Proof Theory, Semantics and Algebra for Normative Systems

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Abstract. This paper reports correspondence results between input/output logic and the theory of joining-systems. The results have the form: every norm (a, x) is logically derivable from a set of norms G if and only if it is in the space of norms algebraically generated by G.

1 Introduction

In their influential book *Normative Systems* [1], Alchourroon and Bulygin conceive a normative system as a deductive mechanism, like black boxes which produce normative statement as output, given we feed them descriptive statements as input. To this tradition belongs as well the input/output logic (I/O logic) of David Makinson and Leon van der Torre in [6–8] and the theory of joining-systems(TJS) proposed by Lars Lindahl and Jan Odelsad in e.g. [4, 5].

Although sharing the same ancestor, I/O logic and TJS have evolved quite separately, and lool very different. I/O logic has a proof theory and a well defined semantics of propositional logic. TJS uses algebra as a tool for modeling normative systems. In this paper, I will show that, nevertheless the two accounts essentially give the same results, and can be seen as "two sides of one and the same coin." The results will illustrate that proof theory, semantics and algebra, as three tools to model normative systems, have their own advantage and disadvantage. Proof theory is neat and easy to be tracked by computers, but hard to be manipulated by human beings. Semantics is intuitive but hard to give us the holistic view of normative systems. Algebra, although it's neither as neat as proof theory nor as intuitive as semantics, it can give rise to holistic results to normative systems in the sense that we can build isomorphisms between the algebraic representation of normative systems. It is their different features that motivate us to use all of them.

The layout of this paper is as follows. In section 2 I will give a brief introduction to I/O logic and TJS. Then, in section 3 I will present two correspondence results between I/O logic and TJS. Section 4 is the section for application of the algebraic tools, illustrating those holistic views we gain by the algebraic representation of normative systems. Section 5 will present some issues for future research.

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2 Background

2.1 Input/Output Logic

In a series of papers [7–9], Makinson and van der Torre developed a class of deontic logic called input/output logic. A gentle and comprehensive introduction can be found in [10] and [14]. In general, the matured version of I/O logic is the constrained version from [8]. For simplicity's sake, the latter one will be put aside, and only two unconstrained I/O logics will be considered in this paper. I start by describing them.

Let $\mathbb{P} = \{p_0, p_1, \ldots\}$ be a countable set of propositional letters and L be the propositional logic built upon $\mathbb{P}$. Throughout this paper L will be the only logic language we talk about. Let G be a set of ordered pairs of formulas of L. A pair $(a, x) \in G$, call it a norm, is read as "given a, it ought to be x". G can be viewed as a function from 2^L to 2^L such that for a set A of formulas, $G(A) = \{x : (a, x) \in G \text{ for some } a \in A\}$.

Makison and van der Torre define the operations out_1 and out_2 as following:

- $out_1(G, A) = Cn(G(Cn(A)))$
- $out_2(G, A) = \bigcap\{Cn(G(V)) : A \subset V, V \text{ is complete}\}$

Here Cn is the classical consequence operator from propositional logic, and a complete set is a set of formulas that is either maxi-consistent or equal to L.

$out_1(G, A)$ and $out_2(G, A)$ are called *simple-minded output* and *basic output* respectively. In [7], simple-minded reusable output and basic reusable output are also defined. I leave them as a topic for future research.

out_1 and out_2 can be fiben a proof theoretic characterization. We say that an ordered pair of formulas is derivable from a set G iff (a, x) is in the least set that includes G, contains the pair (t, t), where t is a tautology, and is closed under a number of rules. The following are the rules we will use:

- SI (strengthening the input): from (a, x) to (b, x) whenever $b \vdash a$
- WO (weakening the output): from (a, x) to (a, y) whenever $x \vdash y$
- AND (conjunction of output): from (a, x),(a, y) to $(a, x \wedge y)$
- OR (disjunction of input): from (a, x),(b, x) to $(a \vee b, x)$

