + X_{J_2} + X_\alpha + X_\beta + X_{\alpha+\beta} + X_{\alpha+2\beta}, \\ \sigma\theta X &= -X_{J_1} + X_{J_2} + Z_{-(\alpha+2\beta)} + Z_\beta + Z_{-(\alpha+\beta)} + Z_{-\alpha} \end{aligned} \tag{4.2}$$

where X_φ and Z_φ denote elements of $\mathfrak{g}_\varphi$. It is directly apparent that $X = J_2$ belongs to $\mathfrak{r}_\theta$. The only other component common to X and $\sigma\theta X$ is in $\mathfrak{g}_\beta$, but it is *a priori* not clear that $X_\beta = Z_\beta$. The dimension of $\mathcal{U}$ is 4 and that of $\mathcal{R}$ is 6, hence $\mathcal{R}_\theta$ is at least 2-dimensional; it is generated by J_2 and $\mathfrak{g}_\beta = \mathbb{R}V$, i.e. $\mathfrak{r}_\theta = \mathrm{Span}\{J_2, V\}$. This proves that the orbit of $u\mathcal{H}$ is open.

The fact that $\mathcal{R}'$ acts freely on $\mathcal{U} = \mathcal{R}/\mathcal{R}_\theta$ proves that $\mathcal{U}$ is locally of group type and since, by definition, $\mathcal{U}$ is only one orbit of $\mathcal{R}$, the space $\mathcal{U}$ is globally of group type. From now on, $\mathcal{M}_o = \mathcal{R}/\mathcal{R}_\theta$ will be identified with $\mathcal{U}$ as homogeneous space, so what we have to find is a group $\tilde{\mathcal{R}}$ which

- acts transitively on $\mathcal{U}$, i.e. $\tilde{\mathcal{R}}u\mathcal{H} = \mathcal{R}u\mathcal{H}$,
- admits a symplectic structure.

It is immediate to see that the algebra $\mathfrak{r}'$ fails to fulfil the symplectic condition. The algebra $\tilde{\mathfrak{r}} = \mathrm{Span}\{A, B, C, D\}$ of a group which fulfils the first condition must at least act transitively on a small neighborhood of $u\mathcal{H}$ and thus be of the form

$$A = J_1 + aJ_2 + a'V \tag{4.3a}$$

$$B = W + bJ_2 + b'V \tag{4.3b}$$

$$C = M + cJ_2 + c'V \tag{4.3c}$$

$$D = L + dJ_2 + d'V. \tag{4.3d}$$

The problem is now to fix the parameters $a, a', b, b', c, c', d, d'$ in such a way that $\mathrm{Span}\{A, B, C, D\}$ is a Lie algebra (i.e. it is closed under the Lie bracket) which admits a symplectic structure and whose group acts transitively on $\mathcal{U}$. We will begin by proving that the surjectivity condition imposes $b = c = d = 0$. Then the remaining conditions for $\tilde{\mathfrak{r}}$ to be an algebra are easy to solve by hand.

First, remark that A acts on the algebra $\mathrm{Span}\{B, C, D\}$ because J_1 does not appears in $[\mathfrak{r}, \mathfrak{r}]$. We can write $\tilde{\mathfrak{r}} = \mathbb{R}A \oplus_{\mathrm{ad}} \mathrm{Span}\{B, C, D\}$ and therefore a general element of the group $\tilde{\mathcal{R}}$ reads $\tilde{r}(\alpha, \beta, \gamma, \delta) = e^{\alpha A} e^{\beta B + \gamma C + \delta D}$ because a subalgebra of a solvable exponential Lie algebra is solvable exponential. Our strategy will be to split this expression in order to get a product $\mathcal{S}\mathcal{R}'$ (which is equivalent to a product $\mathcal{R}'\mathcal{S}$). As Lie algebras, $\mathrm{Span}\{B, C, D\} \subset \mathbb{R}J_2 \oplus_{\mathrm{ad}} \{W, M, L, V\}$. Hence there exist functions w, m, l, v and x of $(\alpha, \beta, \gamma, \delta)$ such that

$$e^{\beta B + \gamma C + \delta D} = e^{xJ_2} e^{wW + mM + lL + vV}. \tag{4.4}$$

