Statistics and Probability Letters 87 (2014) 48–53

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Itô's formula for Walsh's Brownian motion and applications

Hatem Hajri[a,*], Wajdi Touhami[b]

[a] Mathematics Research Unit, University of Luxembourg, Campus Kirchberg, 6 rue Richard Coudenhove-Kalergi, L-1359 Luxembourg, Grand-Duchy of Luxembourg, Luxembourg

[b] Faculté des Sciences de Tunis, Campus Universitaire 2092 - El Manar Tunis, Tunisie

ARTICLE INFO

Article history:
Received 15 July 2013
Received in revised form 28 December 2013
Accepted 29 December 2013
Available online 10 January 2014

Keywords:
Walsh's Brownian motion
Itô's formula
Harmonic functions
Stochastic flows

ABSTRACT

We prove an Itô's formula for Walsh's Brownian motion in the plane with angles according to a probability measure μ on $[0, 2\pi[$. This extends Freidlin–Sheu formula which corresponds to the case where μ has finite support. We also give some applications.

1. Itô's formula for Walsh's Brownian motion

Let $E = \mathbb{R}^2$; we will use polar co-ordinates (r, α) to denote points in E. We denote by $\mathcal{C}(E)$ the space of all continuous functions on E. For $f \in \mathcal{C}(E)$, we define

$$f_\alpha(r) = f(r, \alpha), \quad r > 0,\ \alpha \in [0, 2\pi[.$$

Throughout this paper we fix μ a probability measure on $[0, 2\pi[$, also we define

$$\bar{f}(r) = \int_0^{2\pi} f(r, \alpha)\, \mu(d\alpha), \quad r > 0.$$

Let $(P_t^+)_{t\geq 0}$ be the semigroup of a reflecting Brownian motion on $[0, \infty[$ and let $(P_t^0)_{t\geq 0}$ be the semigroup of a Brownian motion on $[0, \infty[$ killed at 0. Then for $t \geq 0$, define P_t to act on $f \in \mathcal{C}_0(E)$ as follows:

$$P_t f(0, \alpha) = P_t^+ \bar{f}(0),$$
$$P_t f(r, \alpha) = P_t^+ \bar{f}(r) + P_t^0 (f_\alpha - \bar{f})(r)$$

for $r > 0$ and $\alpha \in [0, 2\pi[$. We recall the following

Theorem 1.1 (Barlow et al., 1989). *$(P_t)_{t\geq 0}$ is a Feller semigroup on $\mathcal{C}_0(E)$, the associated process is by definition the Walsh's Brownian motion $(Z_t)_{t\geq 0}$ (with angles distributed as μ).*

We can write $Z_t = (R_t, \Theta_t)$ where the radial part $R_t = |Z_t|$ is a reflected Brownian motion and $(\Theta_t)_{t\geq 0}$ is called the angular process of Z. To well define Θ_t, when $Z_t = 0$, we set by convention $\Theta_t = 0$.

* Corresponding author.
E-mail addresses: Hatem.Hajri@uni.lu (H. Hajri), Wajdi.Maaouia@fst.rnu.tn (W. Touhami).

0167-7152/$ – see front matter © 2014 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.spl.2013.12.021

Notations:

We denote by E_μ, the state space of Walsh's Brownian motion, i.e. $E_\mu = \{(r,\alpha) : r \geq 0, \alpha \in \text{supp}(\mu)\}$. We define the tree-metric ρ on $\mathbb{R}^2$ as follows: for $z_1 = (r_1,\alpha_1), z_2 = (r_2,\alpha_2) \in \mathbb{R}^2$,

$$\rho(z_1,z_2) = (r_1+r_2)1_{\{\alpha_1\neq\alpha_2\}} + |r_1-r_2|1_{\{\alpha_1=\alpha_2\}}.$$

Note that the sample paths of Walsh's Brownian motion are continuous with respect to the tree-topology, since the process cannot jump from one ray to another one without passing through the origin. These paths are also continuous with respect to the relative topology induced by the Euclidean metric on $\mathbb{R}^2$.