The derivation system based on the rules SI, WO and AND is called $deriv_1$. Adding OR to $deriv_1$ gives $deriv_2$. We use $(a, x) \in deriv_i(G)$, or equivalently $x \in deriv_i(G, a)$, to denote the norm (a, x) is derivable from G using rules of derivation system $deriv_i$. Moreover, for a set A of formulas, we use $(A, x) \in deriv_i(G)$, or equivalently $x \in deriv_i(G, A)$ to denote the fact that there exist $a_1 \ldots a_n \in A$ such that $(a_1 \wedge \ldots \wedge a_n, x) \in deriv_i(G)$. In [7], the following completeness theorems for out_1 and out_2 are given:

Theorem 1 ([7]). *Given an arbitrary normative system G and a set A of formulas,*

1. $x \in out_1(G, A)$ *iff* $x \in deriv_1(G, A)$
2. $x \in out_2(G, A)$ *iff* $x \in deriv_2(G, A)$

2.2 Theory of Joining-Systems

An algebraic framework for analyzing normative systems was introduced by Lars Lindahl and Jan Odelstad in their papers [3–5, 12, 13]. The most general form of the theory is called theory of joining-systems(TJS) in [5, 12]. A theory of joining-systems is a triple (B_1, B_2, J) where B_1, B_2 are two ordered algebraic structures and J a relation between B_1 and B_2 satisfying some special conditions. In Lindahl and Odelstad's work, the algebraic structure is usually a Boolean quasi-ordering. In this paper I will work with a Boolean algebra.

Definition 1 (Boolean algebra). *A structure* $\mathfrak{A} = (A, +, \cdot, -, 0, 1)$ *is a Boolean algebra iff it satisfies the following identities:*

(1) $x + y = y + x, \;\; x \cdot y = y \cdot x$
(2) $x + (y + z) = (x + y) + z, \;\; x \cdot (y \cdot z) = (x \cdot y) \cdot z$
(3) $x + 0 = x, \;\; x \cdot 1 = x$
(4) $x + (-x) = 1, \;\; x \cdot (-x) = 0$
(5) $x + (y \cdot z) = (x + y) \cdot (x + z), \;\; x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

We can order the elements of a Boolean algebra by defining $a \leq b$ if $a \cdot b = a$. Here $+$ can be considered as a disjunction, $\cdot$ as a conjunction and $\leq$ as a implication. With this order relation in hand, the *narrowness*($\preceq$) relation between two ordered pairs can be naturally defined as $(a, x) \preceq (b, y)$ iff $b \leq a$ and $x \leq y$. Based on ordered structures, Lindahl and Odelstad define joining-systems as follows:

Definition 2 (Joining-systems, Lindahl and Odelstad's version [12]). *A triple* $(\mathbb{A}, \mathbb{B}, S)$*, where* $\mathbb{A} = (A, \leq)$ *and* $\mathbb{B} = (B, \leq)$ *are ordered structures and* $S \subseteq A \times B$*, is called a joining-system if* S *satisfies the following conditions:*

1. *If* $(a, x) \in S$ *and* $(a, x) \preceq (b, y)$*, then* $(b, y) \in S$*.*
2. *For all* $X \subseteq B$*, if for all* $x \in X$*,*$(a, x) \in S$*, then* $(a, y) \in S$ *for all* $y \in glb(X)$*.*[1]
3. *For all* $X \subseteq A$*, if for all* $x \in X$*,* $(x, b) \in S$*, then* $(y, b) \in S$ *for all* $y \in lub(X)$*.*[2]

In this paper, the major mathematical tool is the joining-systems of Boolean algebra, which is a modified version of Lindahl and Odelstad's joining-systems in the following sense: we let $(1, 1) \in S$ and require the set X in item 2 and 3 above to be finite, of Lindahl and Odelstad's joining-systems.