We are now going to determine $l(\alpha, \beta, \gamma, \delta)$ and study the conditions needed in order for l to be surjective on $\mathbb{R}$. Since J_2 does not appear in any commutator, the Campbell-Baker-Hausdorff formula yields $x = \beta b + \gamma c + \delta d$. From the fact that $[J_2, L] = -L$, we see that the coefficient of L in the left hand side of (4.4) is $-l(1 - e^{-x})/x$. The V-component in the exponential can also get out without changing the coefficient of L. We are left with $\tilde{r}(\alpha, \beta, \gamma, \delta) = e^{\alpha A} e^{xJ_2} e^{yV} e^{w'W + m'M + lL}$ where

w' and m' are complicated functions of (β,γ,δ) and l is given by

$$l(\beta,\gamma,\delta)=\frac{-\delta(\beta b+\gamma c+\delta d)}{1-e^{-\beta b-\gamma c-\delta d}}, \tag{4.5}$$

which is not surjective except when $b=c=d=0$. Taking the inverse a general element of $\tilde{\mathcal{R}}u\mathcal{H}$ reads $\left[e^{-wW-mM-lM}e^{j_1J_1}u\right]$, where the range of l is not the whole $\mathbb{R}$. Since the action of R' is *simply* transitive, $\tilde{\mathcal{R}}$ is not surjective on $\mathcal{R}u\mathcal{H}$.

When $b=c=d=0$, the conditions for (4.3) to be an algebra are easy to solve, leaving only two *a priori* possible two-parameter families of algebras:

Algebra 1.

$$\begin{aligned} A &= J_1+\frac{1}{2}J_2+sV & \qquad [A,B]&=B+sC\\ B &= W & [A,C]&=\frac{3}{2}C\\ C &= M & [A,D]&=2sB+\frac{1}{2}D\\ D &= L+rV & [B,D]&=-rC. \end{aligned}$$

with $r\neq 0$. The general symplectic form on that algebra is given by

$$\omega_1=\begin{pmatrix}0 & -\alpha & -\beta & -\gamma\\ \alpha & 0 & 0 & \frac{2\beta r}{3}\\ \beta & 0 & 0 & 0\\ \gamma & -\frac{2\beta r}{3} & 0 & 0\end{pmatrix}, \tag{4.7}$$

Since $\det\omega=\left(\frac{2\beta r}{3}\right)^2$ we must have $\beta\neq 0$, $r\neq 0$. That algebra will be denoted by $\mathfrak{r}_1$. The analytic subgroup of $\mathcal{R}$ whose Lie algebra is $\mathfrak{r}_1$ is denoted by $\mathcal{R}_1$.

Algebra 2.

$$\begin{aligned} A &= J_1+rJ_2+sV & \qquad [A,B]&=B+sC\\ B &= W & [A,C]&=(r+1)C\\ C &= M & [A,D]&=2sB+(1-r)D\\ D &= L. \end{aligned}$$

There is no way to get a non-degenerate symplectic form on that algebra.

Remark 4.4. One can eliminate the two parameters in algebra $\mathfrak{r}_1$ by the isomorphism

$$\phi=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 4s\\ 0 & 2sr & 1/r & 4s^2/r\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{4.8}$$

which fixes $s=0$ and $r=1$ and transforms $\mathfrak{r}_1$ into the algebra defined by $[A',B']=B'$, $[A',C']=\frac{3}{2}C'$, $[A',D']=\frac{1}{2}D'$, $[B',D']=-C'$.

It is now easy to prove that

Proposition 4.5. *The group $\mathcal{R}_1$ of algebra $\mathfrak{r}_1$ acts transitively on $\mathcal{U}$, i.e. $\mathcal{R}u\mathcal{H}=\mathcal{R}_1u\mathcal{H}$.*

PROOF. The algebra $\mathfrak{r}_1$ can be written $\mathfrak{r}_1 = \mathbb{R}A \oplus_{\mathtt{ad}} \mathbb{R}D \oplus_{\mathtt{ad}} \mathrm{Span}\{B, C\}$, a split extension, hence a general element reads $r_1(\alpha, \beta, \gamma, \delta) = e^{\alpha A} e^{\delta D} e^{\beta W + \gamma M}$. One can use Campbell-Baker-Hausdorff formula to split it into a factor in $\mathcal{R}_\theta$ and one in $\mathcal{R}'$ (where f and g are some auxiliary functions):

$$r_1(\alpha, \beta, \gamma, \delta) = \underbrace{e^{\alpha s V + \frac{\alpha}{2} J_2} e^{\delta r V}}_{\in \mathcal{R}_\theta} \underbrace{e^{\alpha J_1} e^{\left(f(\delta)+\beta\right) W \left(g(\delta)+\gamma\right) M}}_{\text{surjective on } \mathcal{R}'} \tag{4.9}$$

□

The conclusion is that $\mathcal{R}_1$ is the group $\tilde{\mathcal{R}}$ that we were searching for. To summarize, the structure is as follows.

