A classification theorem and a spectral sequence for a locally free sheaf cohomology of a supermanifold

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To our team coach Yu.A. Kirillov on his 70th birthday

Abstract. This paper is based on the paper [3], where two classification theorems for locally free sheaves on supermanifolds were proved and a spectral sequence for a locally free sheaf of modules $\mathcal{E}$ was obtained. We consider another filtration of the locally free sheaf $\mathcal{E}$, the corresponding classification theorem and the spectral sequence, which is more convenient in some cases. The methods, which we are using here, are similar to [2, 3].

The first spectral sequence of this kind was constructed by A.L. Onishchik in [2] for the tangent sheaf of a supermanifold. However, the spectral sequence considered in this paper is not a generalization of Onishchik's spectral sequence from [2].

Mathematics Subject Classification (2010). Primary 32C11; Secondary 58A50.

Keywords. Locally free sheaf, supermanifold, spectral sequence.

1. Main definitions and classification theorems

1.1. Main definitions

Let $(M, \mathcal{O})$ be a supermanifold of dimension $n|m$, i.e. a $\mathbb{Z}_2$-graded ringed space that is locally isomorphic to a superdomain in $\mathbb{C}^{n|m}$. The underlying complex manifold $(M, \mathcal{F})$ is called the *reduction* of $(M, \mathcal{O})$. The simplest class of supermanifolds constitute the so-called *split supermanifolds*. We recall that a supermanifold $(M, \mathcal{O})$ is called split if $\mathcal{O} \simeq \bigwedge_{\mathcal{F}} \mathcal{G}$, where $\mathcal{G}$ is a locally free sheaf of $\mathcal{F}$-modules on M. With any supermanifold $(M, \mathcal{O})$ one can associate a split supermanifold $(M, \tilde{\mathcal{O}})$ of the same dimension which is called

This work was partially supported by MPI Bonn, SFB TR |12, DFG 1388 and by the Russian Foundation for Basic Research (grant no. 11-01-00465a).

the *retract* of $(M,\mathcal{O})$. To construct it, let us consider the $\mathbb{Z}_2$-graded sheaf of ideals $\mathcal{J}=\mathcal{J}_{\bar 0}\oplus\mathcal{J}_{\bar 1}\subset\mathcal{O}$ generated by odd elements of $\mathcal{O}$. The structure sheaf of the retract is defined by

$$\tilde{\mathcal{O}}=\bigoplus_{p\geq 0}\tilde{\mathcal{O}}_p,\ \text{ where }\ \tilde{\mathcal{O}}_p=\mathcal{J}^p/\mathcal{J}^{p+1},\ \ \mathcal{J}^0:=\mathcal{O}.$$

Here $\tilde{\mathcal{O}}_1$ is a locally free sheaf of $\mathcal{F}$-modules on M and $\tilde{\mathcal{O}}_p=\bigwedge^p_{\mathcal{F}}\tilde{\mathcal{O}}_1$. By definition, the following sequences

$$\begin{array}{c}0\to\mathcal{J}\cap\mathcal{O}_{\bar 0}\to\mathcal{O}_{\bar 0}\stackrel{\pi}{\to}\tilde{\mathcal{O}}_0\to 0,\\ 0\to\mathcal{J}^2\cap\mathcal{O}_{\bar 1}\to\mathcal{O}_{\bar 1}\stackrel{\tau}{\to}\tilde{\mathcal{O}}_1\to 0.\end{array}\qquad(1)$$

are exact. Moreover, they are locally split. The supermanifold $(M,\mathcal{O})$ is split iff both sequences are globally split.

Denote by $\mathcal{S}_{\bar 0}$ and $\mathcal{S}_{\bar 1}$ the even and the odd parts of a $\mathbb{Z}_2$-graded sheaf of $\mathcal{O}$-modules $\mathcal{S}$ on M, respectively; by $\Pi(\mathcal{S})$ we denote the same sheaf of $\mathcal{O}$-modules $\mathcal{S}$ equipped with the following $\mathbb{Z}_2$-grading: $\Pi(\mathcal{S})_{\bar 0}=\mathcal{S}_{\bar 1}$, $\Pi(\mathcal{S})_{\bar 1}=\mathcal{S}_{\bar 0}$. A $\mathbb{Z}_2$-graded sheaf of $\mathcal{O}$-modules on M is called *free* (*locally free*) *of rank* $p|q$, $p,q\geq 0$ if it is isomorphic (respectively, locally isomorphic) to the $\mathbb{Z}_2$-graded sheaf of $\mathcal{O}$-modules $\mathcal{O}^p\oplus\Pi(\mathcal{O})^q$. For example, the tangent sheaf $\mathcal{T}$ of a supermanifold $(M,\mathcal{O})$ is a locally free sheaf of $\mathcal{O}$-modules.

