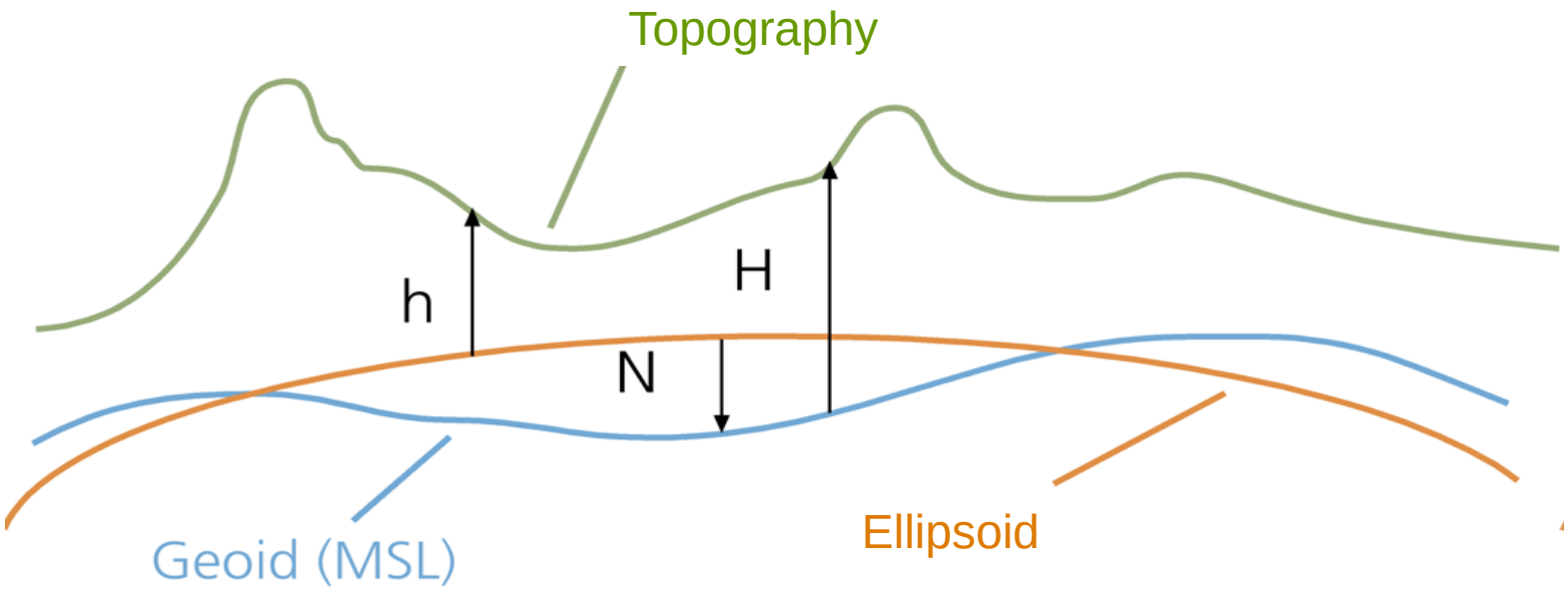
SIND WIR FÜR ZUKÜNFTIGE SCHWEREFELDMISSIONEN GERÜSTET?

Aspekte aus der Schwerefeldforschung am Geodätischen Institut Stuttgart

Matthias Weigelt

Objective: determination of the gravity field of the Earth

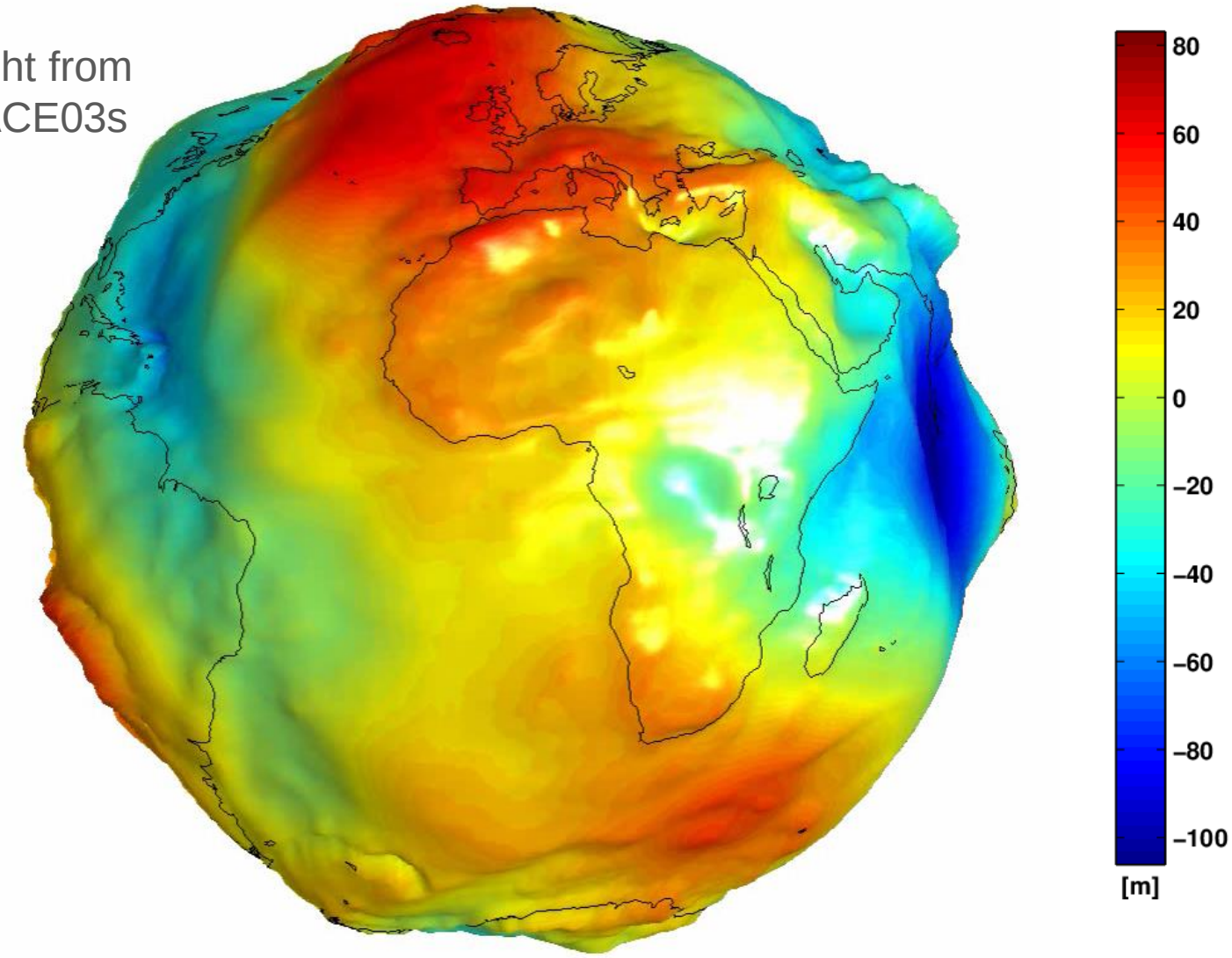
- Almost every geodetic measurement is connected to the plumb-line and/or the geoid.
- The geoid is a equipotential surface at mean sea level:
  a. Every mass has the same potential energy at this level.
  b. It indicates the flow of water (higher → lower potential).
  c. It connects GPS-heights (h) with leveled heights (H).