(i) The AdS space is decomposed into a family of cells: the orbits of a symplectic solvable Lie group $\tilde{\mathcal{R}}$ as in Proposition 4.2 above. Note that these cells may be viewed as the symplectic leaves of the Poisson generalized foliation associated with the left-invariant symplectic structure on $\tilde{\mathcal{R}}$.

(ii) The open $\tilde{\mathcal{R}}$-orbit $\mathcal{M}_o$, endowed with a black hole structure, identifies with the group manifold $\tilde{\mathcal{R}}$.

4.2. Deformation triples for $\mathcal{M}_o$.

4.2.1. *Left-invariant Hilbert function algebras on $\mathcal{R}_0$.* In this section, we present a modified version of the oscillatory integral product (2.2) leading to a left-invariant associative algebra structure on the space of square integrable functions on $\mathcal{R}_0$.

THEOREM 4.6. *Let u and v be smooth compactly supported functions on $\mathcal{R}_0$. Define the following three-point functions:*

$$\begin{aligned} S :=& S_V\big(\cosh(a_1 - a_2)x_0, \cosh(a_2 - a_0)x_1, \cosh(a_0 - a_1)x_2\big) \\ &- \oint_{0,1,2} \sinh\big(2(a_0 - a_1)\big) z_2 \ ; \end{aligned} \tag{4.10}$$

and

$$A := \Big[\cosh\big(2(a_1 - a_2)\big) \cosh(2(a_2 - a_0)) \cosh(2(a_0 - a_1)) \\ \big[\cosh(a_1 - a_2) \cosh(a_2 - a_0) \cosh(a_0 - a_1)\big]^{\dim \mathcal{R}_0 - 2}\Big]^{\frac{1}{2}}.$$

Then the formula

$$u \star_\theta^{(2)} v \ := \ \frac{1}{\theta^{\dim \mathcal{R}_0}} \int_{\mathcal{R}_0 \times \mathcal{R}_0} A \, e^{\frac{2i}{\theta} S} u \otimes v \tag{4.11}$$

extends to $L^2(\mathcal{R}_0)$ as a left-invariant associative Hilbert algebra structure. In particular, one has the strong closedness[11] *property:*

$$\int u \star_\theta^{(2)} v \ = \ \int uv \ .$$

PROOF. The oscillatory integral product (2.2) may be obtained by intertwining the Weyl product on the Schwartz space $\mathcal{S}$ (in the Darboux global coordinates (2.1)) by the following integral operator [**BiMs**]:

$$\tau := \mathcal{F}^{-1} \circ (\phi_\theta^{-1})^\star \circ \mathcal{F} \ ,$$

[11]The notion of strongly closed star product was introduced in [**CFS**] in the formal context.

$\mathcal{F}$ being the partial Fourier transform with respect to the central variable z:

$$\mathcal{F}(u)(a,x,\xi) := \int e^{-i\xi z} u(a,x,z)\mathrm{d}z \;;$$

and ϕ_θ the one parameter family of diffeomorphism(s):

$$\phi_\theta(a,x,\xi) = (a, \frac{1}{\cosh(\frac{\theta}{2}\xi)} x, \frac{1}{\theta}\sinh(\theta\xi)).$$

Set $\mathrm{J} := |(\phi^{-1})^\star \mathrm{Jac}_\phi|^{-\frac{1}{2}}$ and observe that for all $u \in C^\infty \cap L^2$, the function $\mathrm{J}\,(\phi^{-1})^\star u$ belongs to L^2. Indeed, one has

$$\int |\mathrm{J}\,(\phi^{-1})^\star u|^2 = \int |\phi^\star \mathrm{J}|^2\, |\mathrm{Jac}_\phi|\, |u|^2 = \int |u|^2 \;.$$