Let now $\mathcal{E}=\mathcal{E}_{\bar 0}\oplus\mathcal{E}_{\bar 1}$ be a locally free sheaf of $\mathcal{O}$-modules of rang $p|q$ on an arbitrary supermanifold $(M,\mathcal{O})$. We are going to construct a locally free sheaf of the same rank on $(M,\tilde{\mathcal{O}})$. First, we note that $\mathcal{E}_{\mathrm{red}}:=\mathcal{E}/\mathcal{J}\mathcal{E}$ is a locally free sheaf of $\mathcal{F}$-modules on M. Moreover, $\mathcal{E}_{\mathrm{red}}$ admits the $\mathbb{Z}_2$-grading $\mathcal{E}_{\mathrm{red}}=(\mathcal{E}_{\mathrm{red}})_{\bar 0}\oplus(\mathcal{E}_{\mathrm{red}})_{\bar 1}$, by two locally free sheaves of $\mathcal{F}$-modules

$$(\mathcal{E}_{\mathrm{red}})_{\bar 0}:=\mathcal{E}_{\bar 0}/\mathcal{J}\mathcal{E}\cap\mathcal{E}_{\bar 0}\ \text{ and }\ (\mathcal{E}_{\mathrm{red}})_{\bar 1}:=\mathcal{E}_{\bar 1}/\mathcal{J}\mathcal{E}\cap\mathcal{E}_{\bar 1}$$

of ranks p and q, respectively. Further, the sheaf $\mathcal{E}$ possesses the filtration

$$\mathcal{E}=\mathcal{E}_{(0)}\supset\mathcal{E}_{(1)}\supset\mathcal{E}_{(2)}\supset\ldots,\ \text{ where }\mathcal{E}_{(p)}=\mathcal{J}^p\mathcal{E}_{\bar 0}+\mathcal{J}^{p-1}\mathcal{E}_{\bar 1},\ p\geq 1.\qquad(2)$$

Using this filtration, we can construct the following locally free sheaf of $\tilde{\mathcal{O}}$-modules on M:

$$\tilde{\mathcal{E}}=\bigoplus\nolimits_p\tilde{\mathcal{E}}_p,\ \text{ where }\ \tilde{\mathcal{E}}_p=\mathcal{E}_{(p)}/\mathcal{E}_{(p+1)}.$$

The sheaf $\tilde{\mathcal{E}}$ is also a locally free sheaf of $\mathcal{F}$-modules. In other words, $\tilde{\mathcal{E}}$ is a sheaf of sections of a certain vector bundle. The following exact sequence gives a description of $\tilde{\mathcal{E}}$.

$$0\to\tilde{\mathcal{O}}_p\otimes(\mathcal{E}_{\mathrm{red}})_{\bar 0}\to\tilde{\mathcal{E}}_p\to\tilde{\mathcal{O}}_{p-1}\otimes(\mathcal{E}_{\mathrm{red}})_{\bar 1}\to 0.$$

We also have the following two exact sequences, which are locally split:

$$\begin{array}{c}0\to\mathcal{E}_{(1)\bar 0}\to\mathcal{E}_{(0)\bar 0}\stackrel{\alpha}{\to}\tilde{\mathcal{E}}_0\to 0;\\ 0\to\mathcal{E}_{(2)\bar 1}\to\mathcal{E}_{(1)\bar 1}\stackrel{\beta}{\to}\tilde{\mathcal{E}}_1\to 0.\end{array}\qquad(3)$$

The sheaf $\tilde{\mathcal{E}}$ is $\mathbb{Z}$-graded by definition. Unlike the $\mathbb{Z}_2$-grading considered in [3], the natural $\mathbb{Z}_2$-grading is compatible with this $\mathbb{Z}$-grading.

$$(\tilde{\mathcal{E}})_{\bar{0}} := \bigoplus_{p=2k} \tilde{\mathcal{E}}_p, \quad (\tilde{\mathcal{E}})_{\bar{1}} := \bigoplus_{p=2k_1} \tilde{\mathcal{E}}_p.$$

1.2. Classification theorem for locally free sheaves $\mathcal{E}$ on supermanifolds with given $\tilde{\mathcal{E}}$

Our objective now is to classify locally free sheaves $\mathcal{E}$ of $\mathcal{O}$-modules on supermanifolds $(M, \mathcal{O})$ which have the fixed retract $(M, \tilde{\mathcal{O}})$ and such that the corresponding locally free sheaf $\tilde{\mathcal{E}}$ is fixed.