Definition 3 (Joining-systems of Boolean algebras). *A joining-systems of Boolean algebras is a structure* $\mathbb{S} = (\mathfrak{A}, \mathfrak{B}, S)$ *such that* $\mathfrak{A}$*,*$\mathfrak{B}$ *are Boolean algebras and* $S \subseteq A \times B$ *meets the following conditions:*

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[1] Here *glb* is the abbreviation of greatest lower bound. Formally, $glb(X) = \{b : \forall x \in X, b \leq x$ and $\forall a$, if $\forall x \in X$, $a \leq x$, then $a \leq b\}$

[2] *lub* is the abbreviation of least up bound. Formally, $lub(X) = \{a : \forall x \in X, x \leq a$ and $\forall b$, if $\forall x \in X$, $x \leq b$, then $a \leq b\}$

1. $(1,1) \in S$
2. *If* $(a,x) \in S$ *and* $(a,x) \preceq (b,y)$, *then* $(b,y) \in S$.
3. *For all finite* $X \subseteq B$, *if for all* $x \in X$,$(a,x) \in S$, *then* $(a,y) \in S$ *for all* $y \in glb(X)$
4. *For all finite* $X \subseteq A$, *if for all* $x \in X$, $(x,b) \in S$, *then* $(y,b) \in S$ *for all* $y \in lub(X)$

Here we call S a joining-space as Lindahl and Odelstad did in [5]. We can equivalently replace condition (3) and (4) by the following respectively:

3′ If $(a,x) \in S$ and $(a,y) \in S$, then $(a, x \cdot y) \in S$
4′ If $(a,x) \in S$ and $(b,x) \in S$, then $(a+b, x) \in S$

Moreover, we can define joining-space using the standard algebraic terminology of ideal and filter:

Definition 4 (Ideal). *Let* $\mathfrak{A} = (A, +, \cdot, -, 0, 1)$ *be a Boolean algebra and* I *a subset of* A. *For* I *to be an ideal of* $\mathfrak{A}$, *it is necessary and sufficient that the following three conditions be satisfied:*

(1) $0 \in I$
(2) for all $x, y \in I$, $x + y \in I$
(3) for all $x \in I$ *and* $y \in A$, *if* $y \leq x$ *then* $y \in I$

Definition 5 (Filter). *Let* $\mathfrak{A} = (A, +, \cdot, -, 0, 1)$ *be a Boolean algebra and* F *a subset of* A. *For* F *to be a filter of* $\mathfrak{A}$, *it is necessary and sufficient that the following three conditions are satisfied:*

(1) $1 \in F$
(2) for all $x, y \in F$, $x \cdot y \in F$
(3) for all $x \in F$ *and* $y \in A$, *if* $x \leq y$ *then* $y \in F$

Let $F_{\uparrow}(X)$ be the filter generated by X and $I_{\downarrow}(X)$ be the ideal generated by X. Then $I_{\downarrow}(X)(F_{\uparrow}(X))$ is the smallest ideal(filter) containing X, and we have the following proposition:

Proposition 1. *Given a structure* $\mathbb{S} = (\mathfrak{A}, \mathfrak{B}, S)$, S *is a joining space in* $\mathbb{S}$ *if and only if it satisfies the following conditions:*

1. $(1,1) \in S$
2. *For every finite set* $X \subseteq A$, *if* $\forall x \in X$, $(x,b) \in S$, *then* $\forall y \in I_{\downarrow}(X)$, $(y,b) \in S$.
3. *For every finite set* $X \subseteq B$, *if* $\forall x \in X$, $(a,x) \in S$, *then* $\forall y \in F_{\uparrow}(X)$, $(a,y) \in S$.

Proof:
Assume S is a joining space in $\mathbb{S}$. Then trivially we have $(1,1) \in S$.
For the second condition, let X be an arbitrary finite subset of B. Without loss of generality, we can let $X = \{x_1, \ldots, x_n\}$. Suppose $\forall x \in X$, $(a,x) \in S$. Then by

applying clause $3'$ of Definition 3 finitely many times we have $(a, x_1 \cdot \ldots \cdot x_n) \in S$. Since for all $y \in F_{\uparrow}(X)$, $x_1 \cdot \ldots \cdot x_n \leq y$, therefore $(a, y) \in S$.

