Causal Dynamic Inference
No. 368

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Abstract

We suggest a general logical framework for causal dynamic reasoning. As a first step, we introduce a uniform structural formalism and assign it two kinds of semantics, abstract dynamic models and relational models. The corresponding completeness results are proved. As a second step, we extend the structural formalism to a two-sorted state-transition calculus, and prove its completeness with respect to the associated relational semantics.

1 Introduction

The general, though distant, aim of our study consists in singling out and exploring the ultimate logical ingredients of dynamic reasoning. In this paper, however, we will restrict ourselves to a few initial steps in this general quest.

As a starting point, we will contend that, at the most fundamental level, an adequate logical description of a dynamic world can be achieved by defining an appropriate dynamic consequence relation among (propositions denoting) temporally extended events and processes. In this respect, the logical formalism, described below, can be viewed as a particular implementation of the 'dynamic turn in logic' advocated already in [van Benthem, 1996]. Namely, it is a substructural consequence relation defined directly on sequences of propositions. It will be shown, in particular, that the theory of such an inference relation can be developed to the depth and levels comparable to theories of ordinary consequence relations.

A more specific idea we adhere to in this study is that the above mentioned dynamic consequence relation should reflect the basic *causal* relationships among events and processes. In other words, we believe that a natural and systematic description of dynamic domains can be given once we settle what causes or enables what in these domains; the rest of the properties and facts about the domain or situation should follow (at least in principle) by logical means. Furthermore, the rules and postulates that characterize such a causal consequence relation should be viewed not as entailments that follow logically from the meaning of the consequence relation and propositions involved, but rather as nontrivial claims about the (causal) structure of the dynamic world. Accordingly, such postulates provide an indirect, 'functional' description of the concept of causation itself.

The idea that causal relations should constitute the basis of representation and description of dynamic universes is not new; as a matter of fact, it is widely used as a guiding principle in the main fields that deal with such dynamic descriptions, such as linguistic semantics, artificial intelligence and even general theory of computation. Taking only a few examples, it has been persuasively argued in [Steedman, 2005] that the so-called temporal semantics of natural language should not primarily deal with time at all, but rather with representation of causality and goal-directed action (see also [van Lambalgen and Hamm, 2004] for a similar approach). Similarly, a number of causal approaches to reasoning about action and change have been developed in Artificial Intelligence (see, e.g., [Giunchiglia *et al.*, 2004]). In fact, our approach can be seen as a logical counterpart of a qualitative causal modeling in AI developed a long time ago in [Forbus, 1984].

The paper is organized as follows. We introduce first a basic structural calculus of dynamic inference that is defined on sequences of propositions (called processes) and satisfies certain 'sequential' variants of familiar inference rules such as Identity, Monotonicity and Cut. This calculus will be assigned first an abstract monoid-based dynamic semantics, and the completeness results will be proved. Then we will show that the latter semantics can be transformed into more familiar relational, or transition models. Moreover, it will be shown that our dynamic inference constitutes in this respect a generalization of the inference relation of dynamic predicate logic [Groenendijk and Stokhof, 1991].

As a second step, we will introduce a two-sorted extension of the basic formalism in which the underlying set of propositions will be split into transitions (actions) proper, and static propositions (also known as fluents, or tests). This extension of the formalism will be called a state-transition calculus. It will be shown that (under some additional conditions) the state transition calculus is also complete for the corresponding relational semantics.

2 The Structural Dynamic Calculus

The language of the calculus will be a set $L = \{A, B, C, \dots\}$ of propositions that will denote primitive transitions or *events*. Finite sequences of events will be called *processes*, and we will use small letters $a, b, c, \dots$ to denote such processes. The set of all processes will be denoted by L^*. It will include, in particular, an empty sequence denoted by ϵ. As usual, ab will denote the concatenation of sequences a and b from L^* (and similarly for aA, aAb, etc.).

A *dynamic consequence relation* will be a set of rules, or sequents, of the form $a \Vdash b$, where $a, b \in L^*$. The intended meaning of such rules is that a processes a *causes* (or enables) a process b. This consequence relation will be required to satisfy the following postulates:

Identity $\epsilon \Vdash \epsilon$.

Left Monotonicity If $a \Vdash b$, then $Aa \Vdash b$.

Cut If $a \Vdash b$ and $ab \Vdash c$, then $a \Vdash bc$.

Already the very form of our sequents involving *sequences* of propositions (instead of usual sets) indicates that the dynamic consequence relation is substructural, that is, it does not satisfy the usual structural rules for consequence relations such as contraction, permutation and weakening. Note, in particular, that a minimal premise or a conclusion of a sequent is not an empty set, but an empty sequence ϵ. Still, we will show in what follows that the analysis and representation of our dynamic calculus can proceed quite along the same lines as the usual theoretical development for standard consequence relations.

Though all the above postulates constitute a certain weakening of well-known structural rules for a (classical) sequent calculus, we argue that these postulates should rather be viewed as informed claims about the structure of processes and their interactions. In other words, we see the above postulates as assertions that have a non-trivial *content* that jointly determine the structure of the dynamic universe (more exactly, of its representation). Varying or extending these postulates (as will actually be done in subsequent sections) means changing the structure and relations of this universe.

In this respect, it is interesting to note that our postulates preserve locality and continuity of processes. Thus, if the premises of some postulate describe relations among contiguous processes, then its conclusion will also have this property. This feature is especially evident in the form of the Cut postulate, which does not allows us to infer, e.g., $a \Vdash c$ from $a \Vdash b$ and $ab \Vdash c$; such an inference would break continuity and create non-local influences.

Remark. As could be noticed already at this stage, the sequents of our dynamic calculus, though interpreted informally as causal claims, do not correspond precisely to a commonsense notion of causation. A primary witness of this discrepancy is a postulate of Left Monotonicity that allows us to strengthen the premises of a rule with additional propositions (though in a restricted way). Apparent counterexamples to the corresponding strengthening of commonsense causal claims are easy to come by, since *Striking a match causes it to light* obviously does not imply *Putting a match in water and then striking it causes it to light.* Unfortunately, an attempt to establish a precise correspondence between our formal causal rules and their commonsense counterparts would bring us far beyond the scope of the present study. So we mention only that arguments of the above kind are quite familiar to logicians in the form of reservations against taking classical material implication as a representation of commonsense conditionals in general. And in the latter case, such arguments do not deprave the classical implication of its role in logic. Similarly, the assertions made by our dynamic rules could be called *proto-causal* claims. We argue, however, that such proto-causal claims form an essential ingredient of the corresponding (more complex) commonsense causal assertions. Moreover, 'monotonic' causal claims of a similar kind form a basis for a quite successful causal theory of reasoning about action and change in AI (see, e.g., [Giunchiglia *et al.*, 2004; Bochman, 2004]). This theory shows, in particular, that the above-mentioned counterexamples can be successfully dealt with as part of its general nonmonotonic component.

Due to the Horn form of the rules characterizing a dynamic consequence relation, intersection of a set of consequence relations is again a consequence relation. This implies, in particular, that, for any set of sequents, there exists a least dynamic consequence relation containing it.

Now we introduce the basic notion of a theory of a dynamic consequence relation. Intuitively, theories characterize admissible sets of processes (with respect to a given dynamic consequence relation).

Definition 2.1. A set U of processes will be called a *theory* of a dynamic consequence relation $\Vdash$ if, whenever $a \Vdash b$ holds, and s is some process such that $sa \in U$, then $sab \in U$.

Slightly reformulated, a theory is a set of processes such that whenever it includes a process s that ends with a (as its end-segment), and a causes b, then the process sb should also belong to the set.

An important property of our dynamic theories (common with ordinary Tarski theories) is that intersections of theories are again theories of a consequence relation. This immediately implies that, for any set of processes there exists a least theory containing it. In other words, we have a natural closure operator $\mathrm{Th}(V)$ that assigns any set V of processes a unique least theory containing it.

The following simple lemma provides a direct syntactic description of the least theory containing a single process. It will be used in what follows.

Lemma 2.1. $\mathrm{Th}(a) = \{ab \mid a \Vdash b\}$.

Proof. Let T_a denote the set $\{ab \mid a \Vdash b\}$. It is easy to see that any theory containing a should include also T_a. Note also that $a \in T_a$, since $a \Vdash \epsilon$ by Identity and Left Monotonicity. Hence it is sufficient to show that T_a is a theory. Assume that $c \Vdash d$ and $sc \in T_a$ for some process s. Then $sc = ab$, for some b such that $a \Vdash b$. Now, $c \Vdash d$ implies $sc \Vdash d$ by Left Monotonicity, and hence $ab \Vdash d$. Therefore $a \Vdash bd$ by Cut, and consequently $abd = scd \in T_a$. Thus, T_a is a theory of $\Vdash$. This completes the proof. $\square$

3 Dynamic Monoid Semantics

We will introduce first the following very abstract notion of a dynamic semantics.

By a *dynamic frame* we will mean an arbitrary monoid $(P, \cdot, 1)$. In other words, $\cdot$ is an associative binary operation on P, and $u \cdot 1 = 1 \cdot u = u$, for any $u \in P$. Elements of P will be called *paths*. The operation $\cdot$ on P can be viewed as a concatenation of paths. It can be canonically extended to sets of paths as follows: if $U, V \subseteq P$, then $U \cdot V = \{uv \mid u \in U \ \& \ v \in V\}$. This extension is also an associative operation.

In what follows, we will often omit the operation sign $\cdot$ and write ab instead of $a \cdot b$. In addition, $x \preceq y$ will denote the fact that $y = xz$, for some z. It is easy to verify that $\preceq$ is a partial order. We will call it a *prefix relation* on paths.

Definition 3.1. An abstract *dynamic model* is a tuple $D = (P, U, \cdot, 1, \mathcal{V})$, where $(P, \cdot, 1)$ is a dynamic frame, $U \subseteq P$ (called the set of *allowable paths*), while $\mathcal{V}$ is a valuation function assigning each proposition from the language a subset of P, that is, $\mathcal{V}(A) \subseteq P$, for any $A \in L$.

