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Lectures on Algebraic Operads

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Disclaimer

This text is an uncorrected first draft and does not contain any new results. It is due to P. Hilger, who presented it as his Master Thesis, and it is based on a series (50 hours) of (post)doctoral seminars on Algebraic Operads given in 2011 by N. Poncin at the University of Luxembourg in the seminar of the working group 'Algebraic Topology, Geometry and Physics'. These lectures were themselves based upon a preprint of the monograph *Algebraic Operads* by J.-L. Loday and B. Vallette [LV11]. An improved version of the present text is in works.

Introduction

Operads are to algebras, what algebras are to matrices, or, better, to representations. More precisely, an operad encodes a type of algebras. It heaves the algebraic operations of the considered type, their symmetries, their compositions, as well as the specific relations they verify, on a more abstract and universal level, which is best thought of by viewing a universal abstract operation as some tree with a finite number of leaves (or inputs) and one root (or output). To be explicit, to any type of algebras — defined by multilinear operations $m : V^{\times n} \to V$ on a vector space V (over a field of characteristic 0), where n may vary in $\mathbb{N}\setminus\{0\}$, whose 'defining relations' read $\sum_c c(v_1, \ldots, v_n) = 0$, for all $v_i \in V$, where c is a composite of 'generating operations' m — we can associate an operad. The algebras over this operad, i.e. the representations of the operad, form a category, which is equivalent to the category of algebras of the initially considered type.

Operads can be traced back to works that appeared in the fifties and sixties. Let us mention here at least the names of Boardman, MacLane, Stasheff, Vogt... Operads have first been formally introduced by J. Peter May in [May72], who also proposed the denomination 'operad'. This word is actually a contraction of 'operation' and 'monad' (but seems also due to the fact that P. May's mother was an opera singer). Regarding his creation, May wrote in [May97]: "The name 'operad' is a word that I coined myself, spending a week thinking of nothing else."

Operads were initially studied as a tool in homotopy theory, but found some thirty years later interest in a number of other domains like homological algebra, category theory, algebraic geometry, mathematical physics... Among various powerful aspects of operads, let us mention that the operadic language simplifies not only the formulation of the mathematical results but also their proofs, that it allows to gain a more conceptual and deeper insight into classical theorems and to extend them to other types of algebras... E.g. if some construction is possible 'mutatis mutandis' for several types of algebras, it is a very enriching challenge to prove that it goes through for operads ($\star$).

Let us mention the example of homotopy, sh, or infinity algebras [Sta63], which are homotopy invariant extensions of differential graded algebras (see ($\star\star$) below). This property explains their origin in BRST of closed string field theory. One of the prominent applications of Lie infinity algebras (L_∞-algebras) [LS93] is their appearance in Deformation Quantization of Poisson manifolds. The deformation map can be extended from differential graded Lie algebras (DGLAs) to L_∞-algebras and more precisely to a functor from the category $\mathtt{L}_\infty$ to the category `Set`. This functor transforms a weak equivalence into a bijection. When applied to the DGLAs of polyvector fields and polydifferential operators, the latter result, combined with the formality theorem, provides the 1-to-1 correspondence between Poisson tensors and star products.

As suggested above (see ($\star\star$)), homotopy algebras appear when examining whether a compatible algebraic structure on some chain complex can be transferred to a homotopy equivalent complex (V, d_V). For differential graded associative or Lie algebras, the naturally constructed product on V is no longer associative or Lie, but verifies the associativity or Jacobi condition up to some homotopy. We thus obtain a sequence $m_n : V^{\times n} \to V$, $n \in \mathbb{N}^*$, of multilinear maps on the graded vector space V that (have degree $n-2$ and possibly some symmetry properties and) verify a whole sequence of defining relations. We refer to these data as a (an associative or Lie) homotopy algebra structure on V. It was understood quite quickly that the maps m_n can be viewed as the corestrictions of a coderivation on the free graded associative or commutative coalgebra over the suspended space sV and that, more surprisingly, the mentioned sequence of defining relations can be encrypted in the unique requirement that this coderivation be a codifferential. Hence, a (an associative or Lie) homotopy algebra can be interpreted as a codifferential of an appropriate coalgebra.

In their celebrated paper on 'Koszul duality for operads' [GK94], V. Ginzburg and M. Kapranov gave a conceptual approach to a broad family of homotopy algebras and extended the preceding interpretation to any type of homotopy algebra whose corresponding algebra type can be encoded in a so-called Koszul operad (see Remark ($\star$) above).

The present text is intended to be on the one hand sufficiently concise and on the other hand sufficiently complete and detailed to provide a short but understandable introduction to the theory of algebraic operads, featuring elements of Koszul duality and finally portraying the operadic approach to homotopy algebras.

Since the objective is to give an outline of the major aspects of the theory, the proofs are not always given up to the last details, but sometimes only in a sketchy way, concentrating on the most instructive and interesting points. Technical and too far reaching aspects will mostly be omitted, and explanations will be provided in an intuitive, but accurate, manner. This allows to concentrate on presenting the essential aspects, still giving the necessary precision whenever it is needed.

In the first chapter, we give a short summary of representation theory of the symmetric group, which is important for symmetric operads. In particular, the notion of induced representation is treated in detail.

In the second chapter, we recall the concepts of algebras, coalgebras and (co)homology. These being generally well-known basic notions, this chapter can be seen as fixing notations and reminding the concepts appearing in the sequel.

In the third chapter, we deal with twisting morphisms and Koszul morphisms for associative algebras and coalgebras. We take a special interest in the bar and cobar constructions, leading to a model of the considered differential graded associative algebra A. This model is nothing but the bar-cobar resolution of A.

The fourth chapter is dedicated to Koszul duality for associative algebras. The model constructed in the preceding chapter being too large, we replace it, in the special case of Koszul algebras A, by a smaller one, given by the cobar construction $\Omega A^{¡}$ of the Koszul dual coalgebra $A^{¡}$ of A.

In the fifth chapter, we first encounter operads — via their classical definition, which views an operad as a multicategory with a unique object. Moreover, we construct the operads corresponding to associative and to commutative algebras in an intuitive way. Later on, we encounter the partial definition of an operad, which bares a considerable similarity to the classical one.

Chapter six covers the functorial definition of operads. An operad will, in view of this definition, be given as a monad in the category `Vect` of vector spaces, or, equivalently, a monoid in the monoidal category of endofunctors of `Vect` (or still a monoidal structure on an endofunctor of `Vect`). We will then confine ourselves to operads over a Schur functor, which is a specific endofunctor. This allows, in particular, to substitute the equivalent and often advantageous viewpoint of S-modules to the one of endofunctors.

In the seventh chapter, we consider the free operad and also a fourth equivalent definition of operads, namely the combinatorial definition. These two notions will provide a better understanding of the relationship between operads and tree diagrams, which we use throughout this text to represent the abstract universal operations of operads.

Chapter eight is a small excursion to the world of PROPs and other generalizations of operads. PROPs allow to encode the operations and cooperations of bialgebras.

In chapter nine, we deal with operadic twisting and Koszul morphisms. In the main, we transfer all the above-mentioned results for associative algebras (see Chapter 3) to the operadic setting. To do this we have to take some fundamental differences between operads and associative algebras into account. One of the most important ones is that the tensor product of vector spaces is additive with respect to both arguments, left and right. The monoidal 'composition' of endofunctors of `Vect` or of S-modules however, is only additive with respect to its left argument.

Chapter ten outlines Koszul duality for operads. Similar to the preceding chapter, we adapt the whole theory (see Chapter 4) to the operadic framework. For a quadratic Koszul operad P, we will then dispose of a model $P_\infty := \Omega P^{\text{i}}$, which allows to define P_∞-algebras in a conceptual way, as representations of the latter operad.

In the final chapter we deepen the just described operadic approach to infinity algebras and treat the example of associative homotopy algebras in detail. On the one hand, we will, starting from the nonsymmetric associative operad $\mathscr{A}s$, and using the previously introduced cobar construction and Koszul duality, build the operad $\mathscr{A}s_\infty := \Omega\, \mathscr{A}s^{\text{i}}$. On the other hand, we will detail the construction of A_∞-algebras by means of the aforementioned homotopy transfer theorem, and provide a quite direct description of the corresponding operad A_∞. Of course, the two just mentioned approaches to associative infinity algebras should lead to the same concept of sh algebra and the operads $\mathscr{A}s_\infty$ and A_∞ should coincide. The description of these two differential graded nonsymmetric operads in terms of the associahedron or Stasheff polytope will show that this requirement actually holds true.

Chapter 1

Representation theory

In this chapter, we will collect some basic facts from representation theory, which will be useful in the sequel. In particular, we will be interested in representations of the symmetric group, which are important, as they will allow to encode the symmetry properties of the abstract operations of operads.

We are working over a ground field $\mathbb{K}$ of characteristic 0. This convention will be used throughout the whole text, although most constructions remain valid in any characteristic.

1.1 Tensor products

We first recall some facts from tensor algebra which will be needed in the following.

1.1.1 Tensor product of vector spaces

For two vector spaces V and W over a field $\mathbb{K}$, the tensor product $V \otimes W$ can be defined by the following universal property: $\otimes : V \times W \to V \otimes W$ is the bilinear map, such that for any $\mathbb{K}$-vector space U and any bilinear map $b : V \times W \to U$, there exists a unique linear map $\tilde{b} : V \otimes W \to U$, such that $b = \tilde{b} \circ \otimes$, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{\;b\;} & U. \\ {\scriptstyle \otimes}\downarrow & \nearrow_{\tilde{b}} & \\ V \otimes W & & \end{array}$$

In general, a solution of a universal property, if it exists, is unique (up to isomorphism).

In the case of the tensor product of vector spaces, the solution (in the case of finite-dimensional vector spaces) is given by

$$V \otimes W = \mathscr{L}_2(V^* \times W^*, \mathbb{K}),$$

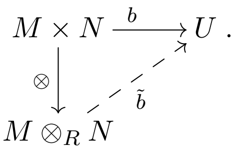
i.e. the tensor product is given by bilinear forms on the the dual.

1.1.2 Tensor product of modules over a commutative ring

We now want to extend the notion of tensor product to modules.

First note that for modules over a *commutative* ring R, the left module structure and the right module structure are in one-to-one correspondence. For instance, given a module M over the commutative ring R with a left module structure $r \cdot m$, we can define a right module structure by $m \cdot r := r \cdot m$, for $m \in M$ and $r \in R$. This gives indeed a right module structure, since $(m \cdot r) \cdot s = s \cdot (r \cdot m) = (sr) \cdot m = m \cdot (sr) = m \cdot (rs)$, for $m \in M$ and $r, s \in R$.

We define the tensor product of two modules M and N over the same commutative ring R among the same lines as previously, by the universal property: $\otimes : M \times N \to M \otimes_R N$ is the R-bilinear map, such that for any R-module U and any R-bilinear map $b : M \times N \to U$, there exists a unique R-linear map $\tilde{b} : M \otimes_R N \to U$, such that $b = \tilde{b} \circ \otimes$, i.e. such that the following diagram commutes:



In order to construct the tensor product $M \otimes_R N$, consider the free R-module $R[M \times N]$ generated by $M \times N$, which is as set

$$R[M \times N] = \Big\{ \sum_{\substack{m \in M \\ n \in N}} r_{(m,n)} e_{(m,n)} \Big\},$$

where only a finite number of coefficients $r_{(m,n)} \in R$ are nonzero. Moreover, consider the R-submodule generated by the elements

$$-e_{(m+m',n)} + e_{(m,n)} + e_{(m',n)}, \; -e_{(m,n+n')} + e_{(m,n)} + e_{(m,n')},$$

$$-e_{(rm,n)} + re_{(m,n)}, \; -e_{(m,rn)} + re_{(m,n)},$$

which represent R-bilinearity. We now define $M \otimes_R N$ as being the quotient of $R[m \times n]$ by this R-submodule. This quotient, which is itself again an R-module, together with the R-bilinear map

$$\otimes : M \times N \ni (m, n) \mapsto m \otimes n = [e_{(m,n)}] \in M \otimes_R N$$

is the solution of the universal property.

1.1.3 Tensor product of modules over a noncommutative ring

In the case of modules over a *noncommutative* ring R, the left and right module structures are not necessarily equivalent. Consider now a right R-module M and a left R-module N, then the tensor product $M \otimes_R N$ is only an abelian group, i.e. a $\mathbb{Z}$-module.

It is defined, as previously, as being the quotient of the free $\mathbb{Z}$-module $\mathbb{Z}[M \times N]$, generated by $M \times N$, by the $\mathbb{Z}$-submodule generated by the elements

$$-e_{(m+m',n)} + e_{(m,n)} + e_{(m',n)}, \; -e_{(m,n+n')} + e_{(m,n)} + e_{(m,n')},$$

$$-e_{(mr,n)} + e_{(m,rn)},$$

which correspond to 'weakened bilinearity'. Note that the last element corresponds to requesting $mr \otimes n = m \otimes rn$, whereas the first two correspond to biadditivity. The tensor product $\mathbb{Z}$-module $M \otimes_R N$ and the natural weakly bilinear map

$$\otimes : M \times N \ni (m, n) \mapsto m \otimes n = [e_{(m,n)}] \in M \otimes_R N$$

are universal. This means that the functor $- \otimes_R N$ from $\texttt{Mod}_R$ to $\texttt{AbGrp}$ is the left adjoint of the functor $\mathrm{Hom}_{\mathbb{Z}}(N, -)$, where the right module structure on $\mathrm{Hom}_{\mathbb{Z}}(N, P)$ is defined by $(fr)(n) = f(rn)$, i.e. we have

$$\mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N, P) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(N, P)),$$

functorially in M and P.

In general it is not possible to define an R-module structure on $M \otimes_R N$. For instance, consider a left R-action on $M \otimes_R N$, which can only be defined by $r \cdot (m \otimes n) = (m \otimes rn)$. Then

$$r \cdot (mr' \otimes n) = mr' \otimes rn = m \otimes (r'r)n,$$

but

$$r \cdot (m \otimes r'n) = m \otimes (rr')n,$$

which are in general not the same, since R is noncommutative.

However, if M is an (S, R)-bimodule, i.e. M is an abelian group equipped with a left S-module structure and a right R-module structure, which are compatible in the sense that $s(mr) = (sm)r$, then we get a left S-module structure on $M \otimes_R N$, where the left S-action is defined by acting by $s \in S$ from the left on the left factor of the product, i.e. $s \cdot (m \otimes n) := sm \otimes n$. Indeed, this does not lead to a contradiction in view of the compatibility of the left and right module structures, since

$$s \cdot (mr \otimes n) = s(mr) \otimes n$$

and

$$s \cdot (m \otimes rn) = sm \otimes rn = (sm)r \otimes n = s(mr) \otimes n.$$

Similarly, if N is an (R, T)-bimodule, i.e. N is an abelian group equipped with a left R-module structure and a right T-module structure, which are compatible in the sense that $r(nt) = (rn)t$, then $M \otimes_R N$ is a right T-module. The right T-action is defined by acting by $t \in T$ from the right on the right factor, i.e. $(m \otimes n) \cdot t := m \otimes nt$.

If M and N each have bimodule structures as above, then the tensor product $M \otimes_R N$ is an (S, T)-bimodule.

1.2 Representations of finite groups

In this section, we will recall the basic definitions and results concerning representation theory of finite groups. In particular, representations of the symmetric group S_n will later play an important role.

1.2.1 Definitions and examples

Definition 1.1: A *representation* (V, ρ) of a finite group G on a finite-dimensional vector space V over a field $\mathbb{K}$ is a group homomorphism

$$\rho : G \to \mathrm{Aut}(V).$$

This definition entails, in particular, that $\rho(e) = \mathrm{id}_V$ and $\rho(gg') = \rho(g)\rho(g')$, for any $g, g' \in G$. Here e denotes the identity element of the group G. As a direct consequence of these two relations, we also get that $\rho(g^{-1}) = (\rho(g))^{-1}$, for any $g \in G$.

Often, the representation space V will be called a representation. Moreover, we will mainly use a more simple notation by writing $g \cdot v$ or gv instead of $\rho(g)(v)$. The previous relations thus become $e \cdot v = v$ and $g \cdot (g' \cdot v) = (gg') \cdot v$, respectively $ev = v$ and $g(g'v) = (gg')v$, for any $g, g' \in G$.

This also justifies the terminology of *G-modules* and *G-actions* as synonyms for *representations.*

Definition 1.2: A *morphism* between two representations V and W is a liner map $\varphi : V \to W$, such that $\varphi(gv) = g\varphi(w)$, for any $g \in G, v \in V$. Such a map is also called a *G-morphism* or a *G-linear map*. The space of G-morphisms between the representations V and W is denoted by $\mathrm{Hom}_G(V, W)$.

Definition 1.3: A *subrepresentation* of a representation V is a vector subspace $W \subset V$, which is invariant under the action of G, i.e. $g \cdot w \in W$, for any $g \in G, w \in W$.

Definition 1.4: A representation V is called *irreducible* if V admits no proper G-invariant subspace.

Let V and W be representations of the group G. Then the direct sum $V \oplus W$ and the tensor product $V \otimes W$ are also representations of G. The n^{th} tensor power $\bigotimes^n V = V^{\otimes n}$, the n^{th} antisymmetric tensor power (or exterior product) $\bigwedge^n V$ and n^{th} symmetric tensor power $\bigvee^n V = S^n V$ are also representations of G, the later two being subrepresentations of the first one. The dual $V^* = \mathrm{Hom}(V, \mathbb{K})$ is a representation of G. Moreover, $\mathrm{Hom}(V, W)$ is a representation via the identification $\mathrm{Hom}(V, W) = V^* \otimes W$.

Examples 1.1: Here are some first examples of representations, which will be useful later.

- The *trivial representation*, where $V = \mathbb{K}$ and the action of G is defined by
$$g \cdot v = v, \text{ for any } v \in V, g \in G.$$
- The *regular representation*, where V is generated by the base vectors $\{e_g : g \in G\}$; elements of V are thus of the form $\sum_{g \in G} k_g e_g$, where $k_g \in \mathbb{K}$. The action of G is given by
$$g' \cdot \sum_{g \in G} k_g e_g = \sum_{g \in G} k_g e_{g'g}.$$
- If $G = S_n$, the *signature representation* is given by the one-dimensional vector space $V = \mathbb{K}$ and the action $g \cdot v = \mathrm{sign}(g)v$.

The following theorem is fundamental in representation theory, as it allows to decompose any (complex) representation into irreducible ones. It thus suffices to concentrate on the study of irreducible representations.

Theorem 1.1: *Any representation V (over a field of characteristic 0) of a finite group can be uniquely decomposed into a direct sum of irreducible representations V_i, i.e.*
$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}.$$

1.2.2 Group algebra

The *group algebra* $\mathbb{K}[G]$ is an important concept, which allows to formulate results of representation theory of finite groups in terms of representations of associative unital algebras.

The group algebra $\mathbb{K}[G]$ consists of the vector space of formal linear combinations $\sum_{g \in G} k_g e_g$ with coefficients $k_g \in \mathbb{K}$, endowed with a (bilinear) multiplication given by $e_g \cdot e_{g'} = e_{gg'}$. Note that he underlying vector space is the same as the representation space for the regular representation.

A representation of the algebra $\mathbb{K}[G]$ on a vector space V is a morphism $\tilde{\rho} : \mathbb{K}[G] \to \mathrm{End}(V)$ of associative unital algebras.

Proposition 1.2: *Representations of the group G and representations of the group algebra $\mathbb{K}[G]$ are in one-to-one correspondence.*

Proof: First, assume that $\rho : G \to \mathrm{Aut}(V)$ is given. We can define $\tilde\rho : \mathbb{K}[G] \to \mathrm{End}(V)$ by $\tilde\rho : \sum_{g\in G} k_g e_g \mapsto \sum_{g\in G} k_g \rho(g)$. Using that ρ is a group morphism, it is easily checked that $\tilde\rho$ is a morphism of associative unital algebras.

Conversely, if $\tilde\rho : \mathbb{K}[G] \to \mathrm{End}(V)$ is given, we can define $\rho : G \to \mathrm{Aut}(V)$ by $\rho : g \mapsto \tilde\rho(e_g)$. Again we can easily check that this defines a group morphism. Moreover, we have to show that the image of ρ really is $\mathrm{Aut}(V)$. Therefore, we have to show that $\tilde\rho(e_g)$ admits an inverse. This inverse is given by $\tilde\rho(e_{g^{-1}})$. Indeed, $\tilde\rho(e_g) \circ \tilde\rho(e_{g^{-1}}) = \tilde\rho(e_g e_{g^{-1}}) = \tilde\rho(e_{gg^{-1}}) = \tilde\rho(1) = \mathrm{id}$ and by the same argument $\tilde\rho(e_{g^{-1}}) \circ \tilde\rho(e_g) = \mathrm{id}$. □

1.3 Induced representation

If H is a subgroup of G and V a representation of G, one can always restrict V in order to obtain a representation of H, denoted by $\mathrm{Res}^G_H V$. This concept of restriction being quite natural, we are more interested in the converse notion. Starting from a representation W of the subgroup H, we want to construct a representation of the group G.

First, recall that G can be partitioned in left cosets gH, given by the right action of H on G. In the following, we will denote the coset classes by $\sigma \in {}^{G}\!/\!_{H}$ and choose a representative g_σ of each class. In particular, we will choose $g_H = e$.

Note that if V is a representation of G and W a H-invariant subspace of V, then, for $g \in G$, the subspace $g \cdot W$ only depends on the left cosets gH, as $(gh) \cdot W = g \cdot (h \cdot W) = g \cdot W$.

Therefore, in order to construct the induced representation V, we will consider for each coset σ a copy $W^\sigma = g_\sigma W$ of W and define

$$V = \mathrm{Ind}^G_H W = \bigoplus_{\sigma\in G/H} W^\sigma = \bigoplus_{\sigma\in G/H} g_\sigma W.$$

In order to define the action of the group G on the representation space $V = \mathrm{Ind}^G_H W$, note that for every $g \in G$ and $\sigma \in G/H$, there exist a unique $\tau \in {}^{G}\!/\!_{H}$ and a unique $h \in H$, such that $gg_\sigma = g_\tau h$. For $g_\sigma w \in g_\sigma W = W^\sigma$ and $g \in G$, we define

$$g \cdot (g_\sigma w) = g_\tau(hw) \in g_\tau W,$$

where g_τ and h are those uniquely given by the previous relation. In particular, this proves the uniqueness of the induced representation.

It remains to prove that the above defined induced representation is really a representation and that by restricting to H, we get the initial action on W back. We have to show that $g' \cdot (g \cdot (g_\sigma w_\sigma)) = (g'g) \cdot (g_\sigma w_\sigma)$, for $g, g' \in G$. If $g' g_\tau = g_\rho h'$ for some $\rho \in {}^{G}\!/\!_{H}$ and $h' \in H$, then $g' \cdot (g \cdot (g_\sigma w_\sigma)) = g' \cdot (g_\tau(hw_\sigma)) = g_\rho(h'(hw_\sigma))$. As $(g'g)g_\sigma = g'(gg_\sigma) = (g'g_\tau)h = g_\rho(h'h)$, we get the requested result. Moreover, for $h \in H$ and $w \in W$, the previously defined action becomes $h \cdot w = h \cdot (ew) = e \cdot (hw) = hw$, which is the initial action of H on W.

An alternative approach to the induced representation is given by

Proposition 1.3: *The induced representation of a representation W of a subgroup H of a group G is*

$$\mathbb{K}[G] \otimes_H W = \mathbb{K}\left[{}^{G}\!/\!_{H}\right] \otimes W$$

(equality of vector spaces), endowed with the canonical G-action.

The tensor product $\mathbb{K}[G] \otimes_H W$ can be viewed as tensor product of vector spaces, or as tensor product over the (not necessarily commutative) ring $\mathbb{K}[H]$. Note that W is endowed with a left $\mathbb{K}[H]$-module structure (since it is a representation), whereas $\mathbb{K}[G]$ admits naturally a right $\mathbb{K}[H]$-module structure and a left $\mathbb{K}[G]$-module structure. These being compatible, it admits a $(\mathbb{K}[G], \mathbb{K}[H])$-bimodule structure. Thus $\mathbb{K}[G] \otimes_H W$ admits a left $\mathbb{K}[G]$-module structure, i.e. it can be viewed as a representation of G.

Proof: Concerning the representation space, we get, using the notations of the previous definition of the induced representation,

$$\begin{aligned}\mathbb{K}[G] \otimes_H W &= \left\{\sum k e_g \otimes w\right\} = \left\{\sum k e_{g_\sigma} h \otimes w\right\} \\ &= \left\{\sum k e_{g_\sigma} \otimes hw\right\} = \left\{\sum k e_{g_\sigma} \otimes w'\right\} \\ &= \bigoplus_\sigma e_{g_\sigma} \otimes W = \bigoplus_\sigma W^\sigma,\end{aligned}$$

on one hand. On the other,

$$\mathbb{K}[G] \otimes_H W = \left\{\sum k e_{g_\sigma} \otimes w'\right\} = \mathbb{K}\left[{}^{G}\!/_{H}\right] \otimes W.$$

Concerning the G-action, we have

$$g \cdot (e_{g_\sigma} \otimes w) = e_{g g_\sigma} \otimes w = e_{g_\tau} \otimes hw,$$

where $g g_\sigma = g_\tau h$ as previously. □

Chapter 2

Algebras, coalgebras and homology

In this chapter, we will collect some simple facts from algebra and homological algebra, which are needed in the following. We will deal with algebras and coalgebras, which are 'dual' to each other, as well as with graded and differential graded structures.

2.1 Algebras

The concepts related to algebras are well known, therefore this section is intended to fix notations and terminology.

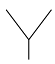
Definition 2.1: An *associative algebra* A is a vector space (over $\mathbb{K}$) together with a linear map
$$\mu : A \otimes A \to A,$$
called *multiplication* (or *product*), which is associative, i.e. verifies $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$.

An associative algebra A is said to be *unital* if there exists a map
$$u : \mathbb{K} \to A,$$
called *unit*, such that $\mu \circ (u \otimes \mathrm{id}) = \mathrm{id} = \mu \circ (\mathrm{id} \otimes u)$.

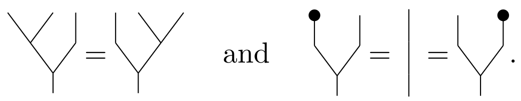
Note that u sends $1_{\mathbb{K}}$ to 1_A and thus $\mathbb{K}$ to $\mathbb{K}1_A \subset A$.

Associativity and unitality can be formulated by means of the following commutative diagrams:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id} \otimes \mu} & A \otimes A \\ {\scriptstyle \mu \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} \mathbb{K} \otimes A & \xrightarrow{u \otimes \mathrm{id}} & A \otimes A & \xleftarrow{\mathrm{id} \otimes u} & A \otimes \mathbb{K} \, . \\ & {\scriptstyle \simeq} \searrow & \downarrow {\scriptstyle \mu} & \swarrow {\scriptstyle \simeq} & \\ & & A & & \end{array}$$

The multiplication can be represented by the tree diagram  and the unit by .

Associativity and unitality are then given by



A *morphism of algebras* is a linear map which respects multiplication and units. Associative algebras and algebra morphism form a category denoted by `Alg`.

Definition 2.2: An associative unital algebra A is called *augmented* if it is an algebra of the form

$$A = \mathbb{K}1_A \oplus \bar{A}.$$

This requirement can be encrypted in the existence of an algebra morphism $\varepsilon : A \to \mathbb{K}$, called *augmentation map*. Looking at the decomposition to be proven, we set $\bar{A} := \ker \varepsilon$ (note that $\bar{A}$ is an ideal), thus obtaining a short exact sequence of algebras

$$0 \to \ker \varepsilon \to A \to \mathbb{K} \to 0.$$

Note that ε sends 1_A to $1_{\mathbb{K}}$. Moreover, $\varepsilon \circ u = \mathrm{id}_{\mathbb{K}}$ and thus the sequence is split, resulting in the required decomposition.

Definition 2.3: The *tensor module* over a vector space V is defined by

$$T(V) = \bigoplus_{n \in \mathbb{N}} V^{\otimes n} = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus \cdots .$$

The tensor module $T(V)$ together with the *concatenation product*, defined by

$$v_1 \cdots v_p \otimes w_1 \cdots w_q = v_1 \cdots v_p w_1 \cdots w_q \in V^{\otimes (p+q)},$$

for elements $v_1 \cdots v_p \in V^{\otimes p}$ and $w_1 \cdots w_q \in V^{\otimes q}$, where the tensor multiplication is partially omitted, defines an associative unital algebra, called *tensor algebra*.

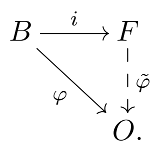
Moreover, this algebra is augmented. The *reduced tensor algebra* is given by the *reduced tensor module*

$$\bar{T}(V) = \bigoplus_{n \in \mathbb{N}^*} V^{\otimes n} = V \oplus V^{\otimes 2} \oplus \cdots$$

and the concatenation product. Note that this is a nonunital associative algebra.

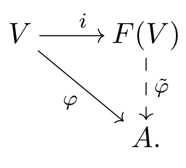
Free objects are the generalization to categories of the notion of a basis in a vector space, in the sense that, if we consider a basis B of a vector space V_1 and a linear map $\ell : B \to V_2$, where V_2 is a second vector space, then then this linear map ℓ can be uniquely extended to a linear map $\tilde{\ell} : V_1 \to V_2$.

Let $\mathtt{C}$ be a category, B a set (called basis), F an object in $\mathtt{C}$ and $i : B \to F$ a function (called canonical injection). This definition should be written using a faithful functor $\mathscr{F} : \mathtt{C} \to \mathtt{Set}$. We say that F is the free object over the basis B (with respect to i) if and only if they satisfy the universal property: For any object O and any function $\varphi : B \to O$, there exists a unique morphism $\tilde{\varphi} : F \to O$, such that $\varphi = \tilde{\varphi} \circ i$, i.e. the following diagram commutes:



Equivalently, one can define a free functor F as being the left adjoint functor to the forgetful functor.

Definition 2.4: The *free associative algebra* over a vector space V is the associative algebra $F(V)$ together with the linear map $i : V \to F(V)$, such that for any associative algebra A and any linear map $\varphi : V \to A$ there exists a unique algebra morphism $\tilde{\varphi} : F(V) \to A$ such that $\varphi = \tilde{\varphi} \circ i$, i.e. the following diagram commutes:



Remark: The tensor algebra is free in the category of associative unital algebras.

Definition 2.5: A *derivation* is a linear map $d : A \to A$ that verifies Leibniz rule, i.e.

$$d(ab) = d(a)b + ad(b),$$

for any $a, b \in A$.

Proposition 2.1: *Any linear map $f : V \to T(V)$ can be uniquely extended to a derivation $d : T(V) \to T(V)$ of the tensor algebra.*

Proof: It suffices to set $d(v_1 \cdots v_p) = \sum_{k=1}^{p} v_1 \cdots f(v_k) \cdots v_p$, for any $v_1 \cdots v_p \in V^{\otimes p}$. □

2.2 Coalgebras

The concept of 'coalgebra' is 'dual' to that of algebra. The (linear) dual of a coalgebra is always an algebra, although the dual of an algebra is a coalgebra only in finite dimension.

Duality must be seen in the sense that the definition of a coassociative counital coalgebra is obtained from that of an associative unital algebra by reversing all the arrows.

Definition 2.6: A *coassociative coalgebra* C is a vector space (over $\mathbb{K}$) together with a linear map

$$\Delta : C \to C \otimes C,$$

called *comultiplication* (or *coproduct*), which is coassociative, i.e. verifies $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$.

A coassociative coalgebra C is said to be *counital* if there exists a map

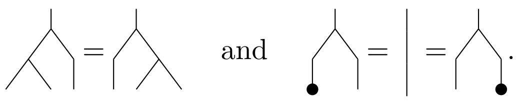
$$\varepsilon : C \to \mathbb{K},$$

called *counit*, such that $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$.

Coassociativity and counitality can be formulated by means of the following commutative diagrams:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes \Delta} \\ C \otimes C & \xrightarrow{\Delta \otimes \mathrm{id}} & C \otimes C \otimes C \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} & & C & & \\ & \swarrow{\scriptstyle\simeq} & \downarrow{\scriptstyle\Delta} & {\scriptstyle\simeq}\searrow & \\ C \otimes \mathbb{K} & \xleftarrow{\mathrm{id} \otimes \varepsilon} & C \otimes C & \xrightarrow{\varepsilon \otimes \mathrm{id}} & \mathbb{K} \otimes C. \end{array}$$

The comultiplication can be represented by the tree diagram  and the counit by . Coassociativity and counitality are then given by



A *morphism of coalgebras* is a linear map which respects comultiplication and counits. Coassociative coalgebras and coalgebra morphisms form a category denoted by `CoAlg`.

Let us detail the dual correspondence between algebras and coalgebras in the finite-dimensional case. Therefore, consider the map

$$\omega : V^* \otimes W^* \to (V \otimes W)^*,$$

which is well-known to be an isomorphism if the vector spaces V and W are finite-dimensional.

Considering now an associative unital algebra A with multiplication $\mu : A \otimes A \to A$ and unit $u : \mathbb{K} \to A$, then

$$\begin{aligned} \Delta &:= {}^t\mu : A^* \to (A \otimes A)^* \simeq A^* \otimes A^* \\ \varepsilon &:= {}^t u : A^* \to \mathbb{K}^* \simeq \mathbb{K} \end{aligned}$$

defines a coassociative counital coalgebra structure on A^*, if A is finite-dimensional.

However, if C is a coassociative counital coalgebra with comultiplication $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to \mathbb{K}$, then

$$\begin{aligned} \mu &:= {}^t\Delta \circ \omega : C^* \otimes C^* \to (C \otimes C)^* \to C^* \\ u &:= {}^t\varepsilon : \mathbb{K}^* \simeq \mathbb{K} \to C^* \end{aligned}$$

defines, in any dimension, an associative unital algebra structure on C^*.

Note that $\mathbb{K}$ is itself a coalgebra, since as a vector space $\mathbb{K}$ is finite-dimensional.

Remark: In the following, we will mention the prefix 'co-' only once.

Definition 2.7: An associative unital coalgebra C is called *coaugmented* if it is a coalgebra of the form

$$C = \bar{C} \oplus \mathbb{K}1_C.$$

This requirement can be encrypted in the existence of a coalgebra morphism $u : \mathbb{K} \to C$, called *coaugmentation map.* This means in particular that u respects the counits ε of C and $\mathrm{id}_{\mathbb{K}}$ of $\mathbb{K}$, i.e. $\varepsilon \circ u = \mathrm{id}_{\mathbb{K}}$. Observe that the augmentation map corresponds to the counit and that the coaugmentation map corresponds to the unit.