Similarly we can prove that the third condition is satisfied.

Now assume S satisfies the three conditions in this proposition. Then obviously $(1, 1) \in S$.

Assume $(a, x) \in S$ and $(a, x) \preceq (b, y)$, then $x \leq y$ and $y \in F_{\uparrow}(x)$, hence $(a, y) \in S$. Moreover we have $b \leq a$ and $b \in I_{\downarrow}(a)$, so we have $(b, y) \in S$.

Assume $(a, x) \in S$ and $(a, y) \in S$. Since $x \cdot y \in F_{\uparrow}(\{x, y\})$, we know $(a, x \cdot y) \in S$.

Similarly we can prove if $(a, x) \in S$ and $(b, x) \in S$, then $(a + b, x) \in S$. Therefore S is a joining space. $\dashv$

Up to now, we have clearly defined what a joining-system and joining space are. But does a joining space always exist? The answer is positive. As the following proposition shows, the largest and the smallest joining space always exists.

Proposition 2. *Given two boolean algebra* $\mathfrak{A}, \mathfrak{B}$*,*

1 *$A \times B$ is the largest joining space of $\mathfrak{A} \times \mathfrak{B}$.*
2 *If $\{S_i | i \in I\}$ is a collection of joining spaces of $\mathfrak{A} \times \mathfrak{B}$, then $S^* = \cap_{i \in I} S_i$ is a joining space of $\mathfrak{A} \times \mathfrak{B}$.*

Proof:

1 It is easy to check that $A \times B$ satisfies the definition of joining space and it is the largest one.
2 For every S_i, we have $(1, 1) \in S_i$, therefore $(1, 1) \in S^*$.
For every finite set $X \subseteq A$, if for every $x \in X$, $(x, b) \in S^*$, then $(x, b) \in S_i$ for every $i \in I$. Therefore $\forall y \in I_{\downarrow}(X)$, $(y, b) \in S_i$. So we must have $(y, b) \in S^*$.
Similarly we can prove the third statement of Proposition 1 is true. Therefore $S^* = \cap_{i \in I} S_i$ is a joining space of $\mathfrak{A} \times \mathfrak{B}$.

3 Correspondence between I/O Logic and TJS

3.1 Basic I/O Logic and TJS

In this section, I will prove that for a set of norms G, a norm (a, x) is entailed by G in basic I/O logic, if and only if it is in the joining space generated by G. To show this, we need to introduce a special Boolean algebra named Lindenbaum-Tarski algebra.

Let $\equiv$ be the provable equivalence relation on L, i.e. for every formula $\phi, \psi \in L$, $\phi \equiv \psi$ iff $\vdash_L \phi \leftrightarrow \psi$. Let $L/_{\equiv}$ be the equivalence classes that $\equiv$ induces on L. For any formula $\phi \in L$, let $[\phi]$ denote the equivalence class contains ϕ.

Definition 6 (Lindenbaum-Tarski algebra). *The Lindenbaum-Tarski algebra for a logic L is a structure $\mathfrak{L} = (L/_{\equiv}, +, \cdot, -, 0, 1)$ where $[\phi] + [\psi] = [\phi \vee \psi]$, $[\phi] \cdot [\psi] = [\phi \wedge \psi]$, $-[\phi] = [\neg\phi]$,$0 = [\bot]$ and $1 = [\top]$.*

For more details of Lindenbaum-Tarski algebra, readers can consult chapter 5 of [2]. It is not hard to check that every Lindenbaum-Tarski algebra is a Boolean algebra. Let G be a set of ordered pairs of formulas of L. Let $G^{\equiv} = \{([a],[x])|(a,x) \in G\}$. Let $\mathbb{S} = (\mathfrak{L}, \mathfrak{L}, S)$ be a joining-systems such that $G^{\equiv} \subseteq S$. By Proposition 2 we know such joining system always exist. Moreover, there must be a smallest joining space G^* such that $G^{\equiv} \subseteq G^*$ and for every joining space S that extends $G^{\equiv}$, $G^* \subseteq S$. Such a G^* is the joining space generated by $G^{\equiv}$, and it satisfies following property:

Proposition 3. *For every* $([a],[x]) \in G^*$*, at least one of the following holds:*

(1) $([a],[x])$ *is* $([1],[1])$
(2) for some $([b],[y]) \in G^*$*,* $([b],[y]) \preceq ([a],[x])$
(3) there exist $([a],[y]),([a],[z]) \in G^*$ *such that* $[x] = [y] \cdot [z]$
(4) there exist $([b],[x]),([c],[x]) \in G^*$ *such that* $[a] = [b] + [c]$

Proof: Suppose $([a],[x]) \in G^*$ and satisfies none of the above four clause, then we can prove $G' = G^* - \{([a],[x])\}$ is a joining system. This contradicts the fact that G^* is the smallest joining system.

With proposition 3 in hand, we can now prove one main correspondence result:

Theorem 2. *The following three propositions are equivalent:*

1 $(a,x) \in deriv_2(G)$
2 $([a],[x]) \in G^*$
3 $x \in out_2(G,a)$

Proof:
$1 \Rightarrow 2$: This can be proved simply by induction one the length of derivation.
$2 \Rightarrow 3$: Assume $([a],[x]) \in G^*$. By proposition 3 we need to deal with four cases.
(i) If $([a],[x])$ is $([1],[1])$, we need to prove $\top \in \cap\{Cn(G^*(V)) : a \in V, V$ is complete$\}$,which is obviously true.
(ii) If for some $([b],[y]) \in G^*$, $([b],[y]) \preceq ([a],[x])$. Then by induction hypotheses we know $y \in \cap\{Cn(G^*(V)) : b \in V, V$ is complete$\}$. Since $[a] \leq [b]$ and $[y] \leq [x]$ we know $x \in Cn(y)$. Hence $x \in \cap\{Cn(G^*(V)) : b \in V, V$ is complete$\}$. Moreover, every complete set V contains a must contain b, therefore $\cap\{Cn(G^*(V)) : b \in V, V$ is complete$\} \subseteq \cap\{Cn(G^*(V)) : a \in V, V$ is complete$\}$. Therefore $x \in \cap\{Cn(G^*(V)) : a \in V, V$ is complete$\}$, $x \in out_2(G,a)$.
(iii) If there exist $([a],[y]),([a],[z]) \in G^*$ such that $[x] = [y] \cdot [z]$. Then by induction hypotheses we know $y \in \cap\{Cn(G^*(V)) : a \in V, V$ is complete$\}$ and $z \in \cap\{Cn(G^*(V)) : a \in V, V$ is complete$\}$. Therefore $y \wedge z \in \cap\{Cn(G^*(V)) : a \in V, V$ is complete$\}$. That is, $y \wedge z \in out_2(G,a)$,$x \in out_2(G,a)$.
(iv) If there exist $([b],[x]),([c],[x]) \in G^*$ such that $[a] = [b] + [c]$. Then by induction hypotheses we know $x \in \cap\{Cn(G^*(V)) : b \in V, V$ is complete$\}$ and $x \in \cap\{Cn(G^*(V)) : c \in V, V$ is complete$\}$. For every complete set V such that $b \vee c \in V$, it must be that either $b \in V$ or $c \in V$. Therefore, for every complete set V that contains $b \vee c$, $x \in Cn(V)$, which means $x \in \cap\{Cn(G^*(V)) : b \vee c \in V, V$ is complete$\}$, i.e. $x \in out_2(G, b \vee c)$, $x \in out_2(G,a)$.
$3 \Rightarrow 1$: This is a special case of observation 2 of [7].