A set of dynamic models will be called a *dynamic (monoid) semantics*.

A dynamic model can be seen as a restriction of a dynamic frame to the set of allowable paths. In other words, it can be viewed as a *partial* monoid in which the concatenation operation $\cdot$ is defined only if it produces an element of U[1]. Note, however, that the valuation function $\mathcal{V}$ also becomes a partial function on this view.

As a preparation, we will extend the valuation function to sequences of propositions (i.e., processes) from L^* as follows:

$$\mathcal{V}(A_1 \dots A_n) \;=\; \mathcal{V}(A_1) \cdot \ldots \cdot \mathcal{V}(A_n).$$

We will extend $\mathcal{V}$ also to an empty sequence by stipulating $\mathcal{V}(\epsilon) = \{1\}$.

Every dynamic semantics $\mathcal{D}$ determines a dynamic consequence relation $\Vdash_{\mathcal{D}}$ defined as follows:

> $a \Vdash_{\mathcal{D}} b \;\equiv\;$ For any $D \in \mathcal{D}$ and any $u, x \in P$, if $ux \in U$ and $x \in \mathcal{V}(a)$, then $uxy \in U$, for some $y \in \mathcal{V}(b)$.

The following simple lemma verifies that the above relation satisfies all the postulates of a dynamic consequence relation.

Lemma 3.1. *$\Vdash_{\mathcal{D}}$ is a dynamic consequence relation.*

[1]This notion also coincides with the notion of a *partially associative operation*, used in [Barwise *et al.*, 1995] (Definition 4.5).

Proof. *Identity* is immediate.

Left Monotonicity. If $Aa \nVdash_{\mathcal{D}} b$, then, for some dynamic model in $\mathcal{D}$, there exist u and x such that $ux \in U$, $x \in \mathcal{V}(Aa)$, but there is no y such that $uxy \in U$ and $y \in \mathcal{V}(b)$. By definition, $x = zt$, where $z \in \mathcal{V}(A)$ and $t \in \mathcal{V}(a)$. Now if $u_1 = uz$, we have $u_1 t = ux \in U$, and $uxy = u_1 ty$, and consequently $a \nVdash_{\mathcal{D}} b$.

Cut. If $a \nVdash_{\mathcal{D}} bc$, then, for some dynamic model in $\mathcal{D}$, there must exist u and x such that $ux \in U$, $x \in \mathcal{V}(a)$, but there is no y such that $uxy \in U$ and $y \in \mathcal{V}(bc)$. But if $a \Vdash_{\mathcal{D}} b$, then $uxz \in U$, for some $z \in \mathcal{V}(b)$, and hence $xz \in \mathcal{V}(ab)$. On the other hand, if it were the case that $uxzy_0 \in U$, for some $y_0 \in \mathcal{V}(c)$, then we would have $zy_0 \in \mathcal{V}(bc)$, contrary to the supposition. Hence $ab \nVdash_{\mathcal{D}} c$. □

If a consequence relation $\Vdash$ coincides with $\Vdash_{\mathcal{D}}$, for some dynamic semantics $\mathcal{D}$, we will say that $\Vdash$ is *generated by* $\mathcal{D}$.

It turns out that, similarly to ordinary consequence relations (see, e.g., [Gabbay, 1976], or [Bochman, 2001]), any dynamic consequence relation can be generated by a canonical dynamic semantics constructed from the set of its theories.

Given a theory U of a consequence relation $\Vdash$, we can construct an associated dynamic model $D_U = (L^*, U, \cdot, \epsilon, \mathcal{V})$, where the dynamic frame is the set of all processes L^* with an operation of concatenation and ϵ as a unit, while $\mathcal{V}$ is a trivial function $\mathcal{V}(A) = \{A\}$. Let $\mathcal{T}_{\Vdash}$ be the set of such dynamic models, for all theories of $\Vdash$. Then we have

Theorem 3.2. *[Representation Theorem] If $\Vdash$ is a dynamic consequence relation, then $\Vdash = \Vdash_{\mathcal{T}_{\Vdash}}$.*

Proof. We have to show that $a \Vdash b$ holds iff for any theory U of $\Vdash$, and any sequence s, if $sa \in U$, then $sab \in U$. The direction from left to right follows directly from the definition of a theory. In the other direction, we take s to be an empty sequence, and choose $U = \mathrm{Th}(a)$. Then $ab \in \mathrm{Th}(a)$ and therefore $a \Vdash b$ by Lemma 2.1. This completes the proof. □

As a result, we obtain that dynamic consequence relations are complete with respect to the dynamic monoid semantics.

Corollary 3.3. *$\Vdash$ is a dynamic consequence relation if and only if it is generated by a dynamic monoid semantics.*

Actually, the above result can be strengthened by making use of the fact that the canonical dynamic semantics $\mathcal{T}_{\Vdash}$, described above, is a dynamic semantics of a very special kind. Its specific properties are reflected in the following definition.

Definition 3.2. A dynamic semantics $\mathcal{D}$ will be called *homogeneous* if

- all its dynamic models involve the same dynamic frame $(P, \cdot, 1)$ and the same valuation function $\mathcal{V}$.
- The function $\mathcal{V}$ is singular: $\mathcal{V}(A) \in P$, for any $A \in L$.

Thus, the variation of models in a homogeneous semantics reduces to the variation of admissible sets U of paths.

Now the Representation Theorem 3.2 immediately implies the following

Corollary 3.4. $\Vdash$ *is a dynamic consequence relation iff it is generated by some homogeneous dynamic semantics.*

The above result will imply a number of consequences important for our study.

Another consequence of the above representation theorem is that dynamic consequence relations are uniquely determined by their theories. Moreover, for more expressive languages the above Representation Theorem can serve as a basis of constructing full-fledged semantics. In this case theories of a consequence relation will play, eventually, the role of its canonical models. In particular, just as for ordinary inference relations, inclusion among consequence relations amounts to inverse inclusion for their sets of theories. Accordingly, imposing further requirements on our consequence relations will amount to imposing additional properties on the associated theories.

Remark. To conclude our discussion in this section, we should mention an important dimension of generality that is embodied in the above notion of a dynamic monoid semantics and, in particular, in the very idea of allowable paths. As the reader may have noticed, the set of allowable paths is not required to be closed with respect to concatenation or sub-paths, so concatenations of allowable paths are not always allowed, and not all parts of allowable paths constitute allowable paths by themselves. Now, an important way to see this consists in viewing a compound process as possibly proceeding 'in one leap' in which we cannot *temporally* (or sequentially) separate its constitutive parts. As an important extreme case, all the transitions in the process may be performed *concurrently*. For instance, a compound process of carrying a table by two people could still be described as a path $a \cdot b$ consisting of two concurrent actions of carrying, respectively, the left and right edge of the table. The causal and inference relations of such a process with other processes cannot be reduced directly to corresponding relations for its two components. Speaking more generally, the general dynamic monoid semantics allows us to accommodate the basic idea of [Pratt, 2003] that, unlike the sequential modeling, the evolution of time and information need not be synchronized.

4 Relational Semantics

In this section we are going to show that our dynamic consequence relations can also be given a (suitably generalized) relational semantics.

We begin with the following well-known notion of a relational semantics.

Definition 4.1. A *relational model* is a triple $M = (S, L, \mathcal{F})$, where S is a set of states, L is a set of propositions, and $\mathcal{F}$ is a function assigning each proposition from L a binary relation on S:

$$\mathcal{F} : L \to \mathcal{P}(S^2).$$

For any binary relation R, $dom(R)$ will denote its domain, and $range(R)$ - its range. The function $\mathcal{F}$ can be canonically extended to sequences of propositions:

$$\mathcal{F}(A_1 \dots A_n) \;=\; \mathcal{F}(A_1) \circ \dots \circ \mathcal{F}(A_n),$$

where $\circ$ denotes the composition of binary relations. We will extend $\mathcal{F}$ also to an empty sequence by stipulating

$$\mathcal{F}(\epsilon) = \{(s, s) \mid s \in S\}.$$

Now, in order to provide a semantics for our dynamic inference, the above notion of a relational model has to be generalized as follows.

Definition 4.2. A *generalized relational model* is a quadruple $M = (S, \checkmark, L, \mathcal{F})$, where $(S, L, \mathcal{F})$ is a relational model, and $\checkmark$ is a subset of states called *termination states*.

The notion of a termination state is actually known in the computational literature. In the latter, it denotes states in which a program, or its parts, successfully terminate, as opposed to other states that do not have computational meaning. In our setting, we will stretch this notion to a general distinction between *real* and *virtual* states, guided by an idea that *atomic transitions in a compound process are not necessary separated by real states.* As we already mentioned at the end of the preceding section, we allow for a possibility that a process b may proceed in one leap, in which we cannot observe, or single out, sequential parts and associated intermediate states. Of course, this idea conflicts somewhat with the very notion of a relational model, where transitions are *defined* as pairs of states. Still, the suggested way out consists in distinguishing between states that are actual (real) and all other, virtual states that may be viewed as purely theoretical constructs without 'physical' meaning. This distinction will also serve as a preparation for the state-transition calculus that will be introduced later.

Now we are ready to formulate the following definition of validity for dynamic inference rules in this semantics:

Definition 4.3. A dynamic rule $a \Vdash b$ will be said to be *valid* in a generalized relational model M if, for any states $s \in S$ and $t \in \checkmark$ such that $(s, t) \in \mathcal{F}(a)$, there exists a state $r \in \checkmark$ such that $(t, r) \in \mathcal{F}(b)$.

$\Vdash_M$ will denote the set of sequents that are valid in a relational model M.

Let $\overrightarrow{\checkmark}$ denote pairs of states that end in $\checkmark$, that is, $\overrightarrow{\checkmark} = \{(s, t) \mid t \in \checkmark\}$. Then the above definition can be compactly written as follows:

$$a \Vdash_M b \ \equiv \ range(\mathcal{F}(a)) \cap \checkmark \subseteq dom(\mathcal{F}(b) \cap \overrightarrow{\checkmark}).$$

To begin with, it is easy to verify the following

Lemma 4.1. *If M is a generalized relational model, then $\Vdash_M$ is a dynamic consequence relation.*

In order to show that this relational semantics is also adequate for dynamic consequence relations, it is sufficient to demonstrate that any dynamic monoid semantics can be transformed into an equivalent relational model.