Therefore, a standard density argument yields the following isometry:

$$T_\theta : L^2(\mathcal{R}_0) \longrightarrow L^2(\mathcal{R}_0) : u \mapsto \mathcal{F}^{-1} \circ m_{\mathrm{J}} \circ (\phi^{-1})^\star \circ \mathcal{F}(u) \;,$$

where m_{J} denotes the multiplication by J. Observing that $T_\theta = \mathcal{F}^{-1} \circ m_{\mathrm{J}} \circ \mathcal{F} \circ \tau$, one has $\star_\theta^{(2)} = \mathcal{F}^{-1} \circ m_{\mathrm{J}} \circ \mathcal{F}(\star_\theta)$. A straightforward computation (similar to the one in [**Bie02**]) then yields the announced formula. □

Remark 4.7. Let us point out two facts with respect to the above formulas:

(i) Note the cyclic symmetry of the oscillating three-point kernel $A\, e^{\frac{2i}{\theta} S}$.

(ii) The above oscillating integral formula gives rise to a strongly closed, symmetry invariant, formal star product on the symplectic symmetric space $(\mathcal{R}_0, \omega, s)$.

Proposition 4.8. *The space $L^2(\mathcal{R}_0)^\infty$ of smooth vectors in $L^2(\mathcal{R}_0)$ of the left regular representation closes as a subalgebra of $(L^2(\mathcal{R}_0), \star_\theta^{(2)})$.*

Proof. First, observe that the space of smooth vectors may be described as the intersection of the spaces $\{V_n\}$ where $V_{n+1} := (V_n)_1$, with $V_0 := L^2(\mathcal{R}_0)$ and $(V_n)_1$ is defined as the space of elements a of V_n such that, for all $X \in \mathfrak{r}_0$, $X.a$ exists as an element of V_n (we endow it with the projective limit topology).

Let thus $a, b \in V_1$. Then, $(X.a) \star b + a \star (X.b)$ belongs to V_0. Observing that $\mathcal{D} \subset V_1$ and approximating a and b by sequences $\{a_n\}$ and $\{b_n\}$ in $\mathcal{D}$, one gets (by continuity of $\star$): $(X.a)\star b + a\star(X.b) = \lim(X.a_n \star b_n + a_n \star (X.b_n)) = \lim X.(a_n \star b_n) = X.(a\star b)$. Hence $a \star b$ belongs to V_1. One then proceeds by induction. □

4.2.2. *Twisted L^2-spinors and deformations of the Dirac operator.* We now follow in our four dimensional setting the deformation scheme presented in [**BDSR**] in the three-dimensional BTZ context.

At the level of the (topologically trivial) open $\mathcal{R}$-orbit, the spin structure over $\mathcal{M}_o$ and the associated spinor $\mathbb{C}^2$-bundle – restriction of the spinor bundle on AdS_n – are trivial. The space of (smooth) spinor fields may then be viewed as $\mathbf{S} := C^\infty(\mathcal{R}_0, \mathbb{C}^2)$, on which the (isometry) group $\mathcal{R}_0$ acts on via the left regular representation. In this setting, the restriction to $\mathcal{M}_o$ of the Dirac operator D on AdS_n may be written as $D = \sum_i \gamma^i\, (\tilde{X}_i + \Gamma_i)$, where

- $\{X_i\}$ denotes an orthonormal basis of $\mathfrak{r}_0 = T_\vartheta(\mathcal{M}_o)$ (w.r.t. the adS-metric at the base point ϑ of $\mathcal{M}_o$);
- for $X \in \mathfrak{r}_0$, $\tilde{X}$ denotes the associated left-invariant vector field on $\mathcal{R}_0$;

- γ^i and Γ_i denote respectively the Dirac γ-endomorphism and the spin-connection element associated with X_i.

In that expression the elements γ^i's and Γ_i's are constant. However, already at the formal level, a left-invariant vector field $\tilde{X}$ as infinitesimal generator of the right regular representation does not in general act on the deformed algebra. In order to cure this problem, we twist the spinor module in the following way.