3.2 Simple-Minded I/O Logic and TJS

The previous section proved a correspondence result between basic I/O logic and TJS. In fact, we can prove a similar result between simple-minded I/O logic and a weaker version of TJS.

Definition 7 (Weak joining-systems). *A weak joining-systems of Boolean algebras is a structure* $\mathbb{S} = (\mathfrak{A}, \mathfrak{B}, S^-)$ *such that* $\mathfrak{A}$,$\mathfrak{B}$ *are boolean algebras and* $S^- \subseteq A \times B$ *satisfies the first three conditions of a joining space. Here we call* S^- *the weak joining space of* $\mathbb{S}$.

Similar to Proposition 2, we can prove the existence of the largest and the smallest weak joining space.

Proposition 4. *Given two Boolean algebra* $\mathfrak{A}, \mathfrak{B}$,

1. $A \times B$ *is the largest weak joining space of* $\mathfrak{A}$ *and* $\mathfrak{B}$.
2. *If* $\{S_i | i \in I\}$ *is a collection of weak joining spaces of* $\mathfrak{A}$ *and* $\mathfrak{B}$, *then* $S^* = \cap_{i \in I} S_i$ *is a weak joining space of* $\mathfrak{A}$ *and* $\mathfrak{B}$.

Let G be a set of ordered pairs of formulas of L and $\mathfrak{L}(\Phi)$ be the Lindenbaum-Tarski algebra of L. Let $G^{\equiv} = \{([a], [x]) | (a, x) \in G\}$ where $[a], [x]$ are the equivalence classes in $\mathfrak{L}(\Phi)$ respective contains a and x. By Proposition 4 we know that there exists a unique smallest weak joining-systems extends $G^{\equiv}$. If we denote it as G^+, then we have the following:

Proposition 5. *For every* $([a], [x]) \in G^+$, *at least one of the following holds:*

(1) $([a], [x])$ *is* $([1], [1])$
(2) for some $([b], [y]) \in G^+$, $([b], [y]) \preceq ([a], [x])$
(3) there exists $([a], [y]), ([a], [z]) \in G^+$ *such that* $[x] = [y] \cdot [z]$

Proof: Similar to the proof of proposition 2.

With Proposition 5 in hand, we can prove the following correspondence result:

Theorem 3. *The following three proposition is equivalent:*

1 $(a, x) \in deriv_1(G)$
2 $([a], [x]) \in G^+$
3 $x \in out_1(G, a)$

Proof:
$1 \Rightarrow 2$: This can be proved simply by induction one the length of derivation.
$2 \Rightarrow 3$: Assume $([a], [x]) \in G^+$.
(i) If $([a], [x])$ is $([1], [1])$, we need to prove $\top \in \cap\{Cn(G(T))\}$,which is obviously true.
(ii) If for some $([b], [y]) \in G^+$, $([b], [y]) \preceq ([a], [x])$. Then by induction hypothesis we know $y \in Cn(G(b))$. Since $[a] \leq [b]$ and $[y] \leq [x]$ we know $x \in Cn(y)$. Hence $x \in Cn(G(b))$.
(iii) If there exists $([a], [y]), ([a], [z]) \in G^+$ such that $[x] = [y] \cdot [z]$. Then by induction hypotheses we know $y \in Cn(G(a))\}$ and $z \in Cn(G(a))\}$. Therefore $y \wedge z \in Cn(G(a))$. That is, $y \wedge z \in out_1(G, a)$,$x \in out_1(G, a)$.
$3 \Rightarrow 1$: This is a special case of observation 1 of [7].

4 Application

In this section, we discuss some of the insights obtained from the algebraic approach to normative systems.

4.1 The Core of a Normative System

In section 2.2 the narrowness relation $\preceq$ is defined as $(a, x) \preceq (b, y)$ iff $b \leq a$ and $x \leq y$. We can further define the strict narrowness relation $\prec$ as $(a, x) \prec (b, y)$ iff $(a, x) \preceq (b, y)$ and not $(b, y) \preceq (a, x)$. A norm (a, x) is minimal in a joining-systems, or normative system, $\mathbb{S}$ iff there is no $(b, y) \in \mathbb{S}$ such that $(b, y) \prec (a, x)$. In [11], such a minimal norm is called a connection from A to B.

