GIS

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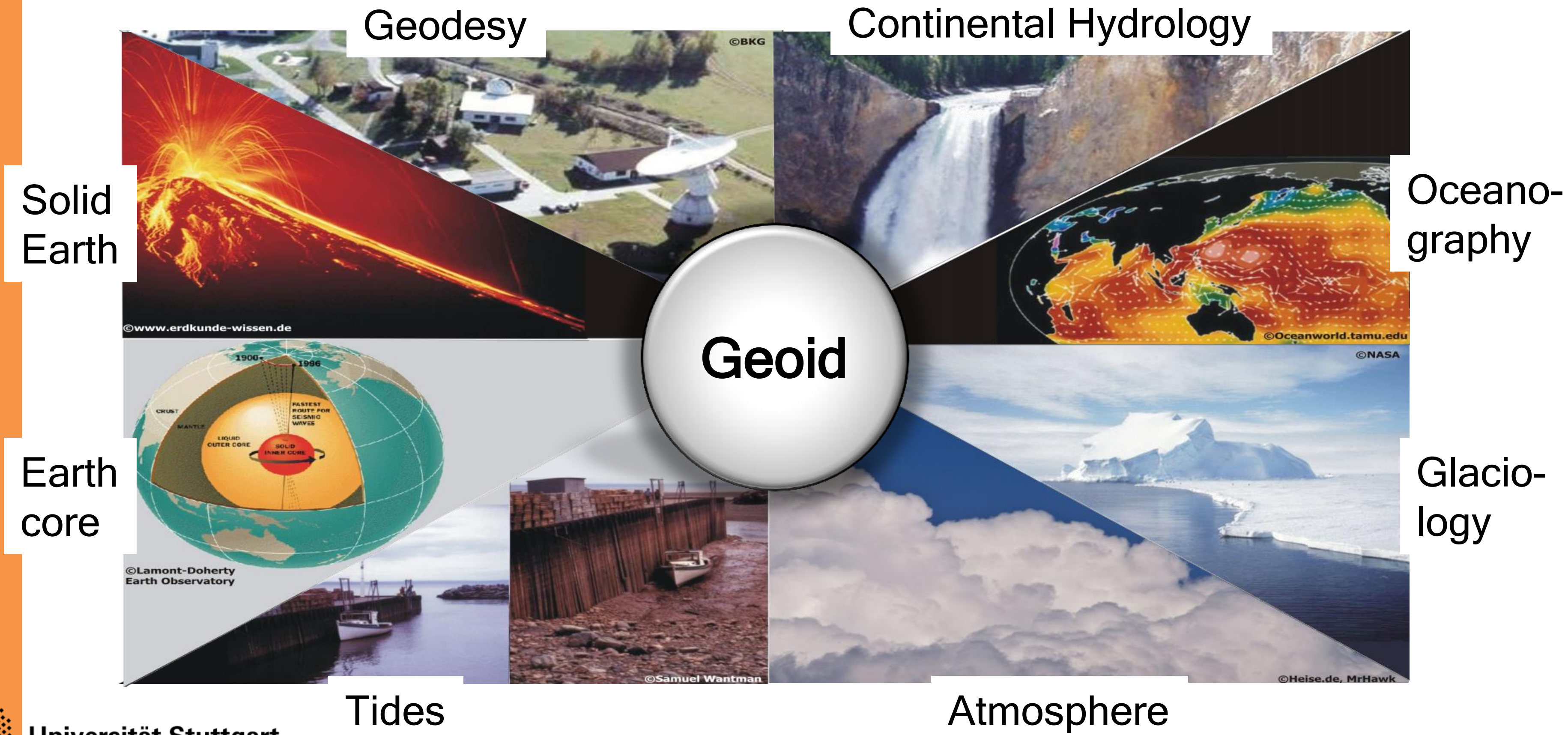
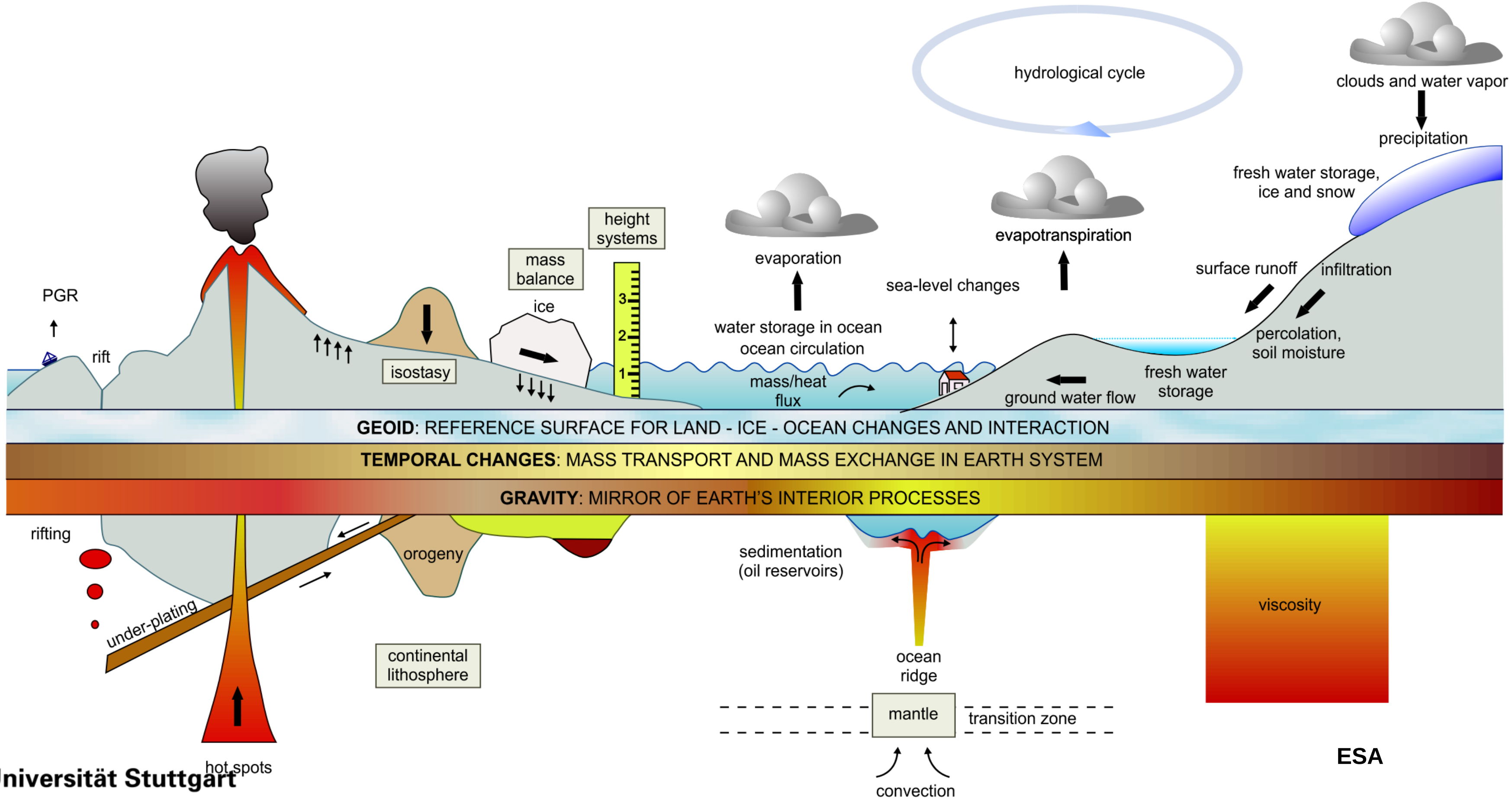
GRACE GRAVITY FIELD SOLUTIONS USING THE DIFFERENTIAL GRAVIMETRY APPROACH