Just like previously, setting $\bar{C} := \ker \varepsilon$, we obtain a split short exact sequence:

$$0 \to \ker \varepsilon \to C \to \mathbb{K} \to 0.$$

Note that $\varepsilon(u(1_{\mathbb{K}})) = 1_{\mathbb{K}}$, thus setting $1_C := u(1_{\mathbb{K}})$ gives $\varepsilon(1_C) = 1_{\mathbb{K}}$.

Examples 2.1:

1. The *tensor coalgebra* $T^c(V)$ is the tensor module $T(V)$ together with the *deconcatenation coproduct* Δ defined by

$$\Delta(v_1 \cdots v_p) = \sum_{k=0}^{p} (v_1 \cdots v_k) \otimes (v_{k+1} \cdots v_p) \in T^c(V) \otimes T^c(V),$$

   for elements $v_1 \cdots v_p \in V^{\otimes p} \subset T^c(V)$. More precisely,

$$\Delta(v_1 \cdots v_p) = 1 \otimes (v_1 \cdots v_p) + v_1 \otimes (v_2 \cdots v_p) + \cdots + (v_1 \cdots v_p) \otimes 1.$$

2. The *reduced tensor coalgebra* $\bar{T}^c(V)$ is the reduced tensor module $\bar{T}(V)$ together with the reduced deconcatenation coproduct $\bar{\Delta}$ defined by

$$\bar{\Delta}(v_1 \cdots v_p) = \sum_{k=1}^{p-1} (v_1 \cdots v_k) \otimes (v_{k+1} \cdots v_p) \in \bar{T}^c(V) \otimes \bar{T}^c(V),$$

   for elements $v_1 \cdots v_p \in V^{\otimes p} \subset \bar{T}^c(V)$.

   Note that

$$\bar{\Delta}(v_1 \cdots v_p) = \Delta(v_1 \cdots v_p) - 1 \otimes (v_1 \cdots v_p) - (v_1 \cdots v_p) \otimes 1.$$

More generally, if (C, Δ) is an augmented coalgebra, then $\bar{C}$ carries the comultiplication $\bar{\Delta}$ defined by $\bar{\Delta}(x) = \Delta(x) - 1 \otimes x - x \otimes 1$, for $x \in \bar{C}$. The fact that the reduced coproduct is valued in $\bar{C} \otimes \bar{C}$ is easily checked by applying $\varepsilon \otimes \mathrm{id}$ and $\mathrm{id} \otimes \varepsilon$ to $\bar{\Delta}(x)$.

Definition 2.8: A *coideal* I of a coalgebra C is a vector subspace $I \subset C$, such that $I \subset \ker \varepsilon$ and $\Delta(x) \in I \otimes C + C \otimes I$, for any $x \in I$.

Remark: The reduced coalgebra $\bar{C} = \ker \varepsilon$ is a coideal of the coalgebra C.
Indeed, $\Delta(x) = \bar{\Delta}(x) + 1 \otimes x + x \otimes 1 \in \bar{C} \otimes \bar{C} + C \otimes \bar{C} + \bar{C} \otimes C \subset \bar{C} \otimes C + C \otimes \bar{C}$, for any $x \in \bar{C}$.

The *iterated coproduct* $\Delta^n : C \to C^{\otimes(n+1)}$ is defined by $\Delta^n = (\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \circ \Delta^{n-1}$, with $\Delta^1 = \Delta$ and $\Delta^0 = \mathrm{id}$. Due to coassociativity, we have that $\Delta^n = (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \circ \Delta^{n-1}$. The reduced iterated coproduct $\bar{\Delta}^n$ is defined exactly among the same lines.

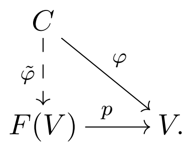
Definition 2.9: Let C be an augmented associative unital coalgebra. We say that C is *conilpotent* if the filtration

$$\mathscr{F}_0 C = \mathbb{K}1_C,$$
$$\mathscr{F}_r C = \mathbb{K}1_C \oplus \{x \in \bar{C} : \bar{\Delta}^n(x) = 0, \forall\, n \geq r\}, \quad r \geq 1,$$

is exhaustive.

Conilpotency means that each element of $\bar{C}$ is annihilated by some power of the reduced coproduct $\bar{\Delta}$. Note also that a nilpotent coalgebra is augmented and hence also unital.

Definition 2.10: The *cofree associative coalgebra* over a vector space V is the nilpotent associative coalgebra $F^c(V)$ together with the linear map $p : F^c(V) \to V$, such that for any nilpotent associative coalgebra C and any linear map $\varphi : C \to V$, with $\varphi(1_C) = 0$, there exists a unique coalgebra morphism $\tilde{\varphi} : C \to F^c(V)$ such that $\varphi = p \circ \tilde{\varphi}$, i.e. the following diagram commutes:



Remark: The tensor coalgebra is free in the category of nilpotent associative coalgebras.

Definition 2.11: Let C be a nilpotent coalgebra. A *coderivation* $d : C \to C$ of coalgebras is a linear map such that

$$\Delta \circ d = (d \otimes \mathrm{id}) \circ \Delta + (\mathrm{id} \otimes d) \circ \Delta.$$

Proposition 2.2: *A coderivation $d : \bar{T}^c(V) \to \bar{T}^c(V)$ is uniquely determined by its corestriction $f : \bar{T}^c(V) \to V$, i.e. $f = \mathrm{pr}_1 \circ d$, where pr_1 is the projection of $\bar{T}^c(V)$ onto V.*

Proof: Consider an element $v \in V$ and note that, due to conilpotency, $\bar{\Delta}v = 0$. Moreover,

$$\bar{\Delta} dv = (d \otimes \mathrm{id})\bar{\Delta} v = 0,$$

and thus $dv \in V$. Therefore, $dv = \mathrm{pr}_1(dv) = fv$.
For an element $vw \in V^{\otimes 2}$, we have

$$\bar{\Delta} d(vw) = (d \otimes \mathrm{id})(v \otimes w) = dv \otimes w + v \otimes dw = fv \otimes w + v \otimes fw = \bar{\Delta}((fv)w + v(fw)),$$

and thus $d(vw) = (fv)w + v(fw) + v'$, where v' is an element of V. Moreover, this element v' is the component of $d(vw)$ in V, i.e. $v' = \mathrm{pr}_1(d(vw)) = f(vw)$. Again, d is completely determined by f.

For an element $vwx \in V^{\otimes 3}$,

$$\begin{aligned}\bar{\Delta}d(vwx) &= (d\otimes \mathrm{id})(v\otimes(wx) + (vw)\otimes x)\\ &= dv\otimes(wx) + d(vw)\otimes x + v\otimes d(wx) + (vw)\otimes dx\\ &= fv\otimes(wx) + (fv)w\otimes x + v(fw)\otimes x + (fvw)\otimes x\\ &\quad + v\otimes(fw)x + v\otimes w(fx) + v\otimes f(wx) + (vw)\otimes fx\\ &= \bar{\Delta}((fv)wx + v(fw)x + vw(fx) + f(vw)x + vf(wx)),\end{aligned}$$

and thus $d(vwx) = (fv)wx + v(fw)x + vw(fx) + f(vw)x + vf(wx) + f(vwx)$.

Similarly, d is completely determined by f for any higher tensor powers. □

2.3 Differential graded algebras and coalgebras

2.3.1 Graded vector spaces

We will now consider $\mathbb{Z}$-graded vector spaces $V = \bigoplus_{n\in\mathbb{Z}} V_n$. Sometimes all the terms of nonpositive degree are $\{0\}$, in this case the considered vector space is $\mathbb{N}$-graded, i.e. $V = \bigoplus_{n\in\mathbb{N}} V_n$. If all the terms of nonnegative degree vanish, i.e. if $V = \bigoplus_{n\in\mathbb{N}} V_{-n}$, we set $V^n := V_{-n}$, so that $V = \bigoplus_{n\in\mathbb{N}} V^n$. If the degrees of the subspaces are denoted by subscripts (resp. superscripts), we say that the space is *homologically* (resp. *cohomologically*) *graded*.

We know that the category `Vect` of vector spaces and linear maps is a (symmetric) monoidal category (i.e. a category with a tenor product, see also definition 6.1). Also the category `grVect` of graded vector spaces and degree 0 linear maps is monoidal. The grading of the product $V\otimes W$ of two graded vector spaces is induced by the gradings of these spaces:

$$V\otimes W = \Big(\bigoplus_i V_i\Big)\otimes\Big(\bigoplus_j W_j\Big) = \bigoplus_{ij}(V_i\otimes W_j) = \bigoplus_n\left(\bigoplus_{i+j=n} V_i\otimes W_j\right) =: \bigoplus_n (V\otimes W)_n.$$

When considering the tensor module $T(V) = \bigoplus_{n\in\mathbb{N}} V^{\otimes n}$, we actually have two gradings, the grading by the number n of factors (given by the tensor power), called the *weight* (or cohomological degree), and the just detailed grading induced by that of V, called the *degree* (or internal degree).

Remark: A graded vector space can also be viewed as a sequence of vector spaces $(V_n)_{n\in\mathbb{Z}}$ instead of a direct sum. This allows to avoid some difficulties, in particular to define the dual of the sequence as the sequence of the duals, whereas the dual of a direct sum is the direct sum of the duals only for finitely many summands.

Definition 2.12: The *suspension* of a graded vector space V is given by

$$sV = \mathbb{K}s\otimes V,$$

where $\mathbb{K}s$ denotes the one-dimensional graded vector space generated by the element s of degree 1. This implies a change of degree: $(sV)_i = V_{i-1}$.

Definition 2.13: The *desuspension* of a graded vector space V is given by

$$s^{-1}V = \mathbb{K}s^{-1}\otimes V,$$

where $\mathbb{K}s^{-1}$ denotes the one-dimensional graded vector space generated by the element s^{-1} of degree -1. This implies a change of degree: $(s^{-1}V)_i = V_{i+1}$.

Definition 2.14: A *braided monoidal category* is a monoidal category equipped with a braiding, i.e. with a family of natural isomorphisms $\gamma_{A,B} : A \otimes B \to B \otimes A$ that satisfy some axiom.

A *symmetric monoidal category* is a braided monoidal category, where the braiding verifies $\gamma_{B,A} = \gamma_{A,B}^{-1}$.

The category `grVect` is symmetric. The braiding is given by the switching map τ with Koszul sign, defined by

$$\tau : V \otimes W \to W \otimes V, \quad \tau(v \otimes w) = (-1)^{\tilde{v}\tilde{w}} w \otimes v.$$

Here $\tilde{v}$ and $\tilde{w}$ denote the degree of the corresponding elements.

Recall that the *Koszul sign rule* states that, in a sign-graded setting, a sign, depending on the degree of the involved elements, appears whenever two elements are interchanged.

2.3.2 Differential graded vector spaces

Definition 2.15: A *differential graded vector space* (or *chain complex*) (V, d) is a graded vector space V together with a linear map d, called *differential*, of degree -1 and satisfying $d^2 = 0$.

$$\cdots \xleftarrow{d} V_{-1} \xleftarrow{d} V_0 \xleftarrow{d} V_1 \xleftarrow{d} V_2 \xleftarrow{d} \cdots$$

A *cochain complex* is given by the cohomological grading $V^n = V_{-n}$. In this case the differential d is of degree 1.

$$\cdots \xrightarrow{d} V^{-1} \xrightarrow{d} V^0 \xrightarrow{d} V^1 \xrightarrow{d} V^2 \xrightarrow{d} \cdots$$

Definition 2.16: A *morphism f of chain complexes* (resp. *cochain complexes*) (V, d) and (W, δ) (or *chain map*, resp. *cochain map*) is a linear map $f : V \to W$ of degree 0 which commutes with differentials, i.e. $f \circ d = \delta \circ f$.

Considering a chain complex (V, d), it is sometimes useful to denote the differential (or *boundary map*) d, in a more explicit way, by $d_n : V_n \to V_{n-1}$. Note that $d^2 = 0$ explicitly reads as $d_n \circ d_{n+1} = 0$, and thus $\operatorname{im} d_{n+1} \subset \ker d_n$. Elements of $\ker d_n$ are called *cycles* and elements of $\operatorname{im} d_{n+1}$ are called *boundaries*.

The *n-th homology group* is by definition

$$H_n(V, d) = {}^{\ker d_n}\!/_{\operatorname{im} d_{n+1}}.$$

We denote $H_\bullet(V, d) = \bigoplus_{n\in\mathbb{Z}} H_n(V, d)$.

Similarly, for a cochain complex (V, d), the differential (or *coboundary map*) d reads, in a more explicit way, $d^n : V^n \to V^{n+1}$, and $d^2 = 0$ becomes $d^n \circ d^{n-1} = 0$, and thus $\operatorname{im} d^{n-1} \subset \ker d^n$. Elements of $\ker d^n$ are called *cocycles* and elements of $\operatorname{im} d^{n-1}$ are called *coboundaries*.

The *n-th cohomology group* is by definition

$$H^n(V, d) = {}^{\ker d^n}\!/_{\operatorname{im} d^{n-1}}.$$

We denote $H^\bullet(V, d) = \bigoplus_{n\in\mathbb{Z}} H^n(V, d)$.

A (co)chain complex is called *acyclic* if its (co)homology is 0 everywhere.

Note that a (co)chain map $f : V \to W$ induces a linear map $f_\sharp$ in (co)homology; if this map $f_\sharp$ is an isomorphism, we say that $f : V \xrightarrow{\sim} W$ is a *quasi-isomorphism*.

A *(co)chain homotopy* between two (co)chain maps $f : (V, d_V) \to (W, d_W)$ and $g : (V, d_V) \to (W, d_W)$ — note that (co)chain complexes, (co)chain maps and (co)chain homotopies form a 2-category — is a map h of degree 1 (resp. -1), such that

$$hd_V + d_W h = f - g.$$

If two (co)chain maps are homotopic, the induced maps in (co)homology coincide. In particular, the application to id and 0 allows showing that a complex is acyclic.

A *homotopy equivalence* between two chain complexes (V, d_V) and (W, d_W) is a chain map $i : V \to W$, such that there exists a chain map $p : W \to V$, such that $p \circ i$ is homotopic to id_V and $i \circ p$ is homotopic to id_W, i.e.

$$h' \circlearrowright (V, d_V) \underset{p}{\overset{i}{\rightleftarrows}} (W, d_W) \circlearrowleft h\ ,$$

$$p \circ i - \mathrm{id}_V = h'd_V + d_V h' \quad \text{and} \quad i \circ p - \mathrm{id}_W = hd_W + d_W h.$$

If $h' = 0$, then the map i is injective and the map p is surjective, and the complex V is called a *deformation retract* of W.

The category of chain complexes (in `Vect`) and chain maps is monoidal. The tensor product $V \otimes W$ of two chain complexes (V, d_V) and (W, d_W) is defined by

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j,$$

which is equipped with the differential

$$d = d_V \otimes \mathrm{id} + \mathrm{id} \otimes d_W.$$

This clearly defines a chain complex with a differential of degree -1.

Instead of considering only chain maps of degree 0, one can also consider chain maps of arbitrary degree r. We denote the space of chain maps $f : V \to W$ of degree r by $\mathrm{Hom}_r(V, W)$, and the space of all chain maps by $\mathrm{Hom}(V, W) = \bigoplus_r \mathrm{Hom}_r(V, W)$. Latter space is again a chain complex, if (V, d_V) and (W, d_W) are chain complexes. The differential ∂ of this complex is defined by

$$\partial f = [d, f] := d_W \circ f - (-1)^r f \circ d_V,$$

for any $f \in \mathrm{Hom}_r(V, W)$. This clearly defines a differential of degree -1, since d_V and d_W are differentials. Indeed,

$$\begin{aligned}\partial^2 f &= [d, [d, f]] = [d, d_W f] - (-1)^r [d, f d_V] \\ &= (d_W^2 - (-1)^{r-1} d_W f d_V) - (-1)^r (d_W f d_V - (-1)^{r-1} f d_V^2) = 0.\end{aligned}$$

2.3.3 Differential graded associative algebras and coalgebras

A *differential graded associative (unital) algebra* (DGAA) is a graded vector space with a compatible associative unital algebra structure and a compatible differential. This means that the bilinear multiplication is of degree 0 and respects the grading, i.e. $\widetilde{ab} = \tilde{a} + \tilde{b}$. Moreover, the differential d is a linear map of degree 0, such that $d^2 = 0$ and d is a derivation for the multiplication, i.e. verifies the graded Leibniz rule.

The definition of a *differential graded associative (unital) coalgebra* (DGAC) is similar to that of a DGAA.

Often the considered differential graded spaces have, in addition to the homological degree, an extra grading, called *weight*. This weight-grading has again to be compatible with the underlying structure. For instance, a weight-graded DGAA is a weight-graded differential graded vector space, together with an associative algebra structure that respects the weight grading, the homological grading, as well as the differential. A weight-graded DGAA A is denoted by $A = \bigoplus_m A_m$, when referring to the homological grading, and by $A = \bigoplus_n A^{(n)}$, when referring to the weight grading.

Chapter 3

Twisting morphisms and Koszul morphisms

This chapter deals with twisting and Koszul morphisms for associative algebras and coalgebras. Moreover, we take a special interest in the bar and cobar constructions, which will finally provide a model (given by the bar-cobar resolution) of the considered differential graded associative algebra.

3.1 Twisting morphisms and twisted tensor complexes

Let (A, μ, u, d_A) be a unital augmented DGAA and $(C, \Delta, \varepsilon, d_C)$ a unital augmented nilpotent DGAC.

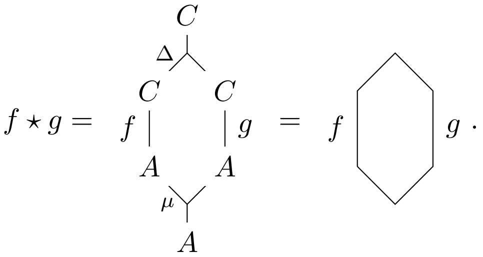
3.1.1 Convolution

We will consider the space $\mathrm{Hom}_{\mathbb{K}}(C, A)$ of $\mathbb{K}$-linear maps from C to A, and equipped it with a bilinear associative operation $\star$, called *convolution*. Equipped with the adequate differential, the considered space will be a unital DGAA.

The convolution $f \star g$ of $f, g \in \mathrm{Hom}_{\mathbb{K}}(C, A)$ is defined by

$$f \star g = \mu \circ (f \otimes g) \circ \Delta,$$

or pictorially,



Concerning degrees, we have $\widetilde{f \star g} = \tilde{f} + \tilde{g}$. The unit for $\star$ is given by $u \circ \mu \in \mathrm{Hom}_{\mathbb{K}}(C, A)$. Indeed, $(u \circ \varepsilon) \star g = g$, for any $g \in \mathrm{Hom}_{\mathbb{K}}(C, A)$, since

$$\mu((u \circ \varepsilon) \otimes g)\Delta c = \mu(u \otimes \mathrm{id})(\mathrm{id} \otimes g)(\varepsilon \otimes \mathrm{id})\Delta c = \mu(u \otimes \mathrm{id})(\mathrm{id} \otimes g)(1 \otimes c) = \mu(u \otimes \mathrm{id})(1 \otimes gc) = gc,$$

for any $c \in C$. Similarly, $f \star (u \circ \varepsilon) = f$, for any $f \in \mathrm{Hom}_{\mathbb{K}}(C, A)$.

The differential ∂, defined by

$$\partial f = [d, f] = d_A f - (-1)^{\tilde{f}} f d_C,$$

is of degree -1 and it can be checked that this is a derivation for the convolution $\star$.

3.1.2 Twisting morphisms

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra, the *Maurer-Cartan form* is a differential one-form ω on G valued in $\mathfrak{g}$, which encodes information about the structure of G and verifies the *Maurer-Cartan equation*

$$d\omega + \tfrac{1}{2}[\omega, \omega] = 0.$$

In the previously constructed convolution DGAA $(\mathrm{Hom}_{\mathbb{K}}(C, A), \star, \partial)$, we can write a similar equation:

$$\partial\alpha + \alpha \star \alpha = 0, \tag{3.1}$$

where $\alpha \in \mathrm{Hom}_{\mathbb{K}}(C, A)$.

Note that we get a condition on the degree of α. If the two terms in (3.1) were of different degree, they would be contained in different terms of the direct sum of the graded space $\mathrm{Hom}_{\mathbb{K}}(C, A)$, and the only possibility, in order to sum to zero, would be that they are both equal to zero. Thus, in order to obtain interesting solutions of (3.1), the two terms have to be of the same degree, i.e. α has to be of degree -1.

Definition 3.1: A *twisting morphism* $\alpha \in \mathrm{Tw}(C, A)$ is a morphism $\alpha \in \mathrm{Hom}_{\mathbb{K}}(C, A)$ of degree -1, that verifies the Maurer-Cartan equation (3.1) and vanishes on units and counits.

The last condition is of technical purpose and can be formulated as

$$\alpha \circ u = 0 \quad \text{and} \quad \varepsilon \circ \alpha = 0,$$

where $u : \mathbb{K} \to C$ denotes the coaugmentation map and $\varepsilon : A \to \mathbb{K}$ the augmentation map. Recalling that $\ker \varepsilon = \bar{A}$, this means that

$$\alpha(\mathbb{K}) = 0 \quad \text{and} \quad \alpha(\bar{C}) \subset \bar{A}.$$

3.1.3 Twisted tensor complex

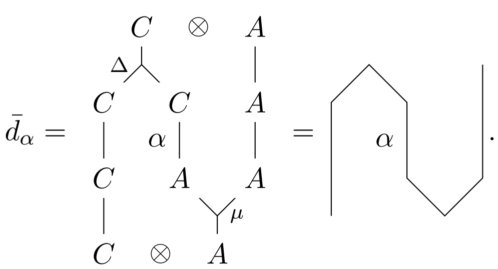
Consider the tensor complex $(C \otimes A, d)$, where $d = d_C \otimes \mathrm{id} + \mathrm{id} \otimes d_A$. Moreover, consider a morphism $\alpha \in \mathrm{Hom}_{\mathbb{K}}(C, A)$ and define

$$\bar{d}_\alpha = (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes \alpha \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id})$$

and

$$d_\alpha = d + \bar{d}_\alpha.$$

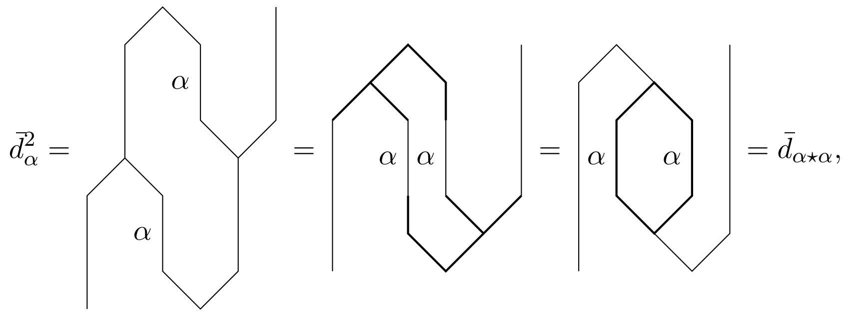
Pictorially,



Lemma 3.1: $d_\alpha^2 = 0$ *if and only if* α *is a twisting morphism.*

Definition 3.2: If $\alpha \in \mathrm{Tw}(C, A)$ is a twisting morphism, then $C \otimes_\alpha A := (C \otimes A, d_\alpha)$ is called *twisted tensor complex.*

Proof of lemma 3.1: We have that $d_\alpha^2 = d^2 + d \circ \bar{d}_\alpha + \bar{d}_\alpha \circ d + \bar{d}_\alpha^2$. Obviously, $d^2 = 0$, since d is a differential. Moreover,



where we used associativity, coassociativity and the definition of the convolution. Similarly, one shows that $d \circ \bar{d}_\alpha + \bar{d}_\alpha \circ d = \bar{d}_{\partial\alpha}$. Finally, $d_\alpha^2 = \bar{d}_{\partial\alpha+\alpha\star\alpha}$, which equals 0 if and only if α verifies the Maurer-Cartan equation (3.1), i.e. α is a twisting morphism. □

Theorem 3.2 (Comparison lemma for twisted tensor complexes):

- *Let $\alpha \in \mathrm{Tw}(C, A)$ and $\alpha' \in \mathrm{Tw}(C', A')$ be twisting morphisms, $f : C \to C'$ a morphism of augmented DGAC and $g : A \to A'$ a morphism of augmented DGAA, such that the following diagram commutes*

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C' & \xrightarrow[\alpha']{} & A'. \end{array}$$

  Then $f \otimes g : C \otimes_\alpha A \to C' \otimes_{\alpha'} A'$ is a chain map, and thus induces a linear map $(f \otimes g)_\sharp$ in homolgy.

- *Under some weight-graded assumptions (necessary for the use of some 'spectral sequences' type argument in the proof), we have: if two of the chain maps f, g and $f \otimes g$ are quasi-isomorphisms, then so is the third.*

3.2 Bar and cobar complexes and adjunction

3.2.1 Bar construction

We detail the bar construction first for augmented (thus also unital) associative algebras (concentrated in degree 0), then for augmented graded associative algebras, and finally for augmented DGAA.

The idea is to encode the multiplication map (resp. the multiplication map and the differential) of an associative algebra (resp. of a DGAA) in a square 0 degree -1 coderivation of the cofree coalgebra $T^c(s\bar{A})$. This coding is realized via a *suspension*, an *extension* and a *summation*. The depicted coderivation will then be the differential of the bar complex $BA := (T^c(s\bar{A}), d_{BA})$.

Let A be an *augmented associative algebra*, we will consider the cofree algebra $T^c(s\bar{A})$, where the use of $\bar{A}$ instead of A is a technical decision due to the fact that we are trying to represent Tw.

To encode the bilinear multiplication $\mu : \bar{A} \otimes \bar{A} \to \bar{A}$ of the ideal $\bar{A}$, we first *suspend*:

$$\begin{array}{ccc} \bar{A} \otimes \bar{A} & \xrightarrow{\mu} & \bar{A} \\ {\scriptstyle s^{-1}\otimes s^{-1}} \uparrow & & \downarrow {\scriptstyle s} \\ s\bar{A} \otimes s\bar{A} & \xrightarrow[\tilde{\mu}]{} & s\bar{A}. \end{array}$$

Since $T^c(s\bar{A})$ is cofree, a coderivation is completely determined by its corestriction $f : T^c(s\bar{A}) \to s\bar{A}$. We set

$$f : T^c(s\bar{A}) \twoheadrightarrow (s\bar{A})^{\otimes 2} \xrightarrow{\tilde{\mu}} s\bar{A}.$$

The unique *extension* of f to a coderivation d_2 is given by

$$d_2(sa_1 \otimes \cdots \otimes sa_n) = \sum_{i=1}^{n-1} (-1)^i sa_1 \otimes \cdots \otimes s\mu(a_i, a_{i+1}) \otimes \cdots \otimes sa_n.$$

It is clear that d_2 is a degree -1 coderivation of $T^c(s\bar{A})$, and easily checked that $d_2^2 = 0$ if and only if μ is associative. Indeed, consider for instance

$$\begin{aligned} d_2(d_2(sa_1 \otimes sa_2 \otimes sa_3)) &= d_2(-s\mu(a_1, a_2) \otimes sa_3 + sa_1 \otimes s\mu(a_2, a_3)) \\ &= -s\mu(\mu(a_1, a_2), a_3) + s\mu(a_1, \mu(a_2, a_3)). \end{aligned}$$

Let A be an *augmented graded associative algebra*, then the construction is similar and the formula for d_2 differs only in the sign:

$$d_2(sa_1 \otimes \cdots \otimes sa_n) = \sum_{i=1}^{n-1} (-1)^{i+\tilde{a}_1+\cdots+\tilde{a}_i} sa_1 \otimes \cdots \otimes s\mu(a_i, a_{i+1}) \otimes \cdots \otimes sa_n.$$

Let A be an *augmented DGAA*, then we deal not only with a bilinear multiplication μ, but also with a linear differential $d : A \to A$, which also must be encoded. Note that $d : \bar{A} \to \bar{A}$, since $\varepsilon : A \to \mathbb{K}$ is a DGAA morphism, where $\mathbb{K}$ is a DGAA concentrated in degree 0 with differential zero, we thus have $\varepsilon \circ d = 0 \circ \varepsilon = 0$, so that $d : \ker \varepsilon = \bar{A} \to \ker \varepsilon = \bar{A}$.

To encode the linear differential $d : \bar{A} \to \bar{A}$, we first *suspend*:

$$\begin{array}{ccc} \bar{A} & \xrightarrow{d} & \bar{A} \\ {\scriptstyle s^{-1}} \uparrow & & \downarrow {\scriptstyle s} \\ s\bar{A} & \xrightarrow[\tilde{d}]{} & s\bar{A}. \end{array}$$

Then we *extend*

$$f : T^c(s\bar{A}) \twoheadrightarrow s\bar{A} \xrightarrow{\tilde{d}} s\bar{A}$$

to a unique coderivation d_1 given by

$$d_1(sa_1 \otimes \cdots \otimes sa_n) = \sum_{i=1}^{n} (-1)^{i-1+\tilde{a}_1+\cdots+\tilde{a}_{i-1}} sa_1 \otimes \cdots \otimes sda_i \otimes \cdots \otimes sa_n.$$

It is clear that d_1 is a degree -1 coderivation of $T^c(s\bar{A})$, and $d_2^2 = 0$ is due to the fact that $d^2 = 0$.

Finally, we *sum*

$$d_{BA} := d_1 + d_2,$$

in order to obtain a new degree -1 coderivation of $T^c(s\bar{A})$, where

$$d_{BA}^2 = d_1^2 + d_1 d_2 + d_2 d_1 + d_2^2 = 0$$

is due to the fact of d being a derivation of μ.

Definition 3.3: $BA := (T^c(s\bar{A}), d_{BA})$ is called the *bar construction* (or the *bar complex*) of A.

Proposition 3.3: *The bar construction BA of the augmented DGAA is a conilpotent DGAC. Actually, B is a functor from the category of augmented DGAAs to the category of conilpotent DGACs:*

$$B : \texttt{augDGAlg} \to \texttt{conilDGCoAlg}.$$

3.2.2 Cobar construction

Consider now a conilpotent DGAC C with differential d_C. The cobar construction is similar to the bar construction. More precisely,

Definition 3.4: $\Omega C := (T(s^{-1}\bar{C}), \delta_{\Omega C})$ is called the *cobar construction* (or the *cobar complex*) of C.

In this definition, the derivation $\delta_{\Omega C}$ is given by

$$\delta_{\Omega C} := \delta_1 + \delta_2,$$

where δ_1 encodes the differential d_C and δ_2 encodes the (reduced) comultiplication $\bar{\Delta}$. The property $d_C^2 = 0$, resp. coassociativity of $\bar{\Delta}$ entail that $\delta_1^2 = 0$, resp. that $\delta_2^2 = 0$.

$$\delta_{\Omega C}^2 = \delta_1^2 + \delta_1\delta_2 + \delta_2\delta_1 + \delta_2^2 = 0$$

is a consequence of the fact that d_C is a coderivation of $\bar{\Delta}$.

Proposition 3.4: *The cobar construction ΩC of the conilpotent DGAC is an augmented DGAA. Actually, Ω is a functor from the category of conilpotent DGACs to the category of augmented DGAAs:*

$$\Omega : \texttt{conilDGCoAlg} \to \texttt{augDGAlg}.$$

3.2.3 Adjunction

Definition 3.5:

- Two functors $F : \texttt{C} \to \texttt{D}$ and $G : \texttt{D} \to \texttt{C}$ are called *adjoint functors*, if there exists a family

$$\eta_{CD} : \mathrm{Hom}_{\texttt{D}}(FC, D) \to \mathrm{Hom}_{\texttt{C}}(C, GD), \quad C \in \texttt{C}, D \in \texttt{D},$$

  of bijections that are natural in C and D.

- A functor $F : \texttt{C} \to \texttt{Set}$ (resp. a contravariant functor $F : \texttt{C} \to \texttt{Set}$) is *representable*, if it is naturally isomorphic to $\mathrm{Hom}_{\texttt{C}}(-, A)$ (resp. $\mathrm{Hom}_{\texttt{C}}(A, -)$), for some object $A \in \texttt{C}$.

Theorem 3.5 (Basic correspondences 1):
If C is nilpotent, then

$$\mathrm{Hom}_{\texttt{grAlg}}(T(s^{-1}\bar{C}), A) \simeq \mathrm{Hom}_{\mathbb{K},-1}(C, A) \simeq \mathrm{Hom}_{\texttt{grCoAlg}}(C, T^c(s\bar{A})). \tag{3.2}$$

This claim is obvious, since algebra morphisms with source the free algebra (respectively coalgebra morphisms with target the cofree coalgebra) are, just as derivations (respectively coderivations), determined by their restrictions (respectively corestrictions).

Theorem 3.6 (Basic correspondences 2):
If C is nilpotent, then

$$\mathrm{Hom}_{\mathtt{DGAlg}}(\Omega C, A) \simeq \mathrm{Tw}(C, A) \simeq \mathrm{Hom}_{\mathtt{DGCoAlg}}(C, BA). \tag{3.3}$$

Here $\simeq$ denotes a bijection which is natural in C and A. It follows that Ω and B are adjoint functors and that Tw *is representable in C and A.*

3.3 Universal twisting morphisms

If we set $A = \Omega C$ in (3.3), the identity $\mathrm{id} : \Omega C = T(s^{-1}\bar{C}) \to \Omega C$ restricts to a twisting morphism $i : C \to \Omega C$, which is the injection of $\bar{C}$ extended by 0 on $\mathbb{K}$. If we set $C = BA$ in (3.3), the identity $\mathrm{id} : BA \to BA = T^c(s\bar{A})$ corestricts to a twisting morphism $\pi : BA \to A$, which is the projection onto $\bar{A}$ viewed as valued in A.

These twisting morphism i and π are called *universal*, since any twisting morphism $\alpha : C \to A$ factors through i and π. More precisely, there exist augmented DGAA and DGAC morphisms f_α and g_α, such that the following diagram commutes:

$$\begin{array}{ccc} & & BA \\ & {\scriptstyle g_\alpha} \nearrow & \downarrow {\scriptstyle \pi} \\ C & \xrightarrow{\ \alpha\ } & A \\ & {\scriptstyle i} \searrow & \uparrow {\scriptstyle f_\alpha} \\ & & \Omega C. \end{array}$$

Proposition 3.7: *The twisted tensor complexes $C \otimes_i \Omega C$ and $BA \otimes_\pi A$ are acyclic.*

Proof: This is proven by means of a chain homotopy between id and 0. □

3.4 Koszul morphisms and bar-cobar resolution

Definition 3.6: A twisting morphism $\alpha \in \mathrm{Tw}(C, A)$ is called a *Koszul morphism*, if the twisted tensor complex $C \otimes_\alpha A$ is acyclic. The set of Koszul morphisms from C to A is denoted by $\mathrm{Kos}(C, A)$.