Now denote by $\mathcal{D}$ the set of all functions $f : E_\mu \to \mathbb{R}$ such that

(i) f is continuous for the tree-topology.
(ii) For all $\alpha \in \text{supp}(\mu)$, f_α is C^2 on $]0,\infty]$ and $f'_\alpha(0+), f''_\alpha(0+)$ exist and are finite.
(iii) For all $K > 0$,

$$\sup_{0<r<K,\alpha\in\text{supp}(\mu)} \Big(|f'_\alpha(r)| + |f''_\alpha(r)|\Big) < \infty.$$

Note that if $f \in \mathcal{D}$, then $\sup_{\alpha\in\text{supp}(\mu)} |f'_\alpha(0+)| < \infty$ and in particular $\int_{\text{supp}(\mu)} |f'_\alpha(0+)|\,\mu(d\alpha)$ is finite. From now on, we will extend any function $g : \text{supp}(\mu) \to \mathbb{R}$ to $[0,2\pi[$ by setting $g = 0$ outside $\text{supp}(\mu)$. For $f \in \mathcal{D}$ and $z = (r,\alpha) \neq 0$, we set $f'(z) = f'_\alpha(r)$ and $f''(z) = f''_\alpha(r)$.

Now define

$$B^Z_t = R_t - R_0 - L_t(R) \tag{1}$$

where $L_t(R)$ stands for the symmetric local time at zero of R, i.e. $L_t(R) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_0^t 1_{\{R_s\leq\epsilon\}}ds$. Then B^Z is an $(\mathcal{F}^Z_t)_t$-Brownian motion (Lemma 2.2 Barlow et al. (1989)). Our main result is the following

Theorem 1.2. *Let $(Z_t)_{t\geq 0}$ be a Walsh's Brownian motion started from z, then for all $f \in \mathcal{D}$, we have*

$$f(Z_t) = f(z) + \int_0^t 1_{\{Z_s\neq 0\}} f'(Z_s)dB^Z_s + \frac{1}{2}\int_0^t 1_{\{Z_s\neq 0\}} f''(Z_s)ds + \left(\int_0^{2\pi} f'_\alpha(0^+)\,\mu(d\alpha)\right) \times L_t(R).$$

When $\text{supp}(\mu)$ is finite this formula is due to Freidlin and Sheu (2000) (see also Hajri (2011) for another proof based on the skew Brownian motion). Our proof is based on standard approximations and Riemann–Stieltjes integration.

Proof. Note that $H_s(\omega) = 1_{\{Z_s(\omega)\neq 0\}} f'(Z_s(\omega))$ is progressively measurable with respect to the filtration $(\mathcal{F}^Z_t)$ and is locally bounded by the assumption (iii) on f, i.e. a.s. $\forall t \geq 0$, $\sup_{s\leq t}|H_s| < \infty$. Thus the stochastic integral with respect to B^Z is well defined. Again by (iii), the integral $\int_0^t 1_{\{Z_s\neq 0\}} f''(Z_s)ds$ is well defined.

First, we will consider the case $z = 0$. Take $f \in \mathcal{D}$, $t > 0$ and for $\varepsilon > 0$, define $\tau^\varepsilon_0 = 0$ and for $n \geq 0$

$$\sigma^\varepsilon_n = \inf\{r \geq \tau^\varepsilon_n : |Z_r| = \varepsilon\}, \qquad \tau^\varepsilon_{n+1} = \inf\{r \geq \sigma^\varepsilon_n : Z_r = 0\}.$$

Set $g(z) = f(z) - f'_\alpha(0+)|z|1_{\{z\neq 0\}}$ where $\alpha = \arg(z)$. Note that $g \in \mathcal{D}$ and $g'_\alpha(0^+) = 0$. We will prove our formula first for g. We have

$$g(Z_t) - g(0) = \lim_{\varepsilon\to 0}\sum_{n=0}^{\infty}\Big(g(Z_{\sigma^\varepsilon_{n+1}\wedge t}) - g(Z_{\sigma^\varepsilon_n\wedge t})\Big) = \lim_{\varepsilon\to 0}\big(A^\varepsilon_t + D^\varepsilon_t\big)$$

where

$$A^\varepsilon_t = \sum_{n=0}^{\infty}\big(g(Z_{\sigma^\varepsilon_{n+1}\wedge t}) - g(Z_{\tau^\varepsilon_{n+1}\wedge t})\big)$$

and

$$D^\varepsilon_t = \sum_{n=0}^{\infty}\big(g(Z_{\tau^\varepsilon_{n+1}\wedge t}) - g(Z_{\sigma^\varepsilon_n\wedge t})\big) = \sum_{n=0}^{\infty}\big(g(Z_{\tau^\varepsilon_{n+1}\wedge t}) - g(Z_{\sigma^\varepsilon_n})\big)1_{\{\sigma^\varepsilon_n\leq t\}}.$$