Universität Stuttgart

Geophysical implications of the gravity field

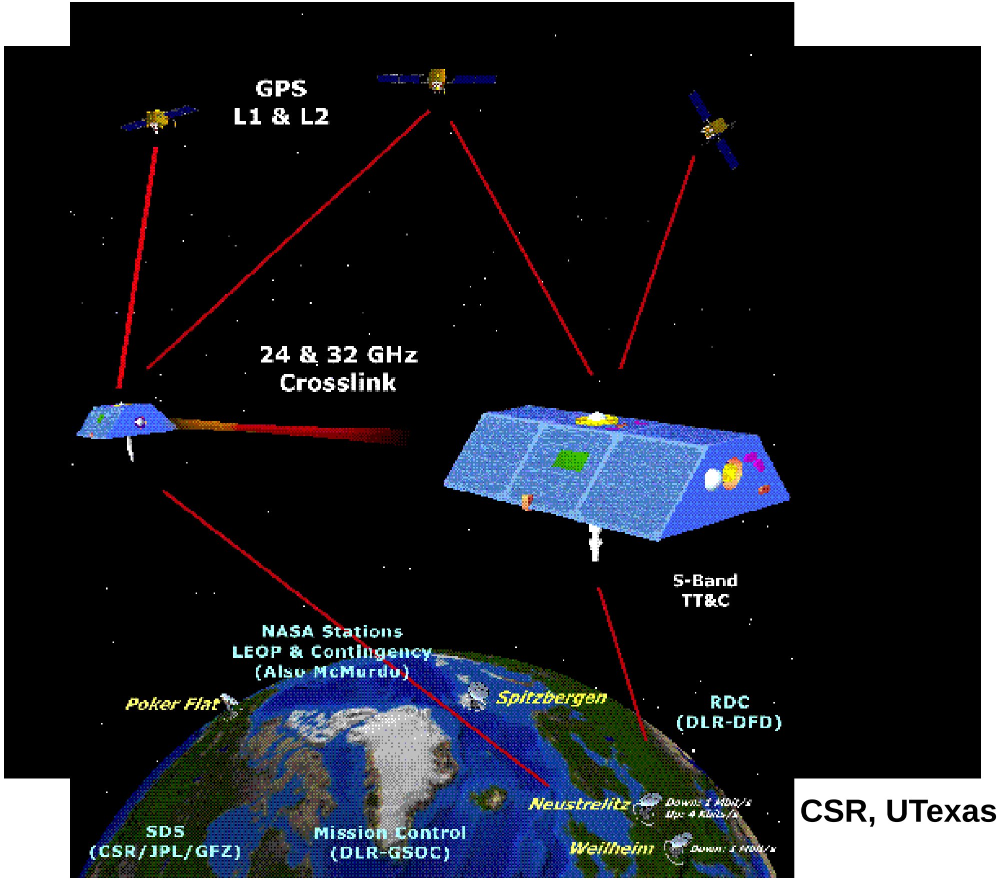


Geophysical implications of the gravity field



GRACE = Gravity Recovery and Climate Experiment

- Initial orbit height: ~ 485 km
- Inclination: ~ 89°
- Key technologies:
  - GPS receiver
  - Accelerometer
  - K-Band Ranging System
- Observed quantity:
  - range ρ
  - range rate $\dot{\rho}$



CSR, UTexas

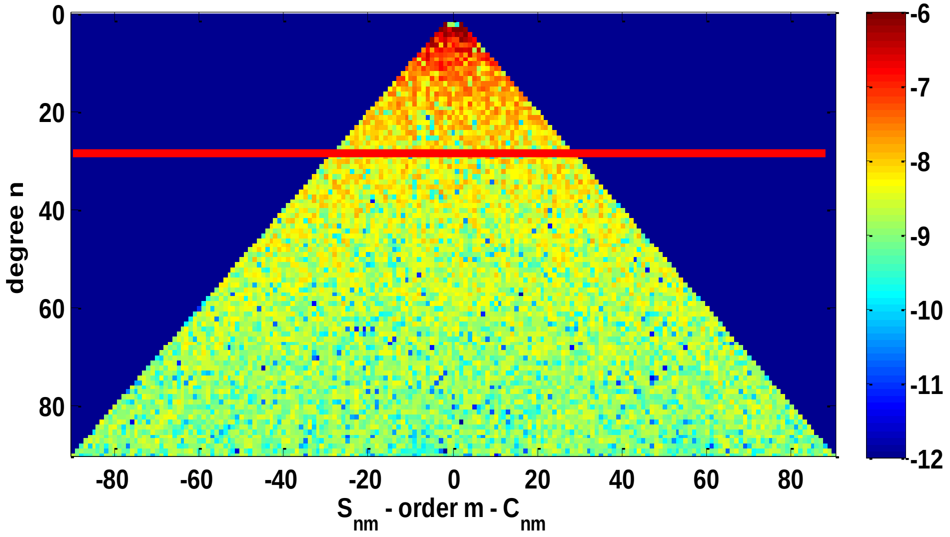
$$V(\lambda,\theta,r) = \frac{GM}{R}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1}\sum_{m=0}^{n}\bar{P}_{nm}(\cos\theta)\left(\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda\right)$$

with

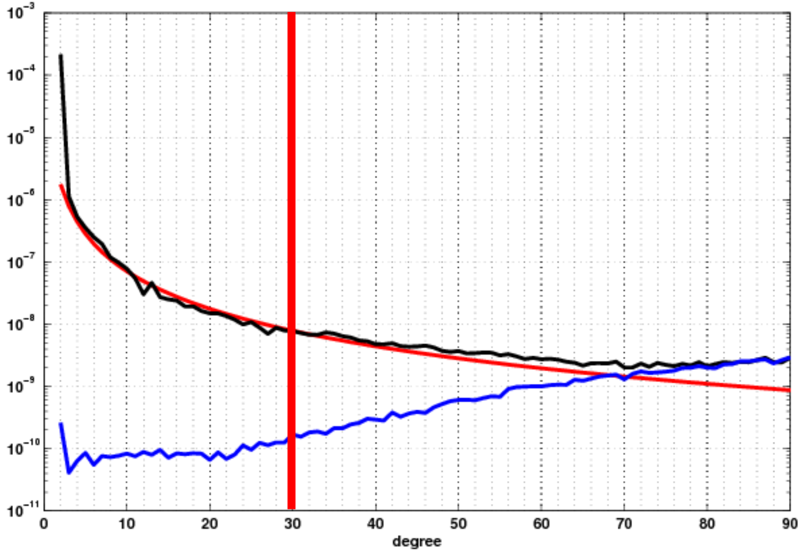
with	GM	gravitational constant times mass of the Earth
	R	radius of the Earth
	λ, θ, r	spherical coordinates of the calculation point
	$\bar{P}_{nm}$	Legendre function
	n, m	degree, order
	$\bar{C}_{nm}, \bar{S}_{nm}$	(unknown) spherical harmonic coefficients