Definition 4.9. Let $\mathrm{d}^r g$ be a right-invariant Haar measure on $\mathcal{R}_0$ and consider the associated space of square integrable functions $L^2_{\mathrm{right}}(\mathcal{R}_0)$. Set

$$\mathcal{H} \ := \ L^2_{\mathrm{right}}(\mathcal{R}_0) \otimes \mathbb{C}^2 \ ;$$

and denote by $\mathcal{H}^\infty$ the space of smooth vectors in $\mathcal{H}$ of the natural right representation of $\mathcal{R}_0$ on $\mathcal{H}$. Then intertwining $\star^{(2)}_\theta$ by the inverse mapping

$$\iota : L^2_{\mathrm{right}}(\mathcal{R}_0) \to L^2(\mathcal{R}_0) : \iota(u)(g) := u(g^{-1})$$

yields a right invariant noncommutative $L^2_{\mathrm{right}}(\mathcal{R}_0)$-bi-module structure (respectively a $(L^2_{\mathrm{right}}(\mathcal{R}_0))^\infty$-bi-module structure) on $\mathcal{H}$ (resp. $\mathcal{H}^\infty$). The latter will be denoted $\mathcal{H}_\theta$ (resp. $\mathcal{H}^\infty_\theta$).

We summarize the main results of this paper in the following:

Theorem 4.10. *The Dirac operator D acts in $\mathcal{H}^\infty_\theta$ as a derivation of the noncommutative bi-module structure. In particular, for all $a \in (L^2_{\mathrm{right}}(\mathcal{R}_0))^\infty$, the commutator $[D, a]$ extends to $\mathcal{H}$ as a bounded operator. In other words, the triple $(L^2_{\mathrm{right}}(\mathcal{R}_0))^\infty, \mathcal{H}^\infty_\theta, D)$ induces on $\mathcal{M}_o$ a pseudo-Riemannian deformation triple.*

5. Conclusions, remarks and further perspectives

To the AdS space we associated a symplectic symmetric space (M, ω, s). That association is *natural* by virtue of the uniqueness property mentioned in Proposition 4.2. The data of any invariant (formal or not) deformation quantization on (M, ω, s) yields then canonically a UDF for the actions of a non-Abelian solvable Lie group. Using it we defined the noncommutative Lorentzian spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ where $\mathcal{A}^\infty := (L^2_{\mathrm{right}}(\mathcal{R}_0))^\infty$ is a noncommutative Fréchet algebra modelled on the space of smooth vectors of the regular representation on square integrable functions on the group $\mathcal{R}_0$. The underlying commutative limit is endowed with a causal black hole structure encoded in the $\mathcal{R}_0$-group action. A first question that this construction raises is that of defining within the present Lorentzian context the notion of causality at the operator algebraic level.

Another direction of research is to analyze the relation between the present geometrical situation and the corresponding one within the quantum group context. Indeed, our universal deformation formulas can be used at the algebraic level to produce nonstandard quantum groups $SO(2, n-1)_q$ via Drinfeld twists. An interesting challenge would then be to study the behaviour of the representation theory under the deformation process.

More generally the somewhat elliptic sentence with which we started the paper may now be better understood if we remark that the physical motivation section and the quantum group framework suggest to study a number of questions related to (noncommutative) singleton physics, in particular:

(i) Since [**FHT, Sta98**] we know that for q even root of unity, there are unitary irreducible finite dimensional representations of the Anti de Sitter groups. Interestingly (cf. [**FGR**] p.122) the "fuzzy 3-sphere" is related to the Wess-Zumino-Witten models and is conjectured to be related to the non-commutative geometry of the quantum group $U_q(\mathrm{sl}_2)$ for $q = e^{2\pi i/(k+2)}, k > 0$, a root of unity.

(ii) The last remark suggests to look more closely at the phenomenon of dimensional reduction which appears in a variety of related problems. In this paper we considered only $n \geq 3$. The reason is that for $n = 2$ the context is in part different: the conformal group of 1+1 dimensional space-time is infinite dimensional, and there are no black holes [**CD07**]. But many considerations remain true, and furthermore many group-theoretical properties find their origin at the 1+1 dimensional level, e.g. the uniqueness of the extension to conformal group [**AFFS**]. Another exemple of dimensional reduction is the fact that the massless UIR of the 2+1 dimensional Poincaré group Di and Rac satisfy Di $\oplus$ Rac $=$ D(HO)$\oplus$D(HO) where D(HO) is the representation D(1/4)$\oplus$D(3/4) of the metaplectic group (double covering of $SL(2,\mathbb{R})$) which is the symmetry of the harmonic oscillator in the deformation quantization approach (see e.g. Section (2.2.4) in [**DS02**]).