For the purposes of the construction that follows, we will use an expression $x \in U \in \mathcal{D}$ as a shorthand for $x \in U$, for an admissible set U of some dynamic model from $\mathcal{D}$. Now, given such an admissible set U, let $U{\downarrow}$ denote the set of prefixes of the elements of U, that is,

$$U{\downarrow} = \{x \mid x \preceq y, \text{ for some } y \in U \in \mathcal{D}\}.$$

Then the set of states of the relational model corresponding to a monoid semantics $\mathcal{D}$ will be a disjoint union of all $U{\downarrow}$, for every admissible set U. As usual, in order to achieve disjointness, each path $x \in U{\downarrow}$ will be indexed with the corresponding set U to become a new object x_U. As a result, we obtain the following construction of a generalized relational model $M_{\mathcal{D}}$:

- $S = \{x_U \mid x \preceq y, \text{ for some } y \in U \in \mathcal{D}\}$;
- $\checkmark = \{u_U \mid u \in U \in \mathcal{D}\}$;
- For any $A \in L$, $\mathcal{F}(A) = \{(x_U, (xy)_U) \mid y \in \mathcal{V}(A) \ \& \ (xy)_U \in S\}$.

The following theorem shows that the resulting relational model validates the same rules as the source dynamic semantics.

Theorem 4.2. *If $M_{\mathcal{D}}$ is the relational model corresponding to the dynamic semantics $\mathcal{D}$, then $\Vdash_{\mathcal{D}} = \Vdash_{M_{\mathcal{D}}}$.*

Proof. Note that, for any process a, $(s,t) \in \mathcal{F}(a)$ if and only if $(s,t) = (x_U, y_U)$, for some x, y such that $y = xz$, for some $z \in \mathcal{F}(a)$, and $y \preceq y_0$, for some $y_0 \in U \in \mathcal{D}$.

Assume first that $a \Vdash_{\mathcal{D}} b$ and $(s,t) \in \mathcal{F}(a)$. Hence $(s,t) = (x_U, y_U)$, for some x, y such that $y = xz$, for some $z \in \mathcal{F}(a)$, and $y \preceq y_0$, for some $y_0 \in U \in \mathcal{D}$. Since U is an admissible set of a dynamic model in $\mathcal{D}$, we have $xzz_1 = yz_1 \in U$, for some $z_1 \in \mathcal{F}(b)$. Let $r = (yz_1)_U$. Then clearly $r \in \checkmark$ and $(t,r) \in \mathcal{F}(b)$. This gives the direction from left to right.

Now assume that $a \Vdash_{M_{\mathcal{D}}} b$ and $ux \in U \in \mathcal{D}$, for some $x \in \mathcal{V}(a)$. Then let us put $s = u_U$ and $t = (ux)_U$. Clearly $t \in \checkmark$ and $(s,t) \in \mathcal{F}(a)$. Since $a \Vdash_{M_{\mathcal{D}}} b$, there exists $r \in \checkmark$ such that $(t,r) \in \mathcal{F}(b)$. By the definition of $\mathcal{F}$, this can happen only if $r = (uxy)_U$, for some $y \in \mathcal{V}(b)$, and therefore $uxy \in U$, which shows that $a \Vdash_{\mathcal{D}} b$ holds. This completes the proof. $\square$

The above result immediately implies that dynamic consequence relations are also complete for the above notion of validity in generalized relational models.

Corollary 4.3. $\Vdash$ *is a dynamic consequence relation iff it coincides with* $\Vdash_M$*, for some generalized relational model* M*.*

As a matter of fact, the above completeness result can be strengthened by exploiting the fact that the dynamic semantics can be safely restricted to homogeneous one (see Corollary 3.4).

Recall that a relational model is *deterministic* if $\mathcal{F}(A)$ is a partial function, that is, for any $A \in P$, if $(s,t) \in \mathcal{F}(A)$ and $(s,r) \in \mathcal{F}(A)$, then $t = r$. Now, it is easy to verify that the relational model corresponding to the canonical dynamic semantics $\mathcal{T}_{\Vdash}$ by the above construction is actually deterministic. Consequently, we immediately obtain that, at this stage, imposing determinism does not produce new valid rules of dynamic inference.

Corollary 4.4. $\Vdash$ *is a dynamic consequence relation if and only if it is generated by a generalized deterministic relational model.*

This result will no longer hold, however, when we will impose some additional conditions on dynamic consequence relations.

5 Sequentiality

A dynamic consequence relation will be called *sequential* if it satisfies

Right Anti-Monotonicity If $a \Vdash bB$, then $a \Vdash b$.

Remark. It should be noted that the above postulate, when compared with Left Monotonicity, makes vivid another essential difference between our consequence relations and common sequent calculi, namely the fact that both antecedents and consequents of our rules are interpreted conjunctively, unlike the usual, disjunctive understanding of succedents in sequents.

The postulates of sequential consequence relations almost coincide with the axiomatization of the calculus $\mathbf{G}_\mu$ presented in [Kanazawa, 1994] with the only exception that the latter does not have the Identity postulate (but appropriately restricts the discussion to non-empty sequences). As has been shown by Kanazawa, such consequence relations can be given a relational semantics of the kind used in dynamic predicate logic of [Groenendijk and Stokhof, 1991]. In what follows we will reproduce this result as an 'unfolding' of the corresponding representation theorem.

To begin with, note that sequentiality can be alternatively characterized by strengthening the Cut rule to the following 'contextual' Cut:

C-Cut If $a \Vdash bd$ and $ab \Vdash c$, then $a \Vdash bc$.

Indeed, the original Cut rule is a special case of C-Cut (when $d = \epsilon$). Moreover, the latter implies Right Anti-Monotonicity: Identity implies $ab \Vdash \epsilon$ by Left Monotonicity, and hence if $a \Vdash bc$ holds, then $a \Vdash b$ follows by C-Cut. In the other direction, given Right Anti-Monotonicity, C-Cut is derivable from Cut, since $a \Vdash bd$ implies $a \Vdash b$.

Sequentiality corresponds to the following restriction of the dynamic monoid semantics:

Definition 5.1. A dynamic semantics $\mathcal{D}$ will be called *sequential* if, for any model $D \in \mathcal{D}$ with an admissible set U, if $u, v \in U$ and $u \preceq v$, then $w \in U$, for any w such that $u \preceq w \preceq v$.

Thus, in a sequential dynamic model the set of admissible paths is interval-closed with respect to the prefix relation $\preceq$. The next lemma shows that such a semantics generates a sequential consequence relation.

Lemma 5.1. *If $\mathcal{D}$ is a sequential dynamic semantics, then $\Vdash_{\mathcal{D}}$ is a sequential consequence relation.*

Proof. We need only to check Right Anti-Monotonicity. If $a \nVdash_{\mathcal{D}} b$, then there exist a dynamic model and $s \in P$ such that $sx \in U$, for $x \in \mathcal{V}(a)$, and there is no y such that $sxy \in U$ and $y \in \mathcal{V}(b)$. Assume that $uxz \in U$, for some $z \in \mathcal{V}(bB)$. Then $z = z_1 z_2$, where $z_1 \in \mathcal{V}(b)$ and $z_2 \in \mathcal{V}(B)$, and hence $sxz_1 \in U$ by sequentiality, which contradicts the supposition. Therefore $a \nVdash_{\mathcal{D}} bB$. □

A theory of a consequence relation will be called *sequential* if it is interval-closed with respect to the prefix relation on processes. As before, the following technical lemma plays the main role in the subsequent representation theorem.

Lemma 5.2. *If $\Vdash$ is a sequential consequence relation, then* $\mathrm{Th}(a)$ *is a sequential theory.*

Proof. Given Right Anti-Monotonicity, Lemma 2.1 implies that if $abB \in \mathrm{Th}(a)$, then $ab \in \mathrm{Th}(a)$. It can be easily seen that this property ensures sequentiality of $\mathrm{Th}(a)$. □

As a result, we can extend the main Representation Theorem 3.2 above to sequential inference and immediately obtain the following completeness result:

Corollary 5.3. *A dynamic consequence relation* $\Vdash$ *is sequential if and only if it is generated by a sequential dynamic semantics.*

As before, we can use Corollary 3.4 to conclude that the sequential dynamic semantics can be safely restricted to a homogeneous one.

Corollary 5.4. *A dynamic consequence relation* $\Vdash$ *is sequential if and only if it is generated by a homogeneous sequential dynamic semantics.*

This strengthening of the general representation result will be used in the next section.

5.1 Relational Dynamic Inference

It turns out that sequential consequence relations can be given a simpler relational semantics. More precisely, sequential dynamic inference amounts to validity in plain relational models, that is, models in which all states are termination ones.

Definition 5.2. A dynamic rule $a \Vdash b$ will be said to be *valid* in a plain relational model M if

$$range(\mathcal{F}(a)) \subseteq dom(\mathcal{F}(b)).$$

The above notion of validity corresponds to what has been called Update-to-Domain Consequence in [van Benthem, 1996], and it describes the inference relation adopted in dynamic predicate logic of [Groenendijk and Stokhof, 1991]. Unfolding the definition, it says that $a \Vdash b$ holds iff, for any $(s, t) \in \mathcal{F}(a)$ there exists $r \in S$ such that $(t, r) \in \mathcal{F}(b)$.

To begin with, we have the following

Lemma 5.5. *If* M *is a relational model, then* $\Vdash_M$ *is a sequential consequence relation.*

The proof amounts to a simple verification of Right Anti-Monotonicity. In order to show that the relational semantics is also adequate for sequential consequence relations, we will make use of Corollary 3.4 and show that any *homogeneous* sequential dynamic semantics can be transformed into an equivalent relational model. Note, however, that the construction below is essentially different from the translation of the general dynamic semantics, described in the preceding section. Most importantly, the construction below creates an *indeterministic model* even from homogeneous dynamic semantics.