Let $(M, \mathcal{O})$ and $(M, \mathcal{O}')$ be two supermanifolds, $\mathcal{E}$, $\mathcal{E}'$ be locally free sheaves of $\mathcal{O}$-modules and $\mathcal{O}'$-modules on M, respectively. Suppose that $\Psi : \mathcal{O} \to \mathcal{O}'$ is a superalgebra sheaf morphism. A vector space sheaf morphism $\Phi_\Psi : \mathcal{E} \to \mathcal{E}'$ is called a *quasi-morphism* if

$$\Phi_\Psi(fv) = \Psi(f)\Phi_\Psi(v), \ f \in \mathcal{O}, \ v \in \mathcal{E}.$$

As usual, we assume that $\Phi_\Psi(\mathcal{E}_{\bar{i}}) \subset \mathcal{E}'_{\bar{i}}$, $\bar{i} \in \{\bar{0}, \bar{1}\}$. An invertible quasi-morphism is called a *quasi-isomorphism*. A quasi-isomorphism $\Phi_\Psi : \mathcal{E} \to \mathcal{E}$ is also called a *quasi-automorphism* of $\mathcal{E}$. Denote by $\mathcal{A}ut\mathcal{E}$ the sheaf of quasi-automorphisms of $\mathcal{E}$. It has a double filtration by the subsheaves

$$\mathcal{A}ut_{(p)(q)}\mathcal{E} := \{\Phi_\Psi \in \mathcal{A}ut\mathcal{E} \mid \Phi_\Psi(v) \equiv v \bmod \mathcal{E}_{(p)}, \ \Psi(f) = f \bmod \mathcal{J}^q \\ \text{for } v \in \mathcal{E}, f \in \mathcal{O}\}, \ p, q \geq 0.$$

We also define the following subsheaf of $\mathcal{A}ut\tilde{\mathcal{E}}$:

$$\widetilde{\mathcal{A}ut}\tilde{\mathcal{E}} := \{\Phi_\Psi \mid \Phi_\Psi \in \mathcal{A}ut(\tilde{\mathcal{E}}), \ \Phi_\Psi \text{ preserves the } \mathbb{Z}\text{-grading of } \tilde{\mathcal{E}}\}. \quad (4)$$

If $\Phi_\Psi \in \widetilde{\mathcal{A}ut}\tilde{\mathcal{E}}$, then $\Psi : \tilde{\mathcal{O}} \to \tilde{\mathcal{O}}$ also preserves the $\mathbb{Z}$-grading. The 0-th cohomology group $H^0(M, \widetilde{\mathcal{A}ut}\tilde{\mathcal{E}})$ acts on the sheaf $\mathcal{A}ut\tilde{\mathcal{E}}$ by the automorphisms $\delta \mapsto a \circ \delta \circ a^{-1}$, where $a \in H^0(M, \widetilde{\mathcal{A}ut}\tilde{\mathcal{E}})$ and $\delta \in \mathcal{A}ut\tilde{\mathcal{E}}$. It is easy to see that this action leaves invariant the subsheaves $\mathcal{A}ut_{(p)(q)}\tilde{\mathcal{E}}$ and hence induces an action of $H^0(M, \widetilde{\mathcal{A}ut}\tilde{\mathcal{E}})$ on the cohomology set $H^1(M, \mathcal{A}ut_{(p)(q)}\tilde{\mathcal{E}})$. The unit element $\epsilon \in H^1(M, \mathcal{A}ut_{(p)(q)}\mathcal{E}')$ is a fixed point with respect to the action of $H^0(M, \mathcal{A}ut\mathcal{E}')$.

Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}$-modules on M. Denote

$$[\mathcal{E}] = \{\mathcal{E}' \mid \mathcal{E}' \text{ is quasi-isomorphic to } \mathcal{E}\}.$$

The total space of the bundle corresponding to a locally free sheaf $\mathcal{E}$ will be denoted $\mathbb{E}$. It is a supermanifold. The locally free sheaf $\tilde{\mathcal{E}}$ corresponding to $\mathcal{E}$ has the following property: The retract $\tilde{\mathbb{E}}$ of $\mathbb{E}$ is the total space of the bundle corresponding to $\tilde{\mathcal{E}}$.