Theorem 3.8 (Fundamental theorem of twisting morphisms):
Under some weight-graded assumptions (necessary to apply the comparison lemma), we have, for a twisting morphism $\alpha \in \mathrm{Tw}(C, A)$, that the following propositions are equivalent:

1. $\alpha \in \mathrm{Kos}(C, A)$*, i.e. $C \otimes_\alpha A$ is acyclic,*
2. $f_\alpha : \Omega C \to A$ *is a quasi-isomorphism,*
3. $g_\alpha : C \to BA$ *is a quasi-isomorphism.*

Proof: Concerning the first equivalence, we have the following commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{\ i\ } & \Omega C \\ {\scriptstyle \mathrm{id}} \downarrow & & \downarrow {\scriptstyle f_\alpha} \\ C & \xrightarrow{\ \alpha\ } & A. \end{array}$$

Applying the comparison lemma to id, f_α and $\mathrm{id} \otimes f_\alpha$, we get that f_α is a quasi-isomorphism if and only if $\mathrm{id} \otimes f_\alpha : C \otimes_i \Omega C \to C \otimes_\alpha A$ is a quasi-isomorphism. Since $C \otimes_i \Omega C$ is acyclic, we get the requested equivalence.

Equivalence between the first and the last statement is obtained similarly. □

Theorem 3.9 (Basic correspondences 3):
If C is nilpotent, then

$$\mathrm{QIso}_{\mathtt{DGAlg}}(\Omega C, A) \simeq \mathrm{Kos}(C, A) \simeq \mathrm{QIso}_{\mathtt{DGCoAlg}}(C, BA). \tag{3.4}$$

Theorem 3.10: *The counit of the bar-cobar adjunction, i.e. the DGAA morphism corresponding to the identity of the DGAC BA, is a quasi-isomorphism of DGAAs:*

$$\Omega BA \xrightarrow[\varepsilon]{\sim} A. \tag{3.5}$$

We say that ε is a *resolution* of A, called *bar-cobar resolution*, and that ΩBA is a *model* of A.

The philosophy of a 'resolution' or a 'model' is that the involved object is disentangled into a simpler object (the model) that can be used to study various aspects of the initial object. For instance, consider an orbifold, which is, roughly, a manifold with possible singularities. A model of this object is given by a smooth manifold, having the same homology as the initial object. In order to study the homology of the initial complicated object, one can thus also study the homology of the nicer model.

The model ΩBA of A, considered here, is 'too big', i.e. not handy enough to be manipulated and studied. Therefore, we will, in the following, replace it by a smaller and handier one.

Chapter 4

Koszul duality for associative algebras

The objective of this chapter is to replace the previously constructed model by a 'smaller' one. However, we will then have to restrict ourselves to specific type of algebras, namely 'quadratic Koszul algebras'.

4.1 Quadratic algebras and coalgebras

Definition 4.1: *Quadratic data* (V, R) consists of a graded vector space V and a graded vector subspace $R \subset V^{\otimes 2}$.

In order to simplify notations, let us first consider the nongraded situation, which can always be obtained by considering a graded vector space V concentrated in degree 0. The graded situation is then completely similar.

Definition 4.2: The *quadratic algebra* $A(V, R)$ associated to the quadratic data (V, R) is the (associative) quotient algebra $T(V)/(R)$, where (R) denotes the (two-sided) ideal generated by R.

Note that

$$(R) = \bigoplus_{n \geq 2} \sum_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j} \tag{4.1}$$

and that

$$\begin{aligned} A(V, R) &= \mathbb{K} \oplus V \oplus V^{\otimes 2}/R \oplus \cdots \oplus V^{\otimes n}/\sum_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j} \oplus \cdots \\ &=: \bigoplus_{n \geq 0} A^{(n)}(V, R). \end{aligned} \tag{4.2}$$

Notice further that the sum $\sum$ in (4.1) is not direct. For instance, if $n = 3$ the considered sum is $V \otimes R + R \otimes V$, considering elements $u, v, w \in V$, such that $u \otimes v \in R$ and $v \otimes w \in R$, then $u \otimes v \otimes w \in V \otimes R \cap R \otimes V$. Moreover, the ideal (R) is homogeneous for the weight grading, which is actually the reason why $A(V, R)$ is graded.

It is clear that the composite map $R \rightarrowtail T(V) \twoheadrightarrow A(V, R)$ vanishes. Furthermore, any algebra morphism $\varphi : T(V) \twoheadrightarrow \mathcal{A}$, such that $R \rightarrowtail T(V) \twoheadrightarrow \mathcal{A}$ vanishes, descends to the quotient, i.e. defines a unique algebra morphism $\tilde{\varphi} : A(V, R) = T(V)/(R) \to \mathcal{A}$, such that the following diagram commutes

$$\begin{array}{ccccc} R & \rightarrowtail & T(V) & \twoheadrightarrow & A(V,R) \\ & & & \searrow^{\varphi} & \downarrow \tilde{\varphi} \\ & & & & \mathcal{A}. \end{array} \qquad (R \xrightarrow{0} A(V,R),\ R \xrightarrow{0} \mathcal{A})$$

This means that $A(V,R)$ is the quotient algebra of $T(V)$ that is universal among all quotient algebras $\mathcal{A}$, such that $R \rightarrowtail T(V) \twoheadrightarrow \mathcal{A}$ vanishes.

The definition of the quadratic coalgebra is ‘dual’ to the definition of the quadratic algebra. More precisely, it is defined as a subcoalgebra, not as a quotient, and by means of the dual composite ${V^{\otimes 2}}/{R} \twoheadleftarrow T^c(V) \leftarrowtail \mathcal{C}$.

Definition 4.3: The *quadratic coalgebra* $C(V,R)$ associated to the quadratic data (V,R) is the subcoalgebra that is universal among all subcoalgebras $\mathcal{C}$ of $T^c(V)$, such that

$${V^{\otimes 2}}/{R} \twoheadleftarrow T^c(V) \leftarrowtail \mathcal{C}$$

vanishes, i.e. such that the following diagram commutes

$$\begin{array}{ccccc} C(V,R) & \rightarrowtail & T^c(V) & \twoheadrightarrow & {V^{\otimes 2}}/{R}\,. \\ \uparrow & \nearrow & & & \\ \mathcal{C} & & & & \end{array} \qquad (C(V,R) \xrightarrow{0} {V^{\otimes 2}}/{R},\ \mathcal{C} \xrightarrow{0} {V^{\otimes 2}}/{R})$$

The quadratic coalgebra is given by

$$\begin{aligned} C(V,R) &= \mathbb{K} \oplus V \oplus R \oplus (V \otimes R \cap R \otimes V) \oplus \cdots \oplus \bigcap_{i+j=n-2} V^{\otimes i} \otimes R \otimes V^{\otimes j} \oplus \cdots \\ &=: \bigoplus_{n \geq 0} C^{(n)}(V,R). \end{aligned} \tag{4.3}$$

Indeed, for instance, the fourth term is exactly what is needed for $C(V,R)$ to be a subcoalgebra. Consider $uvw \in V \otimes R \cap R \otimes V$, where the tensor product is omitted, then

$$\begin{aligned} \Delta(uvw) &= 1 \otimes uvw + u \otimes vw + uv \otimes w + uvw \otimes 1 \\ &\in \mathbb{K} \otimes (V \otimes R \cap R \otimes V) + V \otimes R + R \otimes V + (V \otimes R \cap R \otimes V) \otimes \mathbb{K} \\ &\subset C(V,R) \otimes C(V,R). \end{aligned}$$

Remark: The constructions of the quadratic algebra $A(V,R)$ and the quadratic coalgebra $C(V,R)$ can be extended to the graded setting. Moreover, the so obtained graded algebra, respectively graded coalgebra, can be endowed with the zero differential and thus become a weight-graded DGAA, respectively a weight-graded DGAC. In the following, we will work in this (differential) graded context.

4.2 Koszul dual coalgebra and algebra of a quadratic algebra

Definition 4.4: The *Koszul dual coalgebra* of a quadratic algebra $A = A(V,R)$ is

$$A^{¡} = C(sV, s^2R),$$

i.e. the quadratic coalgebra associated to the shifted quadratic data.

Here s^2R denotes the image of R by $s^2 : V^{\otimes 2} \ni v \otimes w \mapsto sv \otimes sw \in (sV)^{\otimes 2}$. The inverted exclamation mark ¡ is pronounced 'anti-shriek'.

One might expect that the Koszul dual algebra is defined similarly, however it turns out that a similar definition is only possible in finite dimension. The general definition is as follows:

Definition 4.5: The *Koszul dual algebra* of a quadratic algebra $A = A(V, R)$ is defined by

$$A^{!(n)} = s^n (A^{¡*})^{(n)}.$$

The exclamation mark ! is pronounced 'shriek'.

Remark: Note that the dual of an infinite sum is in general not the sum of the duals. Here the dual means that we are taking the dual term by term. Hence, we could also write $A^{!(n)} = s^n \left(A^{¡(n)}\right)^*$.

Let us now take an interest in the previously mentioned finite-dimensional case. The quadratic data (V, R) gives rise to the exact sequence

$$R \rightarrowtail V^{\otimes 2} \twoheadrightarrow V^{\otimes 2}/R.$$

Dualizing gives the exact sequence

$$R^* \twoheadleftarrow (V^*)^{\otimes 2} \leftarrowtail \left(V^{\otimes 2}/R\right)^*,$$

where $\left(V^{\otimes 2}/R\right)^* = \left\{(V^{\otimes 2})^* \ni \alpha : \alpha(R) = 0\right\} = R^\perp$, and we thus get new quadratic data $(V^*, R^\perp)$.

Proposition 4.1: *If V is finite-dimensional, then $A^! = A(V^*, R^\perp)$.*

Sketch of proof: Dualizing the coalgebra $A^¡ = C(sV, s^2R)$ (term by term), we get

$$A^{¡*} = \mathbb{K} \oplus s^{-1}V^* \oplus s^{-2}R^* \oplus \cdots ,$$

where, for instance, the third term $s^{-2}R^* = s^{-2}(V^*)^{\otimes 2}/R^\perp$, thus $A^{¡*} = A(s^{-1}V^*, s^{-2}R^\perp)$. Note also that the definition of the quadratic coalgebra is 'dual' to that of the quadratic algebra. □

Examples 4.1: In the following examples, we consider a finite-dimensional vector space V.

1. Let $R = \{0\}$, then $R^\perp = V^{*\otimes 2}$ and

$$A^¡ = C(sV, 0) = \mathbb{K} \oplus sV,$$
$$A^! = A(V^*, V^{*\otimes 2}) = \mathbb{K} \oplus V^*.$$

2. Let $R = \langle v \otimes w - w \otimes v \rangle$, then $s^2R = \langle sv \otimes sw - sw \otimes sv \rangle$, $R^\perp = S^2V^* \subset V^{*\otimes 2}$ and

$$A^¡ = C(sV, s^2R) = \mathbb{K} \oplus sV \oplus \bigwedge^2 (sV) \oplus \cdots = \bigwedge^c (sV),$$
$$A^! = A(V^*, V^{*\otimes 2}) = \mathbb{K} \oplus V^* \oplus \bigwedge^2 (V^*) \oplus \cdots = \bigwedge (V^*).$$

Definition 4.6: The *Koszul dual algebra* of a quadratic coalgebra $C(V, R)$ is

$$C^¡ = A(s^{-1}V, s^{-2}R).$$

One can verify that $(A^¡)^¡ = A$, $(C^¡)^¡ = C$ and, in finite dimension, $(A^!)^! = A$.

4.3 First (co)homology groups of the bar and cobar constructions in the quadratic case

Consider a quadratic algebra $A = A(V, R)$, which, together with differential $d_A = 0$, is a DGAA, so that we can take an interest in the bar construction

$$\begin{aligned} BA = T^c(s\bar{A}) &= \mathbb{K} \oplus s\bar{A} \oplus (s\bar{A})^{\otimes 2} \oplus \cdots \\ &= \mathbb{K} \oplus s\left(V \oplus {}^{V^{\otimes 2}}\!/\!_{R} \oplus {}^{V^{\otimes 3}}\!/\!_{V \otimes R + R \otimes V} \oplus \cdots\right) \\ &\quad \oplus \left(s\left(V \oplus {}^{V^{\otimes 2}}\!/\!_{R} \oplus {}^{V^{\otimes 3}}\!/\!_{V \otimes R + R \otimes V} \oplus \cdots\right)\right)^{\otimes 2} \oplus \cdots , \end{aligned}$$

with differential $d_{BA} = d_1 + d_2$, where d_1 is the unique coderivation that extends d_A, thus $d_1 = 0$ and $d_{BA} = d_2$, and where d_2 is the unique coderivation that extends the (concatenation) product μ.

In the following we will omit the suspension map s in order to simplify notations. The first terms of the considered bar complex are given by the following diagram.

$$\begin{array}{ccccccc|c} & & & & & & \cdots & \\ 0 & \xleftarrow{d_2} & {}^{V^{\otimes 3}}\!/\!_{V \otimes R + R \otimes V} & \xleftarrow{d_2} & {}^{V^{\otimes 2}}\!/\!_{R} \otimes V \oplus V \otimes {}^{V^{\otimes 2}}\!/\!_{R} & \xleftarrow{d_2} & V \otimes V \otimes V & (3) \\ & & 0 & \xleftarrow{d_2} & {}^{V^{\otimes 2}}\!/\!_{R} & \xleftarrow{d_2} & V \otimes V & (2) \\ & & & & 0 & \xleftarrow{d_2} & V & (1) \\ & & & & & & \mathbb{K} & (0) \\ \hline 3 & & 2 & & 1 & & 0 & \end{array}$$

$\longleftarrow$ syzygy degree $\qquad$ weight $\longrightarrow$

Remarks:

1. Recall that d_2 is in general defined by

$$d_2(sa_1 \otimes \cdots \otimes sa_n) = \sum_{i=1}^{n-1} \pm sa_1 \otimes \cdots \otimes s\mu(a_i, a_{i+1}) \otimes \cdots \otimes sa_n,$$

where we use $\pm$ as a simplified expression for the involved sign. Here, the differential d_2 is defined on tensor products of classes of tensor products.

For instance, d_2 can be applied to elements

$$\begin{gathered} [u] \otimes [v] \otimes [w] \in V \otimes V \otimes V, \\ [u \otimes v] \otimes [w] + [u'] \otimes [v' \otimes w'] \in {}^{V^{\otimes 2}}\!/\!_{R} \otimes V \oplus V \otimes {}^{V^{\otimes 2}}\!/\!_{R}, \\ [u \otimes v \otimes w] \in {}^{V^{\otimes 3}}\!/\!_{V \otimes R + R \otimes V}, \end{gathered}$$

which then gives

$$\begin{gathered} d_2([u] \otimes [v] \otimes [w]) = \pm[u \otimes v] \otimes [w] \pm [u] \otimes [v \otimes w] \in {}^{V^{\otimes 2}}\!/\!_{R} \otimes V \oplus V \otimes {}^{V^{\otimes 2}}\!/\!_{R}, \\ d_2([u \otimes v] \otimes [w] + [u'] \otimes [v' \otimes w']) = \pm[u \otimes v \otimes w] \pm [u' \otimes v' \otimes w'] \in {}^{V^{\otimes 3}}\!/\!_{V \otimes R + R \otimes V}, \\ d_2([u \otimes v \otimes w]) = 0. \end{gathered}$$

2. Note that we are dealing with multiple degrees. The differential d_2 is of degree -1 with respect to the degree induced by the grading of V, called the *homological degree*. The *weight* refers, as usual, to the number of factors in the tensor product and is denoted in parentheses. Moreover, we introduce an additional degree, called *syzygy degree* (lat. syzygia: conjunction without loss of identity), which is defined as being the difference between the weight and the number of involved classes.

   Note further that the bar complex (BA, d_2) is a cochain complex with respect to the syzygy degree, which splits with respect to the weight. We denote $(B^kA)^{(n)}$ the term of BA of syzygy degree k and weight (n).

Theorem 4.2: *Let (V, R) be quadratic data, $A(V, R)$ the associated quadratic algebra, and $A^{¡} = C(sV, s^2R)$ its Koszul dual coalgebra. By the termwise injection*

$$\begin{aligned} A^{¡} &= \mathbb{K} \oplus sV \oplus s^2R \oplus (sV \otimes s^2R \cap s^2R \otimes sV) \oplus \cdots \\ &\rightarrowtail T^c(sV) = \mathbb{K} \oplus sV \oplus (sV)^{\otimes 2} \oplus (sV)^{\otimes 3} \oplus \cdots \end{aligned}$$

into the column corresponding to syzygy degree 0 in the above diagram, $A^{¡}$ is a subcoalgebra of $T^c(sV)$. Hence, the inclusion $A^{¡} \subset B^0A$. More precisely, the first cohomology group of $(B^\bullet A, d_2)$ is

$$H^0(B^\bullet A) = A^{¡}, \quad \textit{i.e.} \quad H^0(B^\bullet A)^{(n)} = (A^{¡})^{(n)}, \quad \forall n \in \mathbb{N}.$$

Proof: For instance, omitting the suspension, we get for $n = 3$:

$$H^0(B^\bullet A)^{(3)} = \ker\left(d_2 : V^{\otimes 3} \to {}^{V^{\otimes 2}}\!/_{R} \otimes V \oplus V \otimes {}^{V^{\otimes 2}}\!/_{R}\right) = R \otimes V \cap V \otimes R = (A^{¡})^{(3)}. \quad \square$$

A similar result holds true for the first homology group of the cobar construction of a quadratic coalgebra. More precisely, if $C = C(V, R)$, then

$$H_0(\Omega_\bullet C) = C^{¡}, \quad \text{i.e.} \quad H_0(\Omega_\bullet C)^{(n)} = (C^{¡})^{(n)}, \quad \forall n \in \mathbb{N}.$$

4.4 Koszul algebras

We now replace, under certain conditions, the ‘big’ resolution $\Omega BA \xrightarrow{\sim} A$, by a smaller one, namely $\Omega A^{¡} \xrightarrow{\sim} A$. In order to obtain such a quasi-isomorphism in $\mathrm{QIso}_{\mathtt{DGAlg}}(\Omega A^{¡}, A)$, we need, by (3.4), a Koszul morphism in $\mathrm{Kos}(A^{¡}, A)$. For a quadratic algebra $A = A(V, R)$, a canonical candidate is

$$\kappa : A^{¡} = C(sV, s^2R) \twoheadrightarrow sV \xrightarrow{s^{-1}} V \rightarrowtail A(V, R) = A.$$

It is clear that κ is a degree -1 morphism that vanishes on units and counits. Since $A^{¡}$ and A are viewed as DGAC and DGAA, respectively, with differential 0, the Maurer-Cartan equation reduces to $\kappa \star \kappa = 0$. Since κ vanishes everywhere except on V, it suffices to check this condition on $R \subset V^{\otimes 2}$:

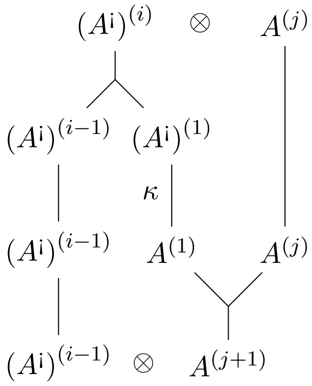
$$\begin{aligned} (\kappa \star \kappa)(rr') &= \mu(\kappa \otimes \kappa)(1 \otimes rr' + r \otimes r' + rr' \otimes 1) \\ &= \mu(\kappa r \otimes \kappa r') = \mu(r \otimes r') = [rr'] = 0, \end{aligned}$$

where we omitted the suspension, tensor products and involved signs. Thus $\kappa \in \mathrm{Tw}(A^{¡}, A)$ is a twisting morphism.

Definition 4.7: Let $A = A(V, R)$ be a quadratic algebra. The twisting morphism $\kappa \in \mathrm{Tw}(A^{¡}, A)$, defined above, defines two twisted tensor complexes $A^{¡} \otimes_\kappa A$ and $A \otimes_\kappa A^{¡}$ called left and right *Koszul complexes* of $A = A(V, R)$.

We are mainly interested in the left Koszul complex, which we will simply call Koszul complex in the following.

Observe that the differential

$d_\kappa = \bar{d}_\kappa =$  ,

with $i + j = n$, of the Koszul complex has the typical shape of a *Koszul differential.*

Theorem 4.3 (Koszul criterion):
Let (V, R) be quadratic data, $A = A(V, R)$ the associated quadratic algebra, $A^{\text{¡}} = C(sV, s^2R)$ the Koszul dual coalgebra, and $\kappa \in \mathrm{Tw}(A^{\text{¡}}, A)$ the twisting morphism defined above. Then the following propositions are equivalent:

1. *$\kappa \in \mathrm{Kos}(A^{\text{¡}}, A)$, i.e. the Koszul complex $A^{\text{¡}} \otimes_\kappa A$ is acyclic,*
2. *the projection $p := f_\kappa : \Omega A^{\text{¡}} \twoheadrightarrow A$ is a quasi-isomorphism of DGAA,*
3. *the injection $i := g_\kappa : A^{\text{¡}} \rightarrowtail BA$ is a quasi-isomorphism of DGAC.*

If these conditions hold true, we say that the quadratic algebra A is a *Koszul algebra.* Moreover, $\Omega A^{\text{¡}} \twoheadrightarrow A$ is then a minimal resolution (i.e. $\Omega A^{\text{¡}}$ is a minimal model) of A, called *Koszul resolution.*

Proof: It suffices to apply the fundamental theorem of twisting morphisms and to check minimality, which comes from the fact that $d_{\Omega A^{\text{¡}}} = \delta_2$, where δ_2 is the differential that encodes the reduced coproduct $\bar{\Delta}$. □

Remarks:

1. Comparing with the first homology and cohomology groups of the bar and cobar constructions of quadratic algebras and coalgebras

$$H^0(B^\bullet A) = A^{\text{¡}}, \quad H_0(\Omega_\bullet A^{\text{¡}}) = A,$$

   we see that A is a Koszul algebra, i.e. $H^\bullet(B^\bullet A) \simeq A^{\text{¡}}$ or $H_\bullet(\Omega_\bullet A^{\text{¡}}) \simeq A$, if and only if $H^n(B^\bullet A) = 0$, respectively $H_n(\Omega_\bullet A^{\text{¡}}) = 0$, for all $n \geq 1$.

2. If $f : A \to A'$ is a quasi-isomorphism between augmented DGAAs, respectively, if $g : C \to C'$ is a quasi-isomorphism between nilpotent DGACs, then $Bf : BA \to BA'$, respectively $\Omega g : \Omega C \to \Omega C'$, is a quasi-isomorphism.

   For any quadratic algebra $A = A(V, R)$, we have

$$\underbrace{\Omega A^{\text{¡}} \xrightarrow{\Omega i} \Omega BA \xrightarrow{\sim} A}_{p} \quad \text{and} \quad \underbrace{A^{\text{¡}} \xrightarrow{\sim} \Omega B \Omega A^{\text{¡}} \xrightarrow{Bp} BA}_{i} .$$

In view of the preceding criterion, A is a Koszul algebra, if and only if one of the maps p, Ωi, i and Bp is a quasi-isomorphism. In that case, all of the considered maps are quasi-isomorphisms.

3. If A is a Koszul algebra, we can replace the 'big' model ΩBA of A by the more handy one $\Omega A^{¡}$.

Chapter 5

Classical definition of operads

We will now give a first definition of operads (the classical definition), using the notion of multicategories. We will also explain why operads can be seen as abstractions of algebras. Moreover, we will provide some examples of operads, in particular we give a detailed construction of the operads corresponding to associative and to commutative algebras. Finally, we will give a second definition of an operad (the partial definition), which mainly differs from the first one by the composition map.

5.1 Multicategories and operads

Categories are made up by objects and morphisms (which can be composed, composition being associative and having units). The morphisms of categories have one input and one output. Multicategory have morphisms with multiple inputs and one output.

Definition 5.1: A *multicategory* $\mathtt{C}$ is made up by

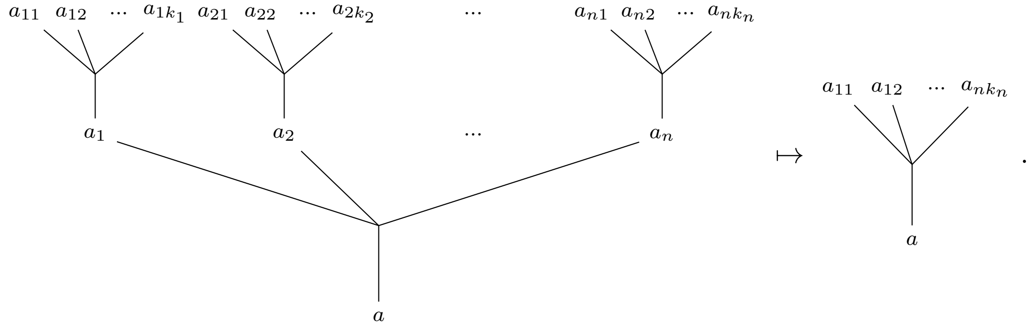
1. a set $\mathtt{C}_0$ of objects denoted by $a, a_1, a_2, \ldots$,
2. for any $n \in \mathbb{N}$, $a_1, \ldots a_n, a$, a set $\mathrm{Hom}(a_1, \ldots, a_n; a)$ of morphisms,
3. a composition map $\gamma_{k_1,\ldots,k_n}$:

$$\begin{aligned}\mathrm{Hom}(a_1, \ldots, a_n; a) \times \mathrm{Hom}(a_{11}, \ldots, a_{1k_1}; a) \times \cdots \times \mathrm{Hom}(a_{n1}, \ldots, a_{nk_n}; a) \\ \to \mathrm{Hom}(a_{11}, \ldots, a_{nk_n}; a) \\ (\theta; \theta_1, \ldots, \theta_n) \mapsto \theta \circ (\theta_1, \ldots, \theta_n),\end{aligned}$$

4. for any a, an identity morphism $1_a \in \mathrm{Hom}(a, a)$,

such that composition is associative and has identities.

It can be helpful to imagine the composition in terms of trees:



The identity is given by the following tree:

The number of inputs is called the *arity*.

Multicategories should not be confused with higher categories. A *morphism of multicategories* is a morphism of categories, i.e. a functor. Small multicategories and morphisms between them form a category `MultiCat`.

An example of a multicategory is an *operad*. An operad is a multicategory with a unique object. As our interest mainly lies in operads, let us give a more explicit definition.

Definition 5.2 (Classical definition of nonsymmetric operads):
A *nonsymmetric operad* (or *operad without symmetry*) P consists of

1. a sequence $(P(n))_{n\in\mathbb{N}}$ of sets, whose elements are called *abstract n-ary operations* of P,
2. for each integers n, $k_1, \dots, k_n$, a map $\gamma_{k_1,\dots,k_n}$:
$$\begin{aligned} P(n) \times P(k_1) \times \cdots \times P(k_n) &\to P(k_1 + \cdots + k_n) \\ (\theta; \theta_1, \dots, \theta_n) &\mapsto \theta \circ (\theta_1, \dots, \theta_n) \end{aligned}$$
called *composition*,
3. an element 1 in $P(1)$ called the *identity*,

satisfying the following associativity and identity properties:
$$\begin{aligned} &\theta \circ (\theta_1 \circ (\theta_{1,1}, \dots, \theta_{1,k_1}), \dots, \theta_n \circ (\theta_{n,1}, \dots, \theta_{n,k_n})) \\ &= (\theta \circ (\theta_1, \dots, \theta_n)) \circ (\theta_{1,1}, \dots, \theta_{1,k_1}, \dots, \theta_{n,1}, \dots, \theta_{n,k_n}) \end{aligned}$$
and
$$\theta \circ (1, \dots, 1) = \theta = 1 \circ \theta.$$

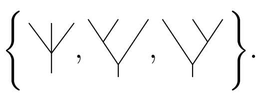
Often, the operad P is ‘enriched’, i.e. the sets $P(n)$ have an additional structure, for instance that of modules over a commutative ring R, vector spaces over a field $\mathbb{K}$, or more generally objects of a symmetric monoidal category `C`. In this case the composition map γ is also required to be a R-multilinear map, a $\mathbb{K}$-multilinear map, or generally a morphism of `C` where the cartesian product is replaced by the tensor product given by the monoidal structure. In the following we will mainly consider operads in the category `Vect`, i.e. operads P, where the sets $P(n)$ are vector spaces and composition is linear.

Operads form a full subcategory `Operad` of the category `MultiCat`.

Remark: Many authors refer to multicategories as *coloured operads*.

Examples 5.1: Let us now consider some examples of operads which will give us a better understanding of this notion. Moreover, these examples will be of importance in the following.

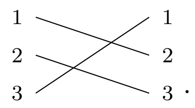
- The *tree operad* $\mathscr{T}$ is made up by the sets $\mathscr{T}(n)$, $n \in \mathbb{N}^*$, of planar trees with 1 root and n leaves. For instance,

  $\mathscr{T}(3) =$ 

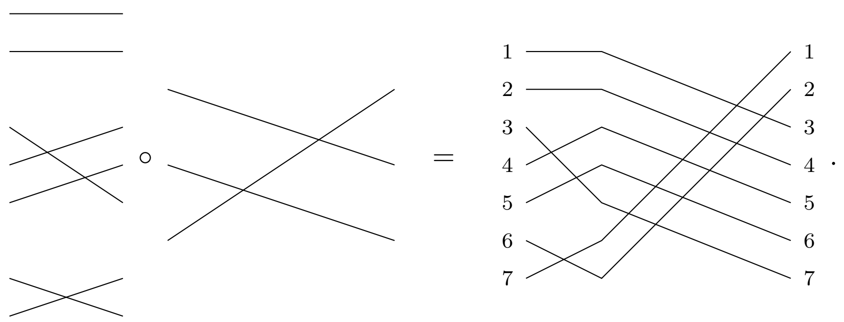
  The composition is just the grafting of the roots of the trees $\theta_1, \dots, \theta_n$ to the leaves $1, \dots, n$ of the tree θ. Let us quote here Boardman and Vogt [BV73]: “[…] the trees are inspired by the attempt to obtain a general composition operation from a collection of indecomposable operations.” The identity is obviously given by $| \in \mathscr{T}(1)$.

- The *endomorphism operad* $\mathscr{E}nd(V)$ over a vector space V is made up by the vector spaces $\mathscr{E}nd(V)(n) = \mathscr{L}_n(V \times \cdots \times V, V) = \mathrm{Hom}(V^{\otimes n}, V)$ of n-linear maps on V, the usual composition and the identity map id_V.

- The *symmetry operad* $\mathscr{S}$ is made up by the sets $\mathscr{S}(n) = S_n$. It is helpful to think of permutations $\sigma \in S_n$ in terms of diagrams. For instance, the permutation $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right) \in S_3$ can be seen as



In order to understand composition, consider the following example. Let $\sigma \in S_3$ be as above, $\tau_1 = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right) \in S_2$, $\tau_2 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right) \in S_3$,and $\tau_3 = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right) \in S_2$. The composite $\sigma \circ (\tau_1, \tau_2, \tau_3)$ is, in terms of diagrams, given by



We thus obtain the permutation

$$\sigma \circ (\tau_1, \tau_2, \tau_3) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 7 & 5 & 6 & 2 & 1 \end{pmatrix} \in S_7.$$

Note that this composite is a permutation of $2+3+2 = 7$ elements, this number is given by the τ_i. Moreover, we see that σ acts on 3 blocks consisting, respectively, of 2, 3 and 2 elements, and that the τ_i act inside these blocks. In general, the composite $\sigma \circ (\tau_1, \ldots, \tau_k)$, where $\sigma \in S_k$ and $\tau_i \in S_{\ell_i}$, is the permutation of $\ell_1 + \cdots + \ell_k$ elements, where σ acts on blocks of respective length $\ell_1, \ldots, \ell_k$, and where the τ_i act inside the i-th block.

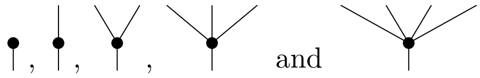
The identity is given by $\mathrm{id} \in S_1 = \mathscr{S}(1)$.

Remark (Tree Guidelines 1):

Tree diagrams are quite helpful to understand some sophisticated notions related to operads. It even turns out that trees are intrinsically linked to operads. Complicated operadic concepts can be reduced to their essence and then be interpreted in terms of trees. Working with trees is in most cases much easier than handling elaborate formulas, and, surprisingly, equivalently rigorous. Therefore, we will occasionally spend some time to define the notions related to trees, to fix the conventions and to explain the relationship between trees and operads.

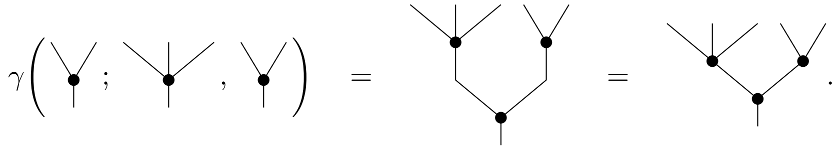
In graph theory, *trees* are usually defined as being acyclic connected graphs (graphs are made up by *vertices* and *edges* joining the vertices). We will slightly modify this definition: the trees, which we consider here, have no external vertices, and thus the external edges become half-edges (sometimes called flags). One of the half-edges will be called *root* and the others are then called *leaves*. The choice of a root endows the considered tree with a natural direction from the leaves (on top) to the root (at the bottom). In the following, if not otherwise mentioned, vertices and edges always refer to internal ones.

There exist some special types of trees. The *trivial tree* |, which has no vertex, and its unique leaf is at the same time its root. *Corollas* are trees having exactly one vertex, the number of leaves can vary in $\mathbb{N}$. For instance,



are corollas. The first one, having no leaves, is sometimes called *stump*. The corolla with n leaves is also called the n-corolla.

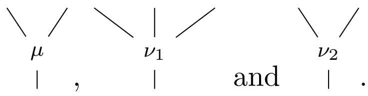
Abstract n-ary operations are usually represented by corollas with n leaves. The leaves correspond to the inputs, whereas the root corresponds to the single output. Composition of these operations is given by *grafting* the corresponding trees. Grafting means that the roots of the trees to be grafted are identified with the leaves of tree on which they are grafted. For instance,



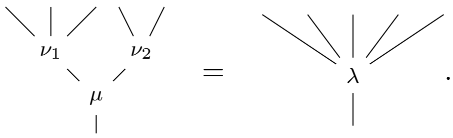
Note that composition with the trivial tree does not change the initial tree, therefore the trivial tree is the identity for this composition by means of grafting.

Often, we will not explicitly draw the vertices of trees (except for the stump).