We will first identify $\lim_{\varepsilon\to 0} D^\varepsilon_t$. We need the following

Lemma 1.3. *Let $h \in \mathcal{D}$, then for all $t \in [\sigma^\varepsilon_n, \tau^\varepsilon_{n+1}]$ we have*

$$h(Z_t) = h(Z_{\sigma^\varepsilon_n}) + \int_{\sigma^\varepsilon_n}^t 1_{\{Z_s\neq 0\}} h'(Z_s)dB^Z_s + \frac{1}{2}\int_{\sigma^\varepsilon_n}^t h''(Z_s)1_{\{Z_s\neq 0\}}\,ds. \tag{2}$$

Proof. Note that $B^Z_{\cdot+\sigma^\varepsilon_n} - B^Z_{\sigma^\varepsilon_n}$ is a Brownian motion independent of $\mathcal{F}^Z_{\sigma^\varepsilon_n}$. Thus by conditioning with respect to $\mathcal{F}^Z_{\sigma^\varepsilon_n}$, it will be sufficient to prove that a Walsh's Brownian motion Y started from a fixed $y \neq 0$ satisfies (2) with $(Y, B^Y, 0, \tau')$ in place of $(Z, B^Z, \sigma_n^\varepsilon, \tau_{n+1}^\varepsilon)$ where τ' is the first hitting time of 0 by Y. Since $L_t(|Y|) = 0$ for all $t \le \tau'$, our claim holds by a simple application of Itô's formula for $|Y|$ using the function $f_{\arg(y)}$. □

Applying the previous lemma, we get

$$\begin{aligned} D_t^\varepsilon &= \sum_{n=0}^{\infty} 1_{\{\sigma_n^\varepsilon \le t\}} \int_{\sigma_n^\varepsilon}^{t\wedge\tau_{n+1}^\varepsilon} \left(g'(Z_s) dB_s^Z + \frac{1}{2} g''(Z_s)\, ds \right) \\ &= \sum_{n=0}^{\infty} \int_{\sigma_n^\varepsilon}^{t\wedge\tau_{n+1}^\varepsilon} \left(g'(Z_s) dB_s^Z + \frac{1}{2} g''(Z_s)\, ds \right) \\ &= \sum_{n=0}^{\infty} \int_0^t 1_{[\sigma_n^\varepsilon, \tau_{n+1}^\varepsilon[}(s) \left(g'(Z_s) dB_s^Z + \frac{1}{2} g''(Z_s)\, ds \right) \\ &= \int_0^t \sum_{n=0}^{\infty} 1_{[\sigma_n^\varepsilon, \tau_{n+1}^\varepsilon[}(s) \left(g'(Z_s)\, dB_s^Z + \frac{1}{2} g''(Z_s)\, ds \right) \end{aligned}$$

where we have used dominated convergence for stochastic integrals (see Revuz and Yor (1999) page 142) in the last line. Using dominated convergence again we see that as $\varepsilon \to 0$, D_t^ε converges in probability to $\int_0^t 1_{\{Z_s \neq 0\}} (g'(Z_s)\, dB_s^Z + \frac{1}{2} g''(Z_s)\, ds)$.