Given a homogeneous sequential dynamic semantics $\mathcal{D}$ with a common frame $(P, \cdot, 1)$, we will construct a relational model $M_{\mathcal{D}}$ as follows:

- The set of states S is the set of all paths P together with all labeled paths from the the disjoint union of $\mathcal{D}$:

$$S \ = \ P \bigcup \{u_U \mid u \in U \in \mathcal{D}\};$$

- For any $A \in L$, $\mathcal{F}(A)$ is a set of all pairs (s,t) from S such that, for some paths $x, y \in P$, $y = x \cdot \mathcal{V}(A)$, and one of the following cases holds:

  1. $s = x$ and $t = y$;
  2. $s = x$ and $t = y_U$, if $y \in U \in \mathcal{D}$;
  3. $s = x_U$ and $t = y_U$, if $x, y \in U \in \mathcal{D}$.

The following theorem shows that the constructed relational model validates the same rules as the source sequential semantics.

Theorem 5.6. *If $M_{\mathcal{D}}$ is a (plain) relational model corresponding to the homogeneous sequential semantics $\mathcal{D}$, then $\Vdash_{\mathcal{D}} = \Vdash_{M_{\mathcal{D}}}$.*

Proof. For any $s \in S$, $\hat{s}$ will denote the underlying path, that is, $\hat{u} = u$ and $\hat{u_U} = u$. Also, to simplify the notation, for a process a, we will use $[a]$ to denote $\mathcal{V}(a)$. Note, in particular, that if $(s,t) \in \mathcal{F}(a)$, then always $\hat{t} = \hat{s}[a]$.

Assume first that $a \Vdash_{\mathcal{D}} b$ and $(s,t) \in \mathcal{F}(a)$. We have to consider two cases. If $t \in P$, then let $r = t[b]$. Clearly, $r \in S$ and $(t,r) \in \mathcal{F}(b)$, as required. So assume now that $t = x_U$, for some $x \in U \in \mathcal{D}$. In this case we have $x = \hat{s}[a]$, and therefore $x[b] = \hat{s}[ab] \in U$ (since $a \Vdash_{\mathcal{D}} b$). Now we put $r = (x[b])_U$. Suppose that $b = B_1 \ldots B_m$. Then we define $t_i = (t[B_1 \ldots B_i])_U$, for every $1 \leq i < m$ (note that sequentiality secures that $\hat{t}_i \in U$). It is easy to see that $t\mathcal{F}(B_1)t_1 \ldots t_{m-1}\mathcal{F}(B_m)r$, which implies $(t,r) \in \mathcal{F}(b)$. This gives the direction from left to right.

Now assume that $a \Vdash_{M_{\mathcal{D}}} b$ and $s[a] \in U$. Let us put $t = (s[a])_U$. If $a = A_1 \ldots A_n$, we define $s_i = s[A_1 \ldots A_i]$ for any $1 \leq i < n$, and then we have $s\mathcal{F}(A_1)s_1 \dots s_{n-1}\mathcal{F}(A_n)t$, which implies $(s,t) \in \mathcal{F}(a)$. Since $a \Vdash_{M_\mathcal{D}} b$, there exists $r \in S$ such that $(t,r) \in \mathcal{F}(b)$. By the definition of $\mathcal{F}$, this can happen only if $r = (s[a][b])_U$, and therefore $s[a][b] \in U$. This completes the proof. $\square$

As an immediate consequence of Corollary 5.4, we conclude with the following completeness result.

Corollary 5.7. *A dynamic consequence relation is sequential if and only if it coincides with a dynamic inference generated by some plain relational model.*

As we noted in the course of the above construction, the relational models obtained by transforming sequential dynamic models are not deterministic, in general. As we are going to see in the next section, this liberty is essential, because restriction of arbitrary relational models to deterministic ones will make valid an additional postulate of dynamic inference.

6 Deterministic Inference

A dynamic consequence relation will be called *deterministic* if it is sequential and satisfies the following postulate:

Cumulativity If $a \Vdash Bc$, then $aB \Vdash c$.

It should be clear that a repeated application of Cumulativity generates the following structural rule for processes:

$$\frac{a \Vdash bc}{ab \Vdash c}$$

Now, in deterministic consequence relations, any rule $a \Vdash bc$ is reducible to a pair of simpler rules:

Lemma 6.1. *If $\Vdash$ is a deterministic consequence relation, then $a \Vdash bc$ if and only if $a \Vdash b$ and $ab \Vdash c$.*

Proof. The direction from left to right follows from Cumulativity and Right Anti-Monotonicity, while the opposite direction follows directly by Cut. $\square$

As a consequence, any rule can now be reduced to a set of rules $a \Vdash A$ involving only singular conclusions (including ϵ). Moreover, this property can be viewed in some sense as a characteristic property of deterministic dynamic relations. In order to show this, let us introduce the following notions. Let us say that a sequent $a \Vdash b$ is *singular*, if b is either an atom B, or an empty sequence ϵ. Then a singular consequence relation, defined below, can be viewed as a restriction of a general dynamic consequence relation to singular sequents.

Definition 6.1. A *singular dynamic consequence relation* is a set of singular sequents that satisfies Left Monotonicity and Identity.

It can be easily verified that a set of singular sequents belonging to an arbitrary dynamic consequence relation forms a singular consequence relation. Moreover, it can be shown that the above two postulates exhaust the structural properties of such an inference relation[2]. So a singular consequence relation is actually a very simple inference relation.

Now, given a singular consequence relation $\Vdash$, we can inductively extend it to arbitrary sequents $a \Vdash b$ as follows: if $a \Vdash b$ is already defined, then we stipulate that $a \Vdash bA$ holds if and only if both $a \Vdash b$ and $ab \Vdash A$ hold. Let $\Vdash^m$ denote the resulting consequence relation. Then the next result shows that any deterministic consequence relation can be viewed as a definitional extension of a singular consequence relation.

Theorem 6.2. $\Vdash$ *is a deterministic consequence relation if and only if* $\Vdash = \Vdash_0^m$*, for some singular consequence relation* $\Vdash_0$*.*

Proof. For the direction from right to left, it is sufficient to show that if $\Vdash_0$ is a singular consequence relation, then $\Vdash_0^m$ is a deterministic consequence relation. Identity is immediate, while both Right Anti-Monotonicity and Cumulativity follow directly from the inductive construction. For the two remaining postulates of deterministic inference, we will show that $\Vdash_0^m$ is closed with respect to each by induction on the number of propositions in the consequents.

Left Monotonicity. Assume that $a \Vdash_0^m bB$. If the consequent is singular (that is, $b = \epsilon$), then $Aa \Vdash_0^m bB$ holds due to the fact that $\Vdash_0$ satisfies Left Monotonicity. Otherwise by construction of $\Vdash_0^m$, we have $a \Vdash_0^m b$ and

[2]A similar fact (though without ϵ) has been mentioned in [van Benthem, 1996].

$ab \Vdash_0^m B$. By the inductive assumption, we have $Aa \Vdash_0^m b$, while $Aab \Vdash_0^m B$ holds by Left Monotonicity for $\Vdash_0$. Therefore $Aa \Vdash_0^m bB$ by the construction.

Cut. Note first that $a \Vdash_0^m b$ and $ab \Vdash_0^m C$ imply $a \Vdash_0^m bC$ by the inductive construction itself. Assume now that $a \Vdash_0^m b$ and $ab \Vdash_0^m cC$. Then $ab \Vdash_0^m c$ and $abc \Vdash_0^m C$ by the inductive construction. The inductive assumption says that $a \Vdash_0^m b$ and $ab \Vdash_0^m c$ jointly imply $a \Vdash_0^m bc$ by Cut. Taken together with $abc \Vdash_0^m C$, this gives us $a \Vdash_0^m bcC$ by the inductive construction.

For the direction from left to right, assume that $\Vdash$ is deterministic, and let $\Vdash_0$ be the set of singular sequents from $\Vdash$. Clearly, $\Vdash_0^m$ is a singular consequence relation. Also, $\Vdash_0^m$ is included in $\Vdash$ (since $\Vdash$ is closed with respect to Cumulativity). Moreover, due to Lemma 6.1, any sequent from $\Vdash$ can be obtained, ultimately, from the singular sequents of $\Vdash_0$ by applying the inductive construction. Thus, $\Vdash$ coincides with $\Vdash_0^m$. This completes the proof. □

Now let us turn to a semantic description. A dynamic monoid semantics of deterministic consequence relations can be defined as follows.

Definition 6.2. A dynamic model $D = (P, U, \cdot, 1, \mathcal{V})$ will be called *deterministic*, if the valuation function $\mathcal{V}$ is singular (that is, $\mathcal{V}(A) \in P$, for any $A \in L$), and the admissible set U is closed with respect to the prefix relation: if $ab \in U$, then $a \in U$. A dynamic monoid semantics will be called *deterministic*, if all its dynamic models are deterministic.

The next result verifies that any deterministic dynamic semantics generates a deterministic consequence relation.

Lemma 6.3. *If $\mathcal{D}$ is a deterministic dynamic semantics, then $\Vdash_{\mathcal{D}}$ is a deterministic consequence relation.*

Proof. Note first that, due to singularity of $\mathcal{V}$, we have $\mathcal{V}(a) \in P$, for any $a \in L^*$. As before, to simplify the notation, we will write $[a]$ instead of $\mathcal{V}(a)$.

We need only to check Cumulativity. If $aB \nVdash_{\mathcal{D}} c$, then there exists a dynamic model with an admissible set U such that $s[aB] \in U$, but $s[aB][c] \notin U$. But then $s[a] \in U$, since $s[a]$ is a prefix of $s[aB]$. Moreover, $s[aB][c] = s[a][Bc]$ by associativity of concatenation, and therefore $s[a][Bc] \notin U$. Thus, $a \nVdash_{\mathcal{D}} Bc$. This completes the proof. □

A theory of a dynamic consequence relation will be called *deterministic* if it is closed with respect to the prefix relation. Clearly, the canonical dynamic model corresponding to a deterministic theory will be a deterministic model. Moreover, we will show that the corresponding canonical dynamic semantics is fully adequate for deterministic consequence relations.