As already mentioned, trees (in particular corollas) are used to represent abstract operations. In general, it is possible that we have to deal with multiple (different) abstract operations having the same number of inputs. In the above example we considered, in particular, two trees with two leaves. Suppose that they correspond to different operations. In order to be able to distinguish the two trees, we decorate the vertex of the tree by the considered operation. If the three operations, which we considered in the above example, are denoted by μ, ν_1 and ν_2, respectively, then the corresponding trees are:



Composing abstract operations gives rise to a new abstract operation. In the above example, we obtained an abstract operation with 5 inputs. Denoting this operation by λ, the equality $\gamma(\mu;\nu_1,\nu_2)=\lambda$ reads as



5.2 Symmetric multicategories and symmetric operads

The action of the symmetric group S_n on $V^{\otimes n}$ can be defined in two different ways. Either, one can define the left S_n-action by

$$\sigma\cdot(v_1\otimes\cdots\otimes v_n)=v_{\sigma_1}\otimes\cdots\otimes v_{\sigma_n}$$

and the right S_n-action by

$$(v_1\otimes\cdots\otimes v_n)\cdot\sigma=v_{\sigma_1^{-1}}\otimes\cdots\otimes v_{\sigma_n^{-1}},$$

or, one can define the left S_n-action by

$$\sigma^* \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma_1^{-1}} \otimes \cdots \otimes v_{\sigma_n^{-1}}$$

and the right S_n-action by

$$(v_1 \otimes \cdots \otimes v_n) \cdot \sigma^* = v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_n},$$

for any $\sigma \in S_n$, $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$.

Remark: In the following we will prefer the latter convention and omit the adjoint symbol $*$. Note further that we are mainly interested in right S_n-actions, therefore also representations of the symmetric group will be considered as right S_n-modules, rather than the previously used left S_n-modules.

In order to define symmetric operads, we will first define symmetric multicategories.

Definition 5.3: A *symmetric multicategory* $\mathtt{C}$ is a multicategory endowed with a family of maps

$$- \cdot \sigma : \mathrm{Hom}(a_1, \ldots, a_n, a) \to \mathrm{Hom}(a_{\sigma_1}, \ldots, a_{\sigma_n}, a),$$

$\sigma \in S_n$, satisfying usual action conditions

$$(\theta \cdot \sigma) \cdot \tau = \theta \cdot (\sigma \circ \tau), \tag{5.1}$$

$$\theta \cdot \mathrm{id} = \theta, \tag{5.2}$$

and the compatibility condition with composition

$$(\theta \cdot \sigma) \circ (\varphi_{\sigma_1} \cdot \pi_{\sigma_1}, \ldots, \varphi_{\sigma_n} \cdot \pi_{\sigma_n}) = (\theta \circ (\varphi_1, \ldots, \varphi_n)) \cdot (\sigma \circ (\pi_{\sigma_1}, \ldots, \pi_{\sigma_n})), \tag{5.3}$$

where θ and φ_i denote morphisms, and σ and π_i denote permutations.

Remarks: Let us detail the axioms in this definition.

1. The conditions (5.1) and (5.2) entail that the maps $- \cdot \sigma$ are bijections. The inverse of $- \cdot \sigma$ is given by $- \cdot \sigma^{-1}$.
2. The equivariance condition (5.3) roughly requires that the action commutes with composition. The precise meaning of this condition will be treated in the Tree Guidelines 2 on page 40.

   Note also that the last permutation in (5.3) is a composite in the symmetry operad.

Definition 5.4 (Classical definition of symmetric operads):
A *symmetric operad* (or *operad with symmetry*) P (in the category $\mathtt{Vect}$) is an operad (in $\mathtt{Vect}$), such that the vector spaces $P(n)$ are endowed with a right S_n-module structure which is equivariant with respect to composition in the sense of (5.3).

Remarks:

- It is also possible to consider (symmetric or nonsymmetric) operads without unit, just by forgetting about all conditions involving the unit in the definitions.
- It always possible to consider a symmetric operad as an operad without symmetry, just by forgetting about all conditions involving symmetry.
- A sequence $(P(n))_{n\in\mathbb{N}}$ of vector spaces $P(n)$ with right S_n-module structures, like in the previous definition, is also called an *S-module P*. We will later deal a lot with such S-modules.

5.3 Morphisms and representations of operads

Operads are important through their representations. In order to define representations of operads, we first have to define morphisms of operads.

Definition 5.5:

- A *morphism* $\varphi : P \to Q$ *of nonsymmetric operads* (in the category `Vect`) consists of a sequence of linear maps $\varphi_n : P(n) \to Q(n)$ that respect composition and units, i.e. $\varphi_n(\theta \circ_P (\theta_1, \ldots, \theta_n)) = \varphi_n(\theta) \circ_Q (\varphi_{k_1}(\theta_1), \ldots, \varphi_{k_n}(\theta_n))$ and $\varphi_1(1_P) = 1_Q$.
- A *morphism* $\varphi : P \to Q$ *of symmetric operads* (in the category `Vect`) consists of a sequence of linear maps $\varphi_n : P(n) \to Q(n)$ that respect composition and units and such that
$$\varphi_n(\theta \cdot \sigma) = \varphi_n(\theta) \cdot \sigma,$$
for $\theta \in P(n)$, $\sigma \in S_n$.

A representation of an operad P is a morphism of operads $\rho : P \to \mathscr{E}nd(V)$. Note that this definition makes sense, since the endomorphism operad admits not only the structure of a nonsymmetric operad, but also the structure of a symmetric operad. Indeed, we can define the S_n-module structure on $\mathscr{E}nd(V)(n) = \mathscr{L}_n(V \times \cdots \times V, V) = \mathrm{Hom}(V^{\otimes n}, V)$ by

$$(\theta \cdot \sigma)(v_1 \otimes \cdots \otimes v_n) = \theta(\sigma \cdot (v_1 \otimes \cdots \otimes v_n)) = \theta(v_{\sigma_1^{-1}} \otimes \cdots \otimes v_{\sigma_n^{-1}}), \tag{5.4}$$

for $\theta \in \mathscr{E}nd(V)(n), \sigma \in S_n, v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$. We will detail equation (5.4) in the Tree Guidelines 2 on page 40.

Let us give the precise definition of a representation of an operad.

Definition 5.6: A *representation of a nonsymmetric operad* P (in `Vect`) on a vector space V is a morphism of nonsymmetric operads $\rho : P \to \mathscr{E}nd(V)$, i.e. made up by a family of linear maps $\rho_n : P(n) \to \mathscr{E}nd(V)(n) = \mathrm{Hom}(V^{\otimes n}, V)$ that respects composition and identity.

Remark: The linear maps $\rho_n : P(n) \to \mathscr{E}nd(V)(n)$ can also be viewed as

$$\tilde{\rho}_n \in \mathrm{Hom}(P(n), \mathrm{Hom}(V^{\otimes n}, V)) \simeq \mathscr{L}_2(P(n) \times V^{\otimes n}, V) = \mathrm{Hom}(P(n) \otimes V^{\otimes n}, V).$$

Definition 5.7: A *representation of a symmetric operad* P (in `Vect`) on a vector space V is a morphism of symmetric operads $\rho : P \to \mathscr{E}nd(V)$, i.e. made up by a family of linear maps $\rho_n : P(n) \to \mathscr{E}nd(V)(n) = \mathrm{Hom}(V^{\otimes n}, V)$ that respects composition and identities, and verifies $\rho_n(\theta \cdot \sigma) = \rho_n(\theta) \cdot \sigma$, for $\theta \in P(n), \sigma \in S_n$.

Remark: The S_n-linear maps ρ_n can be viewed as $\tilde{\rho}_n \in \mathrm{Hom}(P(n) \otimes_{S_n} V^{\otimes n}, V)$. Note that the tensor product, which is over $\mathbb{K}[S_n]$, encodes S_n-linearity.

Indeed,

$$\begin{aligned}\rho_n(\theta \cdot \sigma)(v_1 \otimes \cdots \otimes v_n) &= \tilde{\rho}_n((\theta \cdot \sigma) \otimes (v_1 \otimes \cdots \otimes v_n)) = \tilde{\rho}_n(\theta \otimes (\sigma \cdot (v_1 \otimes \cdots \otimes v_n))) \\ &= \rho_n(\theta)(\sigma \cdot (v_1 \otimes \cdots \otimes v_n)) = (\rho_n(\theta) \cdot \sigma)(v_1 \otimes \cdots \otimes v_n),\end{aligned}$$

where the last equality comes from the symmetric structure on $\mathscr{E}nd(V)$.

As mentioned previously, operads are important through their representations. Indeed, ρ_n associates to abstract n-ary operations $\theta \in P(n)$ concrete n-ary operations on V, i.e. $\rho_n(\theta) \in \mathrm{Hom}(V^{\otimes n}, V)$. Therefore, one can actually get an algebraic structure on V. More precisely, it turns out that to any type of algebras (with operations having one output), one can associate a specific operad. A representation of this operad on a vector space V endows it with corresponding algebraic structure. This justifies the terminology 'algebra over P' and 'P-algebra', and allows understanding that an operad should be viewed as an algebraic theory.

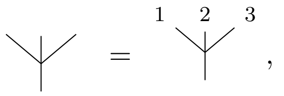
5.4 The commutative and associative operads

We will now construct the operads $\mathscr{C}om$ and $\mathscr{A}ss$ corresponding to commutative and to associative algebras. But first, we should get a better understanding of tree diagrams.

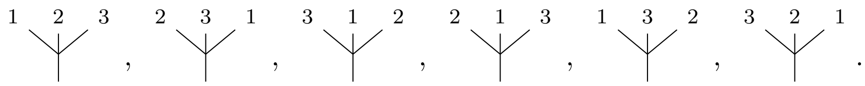
Remark (Tree Guidelines 2): A *planar tree* is a tree with a specified embedding in the plane. Note that every tree can be embedded in the plane. In particular, such an embedding induces a natural ordering (from left to right) on the leaves of the tree, and thus an ordering on the inputs of an abstract operation.

In contrast, a *nonplanar tree* has to be viewed in (3-dimensional) space, where no implicit ordering on the leaves is given. The ordering has thus to be specified explicitly.

For instance, for the planar corolla with 3 leaves, there exists a unique specification for the leaf ordering:



but there are 6 different specifications for the nonplanar 3-corolla:



(5.5)

Note that for a given planar embedding of a nonplanar tree, all other ones are given by permutations of the leaf ordering.

Planar trees are used to describe abstract operations of nonsymmetric operads, where no symmetry is involved, whereas nonplanar trees are used to describe abstract operations of symmetric operads. The symmetric group action on an abstract operation θ can be seen as permutation of the leaf ordering of the corresponding corolla.

In order to get a better idea of this symmetric group action, let us consider the example of a concrete operation θ, which we will think of as an associative ternary multiplication. Applied to elements a, b, c, we have 6 possibilities to define such an operation:

$$\theta_1(a,b,c) = abc,\ \theta_2(a,b,c) = cab,\ \theta_3(a,b,c) = bca,\ \dots$$

We easily see that these 6 possibilities come from a unique underlying operation θ. For instance, if we take $\theta = \theta_1$, then $\theta_2(a,b,c) = \theta(c,a,b)$, $\theta_3(a,b,c) = \theta(b,c,a)$, ... This means that for a fixed operation, the other possibilities are obtained by permuting the inputs.

We also may identify the operations $\theta_1, \dots, \theta_6$ with corollas:

$$\begin{array}{c}1\quad 2\quad 3\\ \diagdown\ |\ \diagup\\ \theta_1\\ |\end{array},\quad \begin{array}{c}1\quad 2\quad 3\\ \diagdown\ |\ \diagup\\ \theta_2\\ |\end{array},\quad \begin{array}{c}1\quad 2\quad 3\\ \diagdown\ |\ \diagup\\ \theta_3\\ |\end{array},\dots,\quad \begin{array}{c}1\quad 2\quad 3\\ \diagdown\ |\ \diagup\\ \theta_6\\ |\end{array},$$

where

$$\begin{array}{c}1\quad 2\quad 3\\ \diagdown\ |\ \diagup\\ \theta_1\\ |\end{array} = \begin{array}{c}1\quad 2\quad 3\\ \diagdown\ |\ \diagup\\ \theta\\ |\end{array},\quad \begin{array}{c}1\quad 2\quad 3\\ \diagdown\ |\ \diagup\\ \theta_2\\ |\end{array} = \begin{array}{c}2\quad 3\quad 1\\ \diagdown\ |\ \diagup\\ \theta\\ |\end{array},\quad \begin{array}{c}1\quad 2\quad 3\\ \diagdown\ |\ \diagup\\ \theta_3\\ |\end{array} = \begin{array}{c}3\quad 1\quad 2\\ \diagdown\ |\ \diagup\\ \theta\\ |\end{array},\dots$$

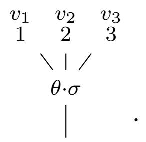
Therefore, we identify

$$\theta_1 = \theta \cdot \mathrm{id},\ \theta_2 = \theta \cdot \sigma,\ \theta_3 = \theta \cdot \sigma',\ \dots,$$

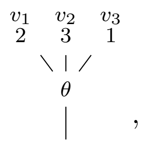
where $\sigma = \left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)$, $\sigma' = \left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right)$, ...

Note that the leaf ordering of $\theta \cdot \sigma$ can be seen as being obtained from the one of the corresponding θ by applying σ^{-1}.

Recall the definition of the symmetric group action on the endomorphism operad (5.4). In terms of tree diagrams, $(\theta \cdot \sigma)(v_1 \otimes \cdots \otimes v_n)$ reads, for $n = 3$ and σ as above, as



Since the input ordering of $\theta \cdot \sigma$ is obtained from the one of θ by applying σ^{-1}, we can obtain the one of θ by applying σ on the input ordering of $\theta \cdot \sigma$. Thus, we get



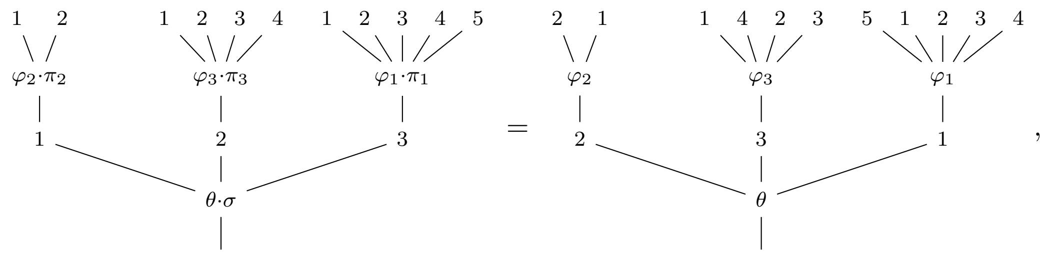
which is the tree diagram corresponding to $\theta(v_3 \otimes v_1 \otimes v_2)$, or generally $\theta(v_{\sigma_1^{-1}} \otimes \cdots \otimes v_{\sigma_n^{-1}})$.

We are now able to understand the equivariance requirement (5.3).

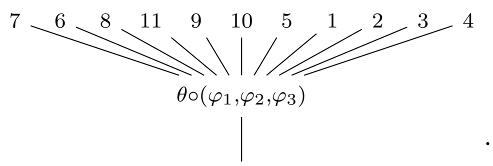
Consider a 3-ary operation θ, a 5-ary operation φ_1, a 2-ary operation φ_2, and a 4-ary operation φ_3, which we think of as corollas with natural input ordering from left to right. Moreover, we consider the permutations $\sigma = \left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)$, $\pi_1 = \left(\begin{smallmatrix}1&2&3&4&5\\5&1&2&3&4\end{smallmatrix}\right)$, $\pi_2 = \left(\begin{smallmatrix}1&2\\2&1\end{smallmatrix}\right)$, and $\pi_3 = \left(\begin{smallmatrix}1&2&3&4\\1&4&2&3\end{smallmatrix}\right)$. Then the LHS

$$(\theta \cdot \sigma) \circ (\varphi_{\sigma_1} \cdot \pi_{\sigma_1}, \varphi_{\sigma_2} \cdot \pi_{\sigma_2}, \varphi_{\sigma_3} \cdot \pi_{\sigma_3})$$

of (5.3) corresponds to the tree diagram



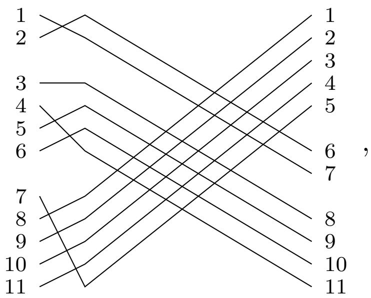
which we may then see as the 11-corolla



For the RHS

$$(\theta \circ (\varphi_1, \varphi_2, \varphi_3)) \cdot (\sigma \circ (\pi_{\sigma_1}, \pi_{\sigma_2}, \pi_{\sigma_3}))$$

of (5.3), we have to consider the permutation $\sigma \circ (\pi_{\sigma_1}, \pi_{\sigma_2}, \pi_{\sigma_3})$ given by

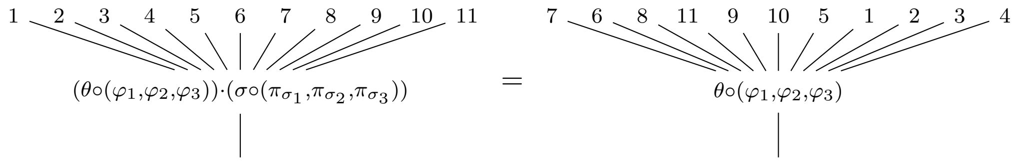


,

i.e.

$$\sigma \circ (\pi_{\sigma_1}, \pi_{\sigma_2}, \pi_{\sigma_3}) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 6 & 8 & 11 & 9 & 10 & 5 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

Therefore,



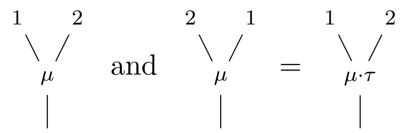
.

Finally, we get, for the LHS and the RHS, twice the same corolla with the same leaf ordering.

In the following, we would like to construct the operads $\mathscr{A}ss$ and $\mathscr{C}om$, corresponding to associative, respectively commutative algebras. Therefore, we will try to identify abstract operadic operations (pictured as trees) with concrete algebraic operations. In our case, this will mostly be the bilinear multiplication μ. The considered algebra will typically be denoted by A, and its elements by $a, a_1, \ldots, a_n, \ldots$

1. The operad $\mathscr{C}om$ is the symmetric operad associated with commutative associative non-unital algebras.

   The two abstract operations

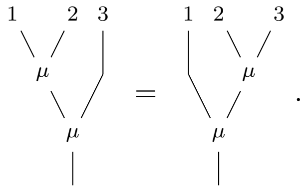
   correspond to

   $$\mu(a_1, a_2) = a_1 a_2 \text{ and } \mu'(a_1, a_2) := (\mu \cdot \tau)(a_1, a_2) = \mu(a_2, a_1) = a_2 a_1.$$

   Here, $\tau \in S_2$ denotes the transposition. Due to commutativity, we should get that the two operations are the same. This is obtained by choosing the trivial action as the symmetric group action. In this case, any permutation acts as identity. Here, we have $\mu \cdot \tau = \mu'$, but due to the trivial action, we also have $\mu \cdot \tau = \mu$, thus we get the desired $\mu = \mu'$. Finally, the two considered 2-ary operations are the same, thus, there is only one unique 2-ary operation, and the vector space $\mathscr{C}om(2)$ is the one-dimensional vector space generated by this operation. We can thus identify $\mathscr{C}om(2) \simeq \mathbb{K}$.

   Moreover, there is only one 1-ary operation, namely id_A, represented by the trivial tree $|$, i.e. the operadic unit. Therefore, also $\mathscr{C}om(1)$ is a one-dimensional vector space, and we have $\mathscr{C}om(1) \simeq \mathbb{K}$.

Operations with 3 or more inputs are obtained by composing 2-ary and 1-ary operations. Due to associativity, we have, for $n = 3$:



Therefore, we can view 3-ary operations as corollas, i.e. we obtain the trees in (5.5). Again, using commutativity, we get that $\mathscr{C}om(3)$ is generated by a unique 3-ary operation. Thus $\mathscr{C}om(3) \simeq \mathbb{K}$.

Similarly, for n-ary operations, $n > 3$, we get $\mathscr{C}om(n) \simeq \mathbb{K}$. Finally,

$$\mathscr{C}om(n) \simeq \mathbb{K}, \quad \text{for } n \geq 1$$

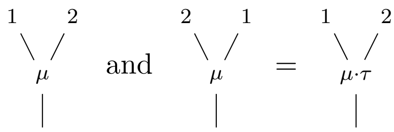
and

$$\mathscr{C}om(0) = 0,$$

since there are no 0-ary operations. The symmetric group action on the spaces $\mathscr{C}om(n)$ is given by the trivial action.

2. The operad $\mathscr{A}ss$ is the symmetric operad associated with associative nonunital algebras.

   As previously, there are no 0-ary operations (i.e. $\mathscr{A}ss(0) = 0$), and we have a unique 1-ary operation id_A, given by the operadic unit. Due to the lack of commutativity, the binary multiplication μ gives rise to two different binary operations



   Thus, $\mathscr{A}ss(2)$ is two-dimensional.

   Concerning ternary operations, we get, using associativity as previously, the 6 corollas in (5.5). As we have seen before, the symmetric group action changes the leaf ordering, i.e. acting by a permutation $\sigma' \in S_3$ on one of these corollas, we obtain another corolla. Since the corollas are only determined by their leaf ordering, we can identify each corolla with an element σ of the symmetric group S_3, or better with a base vector e_σ of the vector space $\mathbb{K}[S_3]$. Thus, $\mathscr{A}ss(3) = \mathbb{K}[S_3]$, and the symmetric group action is obviously the regular action.

   For n-ary operations, $n > 3$, the result is similar. Finally,

$$\mathscr{A}ss(n) \simeq \mathbb{K}[S_n], \quad \text{for } n \geq 1$$

   and

$$\mathscr{A}ss(0) = 0.$$

   The symmetric group action on the spaces $\mathscr{A}ss(n)$ is given by the regular action.

We previously defined an $\mathscr{A}ss$-algebra as an operadic morphism ρ from $\mathscr{A}ss$ to $\mathscr{E}nd(V)$. We will now show, using the above constructed operad $\mathscr{A}ss$, that such a representation actually provides an associative multiplication $\star := \rho_2(\mathrm{id}) \in \mathrm{Hom}(V^{\otimes 2}, V)$. Note that we identify the basis of $\mathbb{K}[S_n]$ with S_n. The proof of associativity is given by the following commutative diagrams:

$$\begin{array}{ccccccc}
S_2 & \times & (S_1 & \times & S_2) & \xrightarrow{\gamma_{\mathscr{S}}} & S_3 \\
\mathrm{id} & & \mathrm{id} & & \mathrm{id} & & \mathrm{id} \\
\downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_3} \\
\star & & \mathrm{id} & & \star & & a_1 \star (a_2 \star a_3) \\
\mathrm{Hom}(V^{\otimes 2}, V) & \otimes & (\mathrm{Hom}(V, V) & \otimes & \mathrm{Hom}(V^{\otimes 2}, V)) & \xrightarrow{\gamma_{\mathscr{E}nd}} & \mathrm{Hom}(V^{\otimes 3}, V)
\end{array}$$

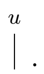
and

$$\begin{array}{ccccccc}
S_2 & \times & (S_2 & \times & S_1) & \xrightarrow{\gamma_{\mathscr{S}}} & S_3 \\
\mathrm{id} & & \mathrm{id} & & \mathrm{id} & & \mathrm{id} \\
\downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_3} \\
\star & & \star & & \mathrm{id} & & (a_1 \star a_2) \star a_3 \\
\mathrm{Hom}(V^{\otimes 2}, V) & \otimes & (\mathrm{Hom}(V^{\otimes 2}, V) & \otimes & \mathrm{Hom}(V, V)) & \xrightarrow{\gamma_{\mathscr{E}nd}} & \mathrm{Hom}(V^{\otimes 3}, V).
\end{array}$$

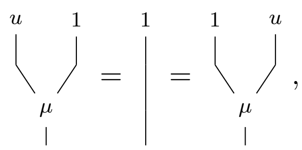
Hence, we have the desired associativity: $(a_1 \star a_2) \star a_3 = \rho_3(\mathrm{id}) = a_1 \star (a_2 \star a_3)$.

3. The operad $u\mathscr{A}ss$ is the symmetric operad associated with associative unital algebras.

   This operad is identical to the operad $\mathscr{A}ss$, except in arity 0. The unit $u : \mathbb{K} \to A, 1_{\mathbb{K}} \mapsto 1_A$ of the considered algebra A is an operation of arity 0. This is quite obvious, since we have no input (in A), but one output, namely the algebraic unit $1_A \in A$. This operation corresponds to the 0-corolla with no leaves, or, by decorating its unique vertex by the considered operation u, to

   Due to unitality $1_A \cdot a = a = a \cdot 1_A$, which corresponds to

   we do not get any additional operations of arity $n \geq 1$. However, since we now also have an operation of arity 0, $u\mathscr{A}ss(0)$ is one-dimensional and we may identify

$$u\mathscr{A}ss(0) \simeq \mathbb{K}[S_0],$$

   where $S_0 = \{\mathrm{id}\}$.

We previously showed that a representation ρ from $\mathscr{A}ss$ to $\mathscr{E}nd(V)$ provides an associative multiplication $\star := \rho_2(\mathrm{id}) \in \mathrm{Hom}(V^{\otimes 2}, V)$. We will now show, in a similar way, that $1 := \rho_0(\mathrm{id}) \in \mathrm{Hom}(V^{\otimes 0}, V) = \mathrm{Hom}(\mathbb{K}, V)$ defines a unit. The proof of unitality is given by the following commutative diagrams:

$$\begin{array}{ccccccc}
S_2 & \times & (S_0 & \times & S_1) & \xrightarrow{\gamma_{\mathscr{S}}} & S_1 \\
\mathrm{id} & & \mathrm{id} & & \mathrm{id} & & \mathrm{id} \\
\rho_2 \downarrow & & \rho_0 \downarrow & & \rho_1 \downarrow & & \rho_1 \downarrow \\
\star & & 1 & & \mathrm{id} & & 1 \star a = a \\
\mathrm{Hom}(V^{\otimes 2}, V) & \otimes & (\mathrm{Hom}(\mathbb{K}, V) & \otimes & \mathrm{Hom}(V, V)) & \xrightarrow{\gamma_{\mathscr{E}nd}} & \mathrm{Hom}(V, V)
\end{array}$$

and

$$\begin{array}{ccccccc}
S_2 & \times & (S_1 & \times & S_0) & \xrightarrow{\gamma_{\mathscr{S}}} & S_1 \\
\mathrm{id} & & \mathrm{id} & & \mathrm{id} & & \mathrm{id} \\
\rho_2 \downarrow & & \rho_1 \downarrow & & \rho_0 \downarrow & & \rho_1 \downarrow \\
\star & & \mathrm{id} & & 1 & & a \star 1 = a \\
\mathrm{Hom}(V^{\otimes 2}, V) & \otimes & (\mathrm{Hom}(V, V) & \otimes & \mathrm{Hom}(\mathbb{K}, V)) & \xrightarrow{\gamma_{\mathscr{E}nd}} & \mathrm{Hom}(V, V).
\end{array}$$

Hence, we have the desired unitality requirement: $1 \star a = a = a \star 1$.

4. The operad $u\mathscr{C}om$ is the symmetric operad associated with commutative associative unital algebras.

   It is identical to the operad $\mathscr{C}om$, except that the space $u\mathscr{C}om(0) \simeq \mathbb{K}$. This is due to the fact that we also have an operation of arity 0, namely the unit.

Remark: Let us notice that the action of the symmetric group encodes the symmetries of the operations of the considered algebraic structure. More precisely, commutativity corresponds to the trivial action, since any permutation of the factors still gives the same result. If no symmetry is present, we get the regular action, since, in general, any permutation of the factors gives another result. Following this idea, anticommutativity should correspond to the signature action.

If no symmetry is present, we can also consider nonsymmetric operads. Following the above philosophy, we have to consider planar trees instead of nonplanar ones. In the case of associative algebras, the multiplication μ corresponds to the unique (planar) corolla with 2 leaves. Similarly, any operation with 3 or more inputs corresponds to a unique corolla, so that all the spaces $\mathscr{A}s(n)$, $n > 0$ are one dimensional. We usually denote the nonsymmetric associative operad by $\mathscr{A}s$.

5.5 Partial definition of operads

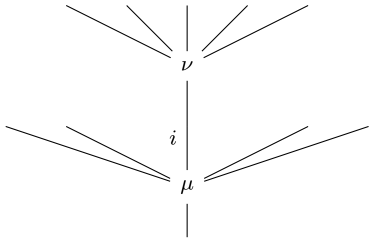
The *partial definition* is an alternative way to define operads. The main difference to the classical definition lies in the composition. We will only consider the case of symmetric operads, since the nonsymmetric case can be obtained by forgetting about symmetry.

Definition 5.8 (Partial definition of operads):
A symmetric operad consists of a sequence $(P(n))_{n\in\mathbb{N}}$ of vector spaces endowed with a right S_n-module structure, *partial composition* maps

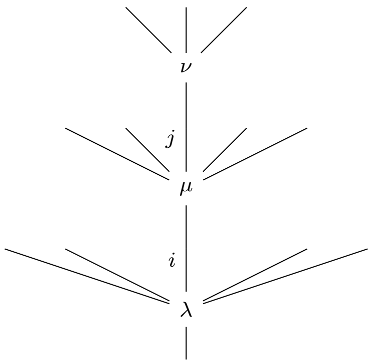
$$- \circ_i - : P(m) \otimes P(n) \to P(m+n-1),$$

for $1 \le i \le m$ and $n \ge 0$, and a unit element $1_P \in P(1)$ satisfying associativity, equivariance and unitality requirements given below.

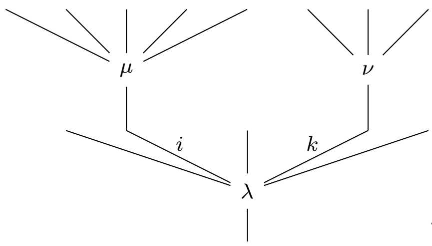
In terms of trees, partial composition $\mu \circ_i \nu$ means that the root of the tree corresponding to ν is grafted onto the i-th leaf of the tree corresponding to μ.



In order to formulate the associativity requirement, notice first that there are two possible cases for partial composition of 3 operations. In terms of trees, these are



and



.

Thus, the associativity requirement reads as

$$\begin{cases} (\lambda \circ_i \mu) \circ_{i+j-1} \nu = \lambda \circ_i (\mu \circ_j \nu), & \text{for } 1 \le i \le \ell, 1 \le j \le m, \\ (\lambda \circ_i \mu) \circ_{m+k-1} \nu = (\lambda \circ_k \nu) \circ_i \mu, & \text{for } 1 \le i < k \le \ell, \end{cases}$$

for any $\lambda \in P(\ell)$, $\mu \in P(m)$, $\nu \in P(n)$.

The unitality requirement is given by

$$\mu \circ_i 1_P = \mu \quad \text{and} \quad 1_P \circ_1 \mu = \mu,$$

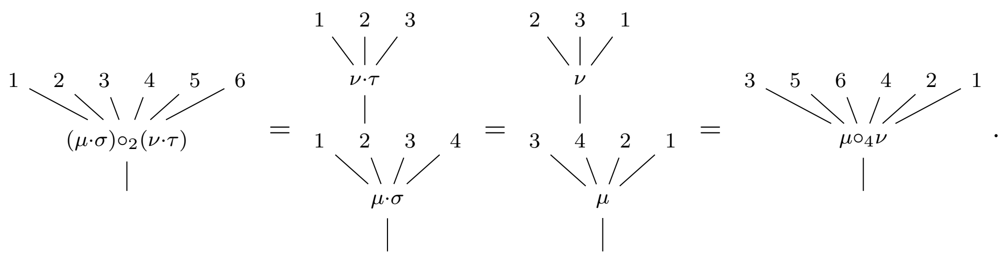
for any $\mu \in P(n)$, $1 \le i \le n$.

In order to be able to formulate the equivariance requirement, we have to define the partial composition in the symmetry operad. For two permutations $\sigma \in S_m$, $\tau \in S_n$, the permutation $\sigma \circ_i \tau \in S_{m+n-1}$ is obtained by inserting τ in the i-th place of σ. Equivariance is now given by

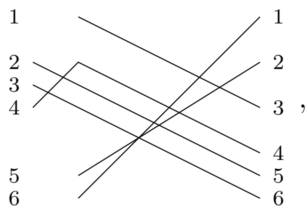
$$(\mu \cdot \sigma) \circ_i (\nu \cdot \tau) = (\mu \circ_{\sigma(i)} \nu) \cdot (\sigma \circ_i \tau),$$

for any $\mu \in P(m)$, $\nu \in P(n)$, $\sigma \in S_m$, $\tau \in S_n$.

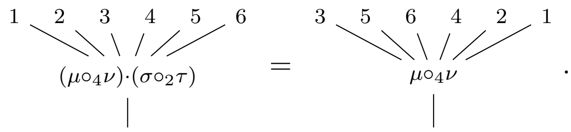
For instance, if $m = 4$, $n = 3$, $\sigma = \left(\begin{smallmatrix}1&2&3&4\\3&4&2&1\end{smallmatrix}\right)$, $\tau = \left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)$, and $i = 2$, $\sigma(i) = 4$, then $(\mu \cdot \sigma) \circ_i (\nu \cdot \tau)$ corresponds to



The permutation $\sigma \circ_i \tau$ is given by



and $(\mu \circ_{\sigma(i)} \nu) \cdot (\sigma \circ_i \tau)$ corresponds to



Finally, we find twice the same tree with the same leaf ordering.

The partial definition is equivalent to the classical definition. Since the main difference between the two definitions lies in the composition maps, we will only detail this aspect. Starting form the composition γ, we can define the partial compositions $- \circ_i -$ by

$$\mu \circ_i \nu = \gamma_{m;1,\ldots,1,n,1,\ldots,1}(\mu; \mathrm{id}, \ldots, \mathrm{id}, \nu, \mathrm{id}, \ldots, \mathrm{id}).$$

Conversely, the composition map γ can be obtained by defining

$$\gamma_{k_1,\ldots,k_n} = (- \circ_1 (\cdots (- \circ_{n-1} (- \circ_n -)) \cdots)).$$

Chapter 6

Functorial definition of operads

In this chapter, we will consider a third equivalent definition of operads. An operad will be given as a monoidal structure on an endofunctor in the category of vector spaces, more precisely on a Schur functor, which is a special kind of endofunctor. This allows, in particular, to substitute the equivalent and often advantageous viewpoint of S-modules to the one of endofunctors.

6.1 Monoids, monads and Schur functors

Let us first explain some category theoretical concepts which are needed in order to give this functorial definition of an operad.

Definition 6.1: A *monoidal category* $\mathtt{C}$ is a category with a bifunctor $\otimes : \mathtt{C} \times \mathtt{C} \to \mathtt{C}$ and an object I called unit, satisfying associativity and unity requirements (up to a natural isomorphism).