Theorem 1.1. *Let $(M, \mathcal{O}')$ be a split supermanifold and $\mathcal{E}'$ be a locally free sheaf of $\mathcal{O}'$-modules on M such that $\mathcal{E}' \simeq \tilde{\mathcal{E}}'$. Then*

$$\{[\mathcal{E}] \mid \tilde{\mathcal{O}} = \mathcal{O}', \ \tilde{\mathcal{E}} = \mathcal{E}'\} \overset{1:1}{\longleftrightarrow} H^1(M, \mathcal{A}ut_{(2)(2)}\mathcal{E}')/H^0(M, \widetilde{\mathcal{A}ut}\mathcal{E}').$$

The orbit of the unit element ϵ, which is ϵ itself, corresponds to $\mathcal{E}'$.

Proof. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}$-modules on $(M,\mathcal{O})$ and $\mathcal{U}=\{U_i\}$ be an open covering of M such that (1) and (3) are split (hence exact) over U_i and $\mathcal{E}|_{U_i}$ are free. In this case, $\tilde{\mathcal{E}}|_{U_i}$ are free sheaves of $\tilde{\mathcal{O}}$-modules. We fix homogeneous bases (even and odd, respectively) $(\hat{e}^i_j)$ and $(\hat{f}^i_j)$ of the free sheaves of $\tilde{\mathcal{O}}$-modules $\tilde{\mathcal{E}}|_{U_i}$, $U_i\in\mathcal{U}$. Without loss of generality, we may assume that $\hat{e}^i_j\in\tilde{\mathcal{E}}_0$ and $\hat{f}^i_j\in\tilde{\mathcal{E}}_1$. We are going to define an isomorphism $\delta_i:\mathcal{E}|_{U_i}\to\tilde{\mathcal{E}}|_{U_i}$.

Let $e^i_j\in\mathcal{E}_{(0)\bar{0}}$ be such that $\alpha(e^i_j)=\hat{e}^i_j$ and $f^i_j\in\mathcal{E}_{(0)\bar{1}}$ be such that $\beta(f^i_j)=\hat{f}^i_j$, see (3). Then (e^i_j,f^i_j) is a local basis of $\mathcal{E}|_{U_i}$. A splitting of (1) determines a local isomorphism $\sigma_i:\mathcal{O}|_{U_i}\to\tilde{\mathcal{O}}|_{U_i}$, see [1]. We put

$$\delta_i(\sum h_je^i_j+\sum g_jf^i_j)=\sum\sigma_i(h_j)\hat{e}^i_j+\sum\sigma_i(g_j)\hat{f}^i_j,\ \ h_j,g_j\in\mathcal{O}.$$

Obviously, δ_i is an isomorphism. We put $\gamma_{ij}:=\sigma_i\circ\sigma_j^{-1}$ and $(g_{ij})_{\gamma_{ij}}:=\delta_i\circ\delta_j^{-1}$. Moreover, $(\gamma_{ij})\in Z^1(\mathcal{U},\mathcal{A}ut_{(2)}\tilde{\mathcal{O}})$, see [1] for more details. We want to show that

$$((g_{ij})_{\gamma_{ij}})\in Z^1(\mathcal{U},\mathcal{A}ut_{(2)(2)}\tilde{\mathcal{E}}).$$

Let us take $v\in\tilde{\mathcal{E}}|_{U_j}$, $v=\sum h_k\hat{e}^i_k+\sum g_k\hat{f}^i_k,\ \ h_j,g_j\in\tilde{\mathcal{O}}$. Then by definition we have

$$\delta_j^{-1}(v)=\sum\sigma_j^{-1}(h_k)e^j_k+\sum\sigma_j^{-1}(g_k)f^j_k.$$

The transition functions of $\tilde{\mathcal{E}}$ may be expressed in $U_i\cap U_j$ as follows:

$$e^j_k=\sum a^k_se^i_s+\sum b^k_sf^i_s,\ f^j_k=\sum c^k_se^i_s+\sum d^k_sf^i_s,\ a^k_s,d^k_s\in\mathcal{O}_{\bar{0}},\ b^k_s,c^k_s\in\mathcal{O}_{\bar{1}}.$$