Now $A_t^\varepsilon = H_t^\varepsilon + K_t^\varepsilon$ with $H_t^\varepsilon = \sum_{n=0}^\infty \big(g(Z_{\sigma_{n+1}^\varepsilon}) - g(0)\big) 1_{\{\sigma_{n+1}^\varepsilon \le t\}}$ and $|K_t^\varepsilon| \le |g(Z_t) - g(0)| 1_{\{|Z_t| \le \varepsilon\}} \to 0$ as $\varepsilon \to 0$. Note that σ_n^ε is $\mathcal{F}^R$-measurable, $\theta_n = \arg(Z_{\sigma_n^\varepsilon})$ is independent of $\mathcal{F}^R$ and in particular σ_n^ε and θ_n are independent. Therefore we obtain

$$\begin{aligned} \mathbb{E}[|H_t^\varepsilon|] &\le \sum_{n=0}^{\infty} \mathbb{E}[|g(\varepsilon, \theta_{n+1}) - g(0)|\ 1_{\{\sigma_{n+1}^\varepsilon \le t\}}] \\ &\le \sum_{n=0}^{\infty} \mathbb{E}[|g(\varepsilon, \theta_{n+1}) - g(0)|]\ \mathbb{P}(\sigma_{n+1}^\varepsilon \le t) \\ &\le \int_0^{2\pi} \frac{|g(\varepsilon, \alpha) - g(0)|}{\varepsilon}\, \mu(d\alpha)\ \varepsilon \sum_{n=0}^{\infty} \mathbb{P}(\sigma_{n+1}^\varepsilon \le t). \end{aligned}$$

But $\varepsilon \sum_{n=0}^\infty \mathbb{P}(\sigma_{n+1}^\varepsilon \le t) \to \mathbb{E}[L_t(R)]$ as $\varepsilon \to 0$ and for $\varepsilon < 1$, we have

$$\frac{|g(\varepsilon, \alpha) - g(0)|}{\varepsilon} \le \sup_{0<r<1, \alpha \in \mathrm{supp}(\mu)} |g'_\alpha(r)| < \infty$$

so that by dominated convergence, as $\varepsilon \to 0$,

$$\int_0^{2\pi} \frac{|g(\varepsilon, \alpha) - g(0)|}{\varepsilon}\, \mu(d\alpha) \to \int_0^{2\pi} |g'_\alpha(0^+)|\, \mu(d\alpha) = 0$$

and finally $H_t^\varepsilon \to 0$ in L^1. Summarizing, we get

$$g(Z_t) = g(0) + \int_0^t 1_{\{Z_s \neq 0\}} g'(Z_s) dB_s^Z + \frac{1}{2} \int_0^t 1_{\{Z_s \neq 0\}} g''(Z_s)\, ds. \tag{3}$$

Note that a.s. $g(Z_t) = f(Z_t) - R_t Y_t$ with $Y_t = f'_{\Theta_t}(0+)\, 1_{\{Z_t \neq 0\}}$. Set

$$C_t = Y_t R_t - \int_0^t Y_s dR_s.$$

We will identify C_t following Prokaj (2009). Put

$$Y_t^\varepsilon = \sum_{k=0}^{\infty} Y_t 1_{[\sigma_k^\varepsilon, \tau_{k+1}^\varepsilon[}(t).$$

Then Y^ε is right-continuous and is constant on every interval $[\sigma_k^\varepsilon, \tau_{k+1}^\varepsilon[$. Since R is continuous, then R is Riemann–Stieltjes integrable with respect to Y^ε almost surely and

$$Y_t^\varepsilon R_t - Y_0^\varepsilon R_0 - \int_0^t Y_s^\varepsilon\, dR_s = \int_0^t R_s\, dY_s^\varepsilon. \tag{4}$$

As $\varepsilon \to 0$, we have $Y_t^\varepsilon \to Y_t$ a.s. Since $|Y^\varepsilon| \le M$ for some $M > 0$, then the dominated convergence for stochastic integrals gives $\int_0^t Y_s^\varepsilon \, dR_s \to \int_0^t Y_s \, dR_s$ in probability. The definition of Y^ε entails that

$$\int_0^t R_s dY_s^\varepsilon = \sum_{k=0}^{\infty} R_{\sigma_k^\varepsilon} Y_{\sigma_k^\varepsilon} 1_{\{\sigma_k^\varepsilon \le t\}} = \varepsilon \sum_{k=0}^{\infty} Y_{\sigma_k^\varepsilon} 1_{\{\sigma_k^\varepsilon \le t\}}.$$