Remark: If the natural isomorphisms mentioned in the above definition are given by identities, then the considered monoidal category is called a *strict monoidal category.*

Examples 6.1: The two examples of monoidal categories considered here are in fact strict monoidal categories.

- The category `Vect` of vector spaces over $\mathbb{K}$ with the usual tensor product $\otimes$ and unit $I = \mathbb{K}$ is a monoidal category.
- The category `End(C)=[C,C]` of endofunctors in `C`, whose objects are functors from `C` to `C` and morphisms are natural transformations, is a monoidal category. The monoidal structure $\otimes$ is given by the composition $\circ$ of endofunctors and the identity I is given by the identity functor.

Definition 6.2: A *monoid* in a monoidal category $(\mathtt{C}, \otimes, I)$ is an object P together with two morphisms $\gamma : P \otimes P \to P$ (called composition) and $i : I \to P$ (called identity), satisfying associativity and unity requirements.

Example 6.2: An associative unital algebra with multiplication γ and unit i is a monoid in $(\mathtt{Vect}, \otimes, \mathbb{K})$.

A monoid can not be defined in an arbitrary category `C`. However, it is always possible to define a monoid in the category `End(C)`, which is always a monoidal category. A monoid in `End(C)` is also called a *monad* (or *triple*) in `C`.

Definition 6.3 (Functorial definition of operads):
An *operad* is a monad in the category `Vect`, i.e. a monoid in the category `End(Vect)` (with monoidal structure $\circ$).

More precisely, an operad P is an object in `End(Vect)`, i.e. a functor $P : \texttt{Vect} \to \texttt{Vect}$ together with two maps $\gamma : P \circ P \to P$ and $i : I \to P$, which are natural transformations, satisfying associativity and unity requirements given by the following commutative diagrams:

$$\begin{array}{ccc} P \circ (P \circ P) \simeq (P \circ P) \circ P & \xrightarrow{\gamma \otimes \mathrm{id}} & P \circ P \\ \downarrow{\scriptstyle \mathrm{id} \otimes \gamma} & & \downarrow{\scriptstyle \gamma} \\ P \circ P & \xrightarrow{\quad\gamma\quad} & P \end{array} \quad \text{and} \quad \begin{array}{ccccc} I \circ P & \xrightarrow{i \otimes \mathrm{id}} & P \circ P & \xleftarrow{\mathrm{id} \otimes i} & P \circ I \, . \\ & {\scriptstyle \simeq} \searrow & \downarrow{\scriptstyle \gamma} & \swarrow {\scriptstyle \simeq} & \\ & & P & & \end{array}$$

Note that, for two endofunctors $P, Q \in \texttt{End(Vect)}$, the composition $\circ$ is obviously defined by

$$(P \circ Q)(V) = P(Q(V)) \text{ and } (P \circ Q)(\ell) = P(Q(\ell)),$$

for any vector space V and any linear map ℓ. It is also possible to define additional operations on endofunctors in `Vect`, namely the tensor product and the direct sum, by

$$(P \otimes Q)(V) = P(V) \otimes Q(V) \text{ and } (P \otimes Q)(\ell) = P(\ell) \otimes Q(\ell),$$

respectively

$$(P \oplus Q)(V) = P(V) \oplus Q(V) \text{ and } (P \oplus Q)(\ell) = P(\ell) \oplus Q(\ell),$$

for any vector space V and any linear map ℓ.

6.2 S-modules

Definition 6.4: An *S-module* P is a sequence $(P_n)_{n \in \mathbb{N}}$ of vector spaces endowed with right S_n-module structures.

In view of the classical definition, operads are defined by means of S-modules. To an S-module P, we can associate an endofunctor $\tilde{P} : \texttt{Vect} \to \texttt{Vect}$, called *Schur functor*, by

$$\tilde{P}(V) = \bigoplus_{n \in \mathbb{N}} P(n) \otimes_{S_n} V^{\otimes n}$$

and

$$\tilde{P}(\ell) = \bigoplus_{n \in \mathbb{N}} \mathrm{id} \otimes_{S_n} \ell^{\otimes n} : \tilde{P}(V) \to \tilde{P}(W),$$

for any vector space V and any linear map $\ell : V \to W$.

A Schur functor is thus a special kind of endofunctor in `Vect`, thus defines an operad in view of the previously given functorial definition. In the following, we would like to limit ourselves to operads given by Schur functors. Showing that Schur functors are in one-to-one correspondence with S-modules will then allow us to use the functorial and the classical definition of operads in an equivalent manner.

In particular, the identification of S-modules and Schur functors should respect the operations $\circ$, $\oplus$ and $\otimes$. Therefore, we first have to define these operations for S-modules.

The *direct sum* of two S-modules P and Q is defined by

$$(P \oplus Q)(n) = P(n) \oplus Q(n),$$

concerning vector spaces, and by $(\mu \oplus \nu) \cdot \sigma = (\mu \cdot \sigma) + (\nu \cdot \sigma)$, concerning the S_n-action. From this definition, it follows that

$$\widetilde{P \oplus Q} = \tilde{P} \oplus \tilde{Q}.$$

The *tensor product* of two S-modules P and Q is defined by

$$(P \otimes Q)(n) = \bigoplus_{i+j=n} \mathrm{Ind}_{S_i \times S_j}^{S_n} P(i) \otimes Q(j),$$

which is a right S_n-module. It can be shown that from this definition, it follows that

$$\widetilde{P \otimes Q} = \tilde{P} \otimes \tilde{Q}.$$

Remark: By proposition 1.3 we get that

$$(P \otimes Q)(n) = \bigoplus_{i+j=n} P(i) \otimes Q(j) \otimes \mathbb{K}\left[{}^{S_n}\!/\!_{S_i \times S_j}\right] = \bigoplus_{i+j=n} P(i) \otimes Q(j) \otimes \mathbb{K}[\mathrm{sh}(i,j)]$$

as vector space, where $\mathrm{sh}(i,j)$ denotes the space of (i,j)-shuffles, i.e. permutations of $i+j=n$ elements, where the first i and the last j elements are respectively in natural order, i.e. permutations $\sigma \in S_{i+j}$ with $\sigma_1 < \ldots < \sigma_i$ and $\sigma_{i+1} < \ldots < \sigma_{i+j}$.

The *composite* of two S-modules P and Q is defined by

$$\begin{aligned}(P \circ Q)(n) &= \bigoplus_{k \in \mathbb{N}} P(k) \otimes_{S_k} Q^{\otimes k}(n) \\ &= \bigoplus_{k \in \mathbb{N}} P(k) \otimes_{S_k} \left(\bigoplus_{i_1 + \cdots + i_k = n} \mathrm{Ind}_{S_{i_1} \times \cdots \times S_{i_k}}^{S_n} Q(i_1) \otimes \cdots \otimes Q(i_k) \right).\end{aligned}$$

This is a right S_n-module if $Q^{\otimes k}(n)$ carries a left S_k-module structure which is compatible with the right S_n-module structure.

Remark: As previously, we get by proposition 1.3 that

$$(P \circ Q)(n) = \bigoplus_{\substack{k \\ i_1 + \cdots + i_k = n}} P(k) \otimes_{S_k} (Q(i_1) \otimes \cdots \otimes Q(i_k)) \otimes \mathbb{K}[\mathrm{sh}(i_1, \ldots, i_k)].$$

This space is spanned by equivalence classes (for the S_k-action) of elements $(\mu; \nu_1, \ldots, \nu_k; \sigma)$, where $\mu \in P(k)$, $\nu_j \in Q(i_j)$ and $\sigma \in \mathbb{K}[\mathrm{sh}(i_1, \ldots, i_k)]$.

The left S_k-module structure on $Q^{\otimes k}(n)$ is explained by the following example. Consider the case $k = 2$, and let $\tau \in S_2$ be the transposition, then the action of τ on

$$Q^{\otimes 2}(n) = \bigoplus_{i+j=n} Q(i) \otimes Q(j) \otimes \mathbb{K}[\mathrm{sh}(i,j)]$$

is given by

$$\tau \cdot (\nu_1, \nu_2, \sigma) = (\nu_2, \nu_1, \sigma'),$$

where $\sigma' = \sigma \circ \left(\begin{smallmatrix} 1 & \cdots & j & j+1 & \cdots & i+j \\ i+1 & \cdots & i+j & 1 & \cdots & i \end{smallmatrix}\right)$. Indeed, for instance, if $i = 3$, $j = 2$ and $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \end{smallmatrix}\right)$, then $\sigma' = \sigma \circ \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 3 & 5 \end{smallmatrix}\right)$. Hence, roughly, the S_k-action on $Q^{\otimes k}(n)$ is given by the action on $Q(i_1) \otimes \cdots \otimes Q(i_k)$ and by changing the shuffle appropriately.

It can be shown that from this definition, it follows that

$$\widetilde{P \circ Q} = \tilde{P} \circ \tilde{Q}.$$

Remark: Operads are abstractions of algebras, however, not all results can be transferred from the algebraic to the operadic setting. The tensor product $\otimes$, providing the monoidal structure on `Vect` is bilinear, whereas the composition $\circ$, providing the monoidal structure on `End(Vect)` is only linear in the left factor. This is best seen in the above given formula for the composite of S-modules, and due to the fact that the right factor Q appears multiple times in this composite.

This weakened form of bilinearity will be the source of several obstructions in the following.

An *S-module morphism* is a sequence of linear maps, commuting with the symmetric group action. S-modules and S-module morphisms form a category S-`Mod`. This category is a monoidal category with monoidal structure given by the composition $\circ$ and the unit S-module $I = (0, \mathbb{K}, 0, 0, \ldots)$.

As the map $\tilde{} : \{S\text{-modules}\} \to \{\text{Schur functors}\}$ respects all operations, we can identify S-modules and Schur functors, provided this map is injective.

In order to proof injectivity, we need the following

Lemma 6.1: *$P(n)$ is the n-multilinear part of $\tilde{P}(\mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n)$.*

Proof: The k-th tensor power $(\mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n)^{\otimes k}$ admits a basis made up by elements of the form $x_{i_1} \cdots x_{i_k}$ (where the tensor product $\otimes$ is omitted). Multilinear means that all the x_i-s are different and n-multilinear thus means that we only consider basis elements of the form $x_{\sigma_1} \cdots x_{\sigma_n}$, $\sigma \in S_n$. The n-multilinear part $\mathscr{M}^n(P)$ is finally given by $\mathscr{M}^n(P) = P(n) \otimes_{S_n} \sum_{\sigma \in S_n} k^\sigma x_{\sigma_1} \cdots x_{\sigma_n}$. Consider now an element of the form $\theta \otimes \tau \cdot (x_1 \cdots x_n) = (\theta \cdot \tau) \otimes (x_1 \cdots x_n)$, which can also be viewed as an element of $P(k) \otimes \mathbb{K}(x_1 \cdots x_n)$, where the latter factor is a one-dimensional vector space. Finally, we can identify the considered element with $\theta \cdot \tau \in P(n)$. □

Injectivity now follows immediately. Indeed, if the two Schur functors $\tilde{P}$ and $\tilde{Q}$ are equal, they have to coincide on every vector space, in particular $\tilde{P}(\mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n) = \tilde{Q}(\mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n)$, for any n, thus their n-multilinear parts are equal, for any n. Finally, $P(n) \simeq Q(n)$, for any n, i.e. $P \simeq Q$.

Remark: We will now confine ourselves to operads given by Schur functors. This allows to view an operad either as an S-module or as a Schur functor, using the most convenient standpoint depending on the situation.

It can be shown that the functorial definition of an operad is 'equivalent' to the classical definition. We will only give a rough description how the classical structure of an operad can be obtained from the functorial one in the nonsymmetric case.

Using the S-module viewpoint, an operad P provides a sequence $(P(n))_{n \in \mathbb{N}}$ of vector spaces. The sequence of linear maps $\gamma_n : (P \circ P)(n) \to P(n)$, where

$$(P \circ P)(n) = \bigoplus_{\substack{k \in \mathbb{N} \\ i_1 + \cdots + i_k = n}} P(k) \otimes P(i_1) \otimes \cdots \otimes P(i_k),$$

gives rise to the composition maps $\gamma_{i_1, \ldots, i_k} : P(k) \otimes P(i_1) \otimes \cdots \otimes P(i_k) \to P(n)$, whereas the sequence of linear maps $i_n : I(n) \to P(n)$, where $I(1) = \mathbb{K}$ and $I(n) = 0$ for $n \neq 1$, gives rise to the identity $i_1 : \mathbb{K} \to P(1), 1 \mapsto i_1(1) =: 1_P$.

6.3 P-algebras

In the classical setting we considered P-algebras, which are representations of an operad P on a vector space, i.e. a sequence of linear maps $\rho_n : P(n) \otimes_{S_n} V^{\otimes n} \to V$ that respects composition and identity. In the functorial setting, we give, using the endofunctor standpoint, the following

Definition 6.5: A *P-algebra* is a vector space V together with a linear map $\gamma_V : P(V) \to V$, such that the following diagrams commute:

$$\begin{array}{ccc} (P \circ P)(V) = P(P(V)) & \xrightarrow{\gamma(V)} & P(V) \\ {\scriptstyle P(\gamma_V)}\downarrow & & \downarrow{\scriptstyle \gamma_V} \\ P(V) & \xrightarrow[\gamma_V]{} & V \end{array} \qquad \begin{array}{ccc} I(V) & \xrightarrow{i(V)} & P(V) \\ & \searrow^{\simeq} & \downarrow{\scriptstyle \gamma_V} \\ & & V. \end{array}$$

The classical and the functorial definition of a P-algebra coincide (if P is a Schur functor). Starting from the functorial definition, we get that a P-algebra is a vector space V together with the linear map

$$\gamma_V : P(V) = \bigoplus_{n \in \mathbb{N}} P(n) \otimes_{S_n} V^{\otimes n} \to V,$$

which is made up by a sequence of linear maps

$$\gamma_{V,n} : P(n) \otimes_{S_n} V^{\otimes n} \to V, \quad n \in \mathbb{N},$$

that respects composition and identity, which is encoded in the commutative diagrams.

Indeed, the triangle diagram encodes that the abstract identity is sent to the concrete one. The square diagram encodes that 'the concrete map associated to abstract composition' (in the upper and right parts of the diagram) and 'composition of concrete maps' (in the left and lower parts) coincide.

Let us roughly explain what happens in the 'composition of concrete maps'. Since composition of Schur functors coincides with the Schur functor associated to the composite of S-modules, we essentially have

$$\begin{aligned} (P \circ P)(V) &= P(k) \otimes P(i_1) \otimes \cdots \otimes P(i_k) \otimes V^{\otimes(i_1 + \cdots + i_k)} \\ &\simeq P(k) \otimes \left(P(i_1) \otimes V^{\otimes i_1}\right) \otimes \cdots \otimes \left(P(i_k) \otimes V^{\otimes i_k}\right) \\ &\xrightarrow{P(\gamma_V) = \mathrm{id} \otimes \gamma_V^{\otimes k}} P(k) \otimes V \otimes \cdots \otimes V = P(k) \otimes V^{\otimes k} \xrightarrow{\gamma_V} V, \end{aligned}$$

where we omitted the direct sums in order to simplify notations.

Definition 6.6: Let (V, γ_V) and (W, γ_W) be two P-algebras. A *P-algebra morphism* $\varphi : (V, \gamma_V) \to (W, \gamma_W)$ is a linear map $\varphi : V \to W$, such that the following diagram commutes:

$$\begin{array}{ccc} P(V) & \xrightarrow{\gamma_V} & V \\ {\scriptstyle P(\varphi)}\downarrow & & \downarrow{\scriptstyle \varphi} \\ P(W) & \xrightarrow[\gamma_W]{} & W. \end{array}$$

P-algebras and P-algebra morphisms form a category P-`Alg`.

Definition 6.7: The *free P-algebra* over a vector space V is the P-algebra $F(V)$ together with the linear map $i : V \to F(V)$, such that for any P-algebra A and any linear map $\varphi : V \to A$ there exists a unique P-algebra morphism $\tilde{\varphi} : F(V) \to A$ such that $\varphi = \tilde{\varphi} \circ i$, i.e. the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{i} & F(V) \\ & \searrow_{\varphi} & \downarrow{\scriptstyle \tilde{\varphi}} \\ & & A. \end{array}$$

If existence of the free P-algebra is proved, we get that F is a functor from `Vect` to P-`Alg` and that $i : V \to F(V)$ is functorial in V, since for any linear map $\ell : V \to W$, there exists a unique P-algebra morphism $F(\ell) : F(V) \to F(W)$, such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{i_V} & F(V) \\ {\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle F(\ell)} \\ W & \xrightarrow[i_W]{} & F(W). \end{array}$$

Existence of the free P-algebra is given by the following

Proposition 6.2: *The free P-algebra over V is the vector space $P(V) = \bigoplus_{n\in N} P(n) \otimes_{S_n} V^{\otimes n}$ given by the Schur functor P, endowed with the P-algebra structure $\gamma_{P(V)} : P(P(V)) \to P(V)$, given by the monoidal composition $\gamma(V) : (P \circ P)(V) \to P(V)$, together with the linear map $i_V : V \to P(V)$, given by $i(V) : I(V) \to P(V)$.*

Remark: Operads are exactly what is needed to construct free algebras.

Example 6.3: We will now revisit the operads $\mathscr{A}ss$ and $\mathscr{C}om$.

1. In view of the previous proposition, the Schur functor $\mathscr{A}ss$ applied to a vector space V should provide the free associative nonunital algebra over V, which is the reduced tensor algebra $\bar{T}(V)$. This means that we should have

$$\mathscr{A}ss(V) = \bigoplus_{n\in\mathbb{N}} \mathscr{A}ss(n) \otimes_{S_n} V^{\otimes n} = \bigoplus_{n\in\mathbb{N}^*} V^{\otimes n} = \bar{T}(V).$$

   Recalling that the tensor product $\otimes_{S_n}$ is actually over $\mathbb{K}[S_n]$, we get that

$$\mathscr{A}ss(n) = \mathbb{K}[S_n],$$

   for $n \geq 1$ and $\mathscr{A}ss(0) = 0$. Hence, we obtain the same result as previously.

   Concerning $u\mathscr{A}ss$, the Schur functor $u\mathscr{A}ss$ applied to a vector space V should provide the free associative unital algebra over V, which is the tensor algebra $T(V)$. This gives again that $u\mathscr{A}ss(n) = \mathbb{K}[S_n]$, for $n \geq 0$.

2. The Schur functor $\mathscr{C}om$ applied to a vector space V should provide the free commutative nonunital algebra over V, which is the reduced symmetric algebra $\bar{S}(V)$. Note that

$$\bar{S}(V) = \bigoplus_{n\in\mathbb{N}^*} S^n V = \bigoplus_{n\in\mathbb{N}^*} \left(V^{\otimes n}\right)_{S_n},$$

   i.e. given by tensors which are invariant under the symmetric group action. This means that we should have

$$\mathscr{C}om(V) = \bigoplus_{n\in\mathbb{N}} \mathscr{C}om(n) \otimes_{S_n} V^{\otimes n} = \bigoplus_{n\in\mathbb{N}^*} S^n V = \bar{S}(V).$$

   In view of the previously obtained form of $\mathscr{C}om$, we should obtain that $\mathbb{K} \otimes_{S_n} V^{\otimes n} = \left(V^{\otimes n}\right)_{S_n}$, where $\mathbb{K}$ is the trivial representation. Indeed, elements of $\mathbb{K} \otimes_{S_n} V^{\otimes n}$ are of the form

$$\sum k \otimes (v_1 \cdots v_n) = \sum k \cdot \sigma \otimes (v_1 \cdots v_n) = \sum k \otimes \sigma \cdot (v_1 \cdots v_n) = \sum k \otimes (v_{\sigma_1^{-1}} \cdots v_{\sigma_n^{-1}}),$$

which is also an element of $(V^{\otimes n})_{S_n}$, and vice versa. Hence, we have

$$\mathscr{C}om(n) = \mathbb{K},$$

for $n \geq 1$ and $\mathscr{C}om(0) = 0$.

Concerning $u\mathscr{C}om$, the Schur functor $u\mathscr{C}om$ applied to a vector space V should provide the free commutative unital algebra over V, which is the symmetric algebra $S(V)$. This gives again that $u\mathscr{C}om(n) = \mathbb{K}$, for $n \geq 0$.

Chapter 7

Free operad and combinatorial definition of operads

The notion of 'free operad' will be important in the following, as it allows to give an operad using only some generating operations, from which all other ones will be freely constructed. A type of algebras can thus be encoded in an operad, which is given as the quotient of a free one (encoding the generating operations) by an operadic ideal (encoding the relations).

Moreover, we provide a fourth equivalent definition of operads, namely the combinatorial definition, which will make the relationship between operads and tree diagrams explicit.

7.1 Free operad

7.1.1 Construction of the free operad

As operads can be regarded as abstractions of algebras, we would like to define the free operad over an S-module in a similar way as we defined the free associative algebra over a vector space. However, due to the lack of linearity in the right factor of the composition of S-modules, this is not possible. Therefore, we will define the free operad using a limiting procedure.

As for any free object, the *free operad* over an S-module M is defined by means of a universal property. Namely, as being the operad $F(M)$ together with the S-module morphism $i : M \to F(M)$, such that for any operad P and any S-module morphism $\varphi : M \to P$, there exists a unique morphism of operads $\tilde{\varphi} : F(M) \to P$, such that $\varphi = \tilde{\varphi} \circ i$, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{i} & F(M) \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \tilde{\varphi}} \\ & & P. \end{array}$$

Equivalently, one can define the free operad functor $S\text{-}\mathtt{Mod} \to \mathtt{Operad}$ as being the left adjoint functor to the forgetful functor $\mathtt{Operad} \to S\text{-}\mathtt{Mod}$.

In order to construct the free operad, we will view the S-module M as a Schur functor and define the sequence of Schur functors $(\mathscr{T}_n M)_{n\in\mathbb{N}}$ by

$$\begin{aligned} \mathscr{T}_0 M &= I \\ \mathscr{T}_1 M &= I \oplus M \\ \mathscr{T}_2 M &= I \oplus (M \circ (I \oplus M)) = I \oplus (M \circ \mathscr{T}_1 M) \\ &\cdots \end{aligned}$$

$$\mathscr{T}_n M = I \oplus (M \circ \mathscr{T}_{n-1} M)$$
$$\dots$$

Remark: In general, we cannot develop the above expressions, since the composition is only left-additive. However, if it were biadditive, we could write $\mathscr{T}_n M = I \oplus M \oplus M^{\circ 2} \oplus \cdots \oplus M^{\circ n}$, which would then give the operadic analogue of the tensor algebra, which is the free associative algebra.

Moreover, we recursively define a sequence $i_n : \mathscr{T}_{n-1} M \to \mathscr{T}_n M$ of natural transformations by

$$i_1 : \mathscr{T}_0 M \to \mathscr{T}_1 M, \quad I \rightarrowtail I \oplus M$$

and

$$i_n : \mathscr{T}_{n-1} M = I \oplus (M \circ \mathscr{T}_{n-2} M) \to \mathscr{T}_n M = I \oplus (M \circ \mathscr{T}_{n-1} M), \quad i_n = \mathrm{id}_I \oplus (\mathrm{id}_M \circ i_{n-1}).$$

Note that i_n is a split monomorphism. A monomorphism is a left-cancellable morphism, i.e. a morphism f, such that $f \circ g = f \circ h \Rightarrow g = h$. In concrete categories, a monomorphism is a slightly weaker concept than an injection, which is itself a slightly weaker concept than a split monomorphism.

Finally, we have a *direct system* $(\mathscr{T}_n M, i_n)$ and we can take the *direct limit* (also called *inductive limit* or *colimit*):

$$\mathscr{T} M = \varinjlim \mathscr{T}_n M = \coprod_n \mathscr{T}_n M \big/ \sim,$$

where the equivalence relation $\sim$ is given by the identification in the disjoint union of $\mathscr{T}_{n-1} M$ and its injection in $\mathscr{T}_n M$. Thus, we can also see $\mathscr{T} M$ as being the increasing union $\bigcup_n \mathscr{T}_n M$. This direct limit $\mathscr{T} M$ will play the role of the free operad over the S-module M.

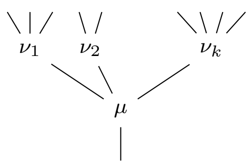
Let us detail another viewpoint, using tree diagrams, of the free operad. In order to do this we need some more information about the relationship between operads and trees.

Remark (Tree Guidelines 3): Recall that the composite $P \circ Q$ of two S-modules P and Q is defined by

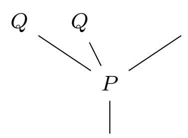
$$(P \circ Q)(n) = \bigoplus_{\substack{k \\ i_1 + \cdots + i_k = n}} P(k) \otimes_{S_k} (Q(i_1) \otimes \cdots \otimes Q(i_k)) \otimes \mathbb{K}[\mathrm{sh}(i_1, \dots, i_k)],$$

and that this space is spanned by (equivalence classes of) elements $(\mu; \nu_1, \dots, \nu_k; \sigma)$, where $\mu \in P(k)$, $\nu_j \in Q(i_j)$ and $\sigma \in \mathbb{K}[\mathrm{sh}(i_1, \dots, i_k)]$.

In order to simplify notations, we will often omit the shuffles in the following. An element $(\mu; \nu_1, \dots, \nu_k)$ will be represented by



, respectively by

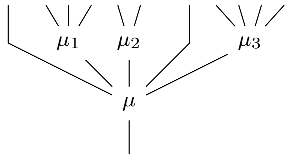


,

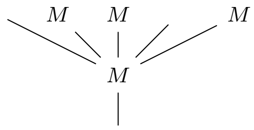
if we are not interested in the chosen operation (and its arity), but only in the corresponding space.

We will now apply this notation to the case of the free operad $\mathscr{T}M$, which is sometimes called the *tree module*.

The unique element id of $\mathscr{T}_0 M = I$ is represented by the trivial tree $|$. If we consider, for instance, an element $(\mu; \mathrm{id}, \mu_1, \mu_2, \mathrm{id}, \mu_3)$ of $M \circ (I \oplus M) \subset \mathscr{T}_2 M$, it can be represented by



, or, more generally, by



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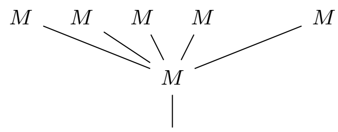
Note that, in particular, elements

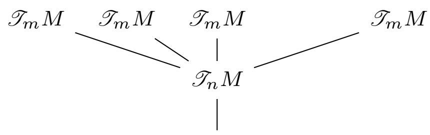


of $M^{\circ 2}$ are of this type, so that $M^{\circ 2} \subset \mathscr{T}_2 M$. In general, $\mathscr{T}_n M$ is the space of trees with n levels at most, whose vertices are labelled by (elements of) M. Furthermore, $M^{\circ n} \subset \mathscr{T}_n M$.

We have, by successive application of the i_k-s, morphisms $i_{n,m} : \mathscr{T}_n M \to \mathscr{T}_m M$. These give rise to a morphism $u : I \to \mathscr{T}M$. Moreover, we have morphisms $j_n : M \circ \mathscr{T}_{n-1} M \to \mathscr{T}_n M$ given by inclusion of the second term in the definition of $\mathscr{T}_n M$. These give rise to a morphism $j : M \to \mathscr{T}M$.

Theorem 7.1: *There is a composition morphism γ, such that $(\mathscr{T}M, \gamma, u)$ is an operad, which together with j is the free operad over M.*

Proof: Composition is defined on elements of $\mathscr{T}M \circ \mathscr{T}M$, and since $\mathscr{T}M = \bigcup_n \mathscr{T}_n M$, it is defined on elements of the form



.

Therefore, we define γ inductively on $\mathscr{T}_n M \circ \mathscr{T}_m M$, by

$$\mathscr{T}_n M \circ \mathscr{T}_m M = (I \oplus (M \circ \mathscr{T}_{n-1} M)) \circ \mathscr{T}_m M \simeq \mathscr{T}_m M \oplus (M \circ (\mathscr{T}_{n-1} \circ \mathscr{T}_m M))$$
$$\xrightarrow{i_{m,n+m} \oplus \mathrm{id}_M \circ \gamma_{n-1,m}} \mathscr{T}_{n+m} M \oplus (M \circ \mathscr{T}_{n+m-1} M) \xrightarrow{\mathrm{id} + j_{n+m}} \mathscr{T}_{n+m} M.$$

Of course, one still has to check that the definition is independent of the choices of n and m, and that all other conditions (associativity, unitality and universality) are verified. □

Remark: Note that in the above definition of the composition map γ, we used left-additivity of the composition $\circ$ of S-modules. Moreover, we used the associativity isomorphism

$$(M \circ \mathscr{T}_{n-1} M) \circ \mathscr{T}_m M \simeq M \circ (\mathscr{T}_{n-1} \circ \mathscr{T}_m M),$$

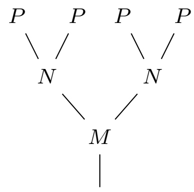
which bares some differences to its algebraic analogue. In particular, when working in a graded context, this associativity isomorphism will lead to Koszul sign, since the switching map is involved. Indeed, the associativity isomorphism identifies the component

$$(M(i) \otimes N(j_1) \otimes N(j_2)) \otimes P(k_1) \otimes P(k_2) \otimes P(k_3) \otimes P(k_4)$$

in $(M \circ N) \circ P$ with the component

$$M(i) \otimes (N(j_1) \otimes P(k_1) \otimes P(k_2)) \otimes (N(j_2) \otimes P(k_3) \otimes P(k_4))$$

in $M \circ (N \circ P)$. Elements of both (identified) components may be pictured as being of the type



.

Example 7.1: Consider the S-module $M = (0, W, 0, \ldots)$, where W is a vector space. The corresponding Schur functor is, applied on a vector space V, $M(V) = W \otimes V$. Note that this functor is linear, i.e. $M(V \oplus V') = M(V) \oplus M(V')$. We can thus write

$$\begin{aligned}
\mathscr{T}_0 M &= I \\
\mathscr{T}_1 M &= I \oplus M \\
\mathscr{T}_2 M &= I \oplus (M \circ (I \oplus M)) = I \oplus M \oplus M^{\circ 2} \\
&\ldots \\
\mathscr{T}_n M &= I \oplus (M \circ \mathscr{T}_{n-1} M) = I \oplus M \oplus \cdots \oplus M^{\circ n} \\
&\ldots
\end{aligned}$$

as Schur functors, or, equivalently,

$$\begin{aligned}
\mathscr{T}_0 M &= (0, \mathbb{K}, 0, \ldots) \\
\mathscr{T}_1 M &= (0, \mathbb{K} \oplus W, 0, \ldots) \\
\mathscr{T}_2 M &= (0, \mathbb{K} \oplus W \oplus W^{\otimes 2}, 0, \ldots) \\
&\ldots \\
\mathscr{T}_n M &= (0, \mathbb{K} \oplus W \oplus \cdots \oplus W^{\otimes n}, 0, \ldots) \\
&\ldots
\end{aligned}$$

as S-modules. Finally, we get

$$\mathscr{T} M = (0, T(W), 0, \ldots),$$

and we recover the tensor algebra $T(W)$, i.e. the free associative algebra over W.

It is possible to introduce a *weight grading* on the free operad $\mathscr{T} M$. This is done by defining the weight of an element $\mu \in M(n)$ to be equal to one, and the weight of the element $\mathrm{id} \in I(1)$ to be zero. The weight of a general element of $\mathscr{T} M$ is then given by the number of operations of M, which it is built from. In terms of trees, the weight is given by the number of vertices (decorated by M). As usually, we denote the space of elements of weight k by $\mathscr{T} M^{(k)}$. In particular, we have that $\mathscr{T} M^{(0)} = I$, $\mathscr{T} M^{(1)} = M$, and that $\mathscr{T} M^{(2)} \subset M^{\circ 2} \subset \mathscr{T}_2 M$.

7.1.2 Free operad and types of algebras

The importance of the free operad lies in the fact that any operad can be given as the quotient of a free operad by an operadic ideal. Indeed, the operad corresponding to some type of algebras can be given as the quotient $\mathscr{T} M / (R)$, where the S-module M is determined by the generating operations of the considered algebra, and $R \subset \mathscr{T} M$ is determined by the relations that these operations verify.

Let us be more precise. An *algebra of type* `P` is given by a vector space A and n-ary operations $\mu_n : A^{\otimes n} \to A$, called *generating operations*, satisfying certain *relations* $r_j = 0$. Further, we assume that the relations are multilinear, i.e. of the form $r_j = \sum_k \varphi_k = 0$, where φ_k is a composite of generating relations (and identities). The elements $r_j = \sum_k \varphi_k$ are called *relators*. The category of algebras of type `P` is denoted by `P-Alg`.

Example 7.2: Let A be an algebra of type associative, i.e. an associative algebra, then there is only one generating operation, namely the binary multiplication $\mu : A^{\otimes 2} \to A$, satisfying the associativity relation

$$-\mu \circ (\mu, \mathrm{id}) + \mu \circ (\mathrm{id}, \mu) = 0.$$

The unique relator r is given by $r = \varphi_1 + \varphi_2 = -\mu \circ (\mu, \mathrm{id}) + \mu \circ (\mathrm{id}, \mu)$.

Let M be the S-module, whose arity n spaces are generated by the n-ary generating operations μ_n, and where the S_n-module structure is given by the symmetries of these operations. Since the relators are composites of these generating relations (and identity), they span a sub-S-module R of the free operad $\mathscr{T}M$. Let (R) denote the operadic ideal of $\mathscr{T}M$ generated by R. The precise definition of operadic ideals is given as follows:

Definition 7.1: An *operadic ideal* I of an operad P is a sub-S-module of P, such that for any family of operations $\{\mu; \nu_1, \ldots, \nu_k\}$ of P, we have that if one of these operations is in I, then the composite $\gamma(\mu; \nu_1, \ldots, \nu_k)$ is also in I.

This way, we have naturally constructed the operad ${}^{\mathscr{T}M}\!/_{(R)}$, which corresponds to algebras of type `P`.

For algebras of type `P`, there exists the notion of free algebras of type `P` over a vector space V. Let P denote the functor $P : V \mapsto P(V)$, which gives the free algebra of type `P` over V. As we have seen in the previous chapter, this functor P is a Schur functor, and more precisely an operad.

By construction $\left({}^{\mathscr{T}M}\!/_{(R)}\right)(V)$ also gives the free algebra of type `P` over the vector space V. Since both constructions are functorial in V, the operads P and ${}^{\mathscr{T}M}\!/_{(R)}$ coincide. We get the following

Proposition 7.2: *A type* `P` *of algebras (whose relations are multilinear) determines an operad* $P = {}^{\mathscr{T}M}\!/_{(R)}$. *Moreover, the category* P-`Alg` *of algebras over this operad is equivalent to the category* `P-Alg` *of algebras of the given type* `P`.