Spectral representations

Spherical harmonic spectrum



Degree RMS



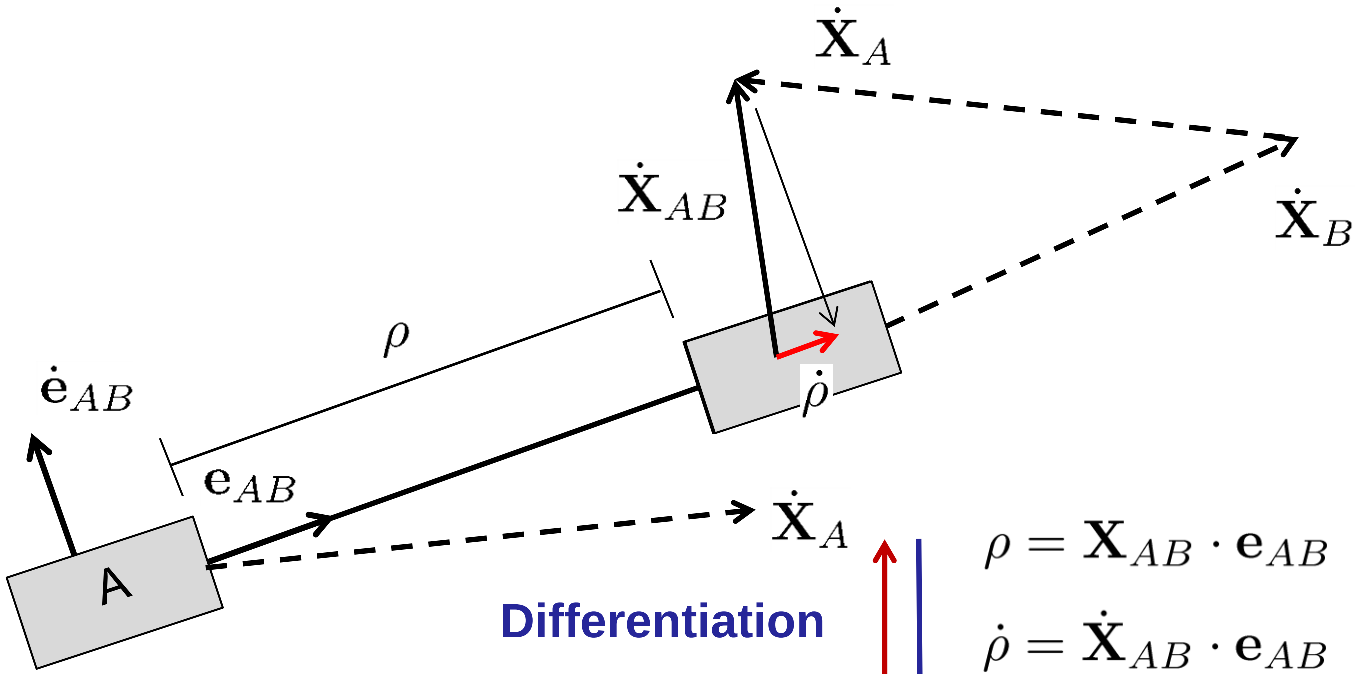
$$c_n = \sqrt{\frac{1}{n}\sum_{m=0}^{n} \bar{C}_{nm}^2 + \bar{S}_{nm}^2}$$

Outline

- GRACE geometry
- Solution strategies
  - Variational equations
  - Differential gravimetry approach
- What about the Next-Generation-GRACE?

Geometry of the GRACE system

Geometry of the GRACE system



Rummel et al. 1978

Differentiation

Integration

$$\rho = \mathbf{X}_{AB} \cdot \mathbf{e}_{AB}$$

$$\dot{\rho} = \dot{\mathbf{X}}_{AB} \cdot \mathbf{e}_{AB}$$

$$\ddot{\rho} = \ddot{\mathbf{X}}_{AB} \cdot \mathbf{e}_{AB} + \dot{\mathbf{X}}_{AB} \cdot \dot{\mathbf{e}}_{AB}$$

$$= \nabla V_{AB}$$

Solution Strategies

Solution strategies

Variational equations

$\rho, \dot{\rho}$ — *Classical*
(Reigber 1989, Tapley 2004)

$\rho, \dot{\rho}, \Delta\rho$ — *Celestial mechanics approach*
(Beutler et al. 2010, Jäggi 2007)

$\rho, \dot{\rho}$ — Short-arc method
(Mayer-Gürr 2006)

…

Numerical integration

In-situ observations

Energy Integral — $\dot{\rho}$
(Han 2003, Ramillien et al. 2010)

Differential gravimetry — $\ddot{\rho}$
(Liu 2010)

LoS Gradiometry — $\frac{\ddot{\rho}}{\rho}$
(Keller and Sharifi 2005)

…

Analytical integration

Equation of motion

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Basic equation: Newton's equation of motion

$$\ddot{\mathbf{X}} = \nabla V + \sum_i \mathbf{g}_i$$

where g_i are all gravitational and non-gravitational disturbing forces

In the general case: *ordinary second order non-linear differential equation*

Double integration yields:

$$\mathbf{X}(t) = f\left(\mathbf{X}(t_0), \dot{\mathbf{X}}(t_0), \bar{C}_{lm}, \bar{S}_{lm}, ...\right) = f(p_i)$$

Linearization

For the solution, linearization using a Taylor expansion is necessary:

$$\mathbf{X}(t, p_i) \approx \mathbf{X}(t, p_i^0) + \sum_i \left.\frac{\partial \mathbf{X}}{\partial p_i}\right|_{p_i^0, t} \Delta p_i$$

Types of partial derivatives:

- initial position $\frac{\partial \mathbf{X}}{\partial x(t_0)}, \frac{\partial \mathbf{X}}{\partial y(t_0)}, \frac{\partial \mathbf{X}}{\partial z(t_0)}$
- initial velocity $\frac{\partial \mathbf{X}}{\partial \dot{x}(t_0)}, \frac{\partial \mathbf{X}}{\partial \dot{y}(t_0)}, \frac{\partial \mathbf{X}}{\partial \dot{z}(t_0)}$

} Homogeneous solution

- residual gravity field coefficients $\frac{\partial \mathbf{X}}{\partial \bar{C}_{nm}}, \frac{\partial \mathbf{X}}{\partial \bar{S}_{nm}}$
- additional parameter …

} Inhomogeneous solution

Homogeneous solution needs the partial derivatives:

$$\frac{\partial \mathbf{X}}{\partial x\left(t_0\right)}, \frac{\partial \mathbf{X}}{\partial y\left(t_0\right)}, \frac{\partial \mathbf{X}}{\partial z\left(t_0\right)}, \frac{\partial \mathbf{X}}{\partial \dot{x}\left(t_0\right)}, \frac{\partial \mathbf{X}}{\partial \dot{y}\left(t_0\right)}, \frac{\partial \mathbf{X}}{\partial \dot{z}\left(t_0\right)}$$

Derivation by integration of the variational equation

Double integration !