Determination of the potatoid

Geoid height from AIUB-GRACE03s

Jäggi et al, 2011

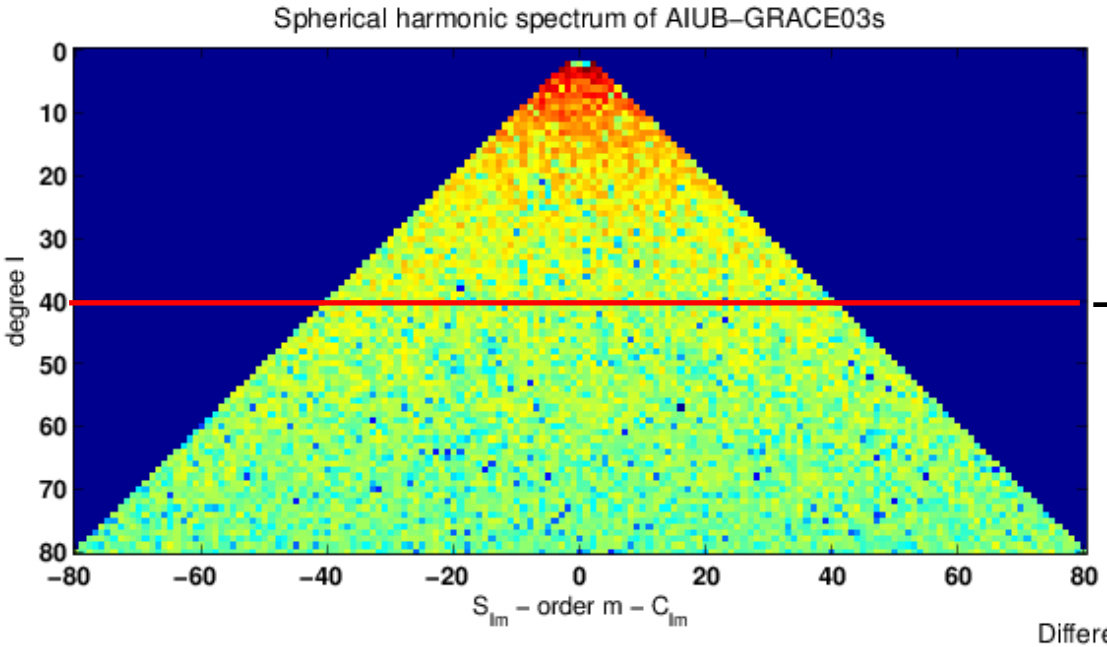


Gravity field modeling

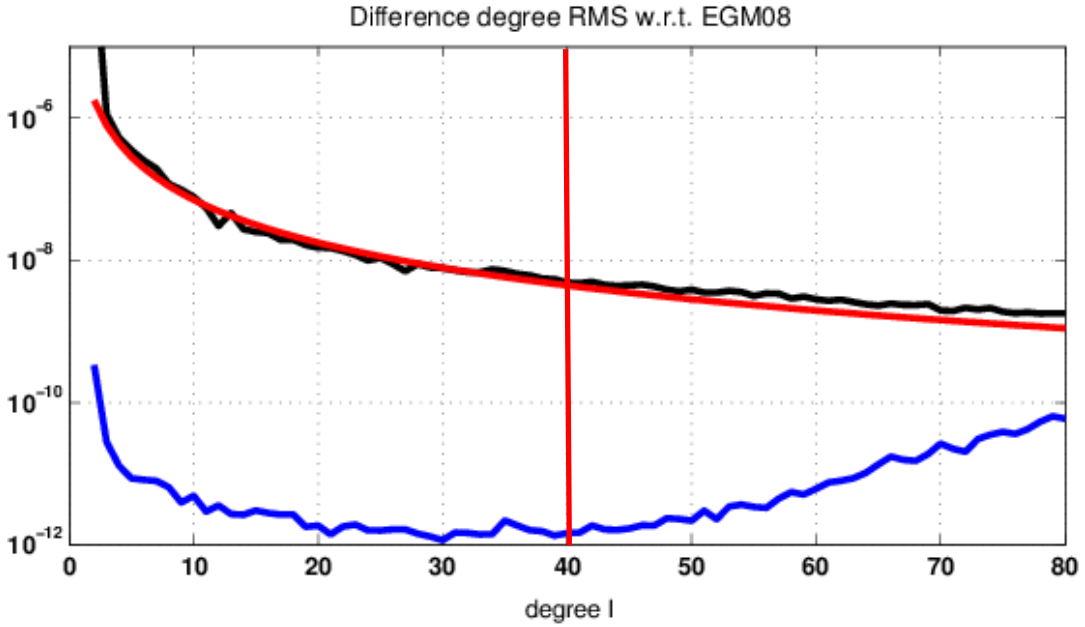
$$V = \frac{GM}{R} \sum_{l=0}^{\infty} \left(\frac{R}{r} \right)^{l+1} \sum_{m=0}^{l} \bar{P}_{lm} \left(\sin \phi \right) \left(\bar{C}_{lm} \cos m\lambda + \bar{S}_{lm} \sin m\lambda \right)$$

with	GM	gravitational constant times mass of the Earth
	R	radius of the Earth
	r, ϕ, λ	spherical coordinates of the calculation point
	$\bar{P}_{lm}$	Legendre function
	l, m	degree, order
	$\bar{C}_{lm}, \bar{S}_{lm}$	(unknown) spherical harmonic coefficients

Spectral representation



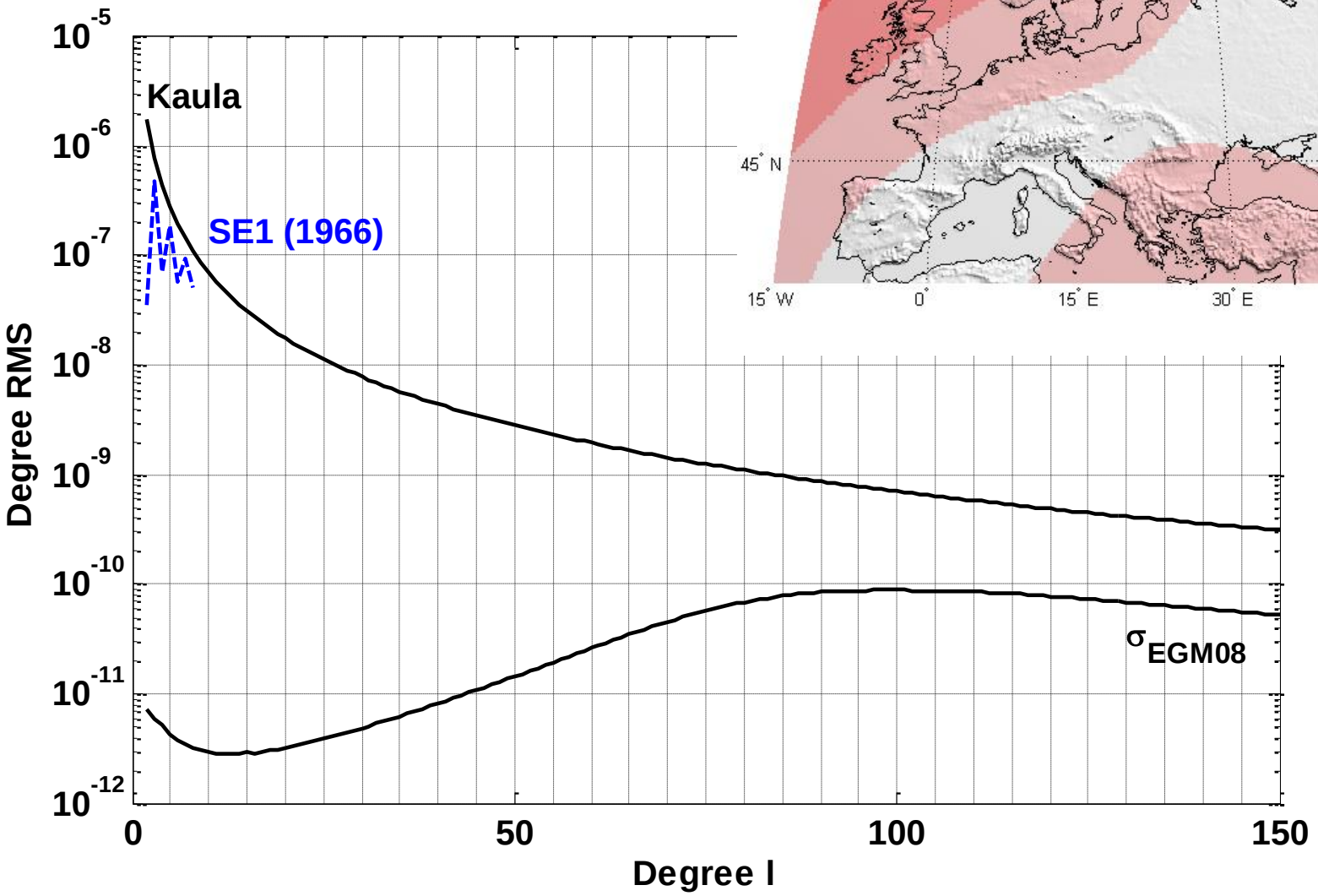
$$c_n = \sqrt{\frac{1}{2n+1}\sum_{m=0}^{n} \bar{C}_{nm}^2 + \bar{S}_{nm}^2}$$



Degree RMS represents the mean signal/error power per degree.