7.2 Combinatorial definition of operads

The content of this section is of multiple interest. We will give a fourth definition for operads, which is equivalent to the ones which we gave before. Moreover, this definition will provide the justification for the previously used representation of abstract operations by means of tree diagrams. More precisely, we will construct a monad of trees, and an operad will then be defined as an algebra over this monad. Another important aspect of this definition is that just by changing the underlying combinatorial objects, it is possible to define generalizations of operads, as for instance PROPs, which we will encounter in the next chapter. Finally, this combinatorial definition of operads is linked to the free operad, since the free operad can also be given by means of the monad of trees which we consider in the combinatorial definition.

As we would like to make the relationship between operads and trees explicit, we shall forget for the moment about all previously given identifications of trees and abstract operations. The definition of trees remains the same as previously, but we will, at the beginning, not label vertices with abstract operations, nor specify any input ordering.

The set of rooted trees will be denoted by RT, for a tree $t \in \mathrm{RT}$, the set of its vertices is denoted by $\mathrm{vert}(t)$, and, for a vertex $v \in \mathrm{vert}(t)$, the set of its input edges is denoted by $\mathrm{in}(v)$. The set of planar rooted trees will be denoted by PT, and the set of planar rooted trees with n leaves by PT_n.

7.2.1 Combinatorial definition of nonsymmetric operads

Let us first consider the nonsymmetric case, which allows best to explain the idea, because we do not have to deal with symmetries. The symmetric case will be dealt with afterwards. Note that a symmetric operad is basically an S-module with composition. A nonsymmetric operad — which is obtained by forgetting about symmetry — is thus a sequence of vector spaces (indexed by the natural numbers $\mathbb{N}$), or an $\mathbb{N}$-graded vector space.

We will take more interest in the category of $\mathbb{N}$-graded vector spaces, which we denote by $\mathbb{N}$-`Mod`. If we define the category $\mathbb{N}$ as the discrete category whose objects are the natural numbers and whose only morphisms are the identity morphisms, the category $\mathbb{N}$-`Mod` coincides with the category `[`$\mathbb{N}$`,Vect]` of functors between $\mathbb{N}$ and `Vect`.

The combinatorial definition of an operad defines an operad as an algebra over a monoidal structure on an endofunctor of $\mathbb{N}$-`Mod` (i.e. over a monoid in the category `End(`$\mathbb{N}$`-Mod)` $=$ `[`$\mathbb{N}$`-Mod,`$\mathbb{N}$`-Mod]`, or over a monad in the category $\mathbb{N}$-`Mod`). The endofunctor in question is

$$\mathcal{T} : \mathbb{N}\text{-}\mathtt{Mod} \to \mathbb{N}\text{-}\mathtt{Mod},$$

defined, for $M = (M_n)_{n\in\mathbb{N}}$, by

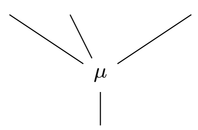
$$\mathcal{T}(M)_n = \bigoplus_{t\in \mathrm{PT}_n} M_t, \ \text{ where } \ M_t = \bigotimes_{v\in\mathrm{vert}(t)} M_{|\,\mathrm{in}(v)|}.$$

Hence, it is natural to think of an element of $\mathcal{T}(M)_n$ as a sum of planar trees with n leaves whose vertices v are decorated by elements of $M_{|\,\mathrm{in}(v)|}$. If $\ell \in \mathrm{Hom}_0(M,N)$, the definition of $\mathcal{T}(\ell) \in \mathrm{Hom}_0(\mathcal{T}(M), \mathcal{T}(N))$ is obvious.

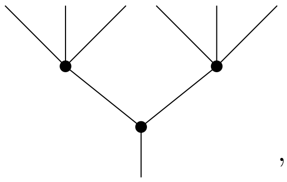
To define a monoidal structure on $\mathcal{T}$, we must define two natural transformations

$$\gamma : \mathcal{T} \circ \mathcal{T} \to \mathcal{T} \quad \text{ and } \quad i : I_{\mathbb{N}\text{-}\mathtt{Mod}} \to \mathcal{T}.$$

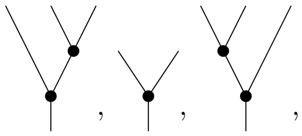
For M given, $i(M)$ consists of the sequence of linear maps $i(M)_n : M_n \to \mathcal{T}(M)_n$, and is defined as follows. The linear map $i(M)_n$ sends $\mu \in M_n$ to the n-corolla with vertex decorated by μ, which is an element of $\mathcal{T}(M)_n$:

$i(M)_n : M_n \ni \mu \mapsto$  $\in \mathcal{T}(M)_n.$

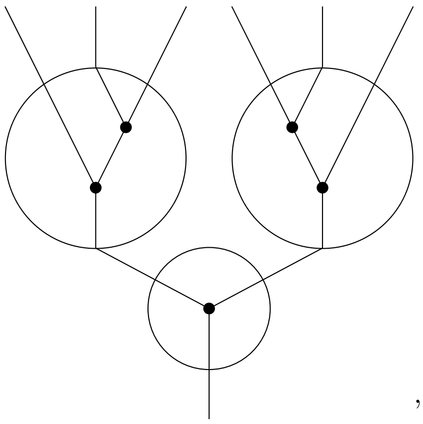
The natural transformation γ is defined using the concept of *substitution of trees*. Note that $\gamma(M) : \mathcal{T}(\mathcal{T}(M)) \to \mathcal{T}(M)$ is made up by a sequence of linear maps $\gamma(M)_n : \mathcal{T}(\mathcal{T}(M))_n \to \mathcal{T}(M)_n$, and that elements of $\mathcal{T}(\mathcal{T}(M))_n$ are (sums of) trees with n leaves, whose vertices are labelled by elements of $\mathcal{T}(M)$, i.e by trees whose vertices are labelled by elements of M. The substitution of trees γ is given by replacing the vertices of the original tree by the corresponding trees, and then viewing the resulting object as an element of $\mathcal{T}(M)$, i.e. as a tree with vertices labelled by elements of M. For instance, omitting decorations by M, the tree



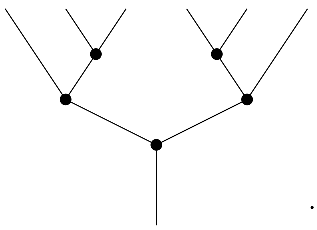
whose vertices are labelled by the trees



will then look like



where the circles indicate the vertices of the original tree. Finally, we get the following tree in $\mathcal{T}(M)$:



Since associativity and unitality constraints are satisfied, $(\mathcal{T}, \gamma, i)$ is a monad in $\mathbb{N}$-$\mathtt{Mod}$.

Definition 7.2 (Combinatorial definition of nonsymmetric operads):
A nonsymmetric operad is an algebra over the monad $(\mathcal{T}, \gamma, i)$. More precisely, a nonsymmetric operad is an $\mathbb{N}$-graded vector space M together with a morphism of $\mathbb{N}$-graded vector spaces $\gamma_M : \mathcal{T}(M) \to M$, that verifies the usual compatibility conditions with γ and i.

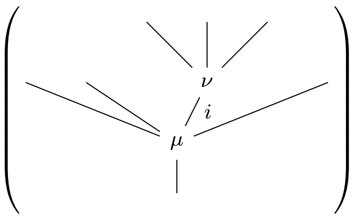
Remark: Note that any monad $(\mathcal{T}, \gamma, i)$ is completely determined by the category of algebras over $\mathcal{T}$ together with the forgetful functor to the underlying category of $\mathcal{T}$.

The preceding combinatorial definition is equivalent to the other definitions of an operad. We will give some details about its equivalence to the partial definition.

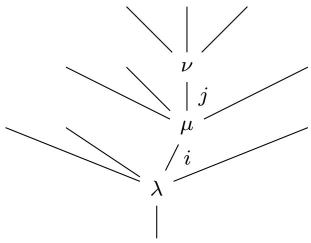
Let (M, γ_M) be a $\mathcal{T}$-algebra. We can define partial composition

$$\circ_i : M_n \otimes M_m \to M_{n+m-1}, \quad 1 \leq i \leq n,$$

for $\mu \in M_n$, $\nu \in M_m$ by

$\mu \circ_i \nu := \gamma_M$  .

Conversely, if the partial compositions $\circ_i$ are given, we define γ_M, for a tree

$t =$  $\in \mathcal{T}(M)_n$,

by

$$\gamma_M(t) := \lambda \circ_i (\mu \circ_j \nu) \in M_n.$$

This way, γ_M may be seen as the contraction along the internal edges of the underlying tree, using the partial compositions.

7.2.2 Combinatorial definition of symmetric operads

We will first detail an alternative viewpoint of S-modules. We define the category S as the category whose objects are $[n] := \{1, \dots, n\}$, and whose morphisms are $\mathrm{Hom}([n], [m]) = \varnothing$, if $n \neq m$, and $\mathrm{Hom}([n], [n]) = S_n$. It is easily seen that the category S-$\mathtt{Mod}$ of S-modules is nothing else than the category $[S,\mathtt{Vect}]$ of functors from the category S to the category $\mathtt{Vect}$. Indeed, $M \in [S,\mathtt{Vect}]$ provides $M(n) \in \mathtt{Vect}$, $n \in \mathbb{N}$, and for $\sigma_n : [n] \to [n]$, $M(\sigma_n) : M(n) \to M(n)$, an automorphism of $M(n)$, so an S_n-module structure on $M(n)$. Moreover, a morphism $\eta : M \to N$ of $[S,\mathtt{Vect}]$ is a natural transformation, so, for $\sigma_n \in \mathrm{Hom}([n], [n]) \subset \mathrm{Mor}\, S$ and $[n] \in S$, we have the following commutative diagram:

$$\begin{array}{ccc} M(n) & \xrightarrow{M(\sigma_n)} & M(n) \\ \eta_n \downarrow & & \downarrow \eta_n \\ N(n) & \xrightarrow{N(\sigma_n)} & N(n), \end{array}$$

so that, for $\mu \in M(n)$, $\eta_n(\mu \cdot \sigma_n) = (\eta_n \mu) \cdot \sigma$, i.e. η provides an S-module morphism $\eta_n : M(n) \to M(n)$, $n \in \mathbb{N}$.

Let now $\mathtt{Bij}$ denote the category of finite sets and bijections between them. Any S-module $M \in [S,\mathtt{Vect}]$ extends to a functor $M \in [\mathtt{Bij},\mathtt{Vect}]$, and any functor of the latter type restricts to an S-module. The restriction is obvious, since, if we know $M(X) \in \mathtt{Vect}$, for any $X \in \mathtt{Bij}$, we know in particular $M(n) := M([n]) = M(\{1, \dots, n\})$. To understand the extension, let us think of $M(2)$ as the space of abstract binary operations obtained, as in the associative case, from a noncommutative concrete binary operation: $a \cdot b = \mu(a, b)$, $b \cdot a = \mu(b, a) = (\mu \cdot \tau)(a, b)$. Hence, $M(2) = \mathbb{K}\mu \oplus \mathbb{K}(\mu \cdot \tau)$. If $X = \{a, b\} = \{b, a\}$, and we define $M(X)$ as the space of abstract binary operations labelled by X, we have no preferred ordering and can use both: $f : 1 \mapsto a, 2 \mapsto b$ and $g : 1 \mapsto b, 2 \mapsto a$, i.e. we consider $\mu(a, b) = (f; \mu)$,

$\mu(b,a) = (g;\mu)$, $(\mu \cdot \tau)(a,b) = (f;\mu \cdot \tau)$, $(\mu \cdot \tau)(b,a) = (g;\mu \cdot \tau)$. In other words, we put all orderings on an equal footing and take

$$\bigoplus_{f \in \mathrm{Bij}([n],X)} M(n)_f.$$

Of course, we then should identify $(g;\mu) \simeq (f;\mu \cdot \tau) = (g \circ \tau;\mu \cdot \tau)$ and $(f;\mu) \simeq (g;\mu \cdot \tau) = (f \circ \tau;\mu \cdot \tau)$. More generally, define on the preceding direct sum the S_n-action

$$(f;\mu) \cdot \sigma_n = (f \circ \sigma_n;\mu \cdot \sigma_n),$$

and set

$$M(X) := \left(\bigoplus_{f \in \mathrm{Bij}([n],X)} M(n)_f \right)_{S_n} \in \mathtt{Vect} \tag{7.1}$$

so to realize the mentioned identifications. This quite natural definition really goes through. Indeed, observe first that if we extend and then reduce M, we recover M. Secondly, if $\sigma \in \mathrm{Bij}(X,Y)$, then we can define the linear map

$$M(\sigma) : M(X) \to M(Y), \text{ by } M(\sigma)[(f;\mu)] = [(\sigma \circ f;\mu)],$$

since $(\sigma \circ f \circ \sigma_n;\mu \cdot \sigma_n) \simeq (\sigma \circ f;\mu)$.

A similar problem, due to the absence of a preferred ordering, appears if we decompose some finite set $X = \{a,b,c\}$ into $\{\{a,b\},c\} =: \{X_b\}_{b \in B}$, and wish to define $\bigotimes_{b \in B} M(X_b)$. The solution is analogous as well:

$$\bigotimes_{b \in B} M(X_b) = ((M(\{a,b\}) \otimes M(\{c\})) \oplus (M(\{c\}) \otimes M(\{a,b\})))_{S_2}.$$

If $n = |B|$, the definition reads in the general case

$$\bigotimes_{b \in B} M(X_b) = \left(\bigoplus_{f \in \mathrm{Bij}([n],B)} M(X_{f(1)}) \otimes \cdots \otimes M(X_{f(n)}) \right)_{S_n}, \tag{7.2}$$

where the S_n-action is defined by $(f;\mu_1,\ldots,\mu_n) \cdot \sigma_n = (f \circ \sigma_n;\mu_{\sigma_n(1)},\ldots,\mu_{\sigma_n(n)})$, so to identify in the quotient, e.g. $\mu \otimes \nu \in M(\{a,b\}) \otimes M(\{c\})$ with $(\mu \otimes \nu) \cdot \tau = \nu \otimes \mu \in M(\{c\}) \otimes M(\{a,b\})$. In fact, we symmetrize the tensor product, so that the order of the factors plays no role.

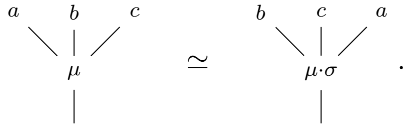
We are now prepared to give the combinatorial definition of symmetric operads. We first define an endofunctor

$$\mathcal{T} : S\text{-}\mathtt{Mod} \to S\text{-}\mathtt{Mod}.$$

Let $M \in S\text{-}\mathtt{Mod} \simeq [S,\mathtt{Vect}]$ and denote its extension to $[\mathtt{Bij},\mathtt{Vect}]$ by M as well. Define $\mathcal{T}(M)$ on $X \in \mathtt{Bij}$ by

$$\mathcal{T}(M)(X) = \bigoplus_{t \in \mathrm{RT}(X)} M(t), \text{ where } M(t) = \bigotimes_{v \in \mathrm{vert}(t)} M(\mathrm{in}(v)).$$

Note that this definition uses (7.1) and (7.2). Here $\mathrm{RT}(X)$ denotes the set of rooted trees whose leaves are labelled by the elements of the finite set X. Therefore, we may think about an element of $\mathcal{T}(M)(X)$ as a rooted tree with leaves labelled by the elements of X and with vertices v decorated by elements $[f;\mu]$ of $M(\mathrm{in}(v))$. For instance, for $n = 3$, the identification $(f;\mu) \simeq (f \circ \sigma;\mu \cdot \sigma)$ in $M(\{a,b,c\})$ can, for $f = \binom{1\,2\,3}{a\,b\,c}$, $\sigma = \binom{1\,2\,3}{2\,3\,1}$, be thought of as



This gives exactly the same identification of trees and abstract operations of a symmetric operad, which we used previously.

Let us observe that not only an S-module M is equivalent to a functor $M \in$ `[Bij,Vect]`, but, moreover, an S-module morphism $\eta : M \to N$, i.e. a family $\eta_n : M(n) \to N(n)$ of S_n-module morphisms, is the same as a morphism of functors (or a natural transformation) $\eta : M \to N$, i.e. a family $\eta_X : M(X) \to N(X)$ of linear maps such that the following diagram commutes:

$$\begin{array}{ccc} M(X) & \xrightarrow{M(\sigma)} & M(Y) \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\ N(X) & \xrightarrow{N(\sigma)} & N(Y). \end{array}$$

Indeed, let η_n be given and define

$$\eta_X : M(X) = \left(\bigoplus_{f \in \mathrm{Bij}([n],X)} M(n)_f \right)_{S_n} \ni [(f;\mu)] \mapsto [(f;\eta_n\mu)] \in N(X).$$

This map is well-defined, since, if we use another representative $(f\sigma_n; \mu\cdot\sigma_n)$, we get $(f\sigma_n; \eta_n(\mu\cdot\sigma_n)) = (f\sigma_n; (\eta_n\mu)\cdot\sigma_n) \simeq (f; \eta_n\mu)$. The commutativity of the diagram is obvious. The converse construction of η_n out of η_X should be clear as well.

If $\eta : M \to N$ denotes now an S-module morphism $\eta_X : M(X) \to N(X)$, we define an S-module morphism $\mathcal{T}(\eta) : \mathcal{T}(M) \to \mathcal{T}(N)$, or better, a linear map $\mathcal{T}(\eta)_X : \mathcal{T}(M)(X) \to \mathcal{T}(N)(X)$ in an obvious way. Hence $\mathcal{T}$ is a functor.

We now define a monoidal structure on $\mathcal{T}$. To define a natural transformation

$$i : I \to \mathcal{T},$$

i.e. a linear map

$$i(M,X) : M(X) \to \mathcal{T}(M)(X),$$

note that the n-corolla is an element of $\mathrm{RT}(X)$ (for $|X| = n$), whenever a labelling of its leaves by the elements of X is given. So, $i(M,X)$ sends an element $[f;\mu] \in M(X)$ to such a corolla, whose vertex is decorated by μ, i.e. to an element of $\mathcal{T}(M)(X)$. Functoriality is easily checked.

As for the natural transformation

$$\gamma : \mathcal{T} \circ \mathcal{T} \to \mathcal{T},$$

i.e the linear map

$$\gamma(M,X) : \mathcal{T}(\mathcal{T}(M))(X) \to \mathcal{T}(M)(X),$$

it is again given by the substitution of trees. Note that this substitution is possible if, for any vertex v of the 'base tree', we are given a tree t_v and a one-to-one correspondence between the leaves of t_v and the input edges of v. The substitution then glues the inputs to the corresponding leaves. It can be verified that this map is functorial in X and M, and that γ and i verify associativity and unitality requirements. Finally, $(\mathcal{T}, \gamma, i)$ is a monad in S-`Mod`.

Definition 7.3 (Combinatorial definition of symmetric operads):
A symmetric operad is an algebra over the monad $(\mathcal{T}, \gamma, i)$. More precisely, a symmetric operad is an S-module M together with a morphism of S-modules $\gamma_M : \mathcal{T}(M) \to M$, that is compatible with γ and i.

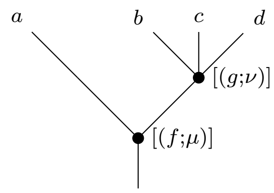
The preceding combinatorial definition is equivalent to the other definitions of an operad. We will give some details about its equivalence to the partial definition.

First,

$$M([n]) = \left(\bigoplus_{f \in S_n} M(n)_f \right)_{S_n} \simeq M(n),$$

the identification being e.g. $[(f;\mu)] = [\mathrm{id}; \mu \cdot f^{-1}] \simeq \mu \cdot f^{-1}$. Therefore, we may think of elements of $M([n]) \simeq M(n)$ as corollas whose leaves are labelled by $[n] = \{1, \dots, n\}$ (from left to right) in natural order. This way, partial compositions $\circ_i$ can be defined, starting from the $\mathcal{T}$-algebra (M, γ_M), just like in the nonsymmetric case.

Conversely, if the partial compositions $\circ_i$ are given, we can define $\gamma_M(X) : \mathcal{T}(M)(X) \to M(X)$ as in the following example. Consider $X = \{a, b, c, d\}$, $n = |X| = 4$,

$t =$  $\in \mathcal{T}(M)(X),$

then we can define $\gamma_M(X)(t)$ by taking the equivalence class of $\mu \circ_2 \nu \in M(4)$ in $M(X)$. This is well-defined, since, if we choose other representatives for $[(f;\mu)]$ and $[(g;\nu)]$, say $(f \circ \sigma_1; \mu \cdot \sigma_1)$ and $(g \circ \sigma_2; \nu \cdot \sigma_2)$, then we have to take $(\mu \cdot \sigma_1) \circ_i (\nu \cdot \sigma_2) = (\mu \circ_{\sigma_1(i)} \nu) \cdot (\sigma_1 \circ_i \sigma_2)$, due to the equivariance property of partial composition. This element belongs to the same class of $M(X)$.

Let us remark that there is a resemblance between the combinatorial definition of an operad and an algebra over an operad. In the functorial definition, we defined an operad as a monad (P, γ, i) in `Vect`, and a P-algebra was defined as an algebra over this monad, i.e. as a vector space V together with a linear map $\gamma_V : P(V) \to V$. In particular, the free P-algebra over V was given by $P(V)$. Here, we constructed a monad $(\mathcal{T}, \gamma, i)$ in S-`Mod`, and defined an operad as an algebra over this monad, i.e. as an S-module M together with an S-module morphism $\gamma_M : \mathcal{T}(M) \to M$. The free operad over M should thus be given by $\mathcal{T}(M)$. Indeed, this is the case for the same reasons as in the algebraic case. The equality $\mathscr{T}M = \mathcal{T}(M)$ then gives the justification for the interpretation of the free operad in terms of trees. In the other direction the free operad provides an alternative approach to the combinatorial definition. The free operad functor $\mathscr{T} : S$-`Mod` $\to$ `Operad` is left adjoint to the forgetful functor `Operad` $\to S$-`Mod`. In general, any pair of adjoint functors gives rise to a monad. Here, we get, by composing the two above functors, a functor $\mathcal{T} : S$-`Mod` $\to S$-`Mod`, which then allows to define the underlying monad of the combinatorial definition.

Chapter 8

PROPs and other generalizations of operads

Operads allow to encode algebraic operations with multiple inputs, but only one output. However, there also exist more general algebraic structures, like, for instance, bialgebras, whose operations have multiple outputs. These can be encoded using PROPs.

8.1 PROPs and bialgebras

The name PROP comes from **pro**duct and **p**ermutation category.

Definition 8.1: A PROP is a symmetric strict monoidal category $(\mathsf{P}, \otimes, I)$, such that the objects are indexed by (or identified with) the set $\mathbb{N}$ of natural numbers, and the monoidal product on objects is given by $m \otimes n = m + n$, hence, the monoidal unit is given by $I = 0$.

Moreover, we assume that this category is enriched over `Vect`, i.e. that the Hom-sets have a vector space structure. For a PROP P, we denote $\mathsf{P}(m, n) := \mathrm{Hom}(m, n)$. Note that the symmetry induces, an (S_m, S_n)-bimodule structure each $\mathsf{P}(m, n)$. Therefore, a PROP P is a sequence $(\mathsf{P}(m, n))_{m,n\in\mathbb{N}}$ of (S_m, S_n)-bimodules with a *horizontal* composition

$$\otimes : \mathsf{P}(m_1, n_1) \otimes \cdots \otimes \mathsf{P}(m_\ell, n_\ell) \to \mathsf{P}(m_1 + \cdots + m_\ell, n_1 + \cdots + n_\ell),$$

a *vertical* composition

$$\circ : \mathsf{P}(m, n) \otimes \mathsf{P}(n, k) \to \mathsf{P}(m, k),$$

and a unit $1_{\mathsf{P}} \in \mathsf{P}(1, 1)$, satisfying associativity, unitality, biequivariance and compatibility conditions.

Remarks: Let us comment on some aspects of the above definition.

- Elements of $\mathsf{P}(m, n)$ will be seen as abstract operations with m outputs and n inputs. The pair (m, n) is called the *biarity*. Such operations may be pictured using graphs with m output edges and n input edges. We will later give a more precise description of the considered graphs.
- The (S_m, S_n)-bimodule structure on $\mathsf{P}(m, n)$ is induced by the symmetry of the category via the identifications $m \simeq 1^{\otimes m}$ and $n \simeq 1^{\otimes n}$. Alternatively, one can also identify objects m with finite sets via $m \simeq [m] := \{1, \ldots, m\}$, then the symmetry condition implies, in particular, that S_m is a subgroup of $\mathrm{Hom}([m], [m]) \simeq \mathsf{P}(m, m)$. Combining this with the horizontal composition, we get the (S_m, S_n)-bimodule structure.

- Note that the horizontal composition map comes from the monoidal product, whereas the vertical composition comes from the categorical composition.
- We have, in fact, units $\mathrm{id}_n \in \mathsf{P}(n,n)$, for any $n \in \mathbb{N}$. These can be obtained by composing $1_{\mathsf{P}} = \mathrm{id}_1$ horizontally n times with itself. Note further that $\mathrm{id}_0 \in \mathsf{P}(0,0)$ is a unit for the horizontal composition.
- The compatibility requirement in the definition is the following compatibility condition between the horizontal and the vertical composition:

$$(\mu \circ \nu) \otimes (\mu' \circ \nu') = (\mu \otimes \mu') \circ (\nu \otimes \nu'),$$

  for any $\mu \in \mathsf{P}(m,n)$, $\nu \in \mathsf{P}(n,k)$, $\mu' \in \mathsf{P}(m',n')$, $\nu' \in \mathsf{P}(n',k')$.
- There exists also a nonsymmetric version of a PROP, called PRO (from **pro**duct category). The definition is similar to the one of a PROP, and can be obtained by forgetting about the symmetry condition.
- One can also define coloured PROPs (and PROs) by replacing the monoid of objects $(\mathbb{N}, +, 0)$ by the free monoid over a finite set. The original definition of a PROP is recovered by taking the free monoid over a singleton.

A *morphism* $f : \mathsf{P} \to \mathsf{Q}$ of PROPs is a sequence $f_{m,n} : \mathsf{P}(m,n) \to \mathsf{Q}(m,n)$, $m,n \in \mathbb{N}$, of biequivariant linear maps, commuting with horizontal and vertical compositions, and respecting identities. PROPs and morphisms of PROPs form a category $\mathtt{PROP}$.

Example 8.1: An important example is the *endomorphism* PROP $\mathsf{End}(V)$ over a vector space V, given by

$$\mathsf{End}(V)(m,n) = \mathrm{Hom}(V^{\otimes n}, V^{\otimes m}).$$

(Note the change of order of m and n.) Horizontal composition is given by the tensor product of linear maps, vertical composition is given by composition of linear maps, and the unit is given by the identity map $\mathrm{id} \in \mathsf{End}(V)(1,1)$.

We are now able to define representations of PROPs:

Definition 8.2: A representation of a PROP P on a vector space V is a morphism

$$\rho : \mathsf{P} \to \mathsf{End}(V)$$

of PROPs. More precisely, it is a sequence

$$\rho_{m,n} : \mathsf{P}(m,n) \to \mathsf{End}(V)(m,n) = \mathrm{Hom}(V^{\otimes n}, V^{\otimes m}),$$

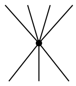
$m,n \in \mathbb{N}$, of biequivariant linear maps, commuting with horizontal and vertical compositions, and respecting identities.

A P-*algebra structure* on a vector space V is then given by a representation of the PROP P on V.

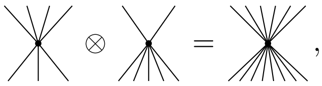
For operads, we considered combinatorial objects, namely trees, to represent abstract operations and their composites. For PROPs, we can also use some combinatorial objects to accomplish the same task. These objects are graphs with some special properties. For abstract operations of biarity (m,n), we consider graphs with m output (half-)edges and n input (half-)edges. We will always put the input edges on top and the output edges at the bottom.

Moreover, we consider oriented graphs; the orientation will only be specified if there is ambiguity, otherwise the orientation is assumed to be given from top to bottom. Furthermore, there are no directed cycles in the considered graphs. Often, we will also label the outputs by $\{1, \ldots, m\}$ and the inputs by $\{1, \ldots, n\}$.

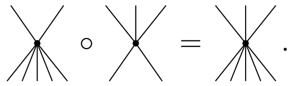
For instance, the following graph can be used to represent an element of $\mathsf{P}(3,4)$:



Horizontal composition of an element of $\mathsf{P}(3,4)$ with an element of $\mathsf{P}(5,2)$ can be seen as



whereas vertical composition of an element of $\mathsf{P}(5,2)$ with an element of $\mathsf{P}(2,3)$ can be seen as



The identities $\mathrm{id}_n \in \mathsf{P}(n,n)$ can be seen as the graph

$$| \ | \ | \ \cdots \ | \ ,$$

i.e. as a union of n trivial trees.

The set of graphs with m input edges and n output edges verifying the above properties — such graphs are also called *directed* (m,n)*-graphs* — is denoted by $\mathscr{G}(m,n)$.

Remark: For operads, we have seen that the identification with combinatorial objects (trees) is justified by the combinatorial definition. For PROPs, there exists also a combinatorial definition, which is, in the main, obtained from the one of operads by changing the underlying combinatorial objects, i.e. replacing trees by (directed) graphs.

While operads may be seen as abstractions of algebras, in the sense that they encode algebraic operations with multiple inputs and one output, and their symmetries, PROPs may be seen as abstractions of bialgebras, in the sense that they encode algebraic operations with multiple inputs and multiple outputs with their symmetries.

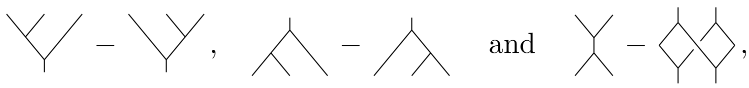
Definition 8.3: An *associative (nonunital) bialgebra* is a vector space B equipped with an associative multiplication $\mu : B \otimes B \to B$ and a coassociative comultiplication $\Delta : B \to B \otimes B$ which are compatible.

Compatibility means that the multiplication μ is a coalgebra morphism or equivalently that the comultiplication is an algebra morphism. This means that $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$, where the dot $\cdot$ denotes the multiplication μ, respectively the multiplication induced on $B \otimes B$, which is given as $(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = (a_1 \cdot b_1) \otimes (a_2 \cdot b_2)$. Finally, the compatibility condition can also be written as

$$\Delta \circ \mu = (\mu \otimes \mu) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta),$$

where τ denotes the switching map given by $\tau(a \otimes b) = b \otimes a$.

The PROP B corresponding to associative (nonunital) bialgebras can be given as the quotient of the free PROP generated by  and , corresponding to μ and Δ, by the PROPic ideal generated by



which encodes associativity, coassociativity and compatibility of μ and Δ.

Similarly, one can construct the PROPs corresponding to Hopf algebras, Lie bialgebras, and other types of algebras with operations having multiple inputs and outputs. For algebras whose operations have only one output, we should get the concept of operads back. Indeed, PROPs can be seen as a generalization of operads, since any operad P gives rise to a unique PROP P, where $\mathsf{P}(1, n) = P(n)$.

8.2 More generalizations of operads

There exist numerous generalizations of operads, which can all be given by means of their underlying combinatorial objects. For a wide-ranging overview, we refer to [Mar08]. We will just give a brief outline of some of these concepts and the corresponding combinatorial objects.

Cyclic operads (introduced by E. Getzler and M. Kapranov) are similar to operads, but make no clear distinction between inputs and output. Roughly speaking, they are operads with an additional symmetry which interchanges the output with one of the inputs. Cyclic operads have the underlying structure of cyclic S-modules, i.e. of S^+-modules; S_n^+ is the group of permutations of $\{0, 1, \ldots, n\}$ and is thus isomorphic to S_{n+1}. The combinatorial objects corresponding to cyclic operads are cyclic (or unrooted) trees.

We have already seen that PROPs are generalizations of operads. However, PROPs are, compared to operads, quite large objects. This can be seen using the underlying combinatorial objects: For operads, there exists only a finite number of trees with n leaves (if one omits composites with 1-corollas), whereas for PROPs, the number of (m, n)-graphs is generally infinite. Therefore, the arity-components of free PROPs are generally infinite-dimensional. This is the reason why smaller versions of PROPs play a quite important role.

Properads (introduced by B. Vallette) form one example of this type. The difference between PROPs and properads is that for properads only connected graphs are allowed. This being a quite small change, properads are still very similar to PROPs. For instance, the endomorphism properad is the same as the endomorphism PROP. Algebras over properads are defined, as usually, as a properad morphism to the endomorphism properad. Note that in the previously treated example of associative bialgebras, one could consider a properad instead of a PROP, since the considered graphs are connected.

Still properads are quite big, a smaller version of PROPs is given by *dioperads*. For dioperads the considered graphs are required to be connected and simply-connected. In particular, Lie bialgebras and infinitesimal bialgebras can be seen as algebras over a dioperad. However associative bialgebras can not be defined as algebras over dioperads, since not all considered graphs are simply-connected.

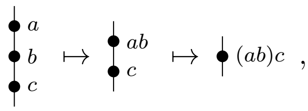
An even smaller version is given by $\frac{1}{2}$PROPs. The considered combinatorial objects are $\frac{1}{2}$graphs. The algebraic structures which can be defined over $\frac{1}{2}$PROPs are typically $\frac{1}{2}$bialgebras.

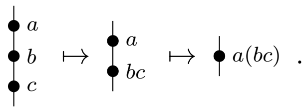
In fact, one has the following chain of inclusions of full subcategories:

$$\texttt{Operad} \subset \tfrac{1}{2}\texttt{PROP} \subset \texttt{diOperad} \subset \texttt{Properad} \subset \texttt{PROP}.$$

Let us remark that not only operads and their generalizations can be defined by means of combinatorial objects, but also associative algebras admit such a description. The graphs to consider for algebras are ladders, which are composites of 1-corollas.

Labelling the vertices by elements of the considered algebra, the multiplication can be seen as contraction along internal edges. Associativity is encoded in the fact that the order in which these contractions are done plays no role:

$\mu \circ (\mu \otimes \mathrm{id})$:


$\mu \circ (\mathrm{id} \otimes \mu)$:


Chapter 9

Operadic twisting morphisms and Koszul morphisms

Operadic twisting and Koszul morphisms will be dealt with, by transferring the corresponding results for associative algebras to the operadic setting.

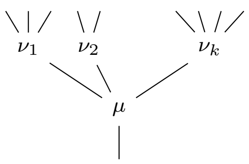
9.1 Infinitesimal composite

Recall that the category $(S\text{-}\mathtt{Mod}, \circ, I)$, where $I = (0, \mathbb{K}, 0, \ldots)$, is a monoidal category. In particular, the composition $\circ$ is — as well as many other involved operations — a (bi)functor.