Let $N(t, \varepsilon)$ be the number of upcrossings of the interval $[0, t]$ by R. So

$$\int_0^t R_s dY_s^\varepsilon = \varepsilon \sum_{j=0}^{N(t,\varepsilon)} f'_{\theta_j}(0+).$$

Since $(f'_{\theta_j}(0+))_{j\ge 0}$ is a sequence of independent random variables having the same law μ, the Bernoulli law of large numbers yields

$$\int_0^t R_s dY_s^\varepsilon = \varepsilon N(t, \varepsilon) \, \frac{\sum_{j=0}^{N(t,\varepsilon)} f'_{\theta_j}(0+)}{N(t, \varepsilon)} \to L_t(R) \, \mathbb{E}[f'_{\theta_0}(0+)].$$

But $\mathbb{E}[f'_{\theta_0}(0+)] = \int_0^{2\pi} f'_\alpha(0+) \, \mu(d\alpha)$ and so $\int_0^t R_s dY_s^\varepsilon \to [\int_0^{2\pi} f'_\alpha(0+) \, \mu(d\alpha)] L_t(R)$ as $\varepsilon \to 0$. Letting $\varepsilon \to 0$ in (4), we obtain

$$C_t = \left(\int_0^{2\pi} f'_\alpha(0+) \, \mu(d\alpha) \right) L_t(R).$$

Using the fact that $1_{\{R_s>0\}} dR_s = 1_{\{R_s>0\}} dB_s^Z$, it follows that

$$\int_0^t Y_s \, dR_s = \int_0^t Y_s \, dB_s^Z = \int_0^t 1_{\{Z_s \ne 0\}} f'_{\Theta_s}(0+) \, dB_s^Z.$$

Now for $z = 0$, our formula follows from (3). The case $z \ne 0$ holds by discussing $t \le \tau$ and $t > \tau$ where τ is the first hitting time of 0 by Z and using the Itô's formula satisfied by $Z_{\cdot+\tau}$ which is a Walsh's Brownian motion started from 0. □

2. Some applications

2.1. Walsh's Brownian motion on graphs

The result of this section can be deduced from Hajri and Raimond (in press) but it has not been announced in that paper. Our purpose is to give an Itô's formula for Walsh's Brownian motion with the more general state space of a graph. This diffusion is defined by means of its infinitesimal generator in Freidlin and Sheu (2000) (see also Chapter 4 in Jehring (2009)). Along an edge it is like a Brownian motion and around a vertex it behaves like a Walsh's Brownian motion.

Let G be a metric graph equipped with the shortest path distance, denote by V, the set of its vertices, and by $\{E_i;\ i \in I\}$ the set of its edges where I is a finite or countable set. To each edge E_i, we associate an isometry $e_i : J_i \to \bar{E}_i$, with $J_i = [0, L_i]$ when $L_i < \infty$ and $J_i = [0, \infty)$ or $J_i = (-\infty, 0]$ when $L_i = \infty$. When $L_i < \infty$, denote $\{g_i, d_i\} = \{e_i(0), e_i(L_i)\}$. When $L_i = \infty$, denote $\{g_i, d_i\} = \{e_i(0), \infty\}$ when $J_i = [0, \infty)$ and $\{g_i, d_i\} = \{\infty, e_i(0)\}$ when $J_i = (-\infty, 0]$. For all $v \in V$, denote $I_v^+ = \{i \in I;\ g_i = v\}$, $I_v^- = \{i \in I;\ d_i = v\}$ and $I_v = I_v^+ \cup I_v^-$. We assume that Card $I_v < \infty$ for all v and that $\inf_{i\in I} L_i > 0$. To each $v \in V$ and $i \in I_v$, we associate a parameter $p_i^v \in [0, 1]$ such that $\sum_{i\in I_v} p_i^v = 1$. Let $p = (p_i^v, v \in V, i \in I_v)$ and denote by $\mathcal{D}_p^G$ the set of all continuous functions $f : G \to \mathbb{R}$ such that for all $i \in I$, $f \circ e_i$ is C^2 on the interior of J_i with bounded first and second derivatives both extendable by continuity to J_i and such that for all $v \in V$

$$\sum_{i\in I_v^+} p_i^v \lim_{r\to 0+} (f \circ e_i)'(r) = \sum_{i\in I_v^-} p_i^v \Big(\lim_{r\to L_i-} (f \circ e_i)'(r) 1_{\{L_i<\infty\}} + \lim_{r\to 0-} (f \circ e_i)'(r) 1_{\{L_i=\infty\}} \Big).$$

For $f \in \mathcal{D}_p^G$ and $x = e_i(r) \in G \setminus V$, set $f'(x) = (f \circ e_i)'(r)$, $f''(x) = (f \circ e_i)''(r)$ and take the following conventions for all $v \in V$, $f'(v) = f''(v) = 0$.