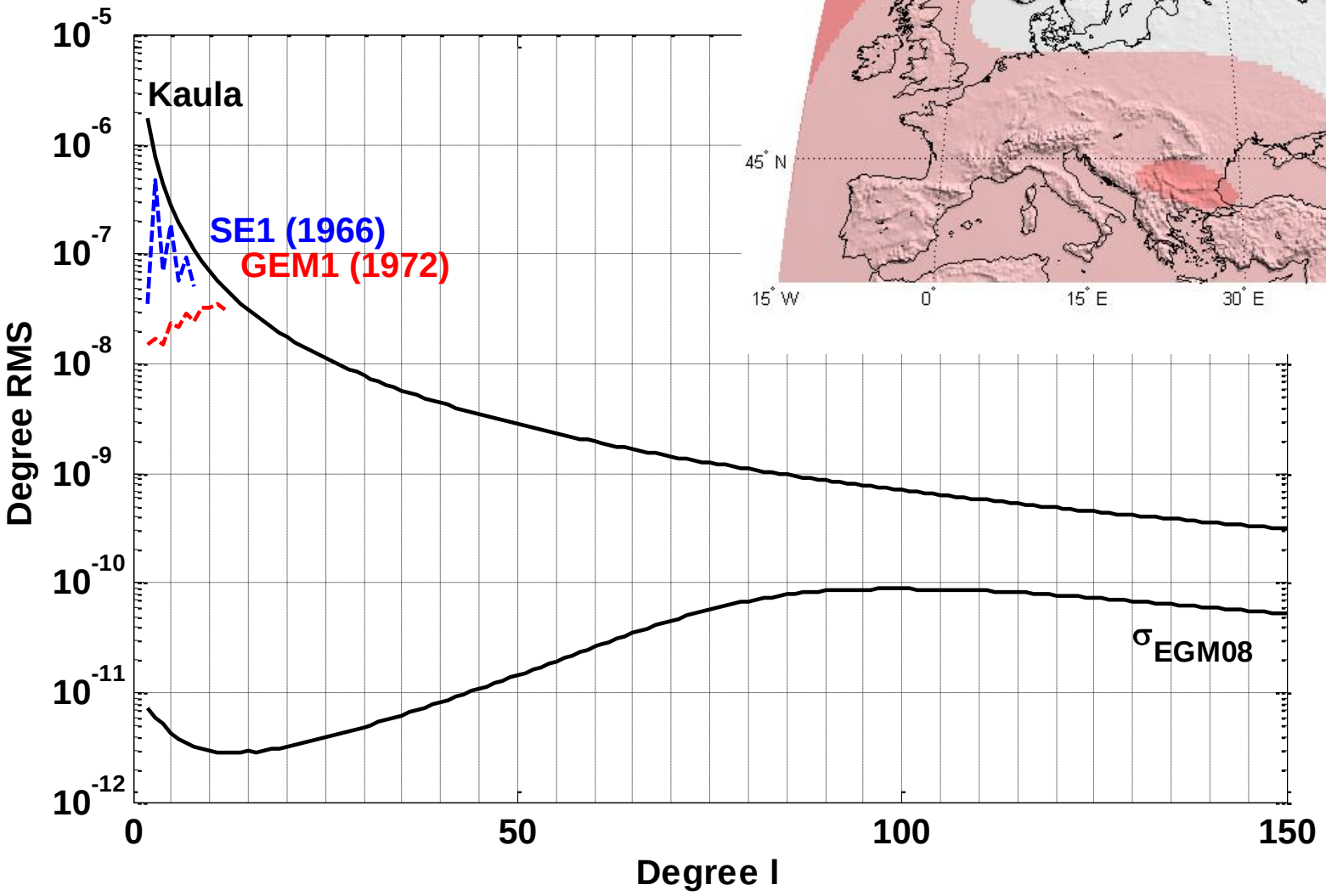
Spatial resolution

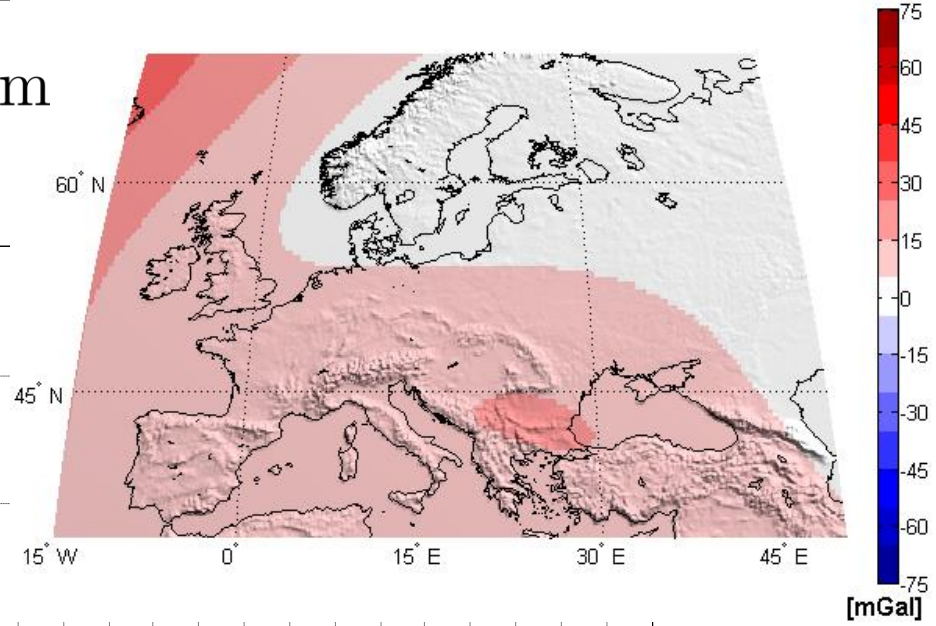
1966: $L_{Max} = 8 \quad \frac{\lambda}{2} = 2500\,\mathrm{km}$



Spatial resolution

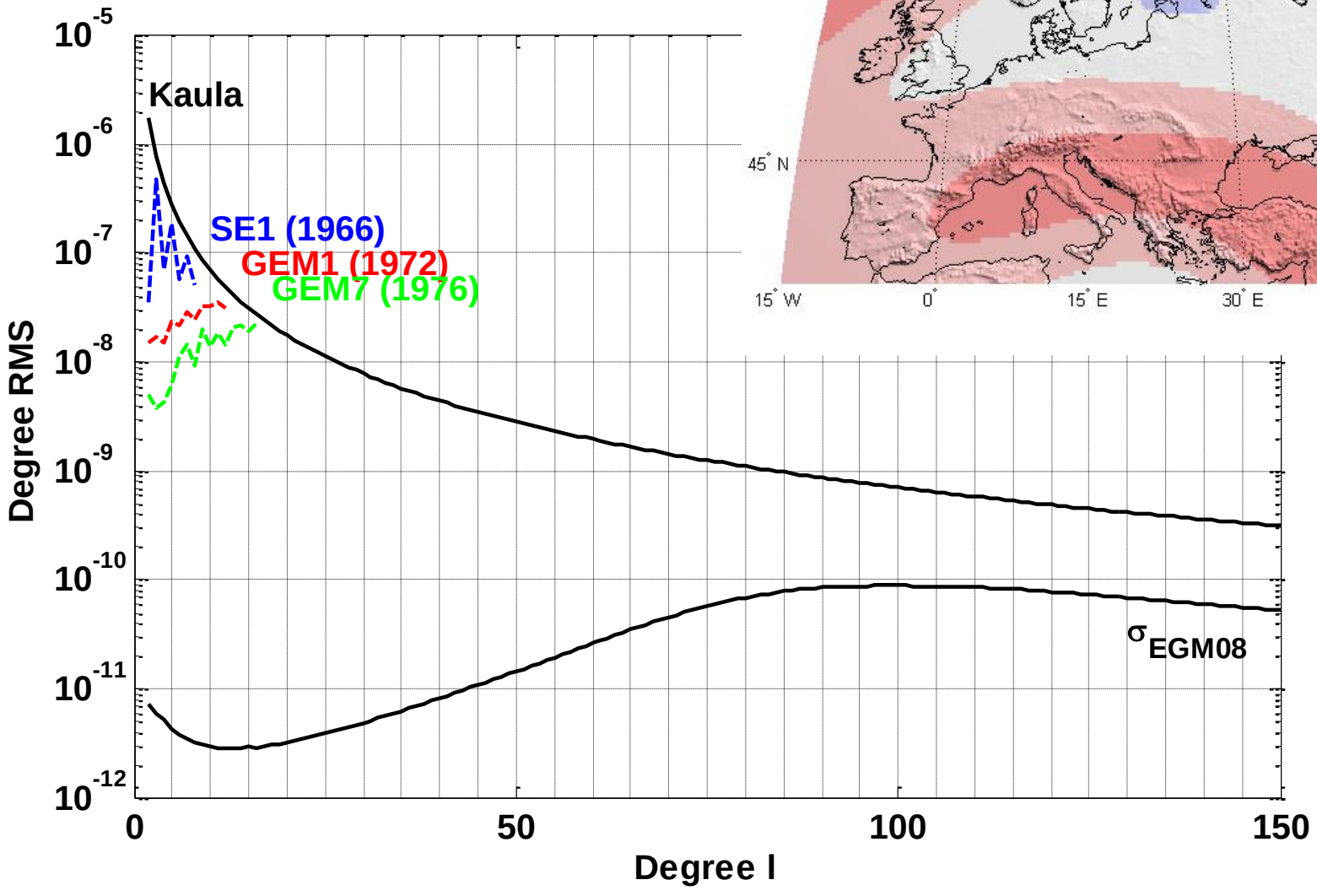
1972: $L_{Max} = 12 \quad \frac{\lambda}{2} = 1600\,\mathrm{km}$