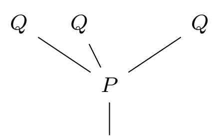
Recall further the definition of the composite $P \circ Q$ of two S-modules P and Q:

$$\begin{aligned}(P \circ Q)(n) &= \bigoplus_k P(k) \otimes_{S_k} \left(\bigoplus_{i_1+\cdots+i_k=n} \mathrm{Ind}^{S_n}_{S_{i_1}\times\cdots\times S_{i_k}} Q(i_1) \otimes \cdots \otimes Q(i_k) \right) \\ &= \bigoplus_{\substack{k \\ i_1+\cdots+i_k=n}} P(k) \otimes_{S_k} (Q(i_1) \otimes \cdots \otimes Q(i_k)) \otimes \mathbb{K}[\mathrm{sh}(i_1, \ldots, i_k)] \\ &= \bigoplus P(k) \otimes Q(i_1) \otimes \cdots \otimes Q(i_k) \otimes \mathrm{sh}(i_1, \ldots, i_k),\end{aligned}$$

where the last line uses a simplified notation. This space is spanned by equivalence classes (for the S_k-action) of elements $(\mu; \nu_1, \ldots, \nu_k; \sigma)$. In the following, we will often simplify the notation by omitting the shuffles in the above considered tensor product; elements will then reads as $(\mu; \nu_1, \ldots, \nu_k)$. Moreover, we will represent elements $(\mu; \nu_1, \ldots, \nu_k)$, by the corresponding tree diagrams (see also Tree Guidelines 3 on page 56)



, or, more generally, by



.

Remember also that the composite of S-modules is additive only in the left factor. However, in order to do homological algebra on S-modules, we need a linearized version of this composite, which will be the infinitesimal composite, constructed in the following.

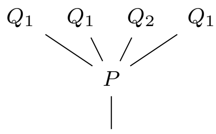
If we consider a polynomial $a + bx + cx^2 + \cdots$, the linear part in x is given by bx, i.e. the term containing exactly one x. The linear part of a composite of S-modules will be defined analogously. For S-modules P, Q_1 and Q_2, we have

$$(P \circ (Q_1 \oplus Q_2))(n) = \bigoplus P(k) \otimes (Q_1(i_1) \oplus Q_2(i_1)) \otimes \cdots \otimes (Q_1(i_k) \oplus Q_2(i_k)).$$

For instance, the term for $k = 2$ is given by

$$\begin{aligned}
&P(n) \otimes (Q_1(i_1) \oplus Q_2(i_1)) \otimes (Q_1(i_2) \oplus Q_2(i_2)) \\
&\quad = (P(n) \otimes Q_1(i_1) \otimes Q_1(i_2)) \oplus (P(n) \otimes Q_1(i_1) \otimes Q_2(i_2)) \\
&\qquad \oplus (P(n) \otimes Q_2(i_1) \otimes Q_1(i_2)) \oplus (P(n) \otimes Q_2(i_1) \otimes Q_2(i_2)),
\end{aligned}$$

where the linear part in Q_2 is made up by the terms containing Q_2 exactly once, i.e. the second and the third term in the above sum. The *linear part* in Q_2 of $P \circ (Q_1 \oplus Q_2)$, denoted by $P \circ (Q_1; Q_2)$, is thus made up by linear combinations of elements of the form



.

Note that $P \circ (Q_1; Q_2)$ is a sub-S-module of $P \circ (Q_1 \oplus Q_2)$. Moreover, this construction defines a functor

$$(P, Q_1, Q_2) \in (S\texttt{-Mod})^{\times 3} \to S\texttt{-Mod} \ni P \circ (Q_1; Q_2).$$

Remarks:

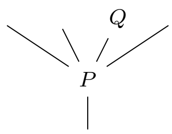
1. The above constructed $P \circ (Q_1; Q_2)$ is linear in P and in Q_2.
2. Taking $Q_1 = Q_2 = Q$ gives rise to

$$P \circ (Q; Q) \rightarrowtail P \circ (Q \oplus Q) \xrightarrow{\mathrm{id}_P \circ (\mathrm{id}_Q + \mathrm{id}_Q)} P \circ Q \,, \tag{9.1}$$

which allows identifying $P \circ (Q; Q)$ with $P \circ Q$.

Definition 9.1:

- The *infinitesimal composite* $P \circ_{(1)} Q$ of two S-modules P and Q is the S-module $P \circ (I; Q)$. Its elements are of the form $(\mu; \mathrm{id}, \dots, \mathrm{id}, \nu, \mathrm{id}, \dots, \mathrm{id})$, generally represented by



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- The corresponding composite $f \circ_{(1)} g$ of two S-module morphisms $f : P_1 \to P_2$ and $g : Q_1 \to Q_2$ is defined by

$$\begin{aligned}
f \circ_{(1)} g : P_1 \circ_{(1)} Q_1 &\to P_2 \circ_{(1)} Q_2, \\
(\mu; \mathrm{id}, \dots, \mathrm{id}, \nu, \mathrm{id}, \dots, \mathrm{id}) &\mapsto (f(\mu); \mathrm{id}, \dots, \mathrm{id}, g(\nu), \mathrm{id}, \dots, \mathrm{id}).
\end{aligned}$$

Instead of linearizing the space $P \circ Q$, we can as well linearize the morphism $f \circ g$. Applying $f \circ g$ to an element $(\mu; \nu_1, \dots, \nu_k)$, consists in applying f to μ and g to every ν_i. Therefore we can identify $f \circ g$ with $f \otimes (g \otimes \cdots \otimes g)$, which leads to the following

Definition 9.2: The *infinitesimal composite* $f \circ' g$ of two S-module morphisms $f : P_1 \to P_2$ and $g : Q_1 \to Q_2$ is defined by

$$f \circ' g = \sum_i f \otimes (\mathrm{id}_{Q_1} \otimes \cdots \otimes \underset{(i)}{g} \otimes \cdots \otimes \mathrm{id}_{Q_1}) : P_1 \circ Q_1 \to P_2 \circ (Q_1; Q_2).$$

Applying $f \circ' g$ to an element $(\mu; \nu_1, \ldots, \nu_k) \in P_1 \circ Q_1$, we get

$$(f \circ' g)(\mu; \nu_1, \ldots, \nu_k) = \sum_{i=1}^{k} \pm (f(\mu); \nu_1, \ldots, g(\nu_i), \ldots, \nu_k),$$

where $\pm$ is a simplified notation for the involved sign.

If $Q_1 = Q_2 = Q$, we can, using (9.1), consider the map

$$P_1 \circ Q \xrightarrow{f \circ' g} P_2 \circ (Q; Q) \simeq P_2 \circ Q.$$

9.2 Differential graded S-modules

Definition 9.3:

- A *graded S-module* P is a sequence $(P_n)_{n\in\mathbb{N}}$ of graded S_n-modules $P(n)$, i.e. of graded vector spaces $(P_p(n))_{p\in\mathbb{Z}}$ endowed with a degree preserving S_n-action. The label n refers to the *arity*, whereas the label p refers to the *degree*.
- A *morphism* $f : P \to Q$ of degree r between graded S-modules P and Q is a sequence $f_n : P(n) \to Q(n)$, $n \in \mathbb{N}$, of degree r S_n-module morphisms, i.e. a sequence of S_n-equivariant linear maps $f_{n,p} : P_p(n) \to Q_{p+r}(n)$, $p \in \mathbb{Z}$. The space of such morphisms is denoted by $\mathrm{Hom}^r_S(P, Q)$.

Remark: The composite product $\circ$ can be extended to graded S-modules by

$$(P \circ Q)_s(n) = \bigoplus_{\substack{k \\ i_1+\cdots+i_k=n \\ q+j_1+\cdots+j_k=s}} P_q(k) \otimes Q_{j_1}(i_1) \otimes \cdots \otimes Q_{j_k}(i_k).$$

Moreover, $I = (0, \mathbb{K}, 0, \ldots)$ can be viewed as a graded S-module concentrated in degree 0. The category $(\mathtt{gr}S\text{-}\mathtt{Mod}, \circ, I)$ of graded S-modules is thus a monoidal category.

Definition 9.4:

- A *differential graded S-module* (P, d) is a graded S-module P endowed with a differential d, i.e an endomorphism $d : P \to P$ of degree -1 of graded S-modules, such that $d^2 = 0$.
- A *morphism* $f : (P, d_P) \to (Q, d_Q)$ of differential graded S-modules is a morphism $f : P \to Q$ of degree 0 of graded S-modules that commutes with the differential, i.e. $f d_P = d_Q f$.

Remark: The composite product $P \circ Q$ of two differential graded S-modules P and Q is a differential graded S-module for the differential

$$d_{P\circ Q} = d_P \circ \mathrm{id}_Q + \mathrm{id}_P \circ' d_Q,$$

where the last term maps $P \circ Q$ to itself, in view of (9.1). The category $(\mathtt{DG}S\text{-}\mathtt{Mod}, \circ, I)$ of differential graded S-modules is a monoidal category.

9.3 Differential graded operads and differential graded cooperads

We know that an operad is a monoidal structure on an S-module. In other words, it is a monoid in the monoidal category $(S\text{-}\mathtt{Mod}, \circ, I)$ of S-modules. Similarly, we have the following

Definition 9.5: A differential graded operad is a monoid (P, d_P, γ, u) in the monoidal category $(\mathtt{DG}S\text{-}\mathtt{Mod}, \circ, I)$. More precisely, (P, d_P) is a differential graded S-module with differential graded S-module morphisms

$$\gamma : P \circ P \to P, \qquad u : I \to P,$$

that verify associativity and unitality constraints.

Remark: The requirement for γ to be a differential graded S-module morphism means that it is a morphism of degree 0, such that

$$d_P\gamma = \gamma d_{P\circ P} = \gamma(d_P \circ \mathrm{id}_P + \mathrm{id}_P \circ' d_P),$$

i.e., on an element $(\mu; \mu_1, \dots, \mu_k)$,

$$d_P(\gamma(\mu; \mu_1, \dots, \mu_k)) = \gamma(d_P\mu; \mu_1, \dots, \mu_k) + \sum_{i=1}^{k} (-1)^{\tilde{\mu} + \sum_{\ell=1}^{i-1} \widetilde{\mu_\ell}} \gamma(\mu; \mu_1, \dots, d_P\mu_i, \dots, \mu_k).$$

This means that d_P is a derivation for γ, which is completely analogous to the algebraic case.

Definition 9.6: A differential graded cooperad is a comonoid $(\mathscr{C}, d_{\mathscr{C}}, \Delta, \varepsilon)$ in the monoidal category $(\mathtt{DG}S\text{-}\mathtt{Mod}, \circ, I)$. More precisely, $(\mathscr{C}, d_{\mathscr{C}})$ is a differential graded S-module with differential graded S-module morphisms

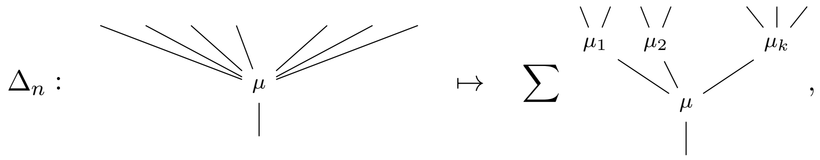
$$\Delta : \mathscr{C} \to \mathscr{C} \circ \mathscr{C}, \qquad \varepsilon : \mathscr{C} \to I,$$

called *decomposition* and *counit*, that verify coassociativity and counitality constraints.

Remark: Note that the decomposition map Δ is given by a sequence

$$\Delta_n : \mathscr{C}(n) \to (\mathscr{C} \circ \mathscr{C})(n) = \bigoplus \mathscr{C}(k) \otimes (\mathscr{C}(i_1) \otimes \cdots \otimes \mathscr{C}(i_k)),$$

$n \in \mathbb{N}$. On an element, this reads as



i.e. $\Delta_n : \mu \mapsto \sum(\mu, \mu_1, \dots, \mu_k)$. Obviously, the μ on the RHS and the μ on the LHS are not the same, we use this quite abusive notation to remind ourselves that the sum on the RHS is the image of μ by Δ_n.

Under this notation, the requirement for Δ to be a differential graded S-module morphism means, in particular, that

$$\Delta d_{\mathscr{C}} = d_{\mathscr{C}\circ\mathscr{C}}\Delta = (d_{\mathscr{C}} \circ \mathrm{id}_{\mathscr{C}} + \mathrm{id}_{\mathscr{C}} \circ' d_{\mathscr{C}})\Delta,$$

i.e., on an element $\mu \in \mathscr{C}(n)$,

$$\begin{aligned}\Delta(d_{\mathscr{C}}(\mu)) &= d_{\mathscr{C}\circ\mathscr{C}} \sum(\mu; \mu_1, \dots, \mu_k) \\ &= \sum(d_{\mathscr{C}}\mu; \mu_1, \dots, \mu_k) + \sum \sum_{i=1}^{k} (-1)^{\tilde{\mu} + \sum_{\ell=1}^{i-1} \widetilde{\mu_\ell}} (\mu; \mu_1, \dots, d_{\mathscr{C}}\mu_i, \dots, \mu_k).\end{aligned}$$

This means that $d_{\mathscr{C}}$ is a coderivation for Δ, which is completely analogous to the coalgebraic case.

9.4 Operadic twisting morphisms

To extend the theory of twisting morphism to operads, we need the linearization of the composition map $\gamma : P \circ P \to P$ of an operad, and of the decomposition map $\Delta : \mathscr{C} \to \mathscr{C} \circ \mathscr{C}$ of a cooperad.

The *infinitesimal composition map* of a (differential graded) operad P is given by

$$\gamma_{(1)} : P \circ_{(1)} P = P \circ (I; P) \xrightarrow{\mathrm{id}_P \circ (u;\mathrm{id}_P)} P \circ (P; P) \xrightarrow{(9.1)} P \circ P \xrightarrow{\gamma} P.$$

The *infinitesimal decomposition map* of a (differential graded) cooperad $\mathscr{C}$ is given by

$$\Delta_{(1)} : \mathscr{C} \xrightarrow{\Delta} \mathscr{C} \circ \mathscr{C} \xrightarrow{\mathrm{id}_{\mathscr{C}} \circ' \mathrm{id}_{\mathscr{C}}} \mathscr{C} \circ (\mathscr{C}; \mathscr{C}) \xrightarrow{\mathrm{id}_{\mathscr{C}} \circ (\varepsilon;\mathrm{id}_{\mathscr{C}})} \mathscr{C} \circ (I; \mathscr{C}) = \mathscr{C} \circ_{(1)} \mathscr{C}.$$

From now on, we will require the differential graded operad (P, d_P, γ, u) to be augmented, i.e. there exists a morphism $\eta : P \to I$ of differential graded operads, i.e. a morphism of differential graded S-modules that respects composition γ and unit u. The differential graded cooperad $(\mathscr{C}, d_{\mathscr{C}}, \Delta, \varepsilon)$ will also be required to be coaugmented, i.e. there exists a morphism $i : I \to \mathscr{C}$ of differential graded cooperads, i.e. a morphism of differential graded S-modules that respects decomposition Δ and counit ε; if necessary, the cooperad $\mathscr{C}$ is also assumed to be conilpotent.

9.4.1 Differential graded convolution operad

We will now construct a differential graded 'convolution' operad structure.

Consider

$$\mathrm{Hom}_{\mathbb{K}}(\mathscr{C}, P) = \left(\mathrm{Hom}_{\mathbb{K}}(\mathscr{C}(n), P(n))\right)_{n\in\mathbb{N}},$$

which is a sequence of graded vector spaces, endowed with an S_n-action that preserves the grading. This action is, for a morphism $f : \mathscr{C}_p(n) \to P_{p+r}(n)$, given by

$$(f \cdot \sigma)(x) = f(x \cdot \sigma^{-1}) \cdot \sigma \in P_{p+r}(n).$$

Therefore, $\mathrm{Hom}_{\mathbb{K}}(\mathscr{C}, P)$ is a graded S-module. We denote

$$\mathscr{H}(n) = \mathrm{Hom}_{\mathbb{K}}(\mathscr{C}, P)(n) = \mathrm{Hom}_{\mathbb{K}}(\mathscr{C}(n), P(n)).$$

In order to make $\mathscr{H}$ a graded operad, we have to define a composition Γ and a unit U. The composition has to be defined as a morphism of graded S-modules

$$\Gamma : (\mathscr{H} \circ \mathscr{H})(n) = \bigoplus \mathscr{H}(k) \otimes \mathscr{H}(i_1) \otimes \cdots \otimes \mathscr{H}(i_k) \to \mathscr{H}(n).$$

This means that applied to an element $(f; g_1, \ldots, g_k)$, with $f \in \mathrm{Hom}_{\mathbb{K}}(\mathscr{C}(k), P(k))$ and $g_j \in \mathrm{Hom}_{\mathbb{K}}(\mathscr{C}(i_j), P(i_j))$, $\Gamma(f; g_1, \ldots, g_k)$ has to be defined to be an element of $\mathrm{Hom}_{\mathbb{K}}(\mathscr{C}(n), P(n))$:

$$\begin{aligned}\Gamma(f; g_1, \ldots, g_k) : \mathscr{C}(n) \xrightarrow{\Delta} (\mathscr{C} \circ \mathscr{C})(n) &\twoheadrightarrow \mathscr{C}(k) \otimes (\mathscr{C}(i_1) \otimes \cdots \otimes \mathscr{C}(i_k)) \\ &\xrightarrow{f\otimes(g_1\otimes\cdots\otimes g_k)} P(k) \otimes (P(i_1) \otimes \cdots \otimes P(i_k)) \\ &\rightarrowtail (P \circ P)(n) \xrightarrow{\gamma} P(n).\end{aligned}$$

Let us admit that all remaining details can be verified and that $(\mathrm{Hom}_{\mathbb{K}}(\mathscr{C}, P), \Gamma, U)$ is a graded operad, called the *graded convolution operad*.

We now endow this operad with the differential ∂ defined by

$$\partial f = [d, f] = d_P \circ f - (-1)^r f \circ d_{\mathscr{C}},$$

for any morphism $f : \mathscr{C}_p(n) \to P_{p+r}(n)$. Since $\partial f : \mathscr{C}_p(n) \to P_{p+r-1}(n)$, ∂ is of degree -1. It can be verified that ∂ is an S-module morphism, that $\partial^2 = 0$, and that Γ and U respect ∂ (in particular, this means that ∂ is a derivation for Γ). Finally, $\mathrm{Hom}_{\mathbb{K}}(\mathscr{C}, P)$ is a differential graded operad, called the *differential graded convolution operad*.

9.4.2 Twisting morphisms and twisted composite complexes

To write down the Maurer-Cartan equation for $\alpha \in \mathrm{Hom}^{-1}_{\mathbb{K}}(\mathscr{C}, P)$, i.e.

$$\partial\alpha + \alpha \star \alpha = \partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0,$$

we need on $\mathrm{Hom}_{\mathbb{K}}(\mathscr{C}, P)$ an associative or a pre-Lie structure $\star$. There exist functors

$$\texttt{(DG)Operad} \to \texttt{(DG)pre-LieAlg} \to \texttt{(DG)LieAlg}$$

that allow to define a pre-Lie structure on the space $\prod_{n\in\mathbb{N}} \mathscr{P}(n)$ of any (DG) operad $\mathscr{P}$ In the case $\mathscr{P} = \mathrm{Hom}_{\mathbb{K}}(\mathscr{C}, P)$ we can define this structure without further details about this functor.

Definition 9.7: For $f, g \in \prod_{n\in\mathbb{N}} \mathrm{Hom}_{\mathbb{K}}(\mathscr{C}(n), P(n)) \simeq \mathrm{Hom}_{\mathbb{K}}(\mathscr{C}, P)$, the convolution is given by

$$f \star g : \mathscr{C} \xrightarrow{\Delta_{(1)}} \mathscr{C} \circ_{(1)} \mathscr{C} \xrightarrow{f \circ_{(1)} g} P \circ_{(1)} P \xrightarrow{\gamma_{(1)}} P.$$

To be able to extend $f \in \mathrm{Hom}^{-1}(\mathscr{C}, P)$ to $\mathrm{Hom}_{\texttt{grOperad}}(\mathscr{T}(s^{-1}\bar{\mathscr{C}}), P)$, so in particular to a morphism of S-modules, we must start from $f \in \mathrm{Hom}^{-1}_S(\mathscr{C}, P)$. It turns out that $\mathrm{Hom}_S(\mathscr{C}, P) := \prod_{n\in\mathbb{N}} \mathrm{Hom}_{S_n}(\mathscr{C}(n), P(n))$ is stable for $\star$ and ∂, and that $(\mathrm{Hom}_S(\mathscr{C}, P), \star, \partial)$ is a DG pre-Lie algebra (that defines a DGLA).

Definition 9.8: An *operadic twisting morphism* $\alpha \in \mathrm{Tw}(\mathscr{C}, P)$ is a solution $\alpha \in \mathrm{Hom}^{-1}_S(\mathscr{C}, P)$ of the Maurer-Cartan equation $\partial\alpha + \alpha \star \alpha = 0$, which verifies

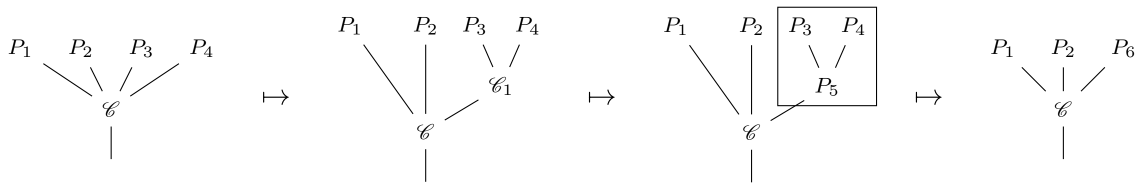
$$\mathscr{C} \xrightarrow{\alpha} P \xrightarrow{\eta} I \quad (\text{composite} = 0) \qquad \text{and} \qquad I \xrightarrow{i} \mathscr{C} \xrightarrow{\alpha} P \quad (\text{composite} = 0).$$

The *composite complex* $(\mathscr{C} \circ P, d_{\mathscr{C}\circ P})$, where $d_{\mathscr{C}\circ P} = d_{\mathscr{C}} \circ \mathrm{id}_P + \mathrm{id}_{\mathscr{C}} \circ' d_P$, is a DG S-module. For $\alpha \in \mathrm{Hom}^{-1}_S(\mathscr{C}, P)$ define $\bar{d}_\alpha$ by

$$\bar{d}_\alpha : \mathscr{C} \circ P \xrightarrow{\Delta_{(1)} \circ \mathrm{id}_P} (\mathscr{C} \circ_{(1)} \mathscr{C}) \circ P \xrightarrow{(\mathrm{id}_{\mathscr{C}} \circ_{(1)} \alpha) \circ \mathrm{id}_P} (\mathscr{C} \circ_{(1)} P) \circ P \simeq \mathscr{C} \circ (P; P \circ P)$$

$$\xrightarrow{\mathrm{id}_{\mathscr{C}} \circ (\mathrm{id}_P; \gamma)} \mathscr{C} \circ (P; P) \simeq \mathscr{C} \circ P.$$

Using tree diagrams, this reads, for instance, as



If $d_\alpha = d_{\mathscr{C}\circ P} + \bar{d}_\alpha$ defines a differential, i.e. if $d_\alpha^2 = 0$, which is the case if and only if $\alpha \in \mathrm{Tw}(\mathscr{C}, P)$, then $\mathscr{C} \circ_\alpha P := (\mathscr{C} \circ P, d_\alpha)$ is a DG S-module called *twisted composite complex*.

The *comparison lemma* remains valid for twisted composite complexes.

9.4.3 Operadic bar and cobar constructions

These constructions are similar to the corresponding ones in the algebraic context. The bar construction is a functor

$$B : \texttt{augDGOperad} \to \texttt{augDGCoOperad},$$

whereas the cobar construction is a functor

$$\Omega : \texttt{augDGCoOperad} \to \texttt{augDGOperad}.$$

The bar and the cobar functor are adjoint functors.

Let us detail the cobar construction. Consider an augmented DG cooperad $(\mathscr{C}, \Delta, \varepsilon, d_{\mathscr{C}})$, i.e., in particular, we have $\mathscr{C} = I \oplus \bar{\mathscr{C}}$. The cobar construction $\Omega\mathscr{C}$ is, similar to the algebraic case, an augmented DG operad structure on $\mathscr{T}(s^{-1}\bar{\mathscr{C}})$, where $\mathscr{T}$ is the functor $\mathscr{T} : S\text{-}\texttt{Mod} \to \texttt{Operad}$ that to any S-module associates the free operad over this S-module. The differential on $\mathscr{T}(s^{-1}\bar{\mathscr{C}})$ is given by the sum $\delta_1 + \delta_2$, where δ_1 extends the differential $d_{\mathscr{C}}$ and δ_2 extends the infinitesimal decomposition $\Delta_{(1)}$. More precisely,

$$s^{-1}\bar{\mathscr{C}} \xrightarrow{s} \bar{\mathscr{C}} \xrightarrow{d_{\mathscr{C}}} \bar{\mathscr{C}} \xrightarrow{s^{-1}} s^{-1}\bar{\mathscr{C}} \rightarrowtail \mathscr{T}(s^{-1}\bar{\mathscr{C}})$$

and

$$s^{-1}\bar{\mathscr{C}} \xrightarrow{s} \bar{\mathscr{C}} \xrightarrow{\bar{\Delta}_{(1)}} \bar{\mathscr{C}} \circ_{(1)} \bar{\mathscr{C}} \xrightarrow{s^{-2}} s^{-1}\bar{\mathscr{C}} \circ_{(1)} s^{-1}\bar{\mathscr{C}} \rightarrowtail s^{-1}\bar{\mathscr{C}} \circ s^{-1}\bar{\mathscr{C}} \rightarrowtail \mathscr{T}(s^{-1}\bar{\mathscr{C}})$$

uniquely extend, since $\mathscr{T}(s^{-1}\bar{\mathscr{C}})$ is free, to derivations δ_1 and δ_2 of $\mathscr{T}(s^{-1}\bar{\mathscr{C}})$.

Finally, $(\Omega\mathscr{C}, \delta_1 + \delta_2)$ is a DG operad.

The definition of the bar construction BP of an augmented DG operad is similar. The basic correspondences and the fundamental theorems detailed in the algebraic context can be extended to the operadic setting. For instance,

Theorem 9.1 (Fundamental theorem of operadic twisting morphisms):
Under some weight-graded assumptions, we have, for an operadic twisting morphism $\alpha \in \mathrm{Tw}(\mathscr{C}, P)$, that the following propositions are equivalent:

1. $\alpha \in \mathrm{Kos}(\mathscr{C}, P)$, *i.e.* $\mathscr{C} \otimes_\alpha P$ *is acyclic,*
2. $f_\alpha \in \mathrm{Hom}_{\texttt{DGOperad}}(\Omega\mathscr{C}, P)$ *is a quasi-isomorphism,*
3. $g_\alpha \in \mathrm{Hom}_{\texttt{DGCoOperad}}(\mathscr{C}, BP)$ *is a quasi-isomorphism.*

Corollary 9.2: *Taking $\mathscr{C} = BP$ (resp. $P = \Omega\mathscr{C}$), we find that $\Omega BP \xrightarrow{\sim} P$ (resp. that $\mathscr{C} \xrightarrow{\sim} B\Omega\mathscr{C}$).*

Chapter 10

Koszul duality for operads

We will adapt the results of Koszul duality for algebras to operads. This will lead, for a quadratic Koszul operad P, to a model $P_\infty := \Omega P^{¡}$, which then allows to define P_∞-algebras (or homotopy P-algebras) as representations of this operad.

10.1 Quadratic operads and cooperads

Definition 10.1: *Operadic quadratic data* (E, R) consists of a graded S-module E and a graded sub-S-module $R \subset \mathscr{T}(E)^{(2)}$.

Here $\mathscr{T}(E)^{(2)}$ refers to the weight 2 part of the free operad $\mathscr{T}(E)$, i.e. to the graded sub-S-module of $\mathscr{T}(E)$, which is spanned by composites of two elements of E.

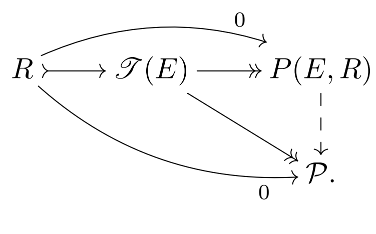
We will use the same terminology as in the algebraic setting and refer to elements of E as generating operations and to elements of R as relations, or better relators.

Definition 10.2: The *quadratic operad* $P(E, R)$ associated to the operadic quadratic data (E, R) is the quotient operad $\mathscr{T}(E)/(R)$, where (R) denotes the operadic ideal generated by $R \subset \mathscr{T}(E)^{(2)}$.

The quadratic operad $P(E, R)$ is the quotient operad of $\mathscr{T}(E)$ that is universal among all quotient operads $\mathcal{P}$ of $\mathscr{T}(E)$, such that the composite

$$R \rightarrowtail \mathscr{T}(E) \twoheadrightarrow \mathcal{P}$$

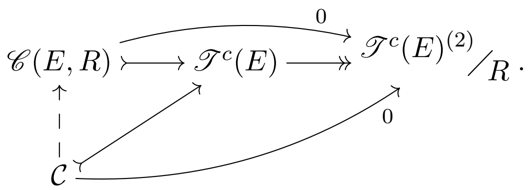
vanishes. More precisely, there exists a unique morphism of operads $P(E, R) \to \mathcal{P}$, such that the following diagram commutes



Definition 10.3: The *quadratic cooperad* $\mathscr{C}(E, R)$ associated to the operadic quadratic data (E, R) is the subcooperad of the cofree cooperad $\mathscr{T}^c(E)$, that is universal among all subcooperads $\mathcal{C}$ of $\mathscr{T}^c(E)$, such that the composite

$$\mathcal{C} \rightarrowtail \mathscr{T}^c(E) \twoheadrightarrow \mathscr{T}^c(E)^{(2)}/R$$

vanishes. More precisely, there exists a unique morphism of cooperads $\mathcal{C} \to \mathscr{C}(E, R)$, such that the following diagram commutes



Remark: The cofree cooperad $\mathscr{T}^c(E)$ is as S-module the same as the free operad $\mathscr{T}(E)$. We refer to the next chapter for more information about decomposition map of the cofree cooperad. A detailed description of the cofree cooperad and its decomposition map can be found in [LV11].

Note that when we are working over graded S-modules, the above defined quadratic operad (respectively cooperad) is not only endowed with an arity grading and a weight grading (coming from the free, respectively, cofree operad), but also with a degree.

10.2 Koszul dual cooperad and operad of a quadratic operad

Definition 10.4: The *Koszul dual cooperad* of a quadratic operad $P = P(E, R)$ is

$$P^{¡} = \mathscr{C}(sE, s^2R),$$

i.e. the quadratic cooperad associated to the shifted operadic quadratic data.

Here sE denotes the shifted S-module, obtained from E by shifting the degree in each arity.

In order to define the Koszul dual operad, we need some preliminary remarks.

First, the *Hadamard product* $P \underset{\mathrm{H}}{\otimes} Q$ of two S-modules is given by $(P \underset{\mathrm{H}}{\otimes} Q)(n) = P(n) \otimes Q(n)$, for any $n \in \mathbb{N}$. The action of the symmetric group is given by the diagonal action, i.e. $(\mu \otimes \nu) \cdot \sigma = (\mu \cdot \sigma) \otimes (\nu \cdot \sigma)$, for any $\mu \in P(n)$, $\nu \in Q(n)$, $\sigma \in S_n$. Moreover, the Hadamard product of operads has a natural operad structure.

Second, the suspension of an operad, obtained by suspending the underlying S-module, is, in general, not an operad. Therefore, we will define an 'operadic suspension'. Let $\mathcal{S} := \mathscr{E}nd(s\mathbb{K})$ be the endomorphism operad over the suspended ground field. This means that $\mathcal{S}(n) = \mathrm{Hom}((s\mathbb{K})^{\otimes n}, s\mathbb{K})$; note that this space contains morphisms of degree $-n+1$. The symmetric group action is given by the signature action. We also denote $\mathcal{S}^{-1} := \mathscr{E}nd(s^{-1}\mathbb{K})$ and $\mathcal{S}^c := \mathscr{E}nd^c(s\mathbb{K})$, where $\mathscr{E}nd^c(s\mathbb{K})$ is the endomorphism cooperad, which is as S-module the same as the endomorphism operad, but equipped with a decomposition map.

Finally, we define the *operadic suspension* of an operad P by $\mathcal{S} \underset{\mathrm{H}}{\otimes} P$. The *operadic desuspension* is given by $\mathcal{S}^{-1} \underset{\mathrm{H}}{\otimes} P$. For a cooperad $\mathscr{C}$, the *cooperadic suspension* is given by $\mathcal{S}^c \underset{\mathrm{H}}{\otimes} \mathscr{C}$. The operadic suspension has the property that a vector space V is equipped with a P-algebra structure, if and only if the suspended vector space sV is equipped with a $\mathcal{S} \underset{\mathrm{H}}{\otimes} P$-algebra structure.

Definition 10.5: The *Koszul dual operad* of a quadratic operad $P = P(E, R)$ is defined by

$$P^! = (\mathcal{S}^c \underset{\mathrm{H}}{\otimes} P^{¡})^*.$$

The dual means here that we take the linear dual in each arity.

Let us mention that the $P^!$ is quadratic in a certain case. More precisely,

Proposition 10.1: *Let $P = P(E, R)$ be a quadratic operad, generated by a reduced S-module E which is of finite dimension in each arity. Then the Koszul dual operad $P^!$ admits the quadratic presentation $P^! = P(s^{-1}\mathcal{S}^{-1} \underset{\mathrm{H}}{\otimes} E^*, R^\perp)$.*

Moreover, we have that, under the assumptions of the previous proposition, $(P^!)^! = P$.

10.3 Koszul operads

For given operadic quadratic data (E, R), we have that $P(E, R)^{(1)} = E$ and $\mathscr{C}(E, R)^{(1)} = E$, and we can define the morphism κ by

$$\kappa : \mathscr{C}(sE, s^2R) \twoheadrightarrow sE \overset{s^{-1}}{\to} E \rightarrowtail P(E, R).$$

This morphism is clearly of degree -1, and verifies (for the same reasons as in the algebraic case) $\kappa \star \kappa = 0$. Therefore, $\kappa \in \mathrm{Tw}$ is an operadic twisting morphism.

This defines a *Koszul complex* $P^{¡} \circ_\kappa P := (P^{¡} \circ P, d_\kappa)$. We thus have a sequence of chain complexes of S_n-modules $((P^{¡} \circ P)(n), d_\kappa)$, called Koszul complexes in arity n.