Let Z be a Walsh's Brownian motion on G associated to the family p and started from z, i.e. if E_i is an adjacent edge to v, then from v, Z "jumps" to E_i with probability p_i^v. We have the following

Proposition 2.1. *There exists a Brownian motion W such that for all $f \in \mathcal{D}_p^G$, we have*

$$f(Z_t) = f(z) + \int_0^t f'(Z_u) dW_u + \frac{1}{2} \int_0^t f''(Z_u) du.$$

Proof. We take $z = v$ a vertex point. We will define W until Z hits another vertex v'. The definition of W should be clear then using the strong Markov property and the successive hitting times of V by Z. Thus we can also assume that G is a star graph (has only one vertex and eventually an infinite number of edges) which we embed in the complex plane just to simplify notations. Let $G^+ = \cup_{i\in I_v^+} E_i$, $G^- = \cup_{i\in I_v^-} E_i$ and

$$W_t = \int_0^t (1_{G^+}(Z_s) - 1_{G^-}(Z_s))dB_s^Z.$$

Now our result holds by applying Theorem 1.2 and taking care of the definition of the derivative of f in the proposition. □

2.2. Harmonic functions on Walsh's Brownian motion

In Jehring (2009), Harmonic functions on Walsh's Brownian motion are studied. The following lemma (Lemma 3.2 in Jehring (2009)) is the most technical point in proving Theorem 3.2 in Jehring (2009). Its proof uses the Excursion theory and non trivial calculations (see Appendix A in Jehring (2009)). We will derive it in few lines from Theorem 1.2.

Lemma 2.2. *Let $h : E \to \mathbb{R}$ be of the form*

$$h(r, \alpha) = m_h(\alpha).r + h(0)$$

for $(r, \alpha) \in E$, where the function $m_h : [0, 2\pi[\to \mathbb{R}$ is an integrable function with respect to μ and satisfies $\int_0^{2\pi} m_h(\alpha)\mu(d\alpha) = 0$. Then $\{h(Z_t), t \geq 0\}$ is a martingale with respect to $(\mathcal{F}^Z)_t$ for every starting point $z \in E$.

Note that when m_h is bounded our formula gives an explicit expression of this martingale:

$$h(Z_t) = h(z) + \int_0^t m_h(\Theta_s)1_{\{Z_s\neq 0\}}dB_s^Z.$$

Proof. By Theorem 1.2, using the function $h^N(r, \alpha) = (m_h(\alpha) \wedge N).r + h(0)$, $N \geq 0$, we see that

$$h^N(Z_t) = h^N(z) + \int_0^t (m_h(\Theta_u) \wedge N)1_{\{Z_u\neq 0\}}dB_u^Z + \int_0^{2\pi} (m_h(\alpha) \wedge N)\mu(d\alpha) \times L_t(R).$$

Thus for all $s \leq t$,

$$\begin{aligned} E[h^N(Z_t)|\mathcal{F}_s^Z] &= h^N(z) + \int_0^s (m_h(\Theta_u) \wedge N)1_{\{Z_u\neq 0\}}dB_u^Z + \int_0^{2\pi} (m_h(\alpha) \wedge N)\mu(d\alpha) \times E[L_t(R)|\mathcal{F}_s^Z] \\ &= h^N(Z_s) - \int_0^{2\pi} (m_h(\alpha) \wedge N)\mu(d\alpha) \times L_s(R) + \int_0^{2\pi} (m_h(\alpha) \wedge N)\mu(d\alpha) \times E[L_t(R)|\mathcal{F}_s^Z]. \end{aligned}$$