$$\frac{\partial^2}{\partial t^2}\left[\begin{array}{ccccc} \frac{\partial x}{\partial x(t_0)} & \frac{\partial x}{\partial y(t_0)} & \cdots & \frac{\partial x}{\partial \dot{y}(t_0)} & \frac{\partial x}{\partial \dot{z}(t_0)} \\ \frac{\partial y}{\partial x(t_0)} & \frac{\partial y}{\partial y(t_0)} & \cdots & \frac{\partial y}{\partial \dot{y}(t_0)} & \frac{\partial y}{\partial \dot{z}(t_0)} \\ \frac{\partial z}{\partial x(t_0)} & \frac{\partial z}{\partial y(t_0)} & \cdots & \frac{\partial z}{\partial \dot{y}(t_0)} & \frac{\partial z}{\partial \dot{z}(t_0)} \end{array}\right] =$$

$$\left[\begin{array}{ccc} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial y} & \frac{\partial^2 V}{\partial x \partial z} \\ \frac{\partial^2 V}{\partial x \partial y} & \frac{\partial^2 V}{\partial y^2} & \frac{\partial^2 V}{\partial y \partial z} \\ \frac{\partial^2 V}{\partial x \partial z} & \frac{\partial^2 V}{\partial y \partial z} & \frac{\partial^2 V}{\partial z^2} \end{array}\right]\left[\begin{array}{ccccc} \frac{\partial x}{\partial x(t_0)} & \frac{\partial x}{\partial y(t_0)} & \cdots & \frac{\partial x}{\partial \dot{y}(t_0)} & \frac{\partial x}{\partial \dot{z}(t_0)} \\ \frac{\partial y}{\partial x(t_0)} & \frac{\partial y}{\partial y(t_0)} & \cdots & \frac{\partial y}{\partial \dot{y}(t_0)} & \frac{\partial y}{\partial \dot{z}(t_0)} \\ \frac{\partial z}{\partial x(t_0)} & \frac{\partial z}{\partial y(t_0)} & \cdots & \frac{\partial z}{\partial \dot{y}(t_0)} & \frac{\partial z}{\partial \dot{z}(t_0)} \end{array}\right]$$

Solution of the inhomogeneous part by the method of the *variation of the constant* (Beutler 2006):

$$\begin{aligned}\mathbf{z}_{p_i}(t) &= \alpha_{p_i,1}\mathbf{z}_1(t) + \alpha_{p_i,2}\mathbf{z}_2(t) + \alpha_{p_i,3}\mathbf{z}_3(t) \\ &+ \alpha_{p_i,4}\mathbf{z}_4(t) + \alpha_{p_i,5}\mathbf{z}_5(t) + \alpha_{p_i,6}\mathbf{z}_6(t)\end{aligned}$$

with $z_i(t)$ being the columns of the matrix of the variational equation of the homogeneous solution at each epoch.

Estimation of (R)$_i$ by solving the equation system at each epoch:

$$\left[\mathbf{z}_1(t)\,\mathbf{z}_2(t)\,\mathbf{z}_3(t)\,\mathbf{z}_4(t)\,\mathbf{z}_5(t)\,\mathbf{z}_6(t)\right]\dot{\alpha}_{p_i} = \begin{pmatrix} \mathbf{0} \\ \frac{\partial \nabla V}{\partial p_i} \end{pmatrix}$$

Application to GRACE

In case of GRACE, the observables are range and range rate:

$$\rho = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2}$$

$$\dot{\rho} = \frac{1}{\rho}\left[(\dot{x}_b - \dot{x}_a)(x_b - x_a) + (\dot{y}_b - \dot{y}_a)(y_b - y_a) + (\dot{z}_b - \dot{z}_a)(z_b - z_a)\right]$$

Chain rule needs to be applied:

$$\frac{\partial \rho}{\partial p_i} = \frac{\partial \rho}{\partial x_a}\frac{\partial x_a}{\partial p_i} + \frac{\partial \rho}{\partial x_b}\frac{\partial x_b}{\partial p_i} + \ldots + \frac{\partial \rho}{\partial z_b}\frac{\partial z_b}{\partial p_i}$$

$$\frac{\partial \dot{\rho}}{\partial p_i} = \frac{\partial \dot{\rho}}{\partial x_a}\frac{\partial x_a}{\partial p_i} + \frac{\partial \dot{\rho}}{\partial x_b}\frac{\partial x_b}{\partial p_i} + \ldots \;\ldots\; \ldots + \frac{\partial \dot{\rho}}{\partial \dot{z}_b}\frac{\partial \dot{z}_b}{\partial p_i}$$

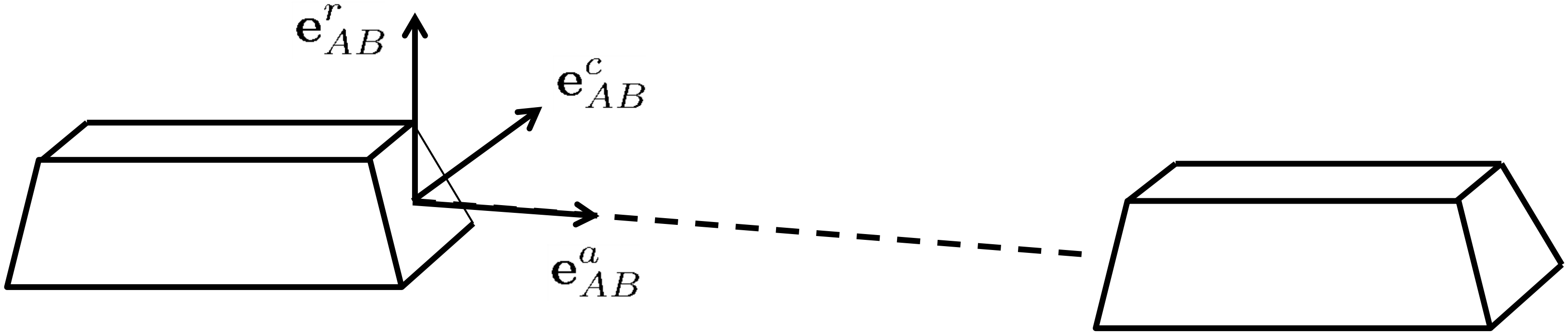
Limitations

- additional parameters
  - compensate errors in the initial conditions
  - counteract accumulation of errors
- outlier detection difficult
- limited application to local areas
- high computational effort
- difficult estimation of corrections to the initial conditions in case of GRACE (twice the number of unknowns, relative observation)