As noticed by [11], the set of all minimal elements of a joining-systems can be viewed as the core of the system. If the joining space is finite, then the whole joining-systems is uniquely determined by its minimal norms. If we know the core of the system, we can logically deduce the whole system. Let for a joining-systems $\mathbb{S}$, let $core(\mathbb{S}) = \{(a, x) \in S | (a, x) \text{ is minimal in } \mathbb{S}\}$ denote the set of all its minimal norms. The following are formal statements about the properties of the core of finite joining-systems.

Observation 1. *For all joining-systems $\mathbb{S} = (A, B, S)$. If S is finite, then $core(\mathbb{S}) \neq \emptyset$*

Proof: The proof is trivial. Due to the fact that S is finite, there is no infinite descending chain on $\prec$.

Observation 2. *For all joining-systems $\mathbb{S} = (A, B, S)$, if S is finite, then for any $(a, x) \in S$, there exists $(b, y) \in core(\mathbb{S})$ such that $(b, y) \preceq (a, x)$.*

Proof: Let (a, x) be an arbitrary norm in S. If $(a, x) \in core(\mathbb{S})$, then $(a, x) \preceq (a, x)$ and we are done. If $(a, x) \notin core(\mathbb{S})$, then (a, x) is not a minimal norm. Hence there exist some (b, y) such that $(b, y) \prec (a, x)$. If $(b, y) \in core(\mathbb{S})$ then we are done. If not, then there exist some (c, z) such that $(c, z) \prec (b, y)$. Since S is finite, this procedure will stop at some point. Then by transitivity of $\preceq$, there must exist some $(a', x') \in core(\mathbb{S})$ such that $(a, x) \preceq (a', x')$.

Observation 3. *For any joining-systems $\mathbb{S} = (A, B, S)$ and $\mathbb{S}' = (A, B, S')$,if both S and S' are finite, then $core(\mathbb{S}) = core(\mathbb{S}')$ iff $S = S'$.*

Proof: The right to left direction is trivial. For the left to right direction. Assume $core(\mathbb{S}) = core(\mathbb{S}')$. For any $(a, x) \in S$, by Observation 2 there exist $(b, y) \in core(\mathbb{S})$ such that $(b, y \prec (a, x))$. By assumption we know $(b, y) \in core(\mathbb{S}')$. Then by the definition of joining space we know $(a, x) \in S'$. Therefore $S \subseteq S'$. Similarly we can prove $S \supseteq S'$.

4.2 Harshness of Normative Systems

Suppose there are two norms (a, x) and $(a, x \wedge y)$, it is reasonable to say that the latter is harsher than the former because the latter demand us to do more than the former under the same situation. For illustration we can let a represent "you are invited to a dinner", x represent "you dress your suit" and y represent "you wash your hair". For similar reasons we can consider $(a \vee b, x)$ to be harsher than (a, x).

In general, $(a, x) \preceq (b, y)$ can intuitively be read as (a, x) is "harsher" than (b, y). Moreover, we can lift this harshness concept to the level of normative system as long as we use joining-systems to represent them.