Letting $N \to \infty$ and using dominated convergence (note that $E[|m_h(\Theta_t)||Z_t|] < \infty$ since $m_h(\Theta_t)$ and $|Z_t|$ are independent), we get $E[h(Z_t)|\mathcal{F}_s^Z] = h(Z_s)$. □

2.3. Stochastic flows in the plane

Let Z be a Walsh's Brownian motion started from 0 and define $W = B^Z$. Then, we have $R_t = W_t - \min_{0\leq u\leq t} W_u$ by Skorokhod Lemma (see Revuz and Yor (1999) page 239), $\sigma(R) = \sigma(W)$ and Θ_t is independent of W for all $t > 0$. This situation is close to Tanaka's equation: if X satisfies

$$X_t = \int_0^t \operatorname{sgn}(X_s)dW_s,$$

then $\sigma(|X|) = \sigma(W)$ and $\operatorname{sgn}(X_t)$ is independent of W for all $t > 0$. Theorem 2 allows thus to define the analogue of Tanaka's equation in the plane. This problem has been considered from the stochastic flows view point in Hajri (2011) for supp(μ) finite.

We say that $(\varphi_{s,t}(z))_{0\leq s\leq t, z\in E}$ is a stochastic flow of mappings (SFM) on E as soon as $\varphi_{s,t}$ is measurable with respect to (z, ω), $\varphi_{s,t}$ and $\varphi_{0,t-s}$ are equal in law, for any sequence $\{[s_i, t_i], 1 \leq i \leq n\}$ of non-overlapping intervals the mappings φ_{s_i,t_i} are independent, and we have the flow property: for all $s \leq t \leq u$ and $z \in E$, a.s. $\varphi_{s,u}(z) = \varphi_{t,u} \circ \varphi_{s,t}(z)$.

Definition 2.3. Let φ be a SFM and W be a Brownian motion. We say that (φ, W) solves Tanaka's equation with angle μ if for all $0 \leq s \leq t, z \in E$ and all $f \in \mathcal{D}$ such that $\int_0^{2\pi} f'_\alpha(0+)\mu(d\alpha) = 0$, a.s.

$$f(\varphi_{s,t}(z)) = f(z) + \int_s^t 1_{\{\varphi_{s,u}(z)\neq 0\}}f'(\varphi_{s,u}(z))dW_u + \frac{1}{2}\int_s^t 1_{\{\varphi_{s,u}(z)\neq 0\}}f''(\varphi_{s,u}(z))du. \quad (5)$$

Following Le Jan and Raimond (2006), we will construct a SFM φ solving this equation: by Kolmogorov extension theorem, there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ on which one can construct a process $(\Theta_{s,t}, W_{s,t})_{0\leq s\leq t<\infty}$ taking values in $[0, 2\pi[\times \mathbb{R}$ such that (i)–(iv) below are satisfied

(i) $W_{s,t} := W_t - W_s$ for all $s \leq t$ and W is a Brownian motion.
(ii) For fixed $s < t$, $\Theta_{s,t}$ and W are independent and the law of $\Theta_{s,t}$ is μ.

(iii) Define $\min_{s,t} = \inf\{W_u; u \in [s,t]\}$. Then for all $s < t$ and $u < v$

$$\mathbb{P}(\Theta_{s,t} = \Theta_{u,v} | \min_{s,t} = \min_{u,v}) = 1.$$

(iv) For all $s < t$ and $\{(s_i, t_i); 1 \le i \le n\}$ with $s_i < t_i$, the law of $\Theta_{s,t}$ knowing $(\Theta_{s_i,t_i})_{1 \le i \le n}$ and W is given by μ on the event $\min_{s,t} \notin \{\min_{s_i,t_i}; 1 \le i \le n\}$ and is given by $\delta_{\Theta_{s_i,t_i}}$ on the event $\{\min_{s,t} = \min_{s_i,t_i}\}$ with $1 \le i \le n$.

Note that (i)–(iv) uniquely define the law of $(\Theta_{s_1,t_1}, \ldots, \Theta_{s_n,t_n}, W)$ for all $s_i < t_i$, $1 \le i \le n$. This family of laws is consistent by construction.