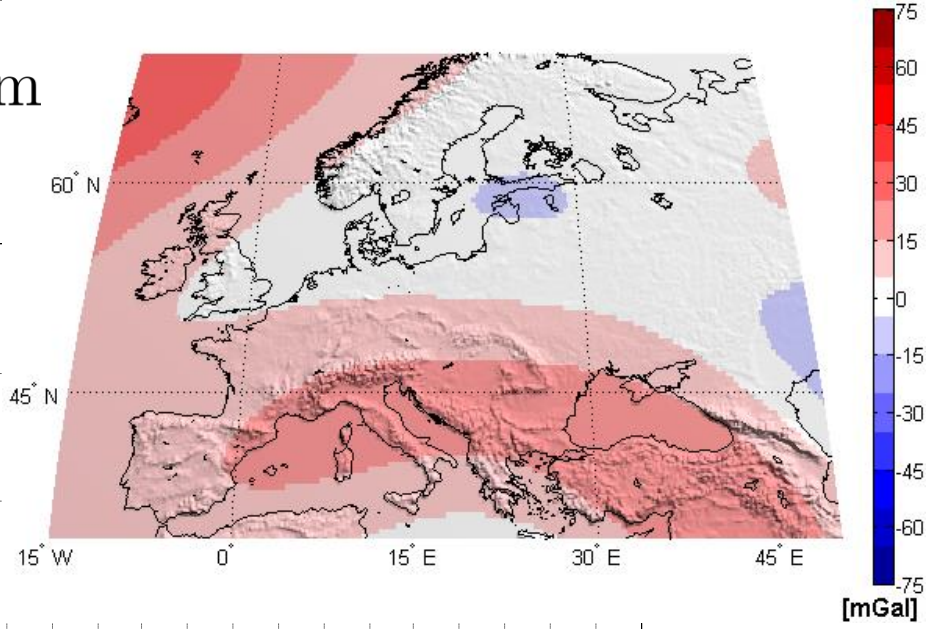




Spatial resolution

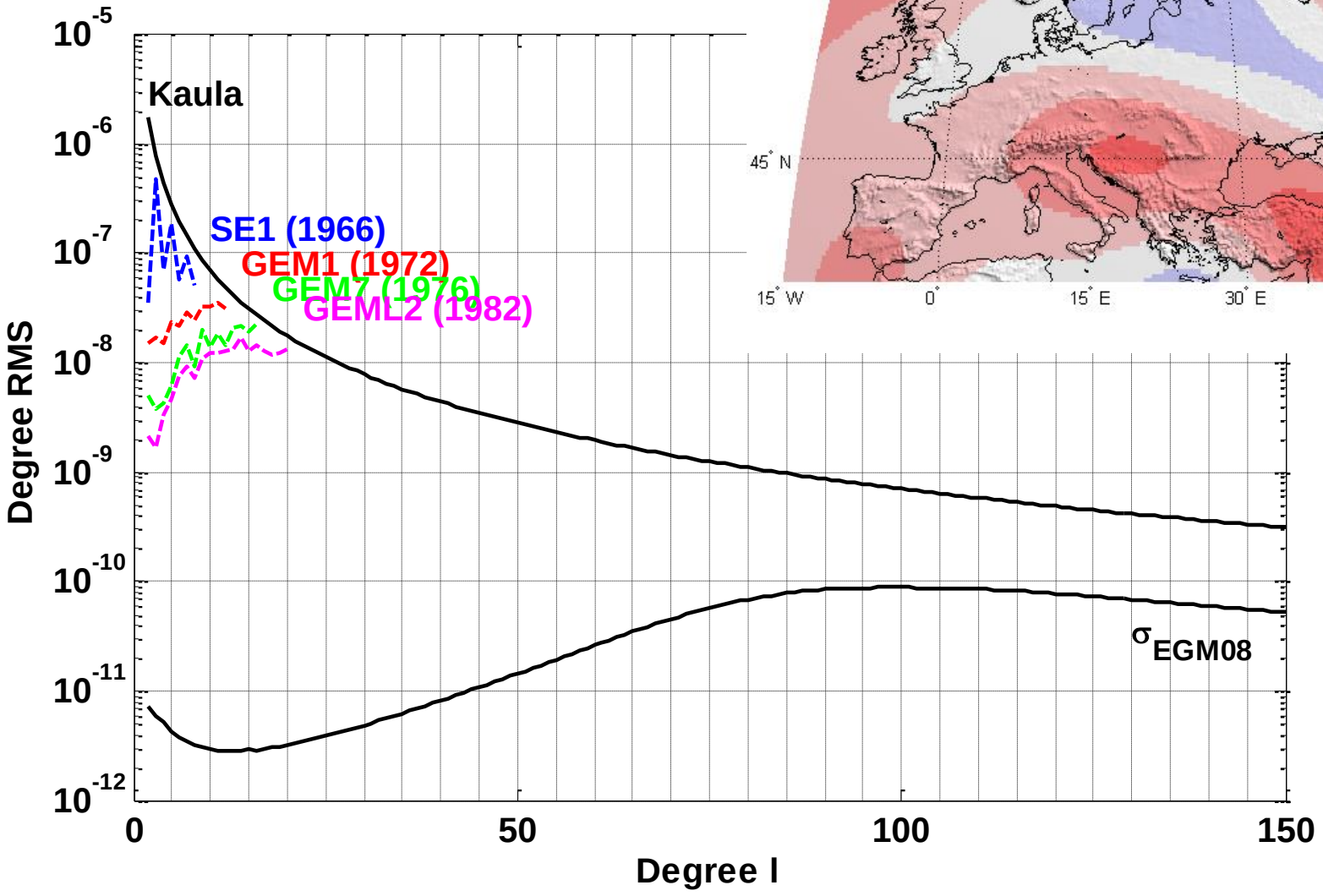
1976: $L_{Max} = 16 \quad \frac{\lambda}{2} = 1250\,\mathrm{km}$