Definition 8 (Harshness). *Let $\mathbb{S} = (A, B, S)$ and $\mathbb{S}' = (A, B, S')$ be two joining-systems, $\mathbb{S}$ is harsher than $\mathbb{S}'$, denote as $\mathbb{S} \lesssim \mathbb{S}'$, iff for all $(a, x) \in core(\mathbb{S})$ there exist $(b, y) \in core(\mathbb{S}')$ such that $(a, x) \preceq (b, y)$ and for all $(b, y) \in core(\mathbb{S}')$ there exist $(a, x) \in core(\mathbb{S})$ such that $(a, x) \preceq (b, y)$.*

Observation 4. *For any joining-systems $\mathbb{S} = (A, B, S)$ and $\mathbb{S}' = (A, B, S')$,if $\mathbb{S} \lesssim \mathbb{S}'$, then $S' \subseteq S$.*

Prove: Assume $(a, x) \in S'$, then there exist $(b, y) \in core(\mathbb{S}')$ such that $(b, y) \preceq (a, x)$. By the definition of harshness there exist $(c, z) \in core(\mathbb{S})$ such that $(c, z) \preceq (b, y)$. There fore $(c, z) \preceq (a, x)$ and $(a, x) \in S$.

This observation shows the more obligation a normative system contains, the harsher it is. Such a result coincides with our intuition quite well.

4.3 Structural Similarity of Normative Systems

For two algebraic structures A and B, if they are isomorphic then they are essentially the same. We can extend the isomorphism of Boolean algebra to joining-systems. But before we do this, we first review the isomorphism of Boolean algebra.

Definition 9 (Isomorphism of Boolean algebra). *For two Boolean algebras $\mathfrak{A} = (A, +, \cdot, -, 0, 1)$ and $\mathfrak{A}' = (A', +, \cdot, -, 0, 1)$ and h a map from A to A'. We say that h is an isomorphism from $\mathfrak{A}$ to $\mathfrak{A}'$ iff for any $x, y \in A$, h satisfies the following conditions:*

1. *h is bijective*
2. *$h(x + y) = h(x) + h(y)$*
3. *$h(x \cdot y) = h(x) \cdot h(y)$*
4. *$h(1) = 1$*

Given an isoporphism h from $\mathfrak{A}$ to $\mathfrak{A}'$, it is easy to check that for all $x, y \in A$ and $x', y' \in A'$, if $h(x) = x'$ and $h(y) = y'$, then $x \leq y$ iff $x' \leq y'$.

Now we extend isomorphism to joining-systems.

Definition 10 (Isomorphism of joining-systems). *For two joining-systems $\mathbb{S} = (A, B, S)$ and $\mathbb{S}' = (A', B', S')$ and h a map from $A \cup B$ to $A' \cup B'$. We say that h is an isomorphism from $\mathbb{S}$ to $\mathbb{S}'$ iff h satisfies the following conditions:*

1. *h is bijective*
2. *the restriction of h on A is an isomorphism from A to A'*
3. *the restriction of h on B is an isomorphism from B to B'*
4. *$(a, x) \in core(\mathbb{S})$ iff $(h(a), h(x)) \in core(\mathbb{S}')$*

If there exist some isomorphism form $\mathbb{S}$ to $\mathbb{S}'$, then we say $\mathbb{S}$ and $\mathbb{S}'$ are isomorphic. Two isomorphic joining-systems can naturally be understood as structurally the same. Although in the last item of the above definition we restrict ourselves to the core of a joining-systems, the correspondence in fact covers the whole system. That is, we have the following observation:

Observation 5. *For any joining-systems $\mathbb{S} = (A, B, S)$ and $\mathbb{S}' = (A, B, S')$,if h is an isomorphism from $\mathbb{S}$ to $\mathbb{S}'$, then for any $(a, x) \in A \times B$, $(a, x) \in S$ iff $(h(a), h(x)) \in S'$.*

5 Conclusion and Future Work

The main contribution of this paper is a correspondence result between input/output logic and the theory of joining-system. These results illustrate that normative systems can be equivalently analyzed using three different tools, proof theory, semantics and algebra. Each tool will give us some special insights of normative systems.

There are a lot of future workto be done. A natural direction is to build a correspondence result between constrained I/O logic and TJS. Another direction is to use more advanced logic and algebra to relate I/O logic and TJS. For example, temporal logic can serve as the basis of I/O logic and Boolean algebra with temporal operator can be the underlying algebra of TJS. Then we can build another correspondence result between the new I/O logic and TJS.

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