Further,

$$\alpha(e^j_k)=\hat{e}^j_k=\sum\pi(a^k_s)\hat{e}^i_s,\quad\beta(f^j_k)=\hat{f}^j_k=\sum\tau(c^k_s)\hat{e}^i_s+\sum\pi(d^k_s)\hat{f}^i_s.$$

We have

$$\begin{aligned}\delta_j\circ\delta_j^{-1}(v)=&\sum_k\gamma_{ij}(h_k)(\sum_s\sigma_i(a^k_s)\hat{e}^i_s+\sum_r\sigma_i(b^k_s)\hat{f}^i_s)+\\ &\sum_k\gamma_{ij}(g_k)(\sum_s\sigma_i(c^k_s)\hat{e}^i_s+\sum_s\sigma_i(d^k_s)\hat{f}^i_s)=\\ &\sum_k h_k(\sum_s\pi(a^k_s)\hat{e}^i_s)+\sum_k g_k(\sum_s\tau(c^k_s)\hat{e}^i_s+\\ &\qquad\sum_s\pi(d^k_s)\hat{f}^i_s)\bmod\tilde{\mathcal{E}}_{(2)}=v\bmod\tilde{\mathcal{E}}_{(2)}.\end{aligned}$$

The rest of the proof is a direct repetition of the proof of Theorem 2 from [3]. □

2. The spectral sequence

2.1. Quasi-derivations

Quasi-derivations were defined in [3]. Let us briefly recall that construction. Consider a locally free sheaf $\mathcal{E}$ on a supermanifold $(M,\mathcal{O})$. An even vector space sheaf morphism $A_\Gamma : \mathcal{E} \to \mathcal{E}$ is called a *quasi-derivation* if $A_\Gamma(fv) = \Gamma(f)v + fA_\Gamma(v)$, where $f \in \mathcal{O}$, $v \in \mathcal{E}$ and Γ is a certain even super vector field. Denote by $\operatorname{Der}\mathcal{E}$ the sheaf of quasi-derivations. It is a sheaf of Lie algebras with respect to the commutator $[A_\Gamma, B_\Upsilon] := A_\Gamma \circ B_\Upsilon - B_\Upsilon \circ A_\Gamma$. The sheaf $\operatorname{Der}\mathcal{E}$ possesses a double filtration

$$\begin{array}{ccccc} \operatorname{Der}_{(0)(0)}\mathcal{E} & \supset & \operatorname{Der}_{(2)(0)}\mathcal{E} & \supset \cdots \\ \cup & & \cup & \\ \operatorname{Der}_{(0)(2)}\mathcal{E} & \supset & \operatorname{Der}_{(2)(2)}\mathcal{E} & \supset \cdots, \\ \vdots & & \vdots & \end{array}$$

where

$$\operatorname{Der}_{(p)(q)}\mathcal{E} := \{A_\Gamma \in \operatorname{Der}\mathcal{E} \mid A_\Gamma(\mathcal{E}_{(r)}) \subset \mathcal{E}_{(r+p)},\ \Gamma(\mathcal{J}^s) \subset \mathcal{J}^{s+q},\ r, s \in \mathbb{Z}\},$$

where $p, q \geq 0$. The map defined by the usual exponential series

$$\exp : \operatorname{Der}_{(p)(q)}\mathcal{E} \to \mathcal{A}ut_{(p)(q)}\mathcal{E},\ p, q \geq 2,$$

is an isomorphism of sheaves of sets, because operators from $\operatorname{Der}_{(p)(q)}\mathcal{E}$, $p, q \geq 2$, are nilpotent. The inverse map is given by the logarithmic series. Define the vector space subsheaf $\operatorname{Der}_{k,k}\tilde{\mathcal{E}}$ of $\operatorname{Der}_{(k)(k)}\tilde{\mathcal{E}}$ for $k \geq 0$ by

$$\operatorname{Der}_{k,k}\tilde{\mathcal{E}} := \quad \{A_\Gamma \in \operatorname{Der}_{(k)(k)}\tilde{\mathcal{E}} \mid A_\Gamma(\tilde{\mathcal{E}}_r) \subset \tilde{\mathcal{E}}_{r+k},\ \Gamma(\tilde{\mathcal{O}}_s) \subset \tilde{\mathcal{O}}_{s+k},\ r, s \in \mathbb{Z}\}.$$