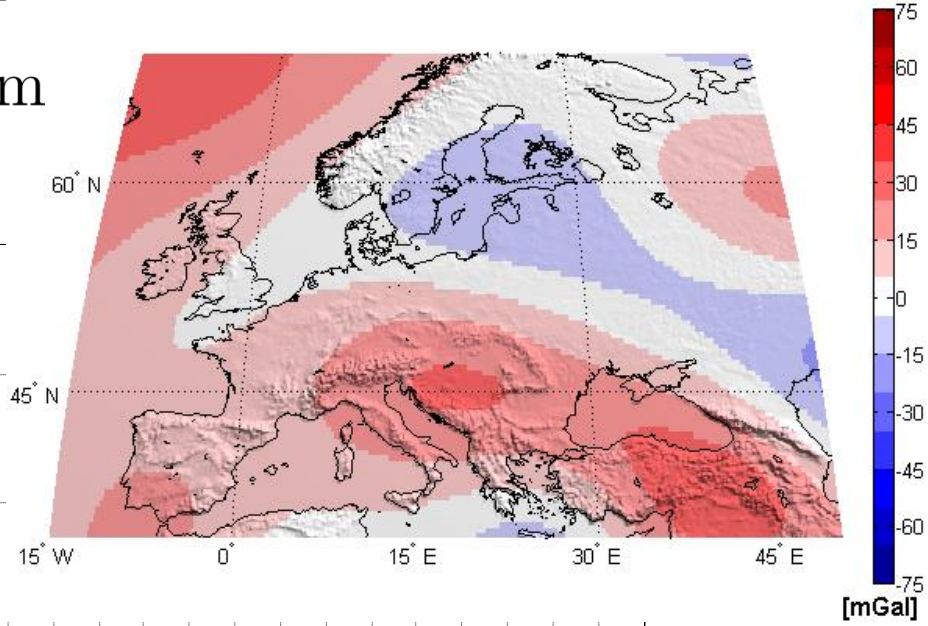




Spatial resolution

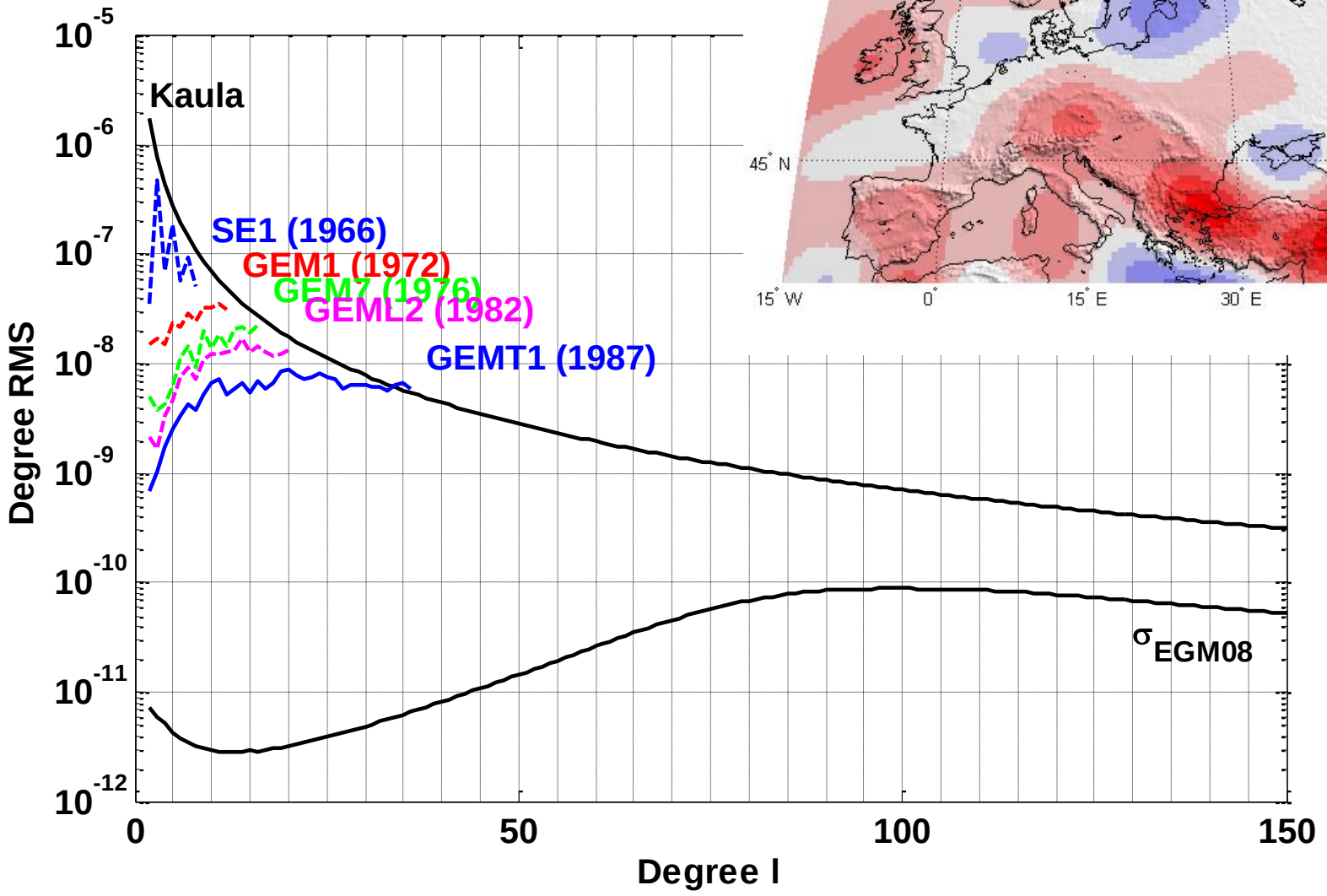
1982: $L_{Max} = 20 \quad \frac{\lambda}{2} = 1000\,\mathrm{km}$