As before, the set of deterministic theories is closed with respect to intersections, so any set of processes is included in a unique least deterministic theory. As a technical preparation for the next representation theorem, the lemma below gives a direct description of the least deterministic theory containing a given process.

Lemma 6.4. *$D(a)$ is a least deterministic theory of a deterministic consequence relation $\Vdash$ containing a process a iff*

$$D(a) = \{bc \mid b \preceq a \ \&\ b \Vdash c\}.$$

Proof. It is easy to check that any deterministic theory containing a should contain also $D(a)$. So we check only that $D(a)$ is a deterministic theory.

Assume that $d \Vdash e$ and $sd \in D(a)$, that is, $sd = bc$, for some b, c such that $b \preceq a$ and $b \Vdash c$. Then $d \Vdash e$ implies $sd \Vdash e$ by Left Monotonicity, and hence $bc \Vdash e$. Given $b \Vdash c$, this implies $b \Vdash ce$ by Cut, and therefore $bce \in D(a)$. But $bce = sde$, and hence $sde \in D(a)$. Thus, $D(a)$ is a theory of $\Vdash$.

Assume that $x \preceq y$ and $y \in D(a)$, Then $y = bc$, for some b, c such that $b \preceq a$ and $b \Vdash c$. Now if $x \preceq b$, then $x \preceq a$, and therefore $x \in D(a)$ (since $x \Vdash \epsilon$). Otherwise $b \prec x$, in which case $x = bd$, for some $d \preceq c$. In this case $b \Vdash c$ implies $b \Vdash d$ by Right Anti-Monotonicity, and hence again $x = bd \in D(a)$. Thus, $D(a)$ is a deterministic theory. This completes the proof. □

Now we are ready to prove the following

Theorem 6.5 (Representation Theorem). *If $\mathcal{T}^d_{\Vdash}$ is a set of deterministic theories of a dynamic consequence relation $\Vdash$, then $\Vdash$ is deterministic if and only if $\Vdash = \Vdash_{\mathcal{T}^d_{\Vdash}}$.*

Proof. The direction from right to left follows from Lemma 6.3. So let $\Vdash$ be a deterministic consequence relation. If $a \Vdash b$, then $a \Vdash_{\mathcal{T}^d_{\Vdash}} b$ directly from the definition of a theory. Assume then that $a \Vdash_{\mathcal{T}^d_{\Vdash}} b$, that is, for any s and any $U \in \mathcal{T}^d_{\Vdash}$, if $sa \in U$, then $sab \in U$. We take $s = \epsilon$ and U to be the least deterministic theory containing a. By Lemma 6.4, we obtain $ab \in D(a)$, and therefore $ab = cd$, for some c, d such that $c \preceq a$ and $c \Vdash d$. Let $a = ce$, for some e. Then $d = eb$, and hence $c \Vdash eb$. By Cumulativity we conclude $ce \Vdash b$, that is $a \Vdash b$. This completes the proof. □

As a result, we can conclude with

Corollary 6.6. *A dynamic consequence relation* $\Vdash$ *is deterministic if and only if it is generated by a deterministic dynamic semantics.*

Note now that any deterministic dynamic semantics is already sequential, and hence corresponds to some relational model. Moreover, we are going to show that deterministic semantics correspond precisely to deterministic relational models in which the accessibility relations are partial functions.

To begin with, we have the following

Lemma 6.7. *If* M *is a deterministic relational model, then* $\Vdash_M$ *is a deterministic consequence relation.*

The proof amounts to a straightforward verification of Cumulativity.

As before, in order to show that deterministic consequence relations are complete with respect to deterministic relational models, we will show that any deterministic dynamic semantics can be transformed into an equivalent deterministic relational model. The construction is actually a simplification of the transformation described earlier for general relational models.

Given a deterministic dynamic semantics $\mathcal{D}$, we will construct the corresponding deterministic relational model $M_{\mathcal{D}}$ as follows:

- $S = \{a_U \mid a \in U \in \mathcal{D}\}$;
- For any $A \in L$, $\mathcal{F}(A) = \{(a_U, b_U) \mid b \in U \in \mathcal{D} \ \&\ b = aA\}$.

It is easy to see that the above relational model is deterministic. The following theorem shows that it validates the same rules as the source dynamic semantics.

Theorem 6.8. *If* $M_{\mathcal{D}}$ *is the relational model corresponding to the deterministic dynamic semantics* $\mathcal{D}$*, then* $\Vdash_{\mathcal{D}} = \Vdash_{M_{\mathcal{D}}}$.

Proof. In our present (simpler) case we have that $(s,t) \in \mathcal{F}(a)$ if and only if $(s,t) = (x_U, y_U)$, for some $x, y \in U \in \mathcal{D}$ such that $y = xa$.

Assume first that and $(s,t) \in \mathcal{F}(a)$. Hence $(s,t) = (x_U, y_U)$, for some $x, y \in U \in \mathcal{D}$ such that $y = xa$. Since U is an admissible set, we have $xab = yb \in U$. Let $r = (yb)_U$. Then clearly $(t,r) \in \mathcal{F}(b)$. This gives the direction from left to right.

Now assume that $a \Vdash_{M_{\mathcal{D}}} b$ and $ma \in U$, for some $m \in L^*$ and $U \in \mathcal{D}$. Since U is prefix-closed, we also have $m \in U$. So let us put $s = m_U$ and $t = (ma)_U$. Clearly $(s,t) \in \mathcal{F}(a)$. Since $a \Vdash_{M_{\mathcal{D}}} b$, there exists $r \in S$ such that $(t,r) \in \mathcal{F}(b)$. By the definition of $\mathcal{F}$, this can happen only if $r = (mab)_U$, and therefore $mab \in U$, which shows that $a \Vdash_{\mathcal{D}} b$ holds. $\square$

Combined with Corollary 6.6, the above correspondence immediately gives us

Corollary 6.9. *A dynamic consequence relation is deterministic if and only if it coincides with a dynamic inference generated by some deterministic relational model.*

This result concludes our study of the basic variety of dynamic inference.

7 State-Transition Calculus

At this stage of our development, we will distinguish between two kinds of events, proper *transitions* (or actions) and *states*, or conditions. So, formally, our language L of events will become a union $Tr \cup St$ of the set $Tr = \{p, q, \dots\}$ of transitions and a disjoint set $St = \{A, B, C, \dots\}$ of state propositions (also called conditions or tests). Still, as before, finite sequences of events (from L^*) will be called *processes*, and we will use the letters $a, b, c, \dots$ to denote such processes. Moreover, such processes will be assumed to satisfy all the postulates of dynamic inference, stated earlier, despite the fact that processes correspond now to mixed sequences of transitions and states. The actual difference with our basic, uniform setting will amount to a stipulation that states form a special kind of events that possesses some additional properties.

The distinction between states and transitions is actually much more subtle than it appears. Intuitively, states correspond to temporally extended occurrences with a relatively stable temporal behavior. As was argued in [Pratt, 2003], though physical states are never completely stationary, we usually assume that during a given state time passes while information remains fixed. Accordingly, states on our understanding include relatively static properties and facts (though extended in time), as well as cases of inertial change.

In a commonsense structuring of the dynamic universe, states are usually seen as boundaries, or limits of transitions and change, in the same sense as points play the role of boundaries of linear geometric segments. And just as in geometry, this role is two-fold. First, boundaries *separate*, and thereby single out, parts of the continuum. But on the other hand, they are *links*, or junctions, that combine these parts into larger pieces. This natural view of the continuum and its boundaries is not too familiar in our modern times, though it can be traced back at least to Aristotle. We owe this situation to the predominant alternative representation, namely the point-based model of the continuum where these two functions are less transparent (though definable).

Now we will turn to a syntactic description. The above understanding of states and transitions is embodied in the following extension of our basic dynamic calculus.

Definition 7.1. A dynamic consequence relation in a two-sorted language will be called an *ST-consequence relation* if it satisfies the following additional rules for static propositions:

S-Identity $A \Vdash A$.

S-Monotonicity If $ab \Vdash c$, then $aAb \Vdash c$.

S-Cut If $a \Vdash A$ and $aAb \Vdash c$, then $ab \Vdash c$.

S-Expansion If $a \Vdash bc$ and $ab \Vdash A$, then $a \Vdash bAc$.

S-Reduction If $a \Vdash bAc$, then $a \Vdash bc$.

As a first example of the acquired possibilities of derivation in the above system, the following lemma shows formally that ST-consequence relations satisfy the permutation and contraction/dilution rules for state propositions.

Lemma 7.1. *The following rules hold for ST-consequence relations:*

Left Contraction *If $aAAb \Vdash c$, then $aAb \Vdash c$;*

Left Permutation *If $aABb \Vdash c$, then $aBAb \Vdash c$.*

Right Dilution *If $a \Vdash bAc$, then $a \Vdash bAAc$;*

Right Permutation *If $a \Vdash bABc$, then $a \Vdash bBAc$.*

Proof. (1) $aA \Vdash A$ by S-Identity and Left Monotonicity, so if $aAAb \Vdash c$, then $aAb \Vdash c$ is derivable by S-Cut.

(2) $B \Vdash B$ by S-Identity, so $BA \Vdash B$ by S-Monotonicity and hence $aBA \Vdash B$ by Left Monotonicity. Now if $aABb \Vdash c$ holds, then $aBABb \Vdash c$ by S-Monotonicity. Together with $aBA \Vdash B$, this gives precisely $aBAb \Vdash c$ by S-cut.

(3) $abA \Vdash A$ holds by S-Identity and Left Monotonicity, so $a \Vdash bAc$ implies $a \Vdash bAAc$ by S-Expansion.