For an even $k \geq 2$, define a map

$$\mu_k : \mathcal{A}ut_{(k)(2)}\tilde{\mathcal{E}} \to \operatorname{Der}_{k,k}\tilde{\mathcal{E}}, \quad \mu_k(a_\gamma) = \bigoplus_q \operatorname{pr}_{q+k} \circ A_\Gamma \circ \operatorname{pr}_q,$$

where $a_\gamma = \exp(A_\Gamma)$ and $\operatorname{pr}_k : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}_k$ is the natural projection. The kernel of this map is $\mathcal{A}ut_{(k+2)(2)}\tilde{\mathcal{E}}$. Moreover, the sequence

$$0 \to \mathcal{A}ut_{(k+2)(2)}\tilde{\mathcal{E}} \longrightarrow \mathcal{A}ut_{(k)(2)}\tilde{\mathcal{E}} \xrightarrow{\mu_k} \operatorname{Der}_{k,k}\tilde{\mathcal{E}} \to 0,$$

where $k \geq 2$ is even, is exact. Denoting by $H_{(k)}(\tilde{\mathcal{E}})$ the image of the natural mapping $H^1(M, \mathcal{A}ut_{(k)(2)}\tilde{\mathcal{E}}) \to H^1(M, \mathcal{A}ut_{(2)(2)}\tilde{\mathcal{E}})$, we get the filtration

$$H^1(M, \mathcal{A}ut_{(2)}\tilde{\mathcal{E}}) = H_{(2)}(\tilde{\mathcal{E}}) \supset H_{(4)}(\tilde{\mathcal{E}}) \supset \ldots.$$

Take $a_\gamma \in H_{(2)}(\tilde{\mathcal{E}})$. We define the *order of* a_γ to be the maximal number k such that $a_\gamma \in H_{(k)}(\tilde{\mathcal{E}})$. The *order of a locally free sheaf* $\mathcal{E}$ of $\mathcal{O}$-modules on a supermanifold $(M, \mathcal{O}_M)$ is by definition the order of the corresponding cohomology class.

2.2. The spectral sequence.

A spectral sequence connecting the cohomology with values in the tangent sheaf $\mathcal{T}$ of a supermanifold $(M, \mathcal{O})$ with the cohomology with values in the tangent sheaf $\mathcal{T}_{\mathrm{gr}}$ of the retract $(M, \tilde{\mathcal{O}})$ was constructed in [2]. Here we use similar ideas to construct a new spectral sequence connecting the cohomology with values in a locally free sheaf $\mathcal{E}$ on a supermanifold $(M, \mathcal{O})$ with the cohomology with values in the locally free sheaf $\tilde{\mathcal{E}}$ on $(M, \tilde{\mathcal{O}})$. Note that our spectral sequence is not a generalization of the spectral sequence obtained in [2] because $\mathcal{T}_{\rm gr}$ is not in general isomorphic to $\tilde{\mathcal{T}}$.

Let $\mathcal{E}$ be a locally free sheaf on a supermanifold $(M, \mathcal{O})$ of dimension $n|m$. We fix an open Stein covering $\mathfrak{U} = (U_i)_{i\in I}$ of M and consider the corresponding Čech cochain complex $C^*(\mathfrak{U}, \mathcal{E}) = \bigoplus_{p\geq 0} C^p(\mathfrak{U}, \mathcal{E})$. The $\mathbb{Z}_2$-grading of $\mathcal{E}$ gives rise to the $\mathbb{Z}_2$-gradings in $C^*(\mathfrak{U}, \mathcal{E})$ and $H^*(M, \mathcal{E})$ given by

$$\begin{aligned} C_{\bar 0}(\mathfrak{U}, \mathcal{E}) &= \bigoplus_{q\geq 0} C^{2q}(\mathfrak{U}, \mathcal{E}_{\bar 0}) \oplus \bigoplus_{q\geq 0} C^{2q+1}(\mathfrak{U}, \mathcal{E}_{\bar 1}),\\ C_{\bar 1}(\mathfrak{U}, \mathcal{E}) &= \bigoplus_{q\geq 0} C^{2q}(\mathfrak{U}, \mathcal{E}_{\bar 1}) \oplus \bigoplus_{q\geq 0} C^{2q+1}(\mathfrak{U}, \mathcal{E}_{\bar 0}).\\ H_{\bar 0}(M, \mathcal{E}) &= \bigoplus_{q\geq 0} H^{2q}(M, \mathcal{E}_{\bar 0}) \oplus \bigoplus_{q\geq 0} H^{2q+1}(M, \mathcal{E}_{\bar 1}),\\ H_{\bar 1}(M, \mathcal{E}) &= \bigoplus_{q\geq 0} H^{2q}(M, \mathcal{E}_{\bar 1}) \oplus \bigoplus_{q\geq 0} H^{2q+1}(M, \mathcal{E}_{\bar 0}). \end{aligned} \tag{5}$$