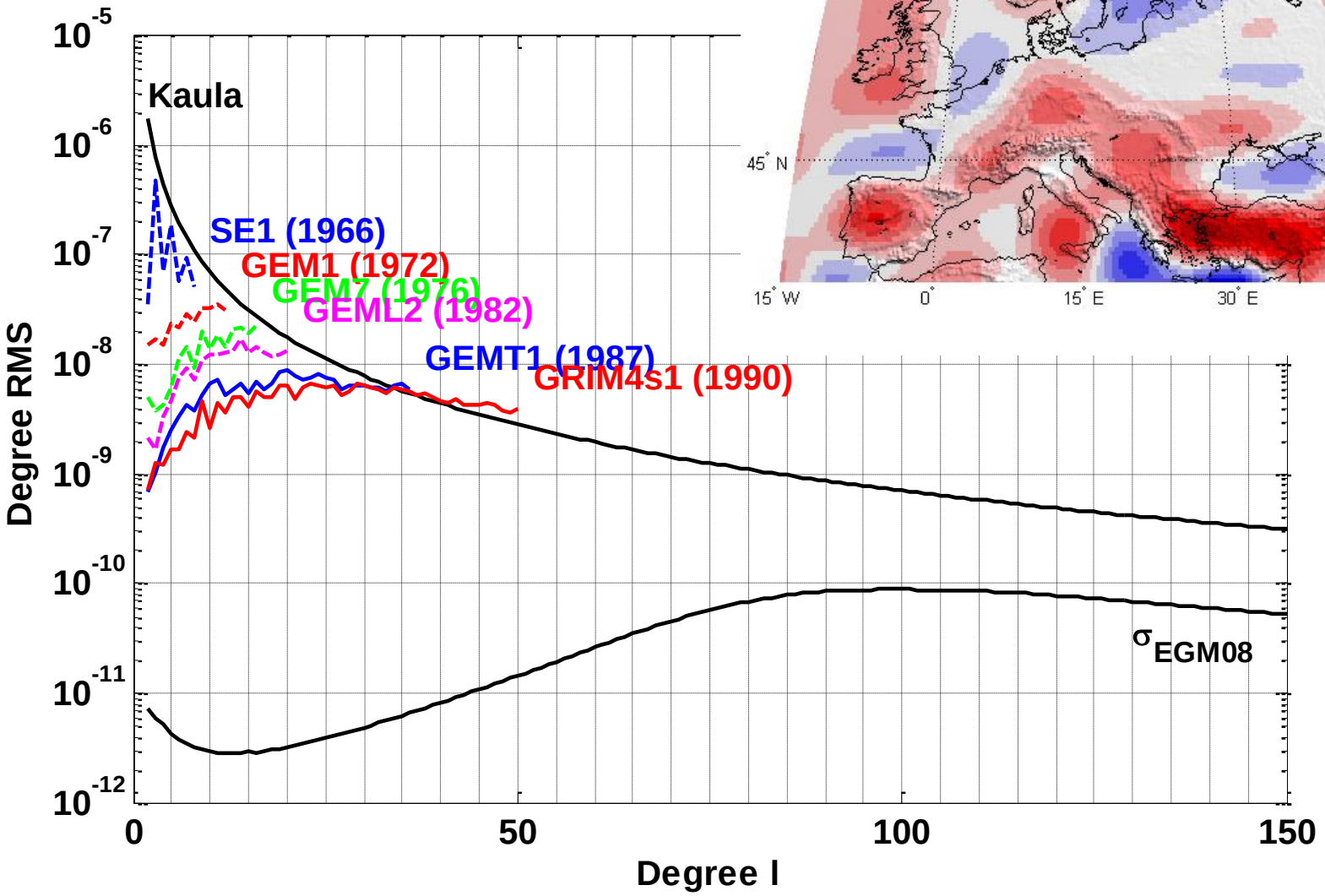
Spatial resolution

1987: $L_{Max} = 36 \quad \frac{\lambda}{2} = 555\,\mathrm{km}$



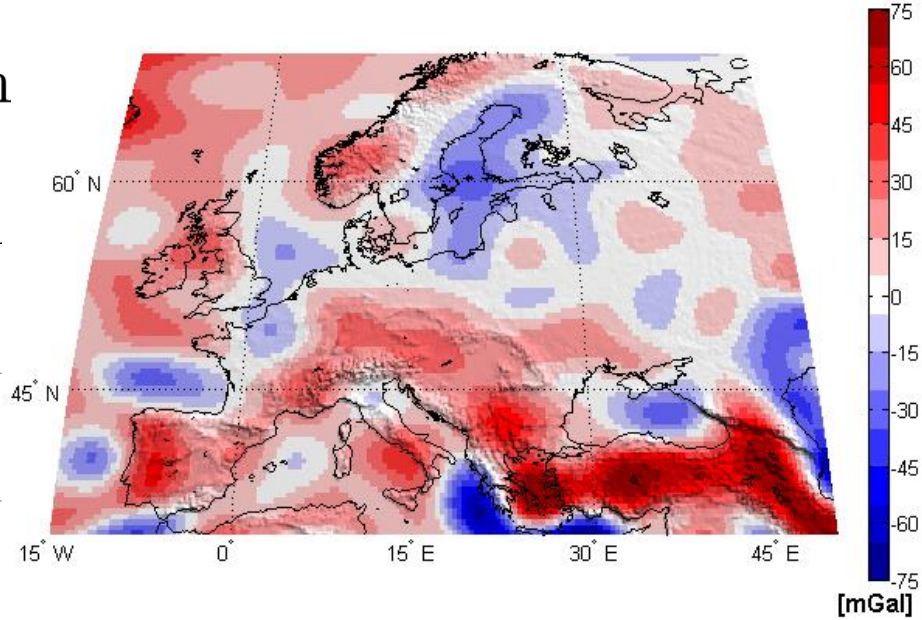
Spatial resolution

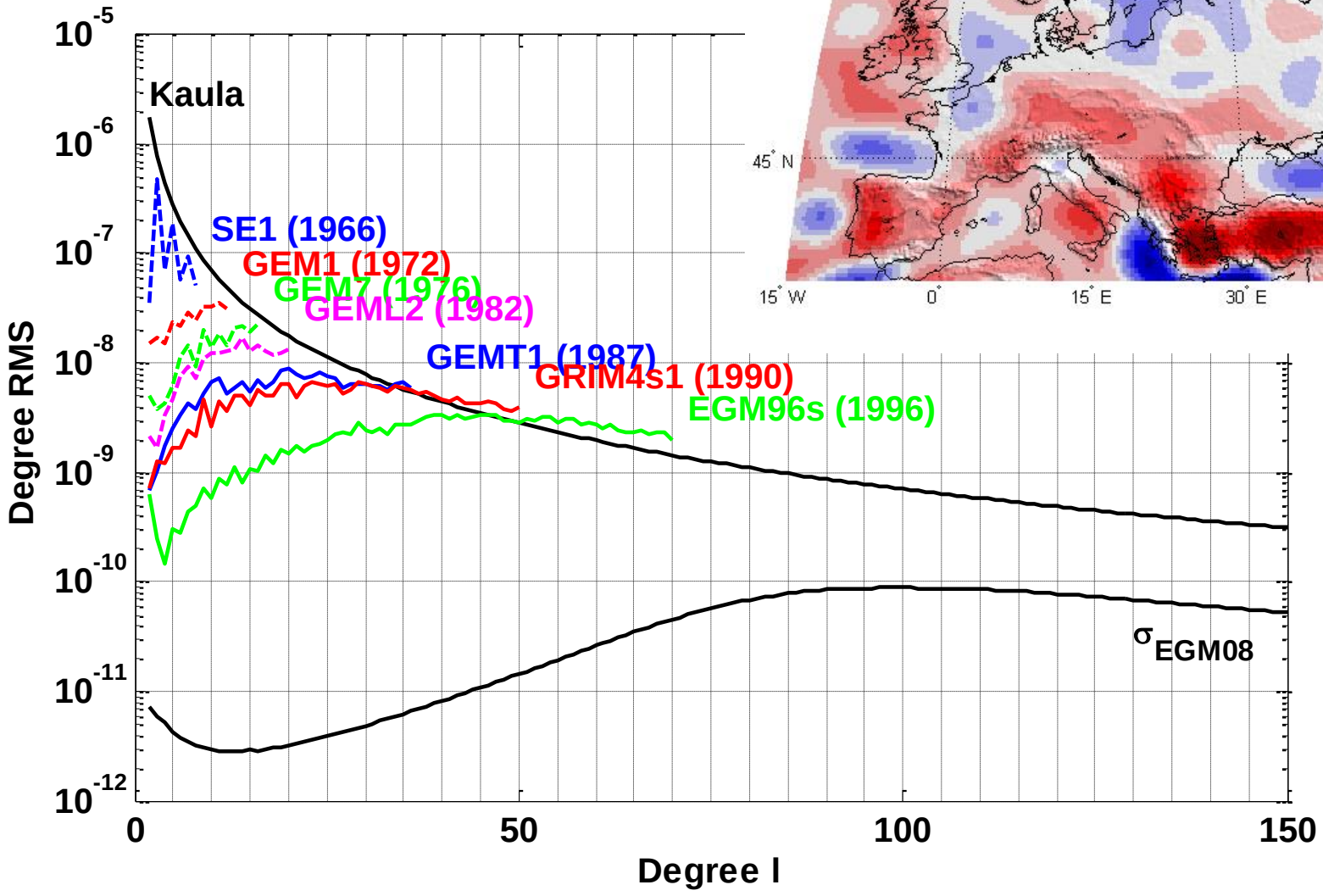
1990: $L_{Max} = 50 \quad \frac{\lambda}{2} = 400\,\mathrm{km}$



Spatial resolution

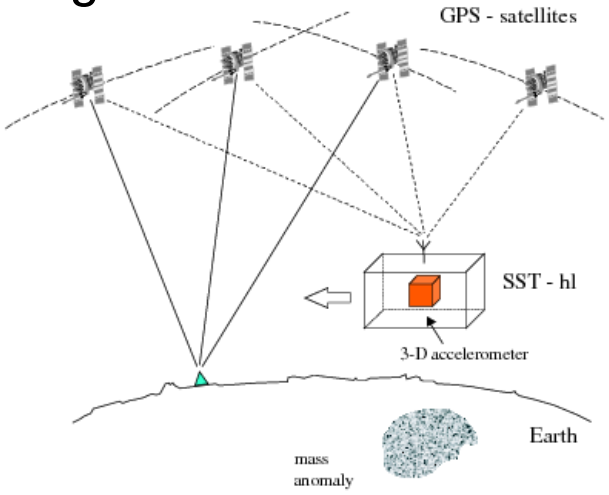
1996: $L_{Max} = 70 \quad \frac{\lambda}{2} = 285\,\mathrm{km}$