(4) $abA \Vdash A$ holds by S-Identity and Left Monotonicity, so $abAB \Vdash A$ by S-monotonicity. Combined with $a \Vdash bABc$, it gives us $a \Vdash bABAc$ by S-Expansion, and hence $a \Vdash bBAc$ by S-Reduction. This completes the proof. □

Note that, due to S-Monotonicity, we actually have that $aAAb \Vdash c$ holds if and only if $aAb \Vdash c$. Similarly, S-Reduction implies that $a \Vdash bAc$ holds iff $a \Vdash bAAc$. Consequently, ST-consequence relations admit all the usual structural rules for static propositions.

Finally, we will mention the following property that will be used in the sequel.

Lemma 7.2. *If s is a sequence of static propositions, then, for any a, $a \Vdash s$ holds iff $a \Vdash A$, for any A that occurs in s.*

Proof. Let $s = A_1 \dots A_n$. Now, the implication from left to right follows by S-Reduction. The other direction can be proved by induction on the length of s. Suppose that the claim holds for $n-1$. Then $a \Vdash A_1 \dots A_{n-1}$. But $a \Vdash A_n$ implies $aA_1 \dots A_{n-1} \Vdash A_n$ by S-Monotonicity, so $a \Vdash A_1 \dots A_{n-1}A_n$ follows by S-Expansion. □

7.1 Trace Calculus

In order to achieve 'freedom of expression' in working with state propositions, we will extend our dynamic calculus to a language that explicitly contains arbitrary *sets* of static propositions. Our main objective in this section will consist in showing that the resulting calculus in the extended language will still satisfy all the postulates of an ST-consequence relation.

Definition 7.2.
- A *trace* is a finite sequence $\alpha = X_0X_1 \dots X_n$, where each X_i is either a proposition, or a set of static propositions.

- An *instantiation* of a trace $\alpha = X_0X_1 \ldots X_n$ is a process $x_1x_2 \ldots x_n$ such that $x_i = X_i$, if X_i is a proposition, or else x_i is a finite sequence of (static) propositions taken from the set X_i.

Traces can be viewed as folded representations of sets of processes having similar behavior. Clearly, ordinary processes can viewed as a special case of traces. Moreover, by the above definition, any process constitutes also a unique instantiation of itself.

As could be expected, due to contraction and permutation properties that hold for static propositions, any two instantiations of a trace that contain the same static propositions will have the same logical properties. It should be noted, however, that traces may include arbitrary, in particular infinite, sets of static propositions. For such traces, no instantiation can provide a complete information about the source trace. Note also that, even for fully finite traces, the number of instantiations is in general infinite due to possible repetitions and varying orders of appearance of static propositions in instantiation sequences.

Definition 7.3. A trace $\alpha = X_0X_1 \ldots X_n$ will be said to *subsume* a trace $\beta = Y_0Y_1 \ldots Y_n$ (notation $\alpha \rhd \beta$) if either $X_i = Y_i$ (if both are propositions), or else $Y_i \subseteq X_i$ for any $i \leq n$.

Using these notions, we will extend the sequents of our calculus to rules that relate general traces by stipulating that such generalized rules will hold when certain ordinary sequents hold that relate corresponding instantiations.

Definition 7.4 (Trace Inference). If α and β are traces, then $\alpha \Vdash \beta$ will be taken to hold if, for any instantiation b of β, there is an instantiation a of α such that $a \Vdash b$ holds.

The first fact that we will note about the above generalization is that the resulting inference relation on traces is monotonic with respect to subsumption.

Lemma 7.3. *For any traces α, β and γ,*

1. *If $\gamma \rhd \alpha$ and $\alpha \Vdash \beta$, then $\gamma \Vdash \beta$.*
2. *If $\alpha \Vdash \beta$ and $\beta \rhd \gamma$, then $\alpha \Vdash \gamma$.*

Proof. Immediate from the fact that if γ subsumes α, then any instantiation of α is an instantiation of γ. □

It turns out to be convenient to view the above defined consequence relation on traces as a plain ST-consequence relation in the extended language obtained by adding arbitrary *sets* of static propositions as new static propositions. We will refer to this extended language as a *trace language*. The following result shows that this view has a solid formal support.

Theorem 7.4. *If* $\Vdash^t$ *is a trace inference corresponding to an ST-consequence relation* $\Vdash$*, then* $\Vdash^t$ *is an ST-consequence relation in the associated trace language, and it is a conservative extension of* $\Vdash$*.*

Proof. The conservativity of $\Vdash^t$ with respect to the source language follows immediately from the fact that the only instantiation of an ordinary process is the process itself.

In the proof below, we will repeatedly use the following construction on instantiations. Suppose that α is a trace, and let $(Z_1, \ldots, Z_k)$ be the list of all sets of static propositions appearing in α (in that order). Then an instantiation of α amounts to replacement of every Z_i in this list with a corresponding sequence of states. Suppose now that a_1 and a_2 are two instantiations of α that correspond, respectively, to two such lists $(s_1, \ldots, s_k)$ and $(t_1, \ldots, t_k)$. Then it should be clear that the 'combined' list $(s_1t_1, \ldots, s_kt_k)$ also corresponds to some instantiation of α. The latter instantiation subsumes, in a sense, both a_1 and a_2. We will denote this instantiation by $a_1{+}a_2$.

Now we will check the validity of all the postulates of an ST-consequence relation for the trace inference. Identity is immediate.

Left Monotonicity. Assume that $\alpha \Vdash \beta$, and let X be either a proposition or a set of static propositions. For any instantiation b of β there exists an instantiation a of α such that $a \Vdash b$. Now if x is an instantiation of X, then $xa \Vdash b$ by Left Monotonicity, and hence $X\alpha \Vdash b$ holds.

Cut. Assume that $\alpha \Vdash \beta$ and $\alpha\beta \Vdash \gamma$, and let b, c be some instantiations of, respectively, β and γ. By the second rule, there are instantiations a_1 of α and b_1 of β such that $a_1b_1 \Vdash c$. Then $a_1(b_1{+}b) \Vdash c$ by S-Monotonicity. Now, the first rule implies that there exists an instantiation a_2 of α such that $a_2 \Vdash b_1{+}b$. Let $a = a_1{+}a_2$. Then $a \Vdash b_1{+}b$ and $a(b_1{+}b) \Vdash c$, therefore $a \Vdash (b{+}b_1)c$ by Cut, and consequently $a \Vdash bc$ by S-Reduction. Thus, $\alpha \Vdash \beta\gamma$ holds.

S-Identity. We need only to check the rule $X \Vdash X$ when X is a set of static propositions. By definition, this amounts to verifying that if s is a finite sequence of static propositions, then $s \Vdash s$ holds. Note first that, if A

occurs in s, then $s \Vdash A$ follows from $A \Vdash A$ by S-Monotonicity. Hence $s \Vdash s$ holds by Lemma 7.2.

S-Monotonicity. Assume that $\alpha\beta \Vdash \gamma$ holds, c is an instantiation of γ, and let X be a single static proposition or a set of static propositions. Then there exist instantiations a of α and b of β such that $ab \Vdash c$. Now if x is an instantiation of X, then $axb \Vdash c$ holds by S-Monotonicity. Therefore $\alpha X\beta \Vdash \gamma$ holds.

S-Cut. Assume that $\alpha \Vdash X$ and $\alpha X\beta \Vdash \gamma$, where X is either a static proposition, or a set of static propositions. Let c be an instantiation of γ. Then $a_1xb \Vdash c$, for some instantiations a_1, x, b of α, X and β, respectively. In addition, $\alpha \Vdash X$ implies that there exists an instantiation a_2 of α such that $a_2 \Vdash x$. Let $a = a_1{+}a_2$. Then $a \Vdash x$ and $axb \Vdash c$ by S-Monotonicity. Now, if x is a single proposition, we obtain $ab \Vdash c$ by S-Cut. The same result also follows in the case when x is a set of static propositions, this time by using Lemma 7.2 and multiple applications of S-Cut. Hence $\alpha\beta \Vdash \gamma$ holds.

S-Expansion. Assume that $\alpha \Vdash \beta\gamma$ and $\alpha\beta \Vdash X$, and let bxc be some instantiation of $\beta X\gamma$. Then there exist instantiations a_1 and b_1 of α and β, respectively, such that $a_1b_1 \Vdash x$. Let $b_0 = b{+}b_1$. Then there exists an instantiation a_2 of α such that $a_2 \Vdash b_0c$. If $a = a_1{+}a_2$, then $a \Vdash b_0c$ and $ab_0 \Vdash x$ by S-Monotonicity, so $a \Vdash b_0xc$ by S-Expansion (multiple applications of S-Expansion in case x is a sequence of static propositions). But then $a \Vdash bxc$ follows by S-Reduction, so $\alpha \Vdash \beta X\gamma$ holds.

S-Reduction. Assume that $\alpha \Vdash \beta X\gamma$, and let bc be an instantiation of $\beta\gamma$. If x is an instantiation of X, then bxc is an instantiation of $\beta X\gamma$, so there exists an instantiation a of α such that $a \Vdash bxc$. But then $a \Vdash bc$ by S-Reduction, so $\alpha \Vdash \beta\gamma$ holds. This completes the proof. □

The above result shows, in effect, that ST-consequence relations provide a necessary and sufficient basis for inference relations on arbitrary traces. This fact will be extensively used in the next section, where we will prove completeness of the corresponding sequential ST-calculus with respect to the relational semantics.

To end this section, we will mention an important property that we will use in what follows, namely that adjacent sets of static propositions can be safely united.

Lemma 7.5. *For any sets X, Y of static propositions,*

1. $\alpha XY\beta \Vdash \gamma$ *if and only if* $\alpha(X \cup Y)\beta \Vdash \gamma$.

2. *$\alpha \Vdash \beta XY\gamma$ if and only if $\alpha \Vdash \beta(X \cup Y)\gamma$.*

Proof. Both the above equivalences can be easily obtained from the following two facts:

1. Any instantiation of $\alpha XY\beta$ is an instantiation of $\alpha(X \cup Y)\beta$;

2. Any instantiation of $\alpha(X\cup Y)\beta$ can be transformed into an instantiation of $\alpha XY\beta$ by using permutation, namely be putting all propositions from $X\backslash Y$ before all propositions from Y.