The filtration (2) for $\mathcal{E}$ gives rise to the filtration

$$C^*(\mathfrak{U}, \mathcal{E}) = C_{(0)} \supset \ldots \supset C_{(p)} \supset \ldots \supset C_{(m+2)} = 0 \tag{6}$$

of this complex by the subcomplexes

$$C_{(p)} = C^*(\mathfrak{U}, \mathcal{E}_{(p)}).$$

Denoting by $H(M, \mathcal{E})_{(p)}$ the image of the natural mapping $H^*(M, \mathcal{E}_{(p)}) \to H^*(M, \mathcal{E})$, we get the filtration

$$H^*(M, \mathcal{E}) = H(M, \mathcal{E})_{(0)} \supset \ldots \supset H(M, \mathcal{E})_{(p)} \supset \ldots. \tag{7}$$

Denote by $\operatorname{gr} H^*(M, \mathcal{E})$ the bigraded group associated with the filtration (7); its bigrading is given by

$$\operatorname{gr} H^*(M, \mathcal{E}) = \bigoplus_{p,q\geq 0} \operatorname{gr}_p H^q(M, \mathcal{E}).$$

By the (more general) Leray procedure, we get a spectral sequence of bigraded groups E_r converging to $E_\infty \simeq \operatorname{gr} H^*(M, \mathcal{E})$. For convenience of the reader, we recall the main definitions here.

For any $p, r \geq 0$, define the vector spaces

$$C^p_r = \{c \in C_{(p)} \,|\, dc \in C_{(p+r)}\}.$$

Then, for a fixed p, we have

$$C_{(p)} = C^p_0 \supset \ldots \supset C^p_r \supset C^p_{r+1} \supset \ldots.$$

The r-th term of the spectral sequence is defined by

$$E_r = \bigoplus_{p=0}^{m} E^p_r,\ r \geq 0,\ \text{where } E^p_r = C^p_r / C^{p+1}_{r-1} + dC^{p-r+1}_{r-1}.$$

Since $d(C^p_r) \subset C^{p+r}_r$, d induces a derivation d_r of E_r of degree r such that $d_r^2 = 0$. Then E_{r+1} is naturally isomorphic to the homology algebra $H(E_r, d_r)$. The

$\mathbb{Z}_2$-grading (5) in $C^*(\mathfrak{U}, \mathcal{E})$ gives rise to certain $\mathbb{Z}_2$-gradings in C_r^p and E_r^p, turning E_r into a superspace. Clearly, the coboundary operator d on $C^*(\mathfrak{U}, \mathcal{E})$ is odd. It follows that the coboundary d_r is odd for any $r \geq 0$.

The superspaces E_r are also endowed with a second $\mathbb{Z}$-grading. Namely, for any $q \in \mathbb{Z}$, set

$$C_r^{p,q} = C_r^p \cap C^{p+q}(\mathfrak{U}, \mathcal{E}), \quad E_r^{p,q} = C_r^{p,q}/C_{r-1}^{p+1,q-1} + dC_{r-1}^{p-r+1,q+r-2}.$$

Then

$$E_r = \bigoplus_{p,q} E_r^{p,q} \text{ and } d_r(E_r^{p,q}) \subset E_r^{p+r,q-r+1} \text{ for any } r,\ p,\ q. \tag{8}$$

Further, for a fixed q, we have $d(C_r^{p,q}) = 0$ for all $p \geq 0$ and all $r \geq m+2$. This implies that the natural homomorphism $E_r^{p,q} \to E_{r+1}^{p,q}$ is an isomorphism for all p and $r \geq r_0 = m+2$. Setting $E_\infty^{p,q} = E_{r_0}^{p,q}$, we get the bigraded superspace

$$E_\infty = \bigoplus_{p,q} E_\infty^{p,q}.$$

Lemma 2.1. *The first two terms of the spectral sequence (E_r) can be identified with the following bigraded spaces:*

$$E_0 = C^*(\mathfrak{U}, \tilde{\mathcal{E}}),\ E_1 = E_2 = H^*(M, \tilde{\mathcal{E}}).$$

More precisely,

$$E_0^{p,q} = C^{p+q}(\mathfrak{U}, \tilde{\mathcal{E}}_p),\ E_1^{p,q} = E_2^{p,q} = H^{p+q}(M, \tilde{\mathcal{E}}_p).$$

We have $d_{2k+1} = 0$ and, hence, $E_{2k+1} = E_{2k+2}$ for all $k \geq 0$.