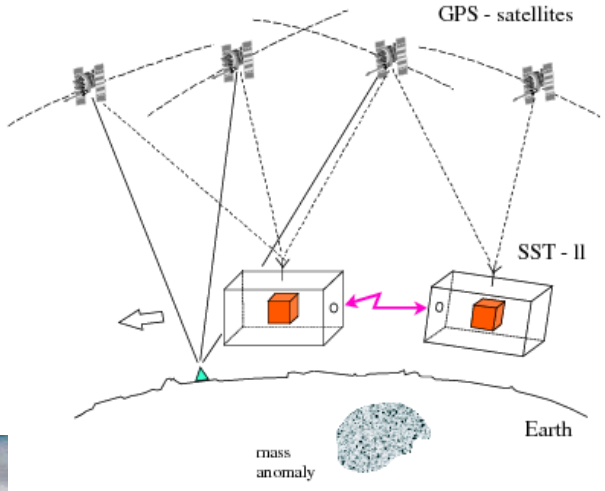
Satellite missions

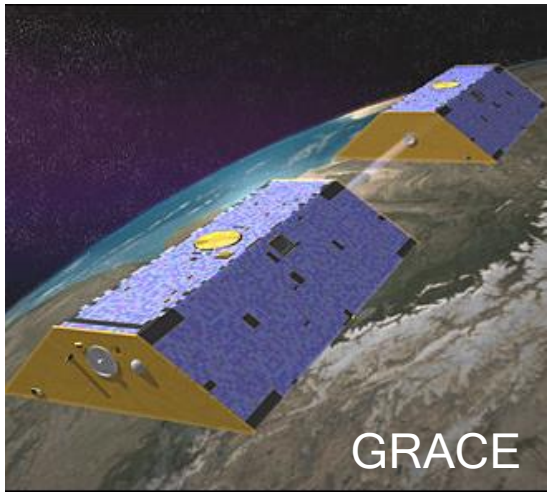
High-low SST



© GFZ-Potsdam

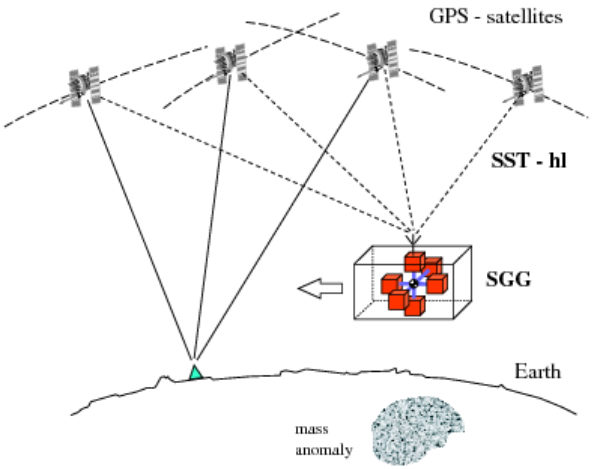
Low-low SST

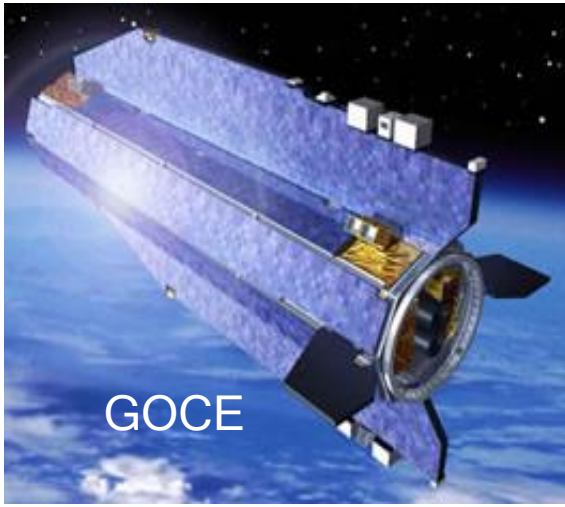




© CSR Texas

Gradiometry

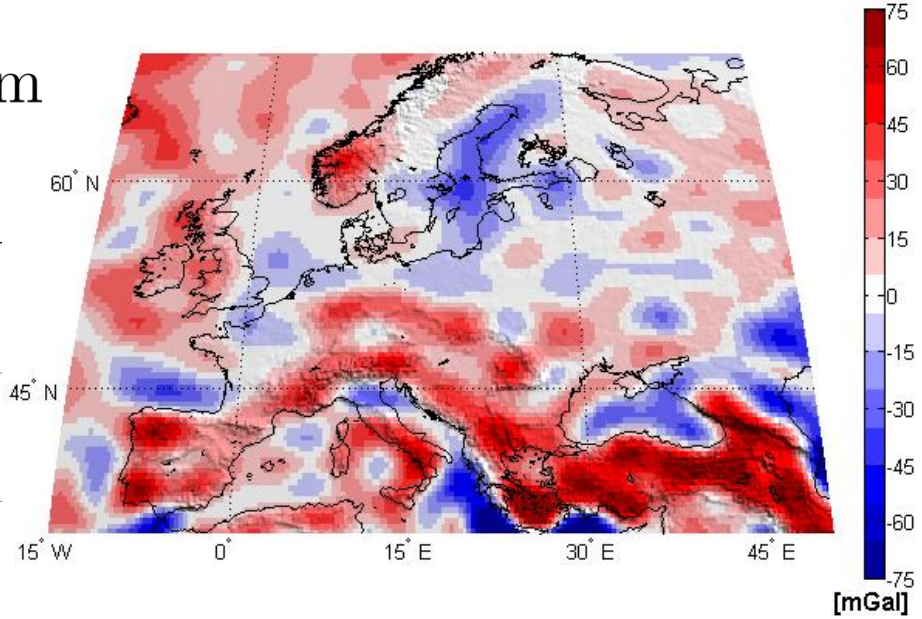


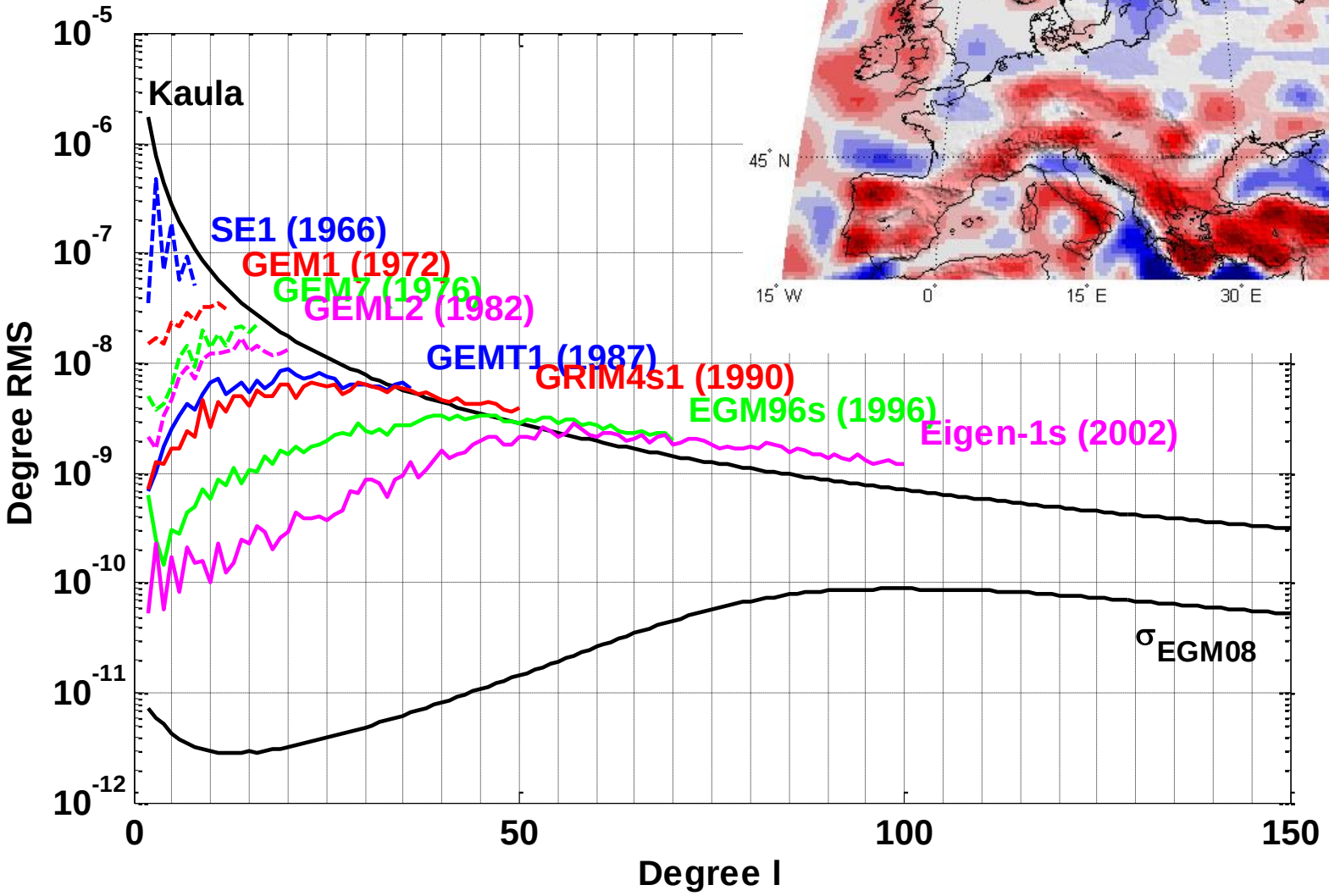