□

Using the above property, any trace can be transformed into a trace in with all adjacent static sets are united. Such traces will be called regular, and they will be used in constructing the canonical model of our calculus.

8 Relational Semantics

In what follows, we will restrict our attention to sequential consequence relations, since they have a simple relational semantics. It should be mentioned, however, that one of the main objectives of our general study consists in exploring the logical properties and representation capabilities of general dynamic inference, not only sequential one. We suspect, however, that it would require a more drastic generalization of the relational semantics than what has been suggested in the present paper.

Relational models for sequential ST-consequence relations can be defined as a straightforward extension of general relational models.

Definition 8.1. A *relational ST-model* is a relational model $M = (S, L, \mathcal{F})$, in which the valuation function $\mathcal{F}$ satisfies the following constraint: for any state proposition $A \in St$, $\mathcal{F}(A) \subseteq Id$ where $Id = \{(s, s) \mid s \in S\}$.

We will keep intact the original definition of dynamic inference in plain relational models. Then the next lemma can be obtained by a straightforward verification of the rules for static propositions.

Lemma 8.1. *If $\mathcal{M}$ is a relational ST-model, then $\Vdash_{\mathcal{M}}$ is a sequential ST-consequence relation.*

In order to show that ST-consequence relations are also complete for this relational semantics, we are going to construct a canonical relational model of a consequence relation. As a matter of fact, it constitutes a generalization of the corresponding construction of a canonical relational model for sequential dynamic inference, described in [Kanazawa, 1994].

8.1 Canonical Model

As a preparation, we will single out the following special kind of traces:

Definition 8.2. A *regular trace* is a finite alternating sequence

$$\alpha = X_0 p_1 X_1 \dots p_n X_n,$$

where each p_i is a transition, and every X_i is a set of static propositions.

Regular traces constitute, in a sense, a canonical structural representation of ST-processes, since they explicitly embody the fact that sub-sequences of static propositions behave essentially as sets. Thus, any process a corresponds to a unique regular trace (that will be denoted by $\hat{a}$) obtained by grouping all maximal segments of static propositions into sets. Moreover, by using contraction and permutation properties of static propositions, it is easy to verify that $\hat{a}$ is inferentially equivalent to a, that is, for any trace β, $\beta \Vdash \hat{a}$ if and only if $\beta \Vdash a$, and $\hat{a} \Vdash \beta$ if and only if $a \Vdash \beta$.

For a regular trace $\alpha = X_0 p_1 X_1 \dots p_n X_n$, $l(\alpha)$ will denote X_n. Also, we will use α^i to denote a regular prefix $X_0 p_1 X_1 \dots p_i X_i$ of α. Note, in particular, that α^0 is X_0.

A *concatenation* $\alpha \cdot \beta$ of regular traces $\alpha = X_0 p_1 X_1 \dots p_n X_n$ and $\beta = Y_0 q_1 Y_1 \dots q_m Y_m$ will be defined only if $\beta^0 \subseteq l(\alpha)$ (that is, $Y_0 \subseteq X_n$), in which case it will be taken to be a (regular) trace

$$\alpha \cdot \beta = X_0 p_1 X_1 \dots p_n X_n q_1 Y_1 \dots q_m Y_m.$$

Finally, we will say that a regular trace $\alpha = X_0 p_1 X_1 \dots p_n X_n$ is *S-closed*, if $A \in X_i$, for any static proposition A such that $\alpha^i \Vdash A$.

It is easy to verify that, for any regular trace $\alpha = X_0 p_1 X_1 \dots p_n X_n$, there exists a least S-closed trace that subsumes α; it is obtained by replacing every X_i in α with the set $X'_i = \{A \in St \mid \alpha^i \Vdash A\}$. In what follows, we will use $cl(\alpha)$ to denote this trace.

In what follows, we will use the following fact about S-closures.

Lemma 8.2. *For any regular trace α and any trace β, $cl(\alpha) \Vdash \beta$ if and only if $\alpha \Vdash \beta$.*

Proof. The implication from right to left follows by subsumption. Assume that $cl(\alpha) \Vdash \beta$, and let $\alpha = X_0 p_1 X_1 \dots p_n X_n$. Then $cl(\alpha) = X'_0 p_1 X_1 \dots p_n X'_n$, where $X'_i = \{A \in St \mid \alpha^i \Vdash A\}$. Using Lemma 7.5, we can safely transform $cl(\alpha)$ into a (non-regular) trace $\alpha' = X_0 Z_0 p_1 X_1 Z_1 \dots p_n X_n Z_n$, where $Z_i = X'_i \backslash X_i$, for any $i \leq n$. Clearly, $Z_i \subseteq X'_i$, and hence $\alpha^i \Vdash Z_i$ by Lemma 7.2. Now let $\gamma^i = X_0 Z_0 p_1 X_1 Z_1 \dots p_i X_i$ and $\delta_i = p_{i+1} \dots p_n X_n Z_n$. Then $\alpha' = \gamma^i Z_i \delta_i$ and hence $\gamma^i Z_i \delta_i \Vdash \beta$. In addition, $\alpha^i \Vdash Z_i$ implies $\gamma^i \Vdash Z_i$ by subsumption, and therefore $\gamma^i \delta_i \Vdash \beta$ by S-Cut. Notice that $\gamma^i \delta_i$ is obtained from α' by removing Z_i. So, repeating this procedure for every Z_j in α', we finally obtain $\alpha \Vdash \beta$. □

In addition to regular traces, we will follow [Kanazawa, 1994] and use *trace-pairs*: sequences of the form $\alpha|\beta$, where α, β are regular traces such that $\alpha \Vdash \beta$. A trace-pair $\alpha|\beta$ will be said to be *S-closed* if $A \in l(\beta^i)$, for any static proposition A such that $\alpha\beta^i \Vdash A$. Moreover, for an arbitrary trace-pair $\alpha|\beta$, we will denote by $cl(\alpha|\beta)$ the trace pair obtained from $\alpha|\beta$ by replacing each $l(\beta^i)$ with

$$\{A \mid A \in St \ \& \ \alpha\beta^i \Vdash A\}.$$

As can be seen from the above definition, $cl(\alpha|\beta)$ has the form $\alpha|\beta'$, where β' subsumes β. Moreover, using S-Expansion, it is easy to verify that if $\alpha \Vdash \beta$, then $\alpha \Vdash \beta'$. So, $cl(\alpha|\beta)$ is indeed a trace-pair. It can be viewed as a least S-closed trace-pair that contains $\alpha|\beta$.

Now, given a dynamic consequence relation $\Vdash$, we will construct a canonical transition model $M_{\Vdash} = (S_{\Vdash}, P, \mathcal{F})$ as follows:

- The set of states $S_{\Vdash}$ is a set of all regular S-closed traces and trace-pairs.
- For any transition $p \in P$, $(s, t) \in \mathcal{F}(p)$ if one of the following cases holds:
  - s and t are traces, and $t = spX$, for some $X \subseteq St$;
  - s is a trace, and $t = cl(\beta|\epsilon)$, for some trace β such that $(s, \beta) \in \mathcal{F}(p)$;
  - $s = \alpha|\beta$ and $t = \alpha|\beta pX$, for some $X \subseteq St$.

- For any state proposition A,

$$\mathcal{F}(A) = \{(\alpha,\alpha) \mid A \in l(\alpha)\} \bigcup \{(\alpha|\beta, \alpha|\beta) \mid A \in l(\beta)\}.$$

It is easy to see that the above construction determines a relational ST-model.

We will prove first the following technical result.

Lemma 8.3. *1. If $s, t \in S_{\Vdash}$ are traces, then $(s,t) \in \mathcal{F}(a)$ iff $t = s \cdot \gamma$, for some trace $\gamma \rhd \hat{a}$.*

2. If $\alpha|\beta \in S_{\Vdash}$, then, for any state s, $(\alpha|\beta, s) \in \mathcal{F}(a)$ iff $s = \alpha|\beta \cdot \gamma$, for some trace $\gamma \rhd \hat{a}$.

3. If s is a trace, then $(s, cl(\beta|\epsilon)) \in \mathcal{F}(a)$ iff $(s,\beta) \in \mathcal{F}(a)$.

Proof. By a simultaneous induction on the length of a. The corresponding inductive bases hold by definition of $\mathcal{F}$. Z below denotes an arbitrary proposition.

1. If $s, t \in S_{\Vdash}$ are traces, then $(s,t) \in \mathcal{F}(aZ)$ iff there is a trace $r \in S_{\Vdash}$ such that $(s,r) \in \mathcal{F}(a)$ and $(r,t) \in \mathcal{F}(Z)$. By the inductive assumption, $(s,r) \in \mathcal{F}(a)$ iff there is a trace $\gamma_1 \rhd \hat{a}$ such that $r = s \cdot \gamma_1$. Also, $(r,t) \in \mathcal{F}(Z)$ iff there is a trace $\gamma_2 \rhd \hat{Z}$ such that $t = r \cdot \gamma_2$. Now let $\gamma = \gamma_1 \cdot \gamma_2$. Clearly, $\gamma \rhd \hat{aZ}$ and $t = s \cdot \gamma$. Moreover, suppose that the last two conditions hold. If Z is a transition, then γ has the form $\gamma_1 pX$, where $\gamma_1 \rhd \hat{a}$. Hence we can define a state r as $t \cdot \gamma_1$, and then we will have $(s,r) \in \mathcal{F}(a)$ and $(r,t) \in \mathcal{F}(Z)$, as required. In case Z is a static proposition, the same outcome is achieved by putting r to be equal to t.