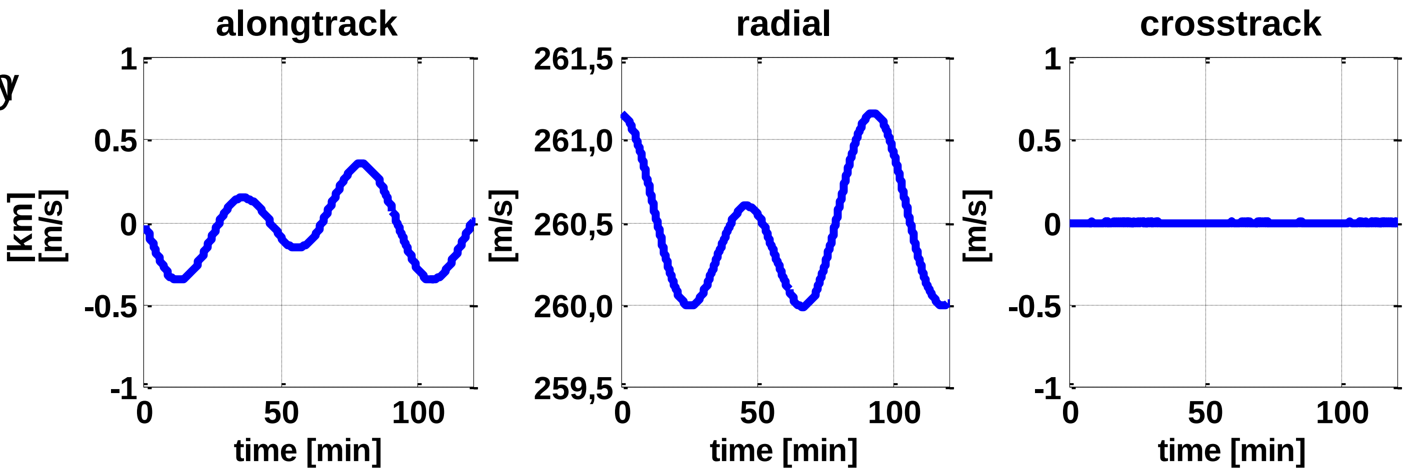
In-situ observations:

Differential gravimetry approach

Instantaneous relative reference frame



Position
Velocity



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Range observables:

$$\mathbf{X}_{AB} = \rho \cdot \mathbf{e}^a_{AB}$$

$$\dot{\mathbf{X}}_{AB} = \dot{\rho} \cdot \mathbf{e}^a_{AB} + \rho \cdot \dot{\mathbf{e}}^a_{AB}$$

$$\ddot{\mathbf{X}}_{AB} = \ddot{\rho} \cdot \mathbf{e}^a_{AB} + 2 \cdot \dot{\rho} \cdot \dot{\mathbf{e}}^a_{AB} + \rho \cdot \ddot{\mathbf{e}}^a_{AB}$$

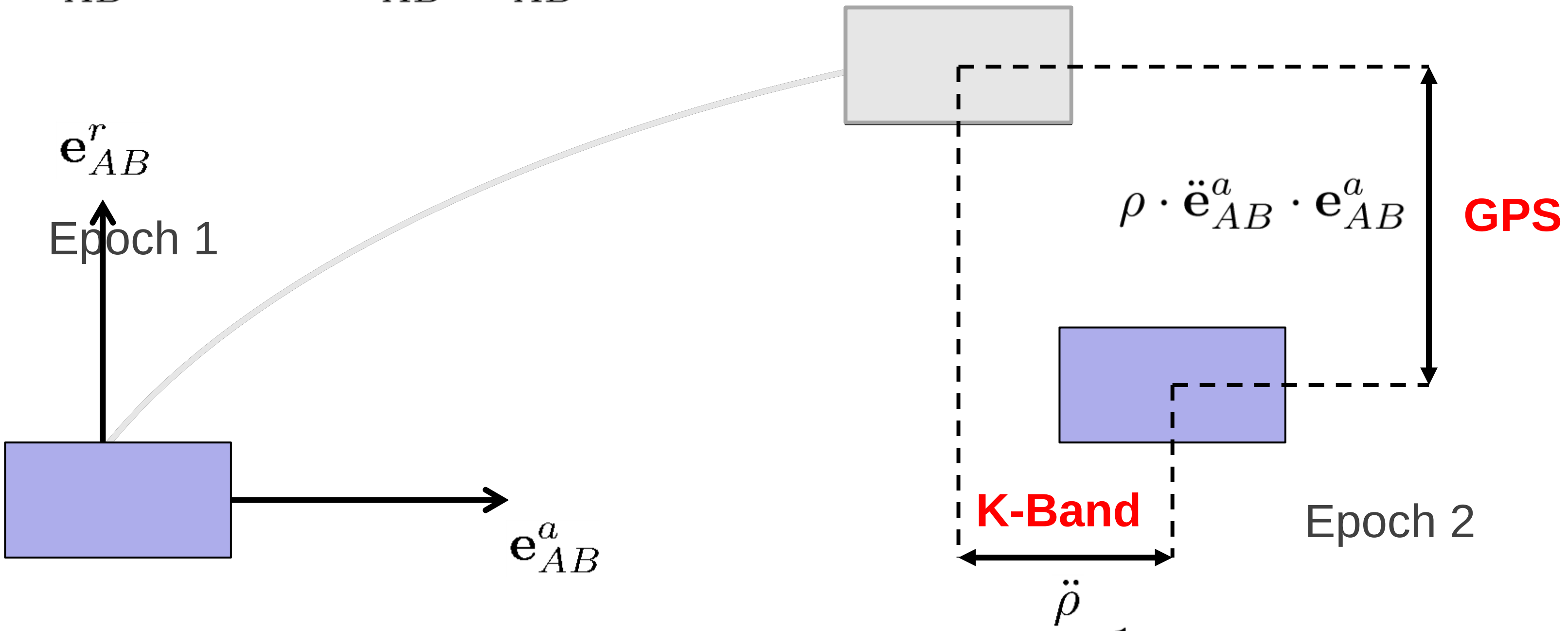
Multiplication with unit vectors:

GRACE

$$\begin{aligned}
\ddot{\mathbf{X}}_{AB} \cdot \mathbf{e}^a_{AB} &= \ddot{\rho} + 0 + \rho \cdot \ddot{\mathbf{e}}^a_{AB} \cdot \mathbf{e}^a_{AB} \\
\ddot{\mathbf{X}}_{AB} \cdot \mathbf{e}^r_{AB} &= 0 + 2 \cdot \dot{\rho} \cdot \|\dot{\mathbf{e}}^a_{AB}\| + \rho \cdot \ddot{\mathbf{e}}^a_{AB} \cdot \mathbf{e}^r_{AB} \\
\ddot{\mathbf{X}}_{AB} \cdot \mathbf{e}^c_{AB} &= 0 + 0 + \rho \cdot \ddot{\mathbf{e}}^a_{AB} \cdot \mathbf{e}^c_{AB}
\end{aligned}$$

$$\ddot{\mathbf{X}}_{AB} \cdot \mathbf{e}^a_{AB} = \ddot{\rho} + \rho \cdot \ddot{\mathbf{e}}^a_{AB} \cdot \mathbf{e}^a_{AB}$$