(iii) What do the degenerate representations Di and Rac become under deformation? Furthermore there may appear, for our nonstandard quantum group $SO(2,n)_q$, new representations that have no equivalent at the undeformed level (e.g. in a way similar to the supercuspidal representations in the p-adic context). These may have interesting physical interpretations.

(iv) We have seen that for q even root of unity $SO(2,n)_q$ has some properties of a compact Lie group. Our cosmological Ansatz suggests that the qAdS black holes are "small." It is therefore natural to try and find a kind of generalized trace that permits to give a finite volume for qAdS. Note that, in contradistinction with infinite dimensional Hilbert spaces, the notions of boundedness and compactness are the same for closed sets in Montel spaces, and that our context is in fact more Fréchet nuclear than Hilbertian. This raises the more general question to define in an appropriate manner the notion of "q-compactness" (or "q-boundedness") for noncommutative manifolds.

(v) Possibly in relation with the preceding question, one should perhaps consider deformation triples in which the Hilbert space is replaced by a suitable locally convex topological vector space (TVS), on which D could be continuous.

(vi) The latter should yield a natural framework for implementing quantum symmetries in deformation triples, since Fréchet nuclear spaces and their duals are at the basis of the topological quantum groups (and their duals) introduced in the 90's, especially in the semi-simple case with preferred deformations (see the review [**BGGS**]). We would thus in fact have quadruples $(\mathcal{A}, \mathcal{E}, D, \mathcal{G})$ where $\mathcal{A}$ is some topological algebra, $\mathcal{E}$ an appropriate TVS, D some (bounded on $\mathcal{E}$) "Dirac" operator and $\mathcal{G}$ some symmetry. [Being in a Lorentzian noncompact framework, we did not address here questions such as the resolvent of D when $\mathcal{E}$ is a Hilbert space, which we did not need at this stage; eventually one may however have to deal with the reasons that motivated the additional requirements on triples in the Riemannian compact context; note that here the restriction to an open orbit was needed in order to have bounded commutators $[D, a]$ in the Hilbertian context, but a good choice of $\mathcal{E}$ could lift the restriction.] That framework should be naturally extendible to the supersymmetric context, which is the one considered in [**FGR**] with modified spectral triples and is natural also for the problems considered here since e.g. Di$\oplus$Rac and D(HO) are UIR of the corresponding supersymmetries.

(vii) If we want to incorporate "everything," the (external) symmetry $\mathcal{G}$ should be the Poincaré group $SO(1,3)\cdot\mathbb{R}^4$ in the ambient Minkowski space (possibly modified by the presence of matter) and $SO(2,3)_q$ in the $q\mathrm{AdS}_4$ black holes, or possibly some supersymmetric extension. The unified (external) symmetry could therefore be something like a groupoid. The latter should be combined in a subtle way (as hinted e.g. in [**St07**]) with the "internal" symmetry associated with the various generations, colors and flavors of (composite) "elementary" particles in a generalized Standard Model, possibly in a noncommutative geometry framework analogous to what is done in [**Co06, CCM, Ba06**]. There would of course remain the formidable task to develop quantized field theories on that background, incorporating composite QED for photons on AdS as in [**FF88**] and some analog construction for the electroweak model (touched in part in [**Frø00**]) and for QCD, possibly making use of some formalism coming from string theory.

(viii) The Gelfand isomorphism theorem permits to realize commutative involutive algebras as algebras of functions on their "spectrum." Finding a noncommutative analog of it has certainly been in the back of the mind of many, since quite some time (see e.g. [**St05**]). We now have theories and many examples of deformed algebras, quantum groups and noncommutative manifolds. The above mentioned quadruples could provide a better understanding of that situation.

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Université catholique de Louvain, département de mathématiques, Chemin du Cyclotron 2, B-1348 Louvain-la-Neuve, Belgium

E-mail address: bieliavsky claessens voglaire @math.ucl.ac.be

Institut de Mathématiques de Bourgogne, Université de Bourgogne, BP 47870, F-21078 Dijon Cedex, France

Current address: Department of Mathematics, Keio University,, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan

E-mail address: Daniel.Sternheimer@u-bourgogne.fr