2. $(\alpha|\beta, t) \in \mathcal{F}(aZ)$ iff there exists a state s such that $(\alpha|\beta, s) \in \mathcal{F}(a)$ and $(s,t) \in \mathcal{F}(Z)$. By the inductive assumption, $(\alpha|\beta, s) \in \mathcal{F}(a)$ iff $s = \alpha|\beta \cdot \gamma$, for some trace $\gamma \rhd \hat{a}$. Hence t is also a trace-pair by the definition of $\mathcal{F}$. Now, if Z is a state proposition, then $(s,t) \in \mathcal{F}(Z)$ holds iff $s = t$ and $A \in l(\gamma)$. But then γ subsumes $(\hat{aZ})$, and $t = \alpha|\beta \cdot \gamma$. Conversely, if $t = \alpha|\beta \cdot \gamma$, for some $\gamma \rhd (\hat{aZ})$, then $\gamma \rhd \hat{a})$, and we have both $(\alpha|\beta, t) \in \mathcal{F}(a)$ (by the inductive assumption) and $(t,t) \in \mathcal{F}(Z)$, as required.

In case Z is a transition, say p, we have $t = \alpha|(\beta \cdot \gamma) \cdot pX$, for some set $X \subseteq St$. Then we let $\gamma_0 = \gamma \cdot pX$, which gives us $t = \alpha|\beta \cdot \gamma_0$ and $\gamma_0 \rhd (\hat{ap})$. Conversely, if the last two conditions hold, then we can define s as $\alpha|\beta \cdot \gamma_1$, where γ_1 is a maximal regular prefix of γ_0 that does not include p. Clearly, $\gamma_1 \rhd \hat{a}$, and hence we have both $(\alpha|\beta, s) \in \mathcal{F}(a)$ and $(s,t) \in \mathcal{F}(p)$, as required.

3. If s is a trace, then $(s, cl(\beta|\epsilon)) \in \mathcal{F}(aZ)$ iff there exists a state t such that $(s,t) \in \mathcal{F}(a)$ and $(t, cl(\beta|\epsilon)) \in \mathcal{F}(Z)$. If Z is a state proposition, then $t = cl(\beta|\epsilon)$ and $\beta \Vdash Z$. Consequently $(s,\beta) \in \mathcal{F}(a)$ by the inductive assumption. But $(s,\beta) \in \mathcal{F}(a)$ and $\beta \Vdash Z$ are jointly equivalent to $(s,\beta) \in \mathcal{F}(aZ)$, and we are done. In case Z is a transition, $(t, cl(\beta|\epsilon)) \in \mathcal{F}(Z)$ can hold only if t is a trace, in which case $(t, cl(\beta|\epsilon)) \in \mathcal{F}(Z)$ iff $(t,\beta) \in \mathcal{F}(Z)$. Now, since $(s,t) \in \mathcal{F}(a)$, we have $(s,\beta) \in \mathcal{F}(aZ)$. Conversely, if $(s,\beta) \in \mathcal{F}(aZ)$, there is a trace t such that $(s,t) \in \mathcal{F}(a)$ and $(t,\beta) \in \mathcal{F}(Z)$. But then $(t, cl(\beta|\epsilon)) \in \mathcal{F}(Z)$, and hence $(s, cl(\beta|\epsilon)) \in \mathcal{F}(aZ)$. This completes the proof. $\square$

Now we are ready to prove the main result of this section:

Theorem 8.4. *If $M_{\Vdash}$ is a canonical model of $\Vdash$, then $\Vdash = \Vdash_{M_{\Vdash}}$.*

Proof. From left to right, assume that $a \Vdash b$ and $(s,t) \in \mathcal{F}(a)$. If $b = \epsilon$, then $(t,t) \in \mathcal{F}(b)$, and we are done. If $b \neq \epsilon$, then let $b = s_0 p_1 s_1 \ldots p_n s_n$, where each s_i is a sequence of static propositions, and each p_i is a transition.

Suppose first that t is a trace. Then s is also a trace, and we have $s \cdot \hat{a} \lhd t$ by the preceding lemma. In this case we define $r = cl(t \cdot \hat{b})$ Note that $a \Vdash b$ implies $a \Vdash s_0$, so $\hat{a} \Vdash s_0$ and consequently $t \Vdash s_0$ by Left Monotonicity and subsumption. Hence $\hat{b}^0 \subseteq l(t)$ (since t is S-closed), and r is well-defined. Also, since t is already S-closed, $r = t \cdot \gamma$, for some $\gamma \rhd \hat{b}$. Consequently, $(t,r) \in \mathcal{F}(b)$ by the preceding lemma, as required.

Suppose now that t is a trace-pair $\alpha|\beta$. Then we will define a state r as $cl(\alpha|\beta \cdot \hat{b})$. By the preceding lemma, we have to show only that $r \in S_{\Vdash}$, that is, r is well-defined, and $\alpha \Vdash \beta\hat{b}$. Two cases should be considered here. Assume first that s is a trace-pair $\alpha|\beta_0$. Then $t = cl(\alpha|\beta_0 \cdot \hat{a})$ by the preceding lemma, hence β subsumes $\beta_0 \cdot \hat{a}$, and therefore $a \Vdash b$ implies $\beta \Vdash \hat{b}$. Since t is S-closed, the latter implies that $\beta \cdot \hat{b}$ is defined, so r is well-defined. Moreover, we also have $\alpha \Vdash \beta$ (since t is a state), and consequently $\alpha \Vdash \beta\hat{b}$ follows by Cut. Otherwise s is a trace, in which case $(s,t) \in \mathcal{F}(a)$ can hold only if there is a state $s_0 = cl(\alpha|\epsilon)$ such that $(s,s_0) \in \mathcal{F}(a_1)$, $(s_0,t) \in \mathcal{F}(a_2)$, and $a = a_1 a_2$. Now, $(s,s_0) \in \mathcal{F}(a_1)$ implies $\alpha \rhd s \cdot \hat{a}_1$. In addition, $(s_0,t) \in \mathcal{F}(a_2)$ implies $\beta \rhd \hat{a}_2$. Consequently, $a \Vdash b$ implies $\alpha\beta \Vdash b$ by subsumption and Left Monotonicity. Then $\alpha\beta \Vdash s_0$ by Right Anti-Monotonicity, and consequently $\beta \cdot \hat{b}$ is defined due to the fact that t is S-closed. Moreover, $\alpha\beta \Vdash b$ obviously implies $\alpha\beta \Vdash \hat{b}$, while the latter and $\alpha \Vdash \beta$ give us $\alpha \Vdash \beta\hat{b}$ by Cut. Thus, we have $\alpha \Vdash \beta\hat{b}$ in both cases, and consequently $(t,r) \in \mathcal{F}(b)$ by the preceding lemma. Therefore $a \Vdash_{M_{\Vdash}} b$ holds. This completes the direction from left to right.

From right to left, suppose that $a \Vdash_{M_{\Vdash}} b$ holds, and let $\tilde{a}$ denote $cl(\hat{a})$. Then we put $s = \tilde{a}^0$ and $t = cl(\tilde{a}|\epsilon)$. Since $\tilde{a}$ is S-closed, we have $t = \tilde{a}|l(\tilde{a})$. Clearly, $\tilde{a} = \tilde{a}^0 \cdot \tilde{a}$ and $\tilde{a} \rhd \hat{a}$. Hence $(s,t) \in \mathcal{F}(a)$ by the preceding lemma, and therefore there must exist a state r such that $(t,r) \in \mathcal{F}(b)$. Using the previous lemma once more, the latter implies $r = \tilde{a}|l(\tilde{a}) \cdot \gamma$, for some $\gamma \rhd \hat{b}$, which is well-defined only if $\tilde{a} \Vdash l(\tilde{a})\gamma$. But then $\tilde{a} \Vdash \gamma$ by S-Reduction (since $l(\tilde{a})$ is a set of static propositions), and therefore $\tilde{a} \Vdash \hat{b}$ by subsumption. Now $\hat{a} \Vdash \hat{b}$ follows by Lemma 8.2, which reduces to $a \Vdash b$ by permutation and contraction properties for static propositions. This completes the proof. □

The above theorem is sufficient for establishing strong completeness of our ST-calculus with respect to the relational semantics:

Corollary 8.5 (Completeness). $\Vdash$ *is a sequential ST-consequence relation if and only* $\Vdash = \Vdash_M$, *for some relational ST-model* M.

9 Conclusions and prospects

As we mentioned in the introduction, the present study constitutes only a first stage in a prospective development of a comprehensive formalism for causal dynamic reasoning. Hopefully, we have succeeded in showing, however, that the suggested substructural formalism has significant representation capabilities and depth that justify further development. Still, a lot of things need to be added to our basic structural calculus in order to fulfil its purpose.

The first required extension of the formalism presented above consists in augmenting the underlying language with appropriate logical connectives. This will raise our abstract structural formalism to a full-fledged logic. Actually, a basic logic of this kind has been suggested in [Bochman and Gabbay, 2010]. It is based on (dynamic versions of) ordinary propositional connectives of conjunction, disjunction and negation, but is shown to have the same expressive capabilities as propositional dynamic logic (PDL).

One of the major objectives of our 'research program' consists in providing a solid logical basis for a general theory of action and change in AI that would be able to cope with temporally extended actions, concurrency and triggered (natural) events. At least, this will require an extension of our calculus to a *concurrent* dynamic logic. In addition, the formalism should be able to cope with traditional AI problems of inertia, frame, ramification and qualification. This would require a corresponding extension to a *non-monotonic* formalism that implements the idea of inertia in a systematic way. These are the subjects of the ongoing work.

Finally, it should be mentioned that our calculus apparently provides an enhanced starting point for studying the process of computation itself. In particular, our main notion of causal inference, being restricted to (voluntary) computing actions, coincides, in effect, with the relation of *enabling* used in event theories of computation (see, e.g, [van Glabbeek and Plotkin, 2009] for the present stage of development). In addition, the main problems in describing concurrent actions and change in AI are actually species of the general task of describing *distributed* processes and distributed computation. All this suggests that a further development of our formalism along these lines could be beneficial for both these fields. Moreover, it will hopefully provide a better understanding of the basic principles of logical reasoning in dynamic domains.

Acknowledgment. We are very grateful to the unanimous referee for the incisive comments and questions concerning this study, even though we still do not have complete answers to some of them.

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