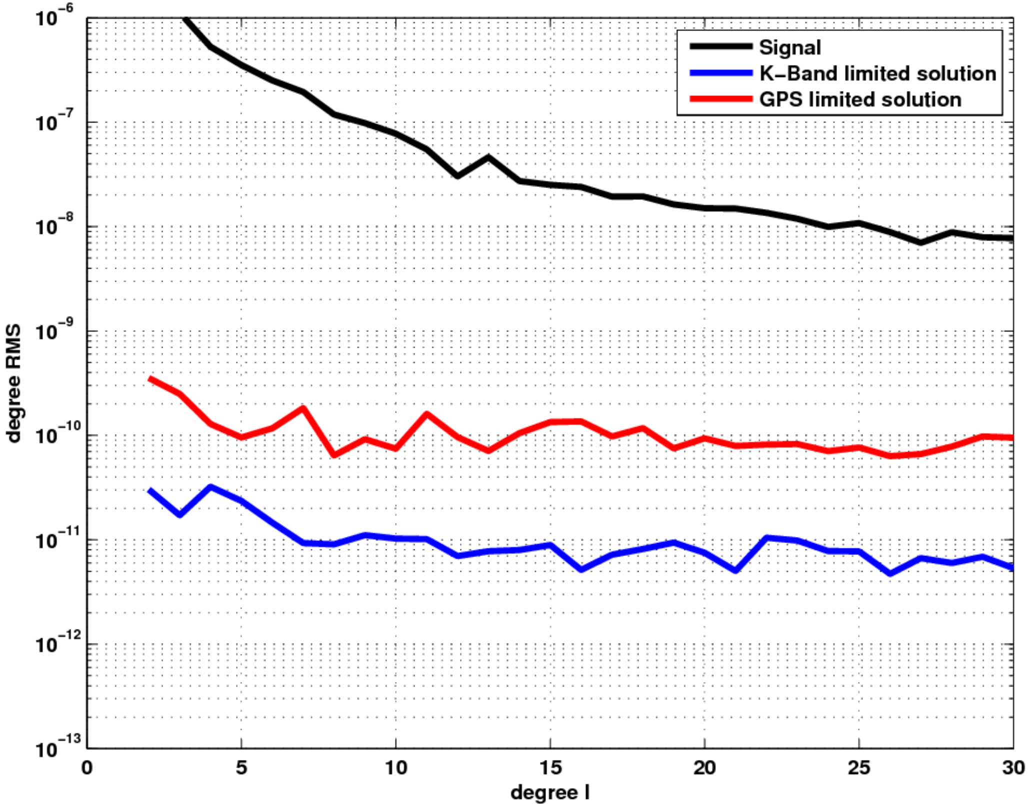
absolute motion neglected!



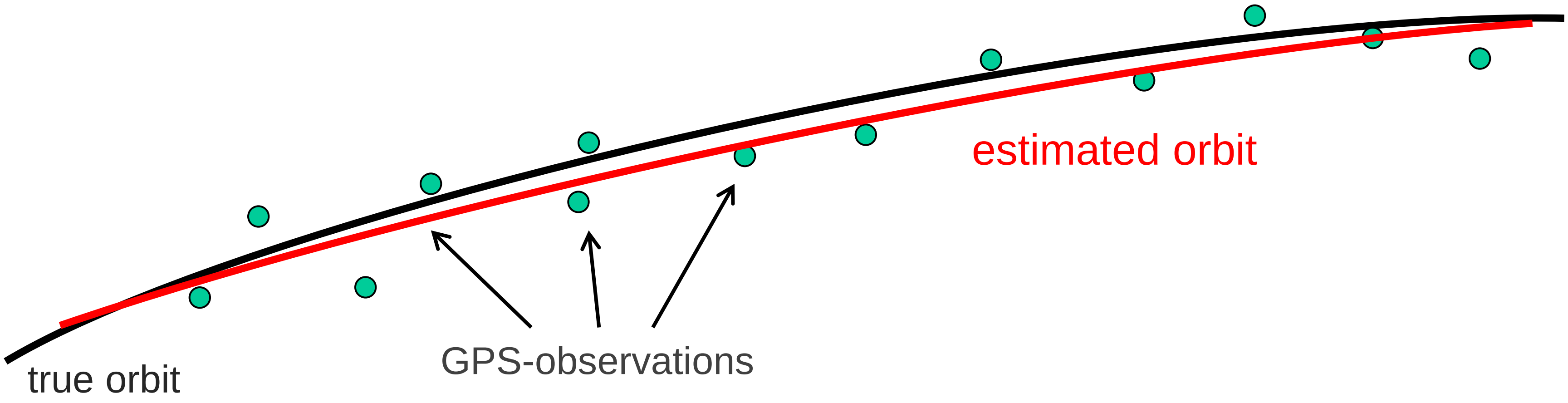
$$\rho \ddot{\mathbf{e}}^a_{AB} \cdot \mathbf{e}_{AB} = \mathbf{X}_{AB} \cdot \ddot{\mathbf{e}}^a_{AB} = -\dot{\mathbf{X}}_{AB} \cdot \dot{\mathbf{e}}^a_{AB} = -\dot{\rho} \|\dot{\mathbf{e}}^a_{AB}\|^2 = -\frac{1}{\rho}\left(\dot{\mathbf{X}}_{AB} \cdot \dot{\mathbf{X}}_{AB} - \dot{\rho}^2\right)$$

Limitation

Combination of highly precise K-Band observations with comparably low accurate GPS relative velocity



- Orbit fitting using the homogeneous solution of the variational equation with a known a priori gravity field