A quadratic operad P is called a *Koszul operad* if the corresponding Koszul complex $P^{¡} \circ_\kappa P$ is acyclic.

Let us mention that there exist many Koszul operads, in particular $\mathscr{A}ss$, $\mathscr{C}om$, $\mathscr{L}ie$ and $\mathscr{P}ois$ are Koszul operads.

Just as we have for Koszul algebras A, a resolution $\Omega A^{¡} \xrightarrow{\sim} A$, we obtain, for Koszul operads P, a resolution $\Omega P^{¡} \xrightarrow{\sim} P$. The operad $\Omega P^{¡}$ is the P_∞-operad. Hence, to a P-algebra structure on a vector space V, given by $P \to \mathscr{E}nd(V)$, corresponds via

$$\begin{array}{ccc} P_\infty := \Omega P^{¡} & \xrightarrow{\sim} & P \\ & \searrow & \downarrow \\ & & \mathscr{E}nd(V) \end{array}$$

a P_∞-algebra (also called homotopy P-algebra) structure on V.

Chapter 11

Infinity algebras over a quadratic Koszul operad

For any operad P, a homotopy P-algebra has been defined as an algebra over the DG operad $P_\infty := \Omega P^{¡}$. On the other hand, homotopy associative algebras or A_∞-algebras have been introduced independently and the corresponding DG operad A_∞ can easily be constructed. The objective of this chapter is to show that the operad $\mathscr{A}s$ is Koszul and that two DG operads $\mathscr{A}s_\infty := \Omega \mathscr{A}s^{¡}$ and A_∞ coincide.

11.1 The operad $\mathscr{A}s$

As neither the generating operation μ of an associative algebra, nor the defining relation $\mu(\mu, \mathrm{id}) = \mu(\mathrm{id}, \mu)$ involve any symmetry, the category of associative algebras can be encrypted into a nonsymmetric operad $\mathscr{A}s$. To emphasise that we are considering a nonsymmetric operad, i.e. an operad whose spaces of n-ary operations are just vector space without S_n-action, we will denote these spaces by $\mathscr{A}s_n$.

Since the free associative algebra over a vector space V is

$$\bar{T}(V) = \bigoplus_{n\in\mathbb{N}^*} V^{\otimes n} = \bigoplus_{n\in\mathbb{N}^*} \mathbb{K}\otimes V^{\otimes n},$$

we see that $\mathscr{A}s_n$ is isomorphic to $\mathbb{K}$. More precisely,

$$\mathscr{A}s_n = \mathbb{K}\mu_n, \quad , n \geq 1,$$

where μ_n is the n-ary operation given by

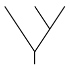
$$\mu_n(a_1, \ldots, a_n) = a_1 \cdots a_n.$$

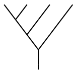
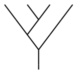
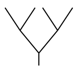
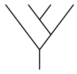
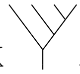
In particular, $\mu_2 = \mu$ and $\mu_1 = \mathrm{id}$.

The operad $\mathscr{A}s$ is quadratic, i.e. it has a presentation $\mathscr{A}s = P(E, R) = {}^{\mathscr{T}(E)}\!\diagup_{(R)}$, where E is the vector space of generating operations and $R \subset \mathscr{T}(E)^{(2)}$ the subspace of defining relations. Clearly, $E = \mathbb{K}\mu$ and $R = \mathbb{K}\,\mathrm{as}$, where $\mathrm{as} := -\mu\circ(\mu, \mathrm{id}) + \mu\circ(\mathrm{id}, \mu)$ is the associator. We also have that

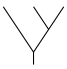
$$\begin{aligned}\mathscr{T}_0(E) = I \subset \mathscr{T}_1(E) &= I \oplus E \\ \subset \mathscr{T}_2(E) &= I \oplus E \circ (I \oplus E) \\ \subset \mathscr{T}_3(E) &= I \oplus E \circ (I \oplus E \circ (I \oplus E)) \subset \cdots\end{aligned}$$

Denoting μ by the 2-corolla , it follows for the first arity-spaces of the free operad $\mathscr{T}(E)$

that $\mathscr{T}(E)_0 = \{0\}$, $\mathscr{T}(E)_1 = \mathbb{K}\,|$, $\mathscr{T}(E)_2 = \mathbb{K}$ , $\mathscr{T}(E)_3 = \mathbb{K}$  $\oplus\ \mathbb{K}$ ,

$\mathscr{T}(E)_4 = \mathbb{K}$  $\oplus\ \mathbb{K}$  $\oplus\ \mathbb{K}$  $\oplus\ \mathbb{K}$  $\oplus\ \mathbb{K}$ , ...

These spaces are visibly isomorphic to the vector spaces spanned by planar binary trees, i.e. $\mathscr{T}(E)_n \simeq \mathbb{K}[\mathrm{PBT}_n]$, where PBT_n denotes the set of planar binary trees with n leaves. Note that the space $\mathscr{T}(E)^{(2)}$ of operations of weight 2 coincides with $\mathscr{T}(E)_3$. In general, $\mathscr{T}(E)^{(n)} = \mathscr{T}(E)_{n+1}$. Moreover,

$R = \mathbb{K}\,\mathrm{as} = \mathbb{K}\Big(-$  $+$  $\Big) \subset \mathscr{T}(E)^{(2)}.$

The operads $\mathscr{A}s$ and $\mathscr{T}(E)/_{(R)}$ coincide, since, in the quotient all n-ary operations given by planar binary trees with n leaves coincide and define a unique n-ary operation μ_n represented by the n-corolla. Composition in the two operads coincides as well.

11.2 The cooperad $\mathscr{A}s^{\text{¡}}$

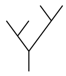
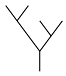
Since $\mathscr{A}s = P(E,R)$, $E = \mathbb{K}\mu$, $R = \mathbb{K}\,\mathrm{as}$, its Koszul dual cooperad $\mathscr{A}s^{\text{¡}} = \mathscr{C}(sE, s^2R)$, $sE = s\mathbb{K}\mu = \mathbb{K}s\mu =: \mathbb{K}\mu^c$, $s^2R = s^2\mathbb{K}\,\mathrm{as}$, is the subcooperad of $\mathscr{T}^c(sE)$ that is universal among all subcooperads $\mathscr{C}$ of $\mathscr{T}^c(sE)$ such that the composite

$$\mathscr{C} \rightarrowtail \mathscr{T}^c(sE) \twoheadrightarrow \mathscr{T}^c(sE)^{(2)}/_{s^2R}$$

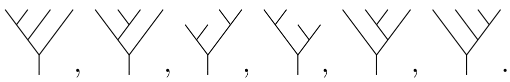
vanishes.

Note first that the cooperation μ^c is of arity 2 and of degree 1. Just as $\mathscr{T}(E)_n \simeq \mathbb{K}[\mathrm{PBT}_n]$, we have $\mathscr{T}^c(sE)_n = \mathscr{T}^c(\mathbb{K}\mu^c)_n \simeq \mathbb{K}[\mathrm{PBT}_n]$ as vector space. We will show that $\mathscr{A}s^{\text{¡}}$ is made up by a family of subspaces $\mathscr{A}s^{\text{¡}}_n = \mathbb{K}\mu^c_n \subset \mathscr{T}^c(\mathbb{K}\mu^c)_n \simeq \mathbb{K}[\mathrm{PBT}_n]$. The definition of the μ^c_n involves a sign that is based on the concept of leveled planar binary trees.

Remark: The vertices of any planar binary tree are arranged in levels. A *leveled planar binary tree* is a planar binary tree having exactly one vertex at each level.

For instance  is not leveled, whereas  and  are the leveled trees corresponding to the considered tree. Among these trees, the first one is leveled upwards.

Hence, the set $\widetilde{\mathrm{PBT}}_4$ of leveled planar binary trees with 4 leaves consists of



Numbering the levels from top to bottom and taking the vertices from left to right, we can assign to an element of $\widetilde{\mathrm{PBT}}_4$ a unique permutations of S_3. For the above leveled trees, the considered permutations are

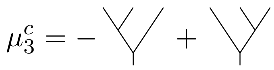
$$[1,2,3],\ [2,1,3],\ [2,3,1],\ [1,3,2],\ [3,1,2],\ [3,2,1].$$

This association is actually a one-to-one correspondence $S_3 \to \widetilde{\mathrm{PBT}}_4$. In general, we have a bijection $S_{n-1} \to \widetilde{\mathrm{PBT}}_n$.

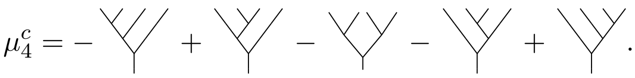
Define now

$$\mu_1^c = | \,, \quad \mu_2^c = \curlyvee, \quad \mu_n^c = - \sum_{t \in \mathrm{PBT}_n} \mathrm{sign}(\tilde{t})\, t, \quad n \geq 3,$$

where $\tilde{t} = t$ is t is already leveled, and $\tilde{t}$ is the upward leveled tree corresponding to t, if t is not leveled. Of course, the signature of a leveled tree is obtained by identifying the leveled tree with the corresponding permutation. For instance,



and



In order to show that the family of vector spaces $\mathbb{K}\mu_n^c$ forms a subcooperad of the cofree cooperad $\mathscr{T}^c(\mathbb{K}\mu^c)$, we need some more information about the cofree cooperad.

Remark (The cofree cooperad): The *cofree cooperad* $\mathscr{T}^c(M)$ over an $\mathbb{N}$-graded vector space (resp. S-module in the symmetric case) M is defined by the usual universal property defining cofree objects. As a vector space (resp. S-module), it is equal to $\mathscr{T}(M)$. The main difference lies in the decomposition map $\Delta : \mathscr{T}^c(M) \to \mathscr{T}^c(M) \circ \mathscr{T}^c(M)$. The idea behind this map is to decompose any operation of the cofree cooperad in all possible ways, such that composing again gives the initial operation back. In particular, for an operation μ, one has that

$$\Delta(\mu) = (\mu; \mathrm{id}, \ldots, \mathrm{id}) + \bar{\Delta}(\mu) + (\mathrm{id}; \mu),$$

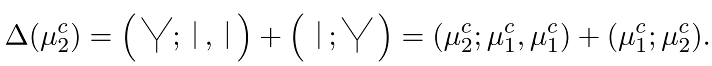
where $\bar{\Delta}$ takes the nontrivial decompositions into account.

In terms of trees, the map Δ consists in degrafting the initial tree by means of cutting. This cutting has, of course, to be done such that grafting again gives the initial tree back. Moreover, cutting the initial tree into smaller trees is done such that the root of the first one of the obtained trees is the root of the initial tree, and such that the leaves of the latter trees are the leaves of the initial tree. The example below will clarify the idea.

For further information about the cofree cooperad and the decomposition map, we refer to [LV11].

In our present situation, where Δ is the decomposition map of $\mathscr{T}^c(\mathbb{K}\mu^c)$, we have, for instance,

$$\Delta(\mu_1^c) = (\,|\,;\,|\,) = (\mu_1^c; \mu_1^c),$$



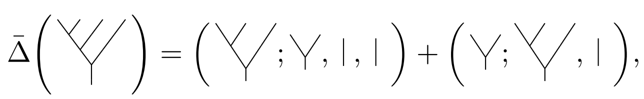
For μ_3^c, consider first

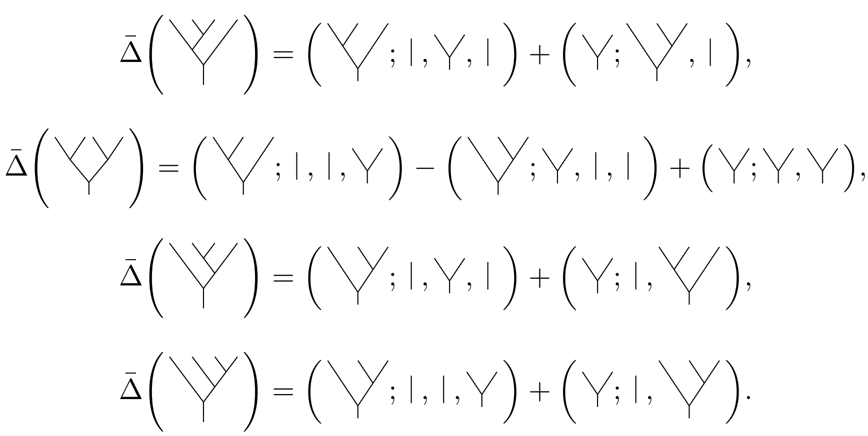
Thus

$$\Delta(\mu_3^c) = (\mu_3^c; \mu_1^c, \mu_1^c, \mu_1^c) - (\mu_2^c; \mu_2^c, \mu_1^c) + (\mu_2^c; \mu_1^c, \mu_2^c) + (\mu_1^c; \mu_3^c),$$

where the signs come from the definition of μ_3^c.

For μ_4^c consider





Notice the negative sign in the third line, which is a Koszul sign coming from the fact that μ^c is of degree 1. This sign can be seen appearing in the precise definition of the decomposition map, which we did not give here. Using the definition of μ_4^c, it follows that

$$\begin{aligned}\Delta(\mu_4^c) = {} & (\mu_4^c;\mu_1^c,\mu_1^c,\mu_1^c,\mu_1^c) + (\mu_3^c;\mu_2^c,\mu_1^c,\mu_1^c) - (\mu_3^c;\mu_1^c,\mu_2^c,\mu_1^c) + (\mu_3^c;\mu_1^c,\mu_1^c,\mu_2^c) \\ & + (\mu_2^c;\mu_3^c,\mu_1^c) + (\mu_2^c;\mu_1^c,\mu_3^c) - (\mu_2^c;\mu_2^c,\mu_2^c) + (\mu_1^c;\mu_4^c).\end{aligned}$$

It may appear surprising that we use linearity in the RHS, but in fact we are just applying the definition of the decomposition map.

Generalizing the preceding computations, we obtain

$$\Delta(\mu_n^c) = \sum_{i_1+\cdots+i_k=n} \pm(\mu_k^c;\mu_{i_1}^c,\ldots,\mu_{i_k}^c).$$

Hence, $\Delta\left((\mathbb{K}\mu_n^c)_{n\in\mathbb{N}^*}\right) \subset (\mathbb{K}\mu_n^c)_{n\in\mathbb{N}^*} \circ (\mathbb{K}\mu_n^c)_{n\in\mathbb{N}^*}$, so that $(\mathbb{K}\mu_n^c)_{n\in\mathbb{N}^*}$ is a subcooperad of $\mathscr{T}^c(\mathbb{K}\mu^c)$. Moreover, the composite

$$(\mathbb{K}\mu_n^c)_{n\in\mathbb{N}^*} \rightarrowtail \mathscr{T}^c(\mathbb{K}\mu^c) \twoheadrightarrow \mathscr{T}^c(\mathbb{K}\mu^c)^{(2)}\big/ s^2\mathbb{K}\,\mathrm{as}$$

clearly vanishes, since the projection onto $\mathscr{T}^c(\mathbb{K}\mu^c)^{(2)}$ vanishes everywhere except on μ_3^c, but μ_3^c vanishes in the considered quotient. By universality, this operad coincides with $\mathscr{A}s^{\text{i}}$, i.e. $\mathscr{A}s^{\text{i}}_n = \mathbb{K}\mu_n^c$, $n \geq 1$.

It can be proven that the complex $(\mathscr{A}s^{\text{i}} \circ \mathscr{A}s, d_\kappa)$, where

$$\kappa : \mathscr{A}s^{\text{i}} = \mathscr{C}(\mathbb{K}\mu^c, s^2\mathbb{K}\,\mathrm{as}) \twoheadrightarrow \mathbb{K}\mu^c = s\mathbb{K}\mu \xrightarrow{s^{-1}} \mathbb{K}\mu \rightarrowtail P(\mathbb{K}\mu, \mathbb{K}\,\mathrm{as}) = \mathscr{A}s,$$

is acyclic, so that $\mathscr{A}s$ is a quadratic Koszul operad. Moreover,

$$\mathscr{A}s^! = \left(\mathcal{S}^c \underset{\mathrm{H}}{\otimes} \mathscr{A}s^{\text{i}}\right)^* = \mathscr{A}s.$$

11.3 A_∞-algebras

A_∞-algebras have been introduced by Jim Stasheff in [Sta63]. If (V, d_V) is a deformation retract of (A, d_A), i.e. if both chain complexes are homotopy equivalent and the homotopy of V vanishes, then a DGAA structure on (A, d_A) induces an A_∞-structure on (V, d_V).

$$(V, d_V) \underset{p}{\overset{i}{\rightleftarrows}} (A, d_A) \circlearrowleft h$$

More precisely, if one tries to transfer the DGAA structure on (A, d_A) to (V, d_V), the most natural definition for the binary multiplication map on V is

$$m_2(u, v) := p\mu(i(u), i(v)),$$

where μ is the multiplication on A. However, this operation m_2 is not associative in general, but only 'associative up to higher homotopy'. This means that there exists a ternary operation m_3, such that the associativity condition is replaced by

$$m_2 \circ (m_2, \mathrm{id}) - m_2 \circ (\mathrm{id}, m_2) = \partial m_3,$$

where $\partial m_3 := d_V m_3 + m_3 d_{V^{\otimes 3}} = d_V m_3 + m_3(d_V \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes d_V \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes d_V)$. Similarly, we will get an operation m_4 of arity 4 when trying to write down conditions involving the operations m_2 and m_3. This whole process will continue and lead to an infinite sequence of operations and conditions. This structure will then be called an A_∞-algebra or homotopy associative algebra.

For the above mentioned deformation retract, we have that a DGAA structure on (A, d_A) induces an A_∞-structure on (V, d_V). However, an A_∞-structure on (A, d_A) will induce an A_∞-structure on (V, d_V). This transfer theorem can be extended to other types, and is one of the most important properties of infinity algebras. For instance, if (L, d_L) and (V, d_V) are homotopy equivalent chain complexes, a Lie infinity (L_∞) structure on (L, d_L) induces an L_∞-structure on (V, d_V).

Let us give a more precise description of A_∞-algebras. An A_∞-algebra is a graded vector space A endowed with a family of maps $m_k \in \mathrm{Hom}(A^{\otimes k}, A)$ of degree $k-2$, $k \geq 1$, that verify the following family of conditions

$$\sum_{\substack{p+q+r=n \\ p+1+r=k \\ k,q\geq 1}} (-1)^{p+qr} m_k(\underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{(p)}, m_q, \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{(r)}) = 0, \quad n \geq 1. \tag{11.1}$$

If we view the operations as maps

$$m_k : (sA)^{\otimes k} \to sA,$$

they all become maps of degree -1. These m_k define a map

$$m : \bar{T}^c(sA) = \bigoplus_{k\geq 1} (sA)^{\otimes k} \to sA,$$

which — since $\bar{T}^c(sA)$ is the cofree coalgebra over sA — extends uniquely to a degree -1 coderivation

$$m \in \mathrm{CoDer}_{-1}(\bar{T}^c(sA)).$$

The astonishing fact is that the family of relations (11.1) is encrypted in the unique condition $m \circ m = 0$. The converse result is true as well: to any codifferential $m \in \mathrm{CoDiff}_{-1}(\bar{T}^c(sA))$ corresponds a unique A_∞-structure on A.

This correspondence of infinity structures and codifferentials (coalgebraic approach to infinity algebras) has an algebraic variant in finite dimension (algebraic approach) and can be extended to other types of algebras. This generalization is the celebrated Ginzburg-Kapranov result:

Theorem 11.1 (Ginzburg-Kapranov [GK94]):
Let P be a quadratic Koszul operad. A P_∞-structure on a graded vector space V, in the sense of a representation on V of the differential graded operad $P_\infty := \Omega P^{¡}$, is equivalent (in the finite-dimensional setting) to an endomorphism of the free graded $P^!$-algebra over sV^*, which is of degree 1, squares to 0, and is a derivation with respect to any binary operation in $P^!$. Hence,

$$P_\infty\text{-structures on } V \leftrightarrow m \in \mathrm{Der}_1\left(\mathscr{F}^{gr}_{P^!}(sV^*)\right), m^2 = 0.$$

Similarly, a P_∞-structure on V (here, no finite-dimensional requirement is needed) is equivalent to an endomorphism of $\mathscr{F}^{gr,c}_{P^!}(sV)$, which is of degree -1, squares to 0, and is a coderivation. Hence,

$$P_\infty\text{-structures on } V \leftrightarrow m \in \mathrm{CoDer}_{-1}\left(\mathscr{F}^{gr,c}_{P^!}(sV)\right), m^2 = 0.$$

The derivation requirement in this theorem means that in the case $\mathscr{A}s^! = \mathscr{A}s$, $\mathscr{L}ie^! = \mathscr{C}om$, or $\mathscr{P}ois^! = \mathscr{P}ois$, the endomorphism be a derivation of the associative, the commutative, or the Lie and commutative products, respectively.

11.4 The operad A_∞

Just as associative algebras are algebras over a naturally constructed nonsymmetric operad, A_∞-algebras can be viewed as algebras over a quite obvious nonsymmetric operad A_∞, which we will now describe.

Let $(A, m_1, m_2, \ldots)$, $m_k \in \mathrm{Hom}(A^{\otimes k}, A)$, $\deg m_k = k-2$, be an A_∞-algebra. For $n = 1$, the relation (11.1) reads $m_1 \circ m_1 = 0$, so that $d := -m_1 \in \mathrm{End}_{-1}(A)$ endows the graded vector space A with a chain complex structure. Hence, $A^{\otimes n}$ is a chain complex for the differential $d_{A^{\otimes n}} = \sum_{p+1+r=n} \mathrm{id}^{\otimes p} \otimes d \otimes \mathrm{id}^{\otimes r}$. This entails that $\mathrm{Hom}(A^{\otimes n}, A)$ is a chain complex for the differential $\partial = [d, -] = [-m_1, -]$. Therefore,

$$\partial m_n = -m_1(m_n) + (-1)^{n-2} m_n \left(\sum_{p+1+r=n} \mathrm{id}^{\otimes p} \otimes m_1 \otimes \mathrm{id}^{\otimes r} \right),$$

and the relations (11.1), for $n \geq 2$, read

$$\partial m_n = \sum_{\substack{p+q+r=n \\ p+1+r=k \\ k,q\geq 2}} (-1)^{p+qr} m_k(\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}). \tag{11.2}$$

As an A_∞-algebra is a chain complex (A, d) endowed with operations $m_n \in \mathrm{Hom}(A^{\otimes n}, A)$ of degree $n-2$, $n \geq 2$, that verify the relations (11.2), the corresponding operad is a nonsymmetric DG operad A_∞. Its generating operations, i.e. the operations which are not composites of other ones, are μ_n, as usually represented by corollas, and correspond to the m_n, $n \geq 2$. It is clear that the differential of the operad A_∞, is defined by (11.2). Hence, no relations must be encoded via quotiening by some operadic ideal, and A_∞ is the free DG operad $\mathscr{T}\left(\bigoplus_{n\geq 2} \mathbb{K}\mu_n\right)$ together with the mentioned differential ∂. Of course, we should check that ∂ respects the grading (i.e. is of degree -1) and composition (i.e. is a derivation). The first condition follows immediately from (11.2), since $k-2+q-2 = n-3$, and the second is part of the definition of ∂.

On one hand, we have

$$\begin{aligned} m \in \mathrm{CoDiff}_{-1}\left(\bar{T}^c(sA)\right) &\leftrightarrow A_\infty\text{-structures on } A \\ &\leftrightarrow \text{ representations of the DG operad } A_\infty \text{ on } (A, d). \end{aligned}$$

On the other hand, it follows from Ginzburg-Kapranov that

$$\begin{aligned} m \in \mathrm{CoDiff}_{-1}\left(\bar{T}^c(sA)\right) &\leftrightarrow \mathscr{A}s_\infty\text{-structures on } A \\ &\leftrightarrow \text{ representations of the DG operad } \mathscr{A}s_\infty = \Omega \mathscr{A}s^{¡} \text{ on } A. \end{aligned}$$

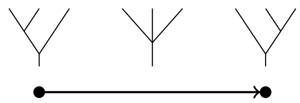
This suggest already that the operads A_∞ and $\mathscr{A}s_\infty$ coincide. In the following, we will provide a direct proof of the fact that $A_\infty = \mathscr{A}s_\infty$. The conclusion will actually follow from the description of the two operads in terms of the associahedron.

11.5 Stasheff polytope or associahedron

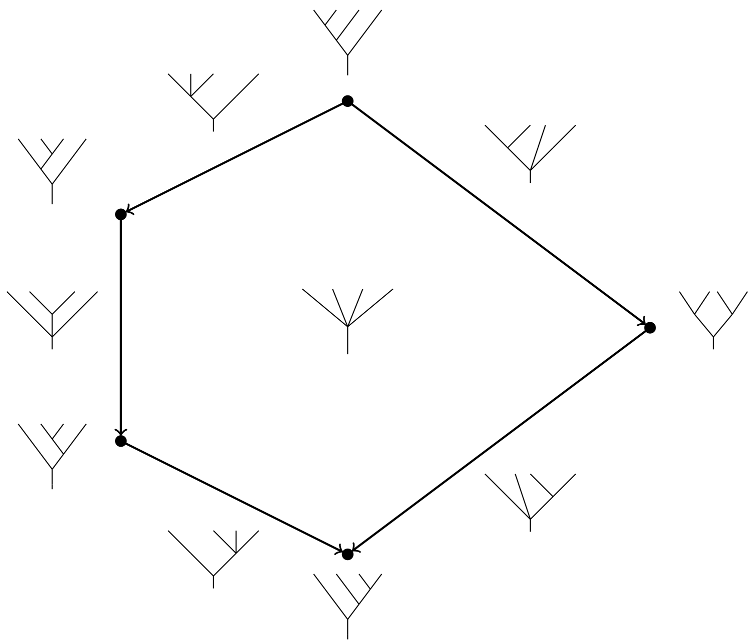
The polytope $\mathcal{K}^n$, $n \geq 0$ is a cell complex of dimension n that is homeomorphic to a ball and whose cells are in one-to-one correspondence with planar trees with $n+2$ leaves. Note that the set PT_m of planar trees with m leaves is graded by the number of vertices; we denote the set of planar trees with m leaves and ℓ vertices by $\mathrm{PT}_{m,\ell}$. In fact, we have that the cells of dimension k of the polytope are in bijection with elements of $\mathrm{PT}_{n+2,n+1-k}$. The k-chains, whose space is denoted by $\mathcal{C}_k(\mathcal{K}^n)$, are formal linear combinations with coefficients in $\mathbb{K}$ of the k-cells.

$\mathcal{C}_\bullet(\mathcal{K}^0)$:

$\mathcal{C}_\bullet(\mathcal{K}^1)$:

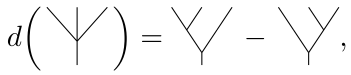


$\mathcal{C}_\bullet(\mathcal{K}^2)$:

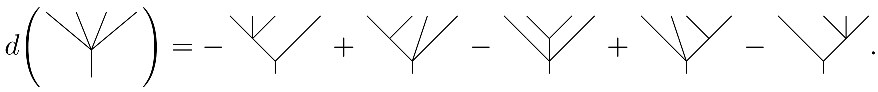


Observe that we pass from the top 0-cell to the bottom 0-cell by transforming 'left leaves' into 'right leaves' in two different ways.

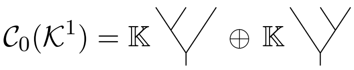
Of course, the boundary operator d of $\mathcal{C}_\bullet(\mathcal{K}^n)$ assigns to any chain or cell the cell boundary. For instance, for the n-cell identified with the n-corolla, we have, for $n = 1$,



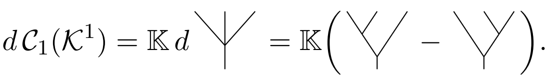
and, for $n = 2$,



Note that $\mathcal{C}_1(\mathcal{K}^1)$ does not contain any 1-cycle, so that the homology vanishes in degree 1. On the other hand, any 0-chain in



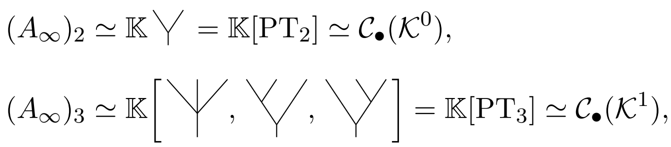
is a 0-cycle, whereas the 0-boundaries are



It follows that $H_0(\mathcal{K}^1) \simeq \mathbb{K}$.

11.5.1 Description of the operad A_∞ in terms of the associahedron

Roughly, the DG operad A_∞ is made up by a family $(A_\infty)_n$ of graded vector spaces, a composition map, and a differential. Since A_∞ is the free graded operad $\mathscr{T}\left(\bigoplus_{n\geq 2} \mathbb{K}\mu_n\right)$, where μ_n is identified with the n-corolla, it is clear that



$$\dots$$

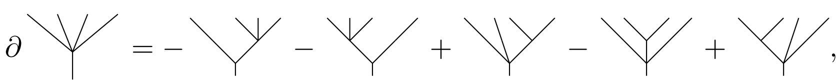
$$(A_\infty)_n \simeq \mathbb{K}[\mathrm{PT}_n] \simeq \mathcal{C}_\bullet(\mathcal{K}^{n-2}),$$

$$\dots$$

Note that composition in A_∞ is encrypted in the preceding description. As for the differential of A_∞, it is given by (11.2). For instance,

$$\partial\mu_4 = -(\mu_2; \mathrm{id}, \mu_3) - (\mu_2; \mu_3, \mathrm{id}) + (\mu_3; \mathrm{id}, \mathrm{id}, \mu_2) - (\mu_3; \mathrm{id}, \mu_2, \mathrm{id}) + (\mu_3; \mu_2, \mathrm{id}, \mathrm{id}),$$

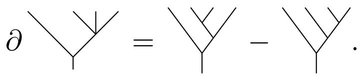
i.e.



and

$$\begin{aligned}\partial(\mu_2; \mathrm{id}, \mu_3) &= (\partial\mu_2; \mathrm{id}, \mu_3) + (\mu_2; \mathrm{id}, \partial\mu_3)\\ &= (\mu_2; \mathrm{id}, (\mu_2; \mu_2, \mathrm{id}) - (\mu_2; \mathrm{id}, \mu_2))\\ &= (\mu_2; \mathrm{id}, (\mu_2; \mu_2, \mathrm{id})) - (\mu_2; \mathrm{id}, (\mu_2; \mathrm{id}, \mu_2)),\end{aligned}$$

i.e.



Hence, the differential ∂ of $(A_\infty)_4$ (and more generally of $(A_\infty)_n$) coincides with the boundary operator d of $\mathcal{C}_\bullet(\mathcal{K}^2)$ (and more generally of $\mathcal{C}_\bullet(\mathcal{K}^{n-2})$).

11.5.2 Description of the operad $\mathscr{A}s_\infty$ in terms of the associahedron

The cobar construction of an augmented DG cooperad has been described previously. Recall that, in general, $\Omega\mathscr{C} = \mathscr{T}(s^{-1}\bar{\mathscr{C}})$, and $\delta = \delta_1 + \delta_2$, where δ_1 (resp. δ_2) is the extension of $d_{\mathscr{C}}$ (resp. $\bar{\Delta}_{(1)}$) to a derivation of $\mathscr{T}(s^{-1}\bar{\mathscr{C}})$. However, in our situation $\mathscr{C} = \mathscr{A}s^{\text{¡}} = \mathscr{C}(sE, s^2R)$ and $d_{\mathscr{C}} = 0$. Hence, roughly, the DG operad $\mathscr{A}s_\infty = \Omega\mathscr{A}s^{\text{¡}}$ is made up by a family of graded vector spaces $(\mathscr{A}s_\infty)_n$, a composition map, and the differential $\delta = \delta_2$.

Since $(\mathscr{A}s_\infty)_n = \mathscr{T}(s^{-1}\overline{\mathscr{A}s^{\text{¡}}})_n$ is the space of n-ary operations of the free graded operad over $\overline{\mathscr{A}s^{\text{¡}}} = \mathbb{K}\mu_2^c \oplus \mathbb{K}\mu_3^c \oplus \cdots$, μ_n^c being the unique n-ary operation (represented by the n-corolla) in $\mathscr{A}s^{\text{¡}}$, it is clear that

$$(\mathscr{A}s_\infty)_n \simeq \mathbb{K}[\mathrm{PT}_n] \simeq \mathcal{C}_\bullet(\mathcal{K}^{n-2}).$$

If we now prove that the differential δ of $(\mathscr{A}s_\infty)_n$ coincides with the boundary operator d of $\mathcal{C}_\bullet(\mathcal{K}^{n-2})$, we can conclude that $A_\infty \simeq \mathscr{A}s_\infty$ as DG operads. As mentioned, δ is a derivation for composition in $\mathscr{A}s_\infty$, whereas d (resp. ∂) is a derivation for composition of trees (resp. in A_∞). It therefore suffices to prove that δ and d coincide on generators μ_n^c (identified with the n-corolla), $n \geq 2$.

On μ_n^c, δ coincides with $\bar{\Delta}_{(1)}$, where

$$\Delta_{(1)} : \mathscr{C} \xrightarrow{\Delta} \mathscr{C} \circ \mathscr{C} \simeq \mathscr{C} \circ (\mathscr{C}; \mathscr{C}) \xrightarrow{\mathrm{id} \circ (\varepsilon; \mathrm{id})} \mathscr{C} \circ (I; \mathscr{C}) = \mathscr{C} \circ_{(1)} \mathscr{C}.$$

In other words, $\Delta_{(1)}$ is Δ followed by a replacement of all but one 'upper' elements of $\mathscr{C}$ by id, and $\bar{\Delta}_{(1)}$ is similarly obtained from $\bar{\Delta}$ (recall that $\bar{\Delta}(\mu) = \Delta(\mu) - (\mu; \mathrm{id}, \ldots, \mathrm{id}) - (\mathrm{id}; \mu)$). Thus, using the formula

$$\Delta(\mu_n^c) = \sum_{i_1 + \cdots + i_k = n} \pm (\mu_k^c; \mu_{i_1}^c, \ldots, \mu_{i_k}^c),$$

we get

$$\delta(\mu_n^c) = \bar{\Delta}_{(1)}(\mu_n^c) = \sum_{\substack{p+q+r=n \\ p+1+r=k \\ k,q \geq 2}} \pm (\mu_k^c; \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{(p)}, \mu_q^c, \underbrace{\mathrm{id}, \ldots, \mathrm{id}}_{(r)}) = d(\mu_n^c), \quad n \geq 2,$$

where the conditions $k, q \geq 2$ come from the fact that we linearize $\bar{\Delta}$. We have thus proved that ∂ and d coincide.

Finally, we have proved the following

Theorem 11.2: *The operads A_∞ and $\mathscr{A}s_\infty := \Omega\mathscr{A}s^{\text{¡}}$ coincide. Moreover, the categories of A_∞-algebras and $\mathscr{A}s_\infty$-algebras are the same.*

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