For $0 \le s \le t, z \in E$, define

$$\tau_s(z) = \inf\{r \ge s : \ W_{s,r} = -|z|\}$$

and

$$\varphi_{s,t}(z) = \big(|z| + W_{s,t}, \arg(z)\big) 1_{\{t \le \tau_s(z)\}} + \big(W_t - \min_{s,t}, \Theta_{s,t}\big) 1_{\{t > \tau_s(z)\}}.$$

Following Le Jan and Raimond (2006), one can prove that φ is a SFM such that (φ, W) satisfies (5). It is also possible to extend (5) to stochastic flows of kernels as in Le Jan and Raimond (2006). A stochastic flow of kernels (SFK) $K = (K_{s,t};\ s \le t)$ is the same as a SFM, but the mappings are replaced by kernels, and the flow property being now that for all $z \in E, s \le t \le u$, a.s. $K_{s,u}(x) = K_{s,t}K_{t,u}(x)$. For example

$$K^W_{s,t}(z) = E[\delta_{\varphi_{s,t}(z)} | W] = \delta_{\left(|z| + W_{s,t}, \arg(z)\right)} 1_{\{t \le \tau_s(z)\}} + \int_0^{2\pi} \delta_{\left(W_t - \min_{s,t}, \alpha\right)} \mu(d\alpha) 1_{\{t > \tau_s(z)\}}$$

is a SFK. By conditioning with respect to W in (5), we see that (K^W, W) is solution of the following equation.

Definition 2.4. Let K be a SFK and W be a Brownian motion. We say that (K, W) solves Tanaka's equation with angle μ if for all $0 \le s \le t, z \in E$ and all $f \in \mathcal{D}$ such that $\int_0^{2\pi} f'_\alpha(0+)\mu(d\alpha) = 0$, a.s.

$$K_{s,t}f(z) = f(z) + \int_s^t K_{s,u}(f' 1_{\mathbb{R}^2 \setminus \{0\}})(z) dW_u + \frac{1}{2} \int_s^t K_{s,u}(f'' 1_{\mathbb{R}^2 \setminus \{0\}})(z) du. \tag{6}$$

Note that, now we have two solutions to (6): (δ_φ, W) and (K^W, W). In Hajri (2011), when $\mathrm{supp}(\mu)$ is finite a complete description of the laws of all SFK's K solutions of (6) is given under some further assumptions on stochastic flows (see the definitions in Le Jan and Raimond (2004)). Under these assumptions, it should be possible to prove in our setting that φ is the law-unique SFM solution of (5), K^W is the unique Wiener (i.e. $\sigma(W)$-measurable) flow solution of (6) and to give a complete classification of solutions to (6).

Acknowledgements

The research of the first author is supported by the National Research Fund, Luxembourg, and cofunded under the Marie Curie Actions of the European Commission (FP7-COFUND).

References

Barlow, M., Pitman, J., Yor, M., 1989. On Walsh's Brownian motions. Lecture Notes in Math. 1372, 275–293.
Freidlin, M., Sheu, S., 2000. Diffusion processes on graphs: stochastic differential equations, large derivation principle. Probab. Theory Related Fields 116 (2), 181–220.
Hajri, H., 2011. Stochastic flows related to Walsh Brownian motion. Electron. J. Probab. 16 (58), 1563–1599.
Hajri, H., Raimond, O., 2014. Stochastic flows on metric graphs. Electron. J. Probab. http://arxiv.org/abs/1305.0839 (in press).
Jehring, K.E., 2009. Harmonic functions on Walsh's Brownian motion, Ph.D. dissertation, Univ. of California Saint Diego.
Le Jan, Y., Raimond, O., 2004. Flows, coalescence and noise. Ann. Probab. 32 (2), 1247–1315.
Le Jan, Y., Raimond, O., 2006. Flows associated to Tanak's SDE. ALEA Lat. Am. J. Probab. Math. Stat. 1, 21–34.
Prokaj, V., 2009. Unfolding the Skorohod reflection of a semimartingale. Statist. Probab. Lett. 79, 534–536.
Revuz, D., Yor, M., 1999. Continuous Martingales and Brownian Motion. Springer Verlag, Berlin.