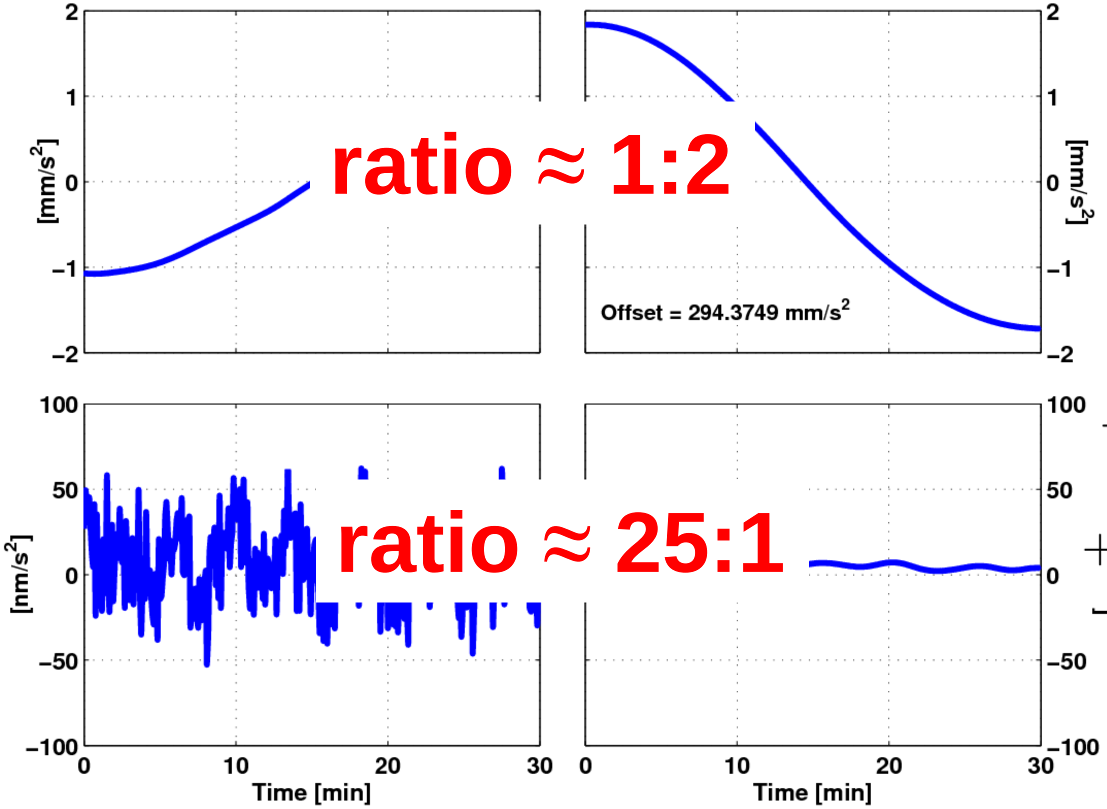


- Avoiding the estimation of empirical parameters by using short arcs

Residual quantities

$\ddot{\rho}$

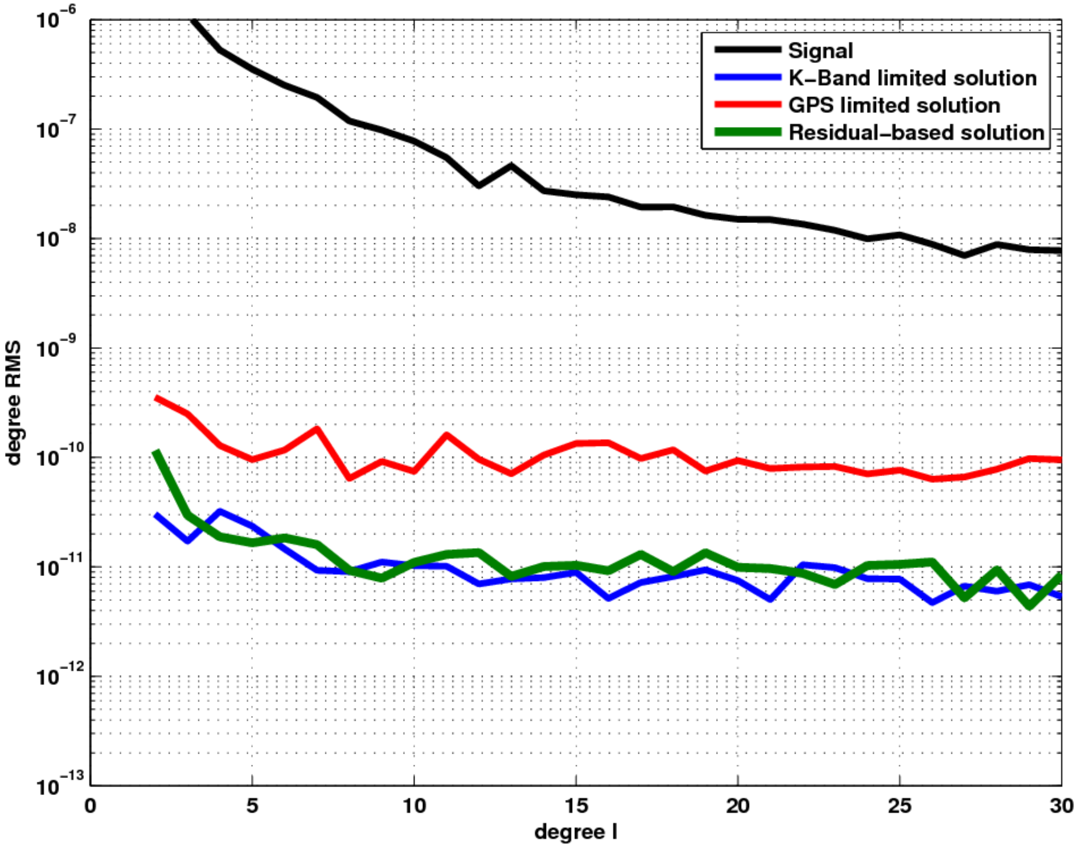
$\ddot{\rho} - \ddot{\rho}_0$



$$-\frac{1}{\rho}\left(\dot{\mathbf{X}}_{AB}\cdot\dot{\mathbf{X}}_{AB}-\dot{\rho}^2\right)$$

$$-\frac{1}{\rho}\left(\dot{\mathbf{X}}_{AB}\cdot\dot{\mathbf{X}}_{AB}-\dot{\rho}^2\right)+\frac{1}{\rho_0}\left(\dot{\mathbf{X}}_{AB,0}\cdot\dot{\mathbf{X}}_{AB,0}-\dot{\rho_0}^2\right)$$