Proof. The proof is similar to the proof of Proposition 3 in [2]. □

Lemma 2.2. *There is the following identification of bigraded algebras:*

$$E_\infty = \operatorname{gr} H^*(M, \mathcal{E}),\ \textit{where } E_\infty^{p,q} = \operatorname{gr}_p H^{p+q}(M, \mathcal{E}).$$

If M is compact, then $\dim H^k(M, \mathcal{E}) = \sum_{p+q=k} \dim E_\infty^{p,q}$.

Proof. The proof is a direct repetition of the proof of Proposition 4 in [2]. □

Now we prove our main result concerning the first non-zero coboundary operators among $d_2,\ d_4, \ldots$. Assume that the isomorphisms of sheaves $\delta_i : \mathcal{E}|U_i \to \tilde{\mathcal{E}}|U_i$ from Theorem 1.1 are defined for each $i \in I$. By Theorem 1.1, a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$ on M corresponds to the cohomology class a_γ of the 1-cocycle $((a_\gamma)_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)(2)}\tilde{\mathcal{E}})$, where $(a_\gamma)_{ij} = \delta_i \circ \delta_j^{-1}$. If the order of $(a_\gamma)_{ij}$ is equal to k, then we may choose $\delta_i,\ i \in I$, in such a way that $((a_\gamma)_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(k)(2)}\tilde{\mathcal{E}})$. We can write $a_\gamma = \exp A_\Gamma$, where $A_\Gamma \in C^1(\mathfrak{U}, \operatorname{Der}_{(k)(2)} \tilde{\mathcal{E}})$.

We will identify the superspaces (E_0, d_0) and $(C^*(\mathfrak{U}, \tilde{\mathcal{E}}), d)$ via the isomorphism of Lemma 2.1. Clearly, $\delta_i : \mathcal{E}_{(p)}|U_i \to \tilde{\mathcal{E}}_{(p)}|U_i = \sum_{r \geq p} \tilde{\mathcal{E}}_r|U_i$ is an isomorphism of sheaves for all $i \in I,\ p \geq 0$. These local sheaf isomorphisms permit us to define an isomorphism of graded cochain groups

$$\psi : C^*(\mathfrak{U}, \mathcal{E}) \to C^*(\mathfrak{U}, \tilde{\mathcal{E}})$$

such that

$$\psi : C^*(\mathfrak{U}, \mathcal{E}_{(p)}) \to C^*(\mathfrak{U}, (\tilde{\mathcal{E}})_{(p)}),\ p \geq 0.$$

We put

$$\psi(c)_{i_0\ldots i_q} = \delta_{i_0}(c_{i_0\ldots i_q})$$

for any $(i_0, \ldots, i_q)$ such that $U_{i_0} \cap \ldots \cap U_{i_q} \neq \emptyset$. Note that ψ is not an isomorphism of complexes. Nevertheless, we can explicitly express the coboundary d of the complex $C^*(\mathfrak{U}, \mathcal{E})$ by means of d_0 and a_γ.

The following theorem permits to calculate the spectral sequence (E_r) whenever d_0 and the cochain a_γ are known. It also describes certain coboundary operators $d_r,\ r \geq 1$.

Theorem 2.3. *For any $c \in C^*(\mathfrak{U}, \tilde{\mathcal{E}}_q) = E_0^q$, we have*

$$(\psi(d\psi^{-1}(c)))_{i_0\ldots i_{q+1}} = (d_0 c)_{i_0\ldots i_{q+1}} + ((a_\gamma)_{i_0 i_1} - \mathrm{id})(c_{i_1\ldots i_{q+1}}).$$

Suppose that the locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$ on M has order k and denote by a_γ the cohomology class corresponding to $\mathcal{E}$ by Theorem 1.1. *Then $d_r = 0$ for $r = 1, \ldots, k-1$, and $d_k = \mu_k(a_\gamma)$.*

Proof. The proof is similar to the proof of Theorem 7 in [3]. □

Acknowledgment

The author is very grateful to A. L. Onishchik for useful discussions.

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