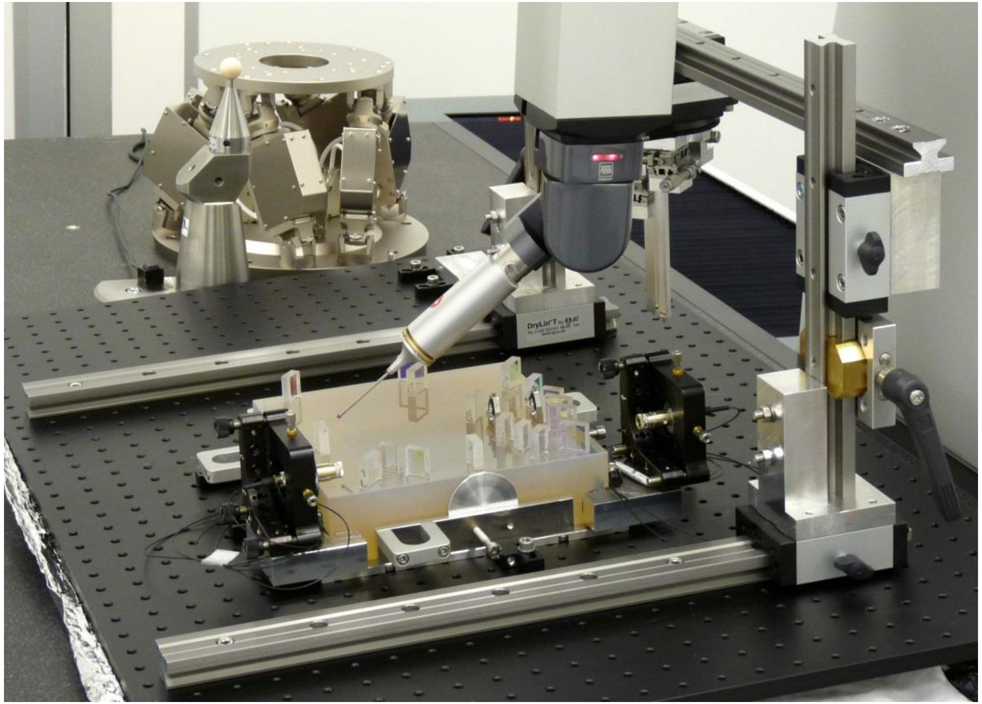
Approximated solution



Next generation GRACE

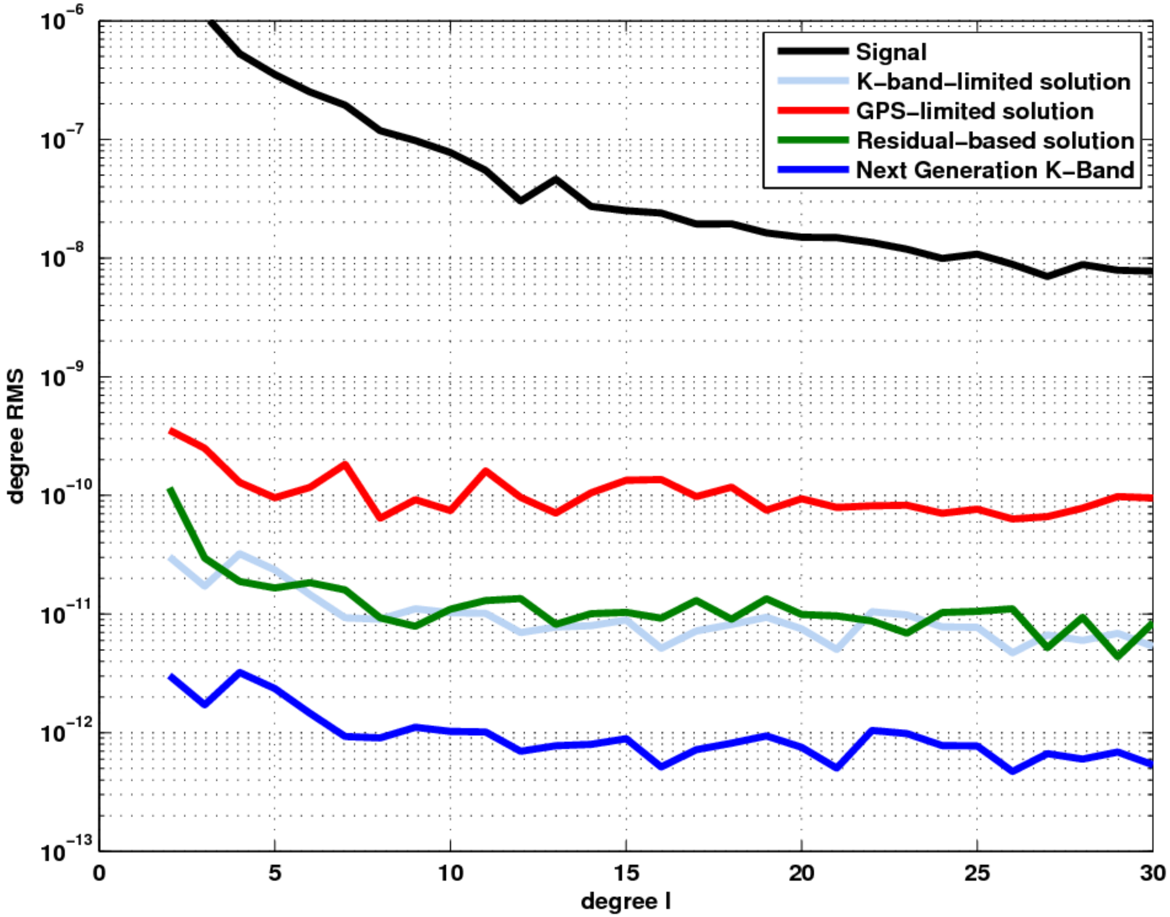
Next generation GRACE

- New type of intersatellite distance measurement based on laser interferometry
- Noise reduction by a factor 10 expected



M. Dehne, Quest

Solution for next generation GRACE



Variational equations for velocity term

- Reduction to residual quantity insufficient
- Modeling the velocity term by variational equations:

$$f = -\frac{1}{\rho}\left(\dot{\mathbf{X}}_{AB} \cdot \dot{\mathbf{X}}_{AB} - \dot{\rho}^2\right)$$

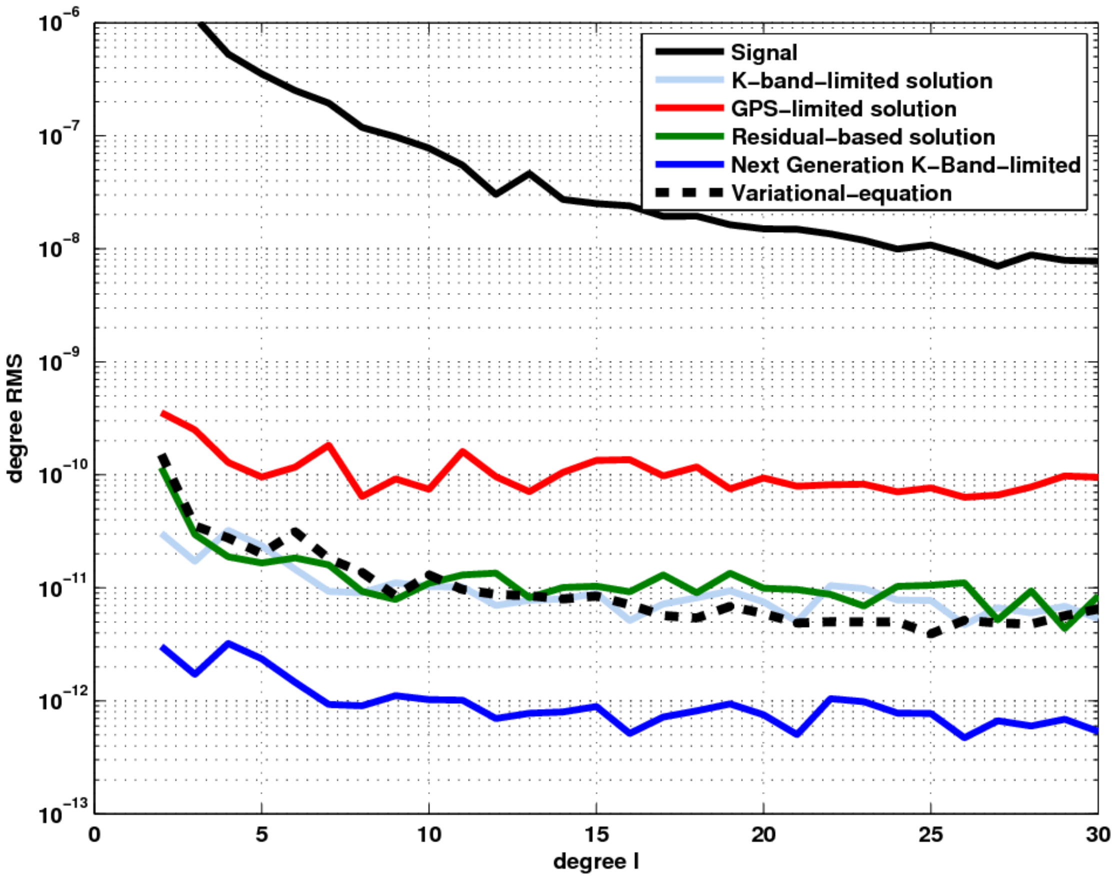
$$\frac{\partial f}{\partial p_i} = \frac{\partial f}{\partial x_a}\frac{\partial x_a}{\partial p_i} + \frac{\partial f}{\partial x_b}\frac{\partial x_b}{\partial p_i} + \dots \; \dots \; \dots + \frac{\partial f}{\partial \dot{z}_b}\frac{\partial \dot{z}_b}{\partial p_i}$$

- Application of the method of the variations of the constants

Results

- Only minor improvements by incorporating the estimation of corrections to the spherical harmonic coefficients due to the velocity term
- Limiting factor is the orbit fit to the GPS positions

→ additional estimation of corrections to the initial conditions necessary



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Summary

- The primary observables of the GRACE system (range & range rate) are connected to gravity field quantities through variational equations (numerical integration) or through in-situ observations (analytical integration).
- Variational equations pose a high computational effort.
- In-situ observations demand the combination of K-band and GPS information.
- Next generation GRACE instruments pose a challenge to existing solution strategies.