© ESA

Spatial resolution

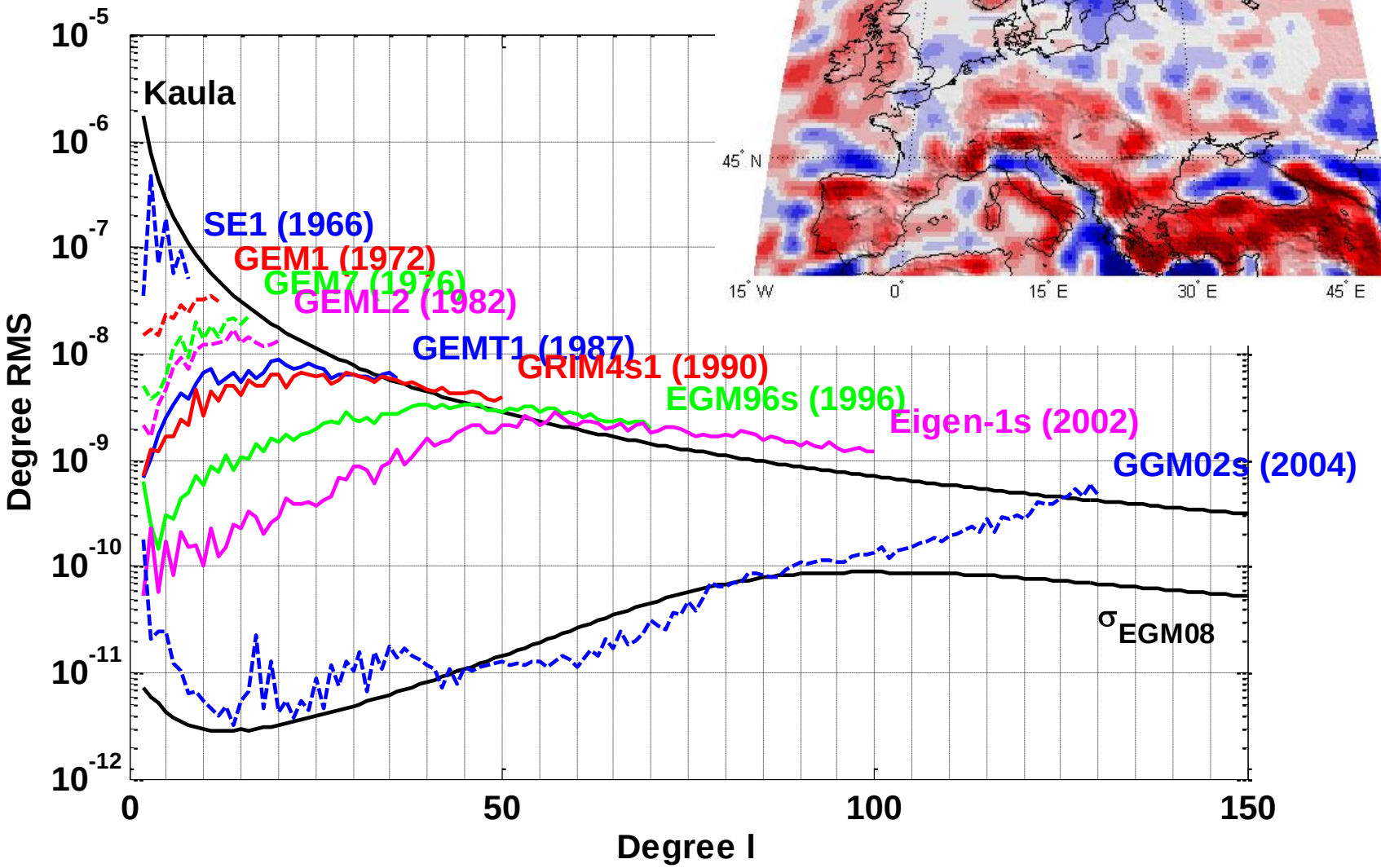
2002: $L_{Max} = 100 \quad \frac{\lambda}{2} = 200\,\mathrm{km}$



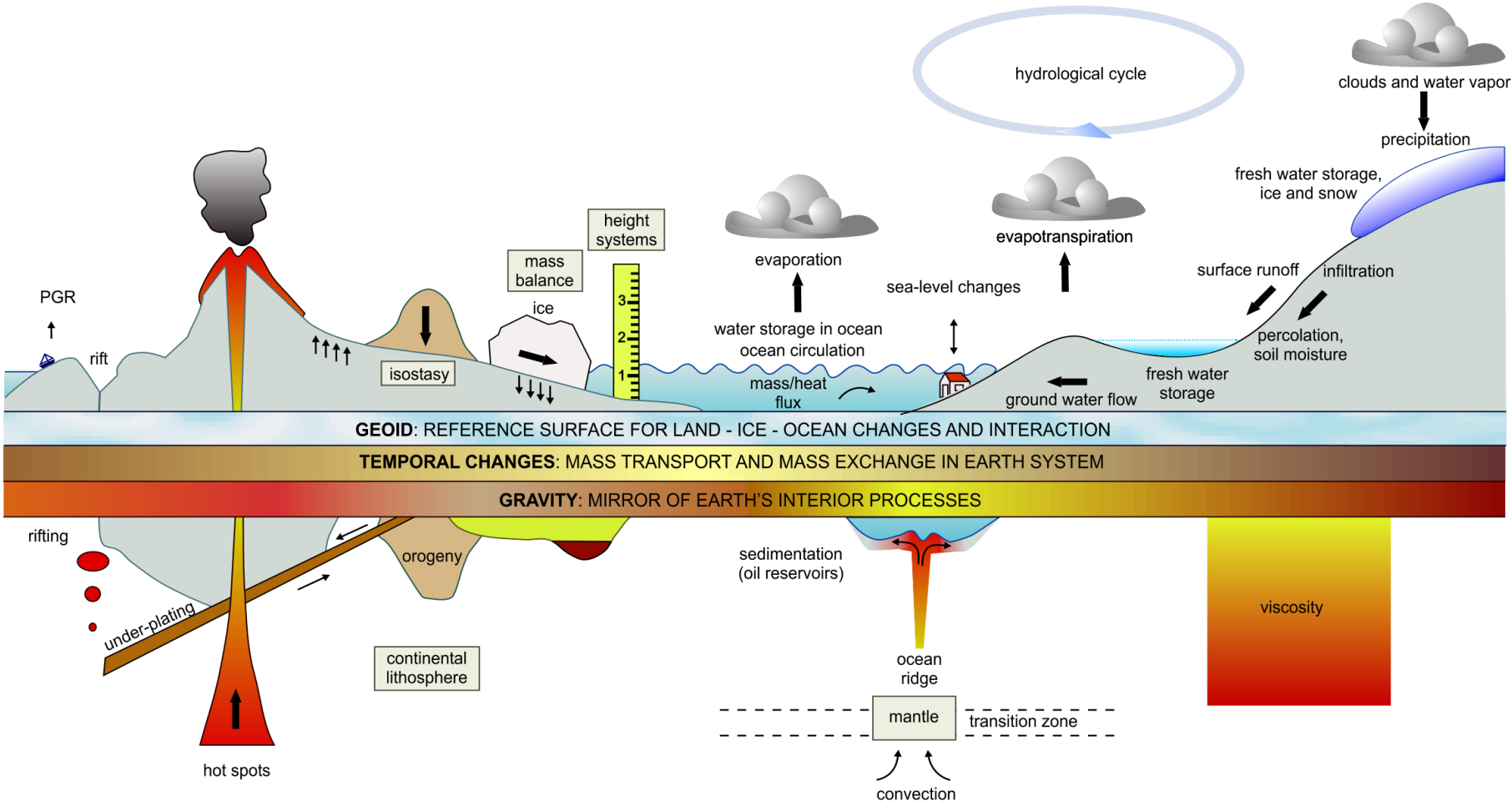


Spatial resolution

2004: $L_{Max} = 130 \quad \frac{\lambda}{2} = 155\,\mathrm{km}$



Geophysical implications of the gravity field



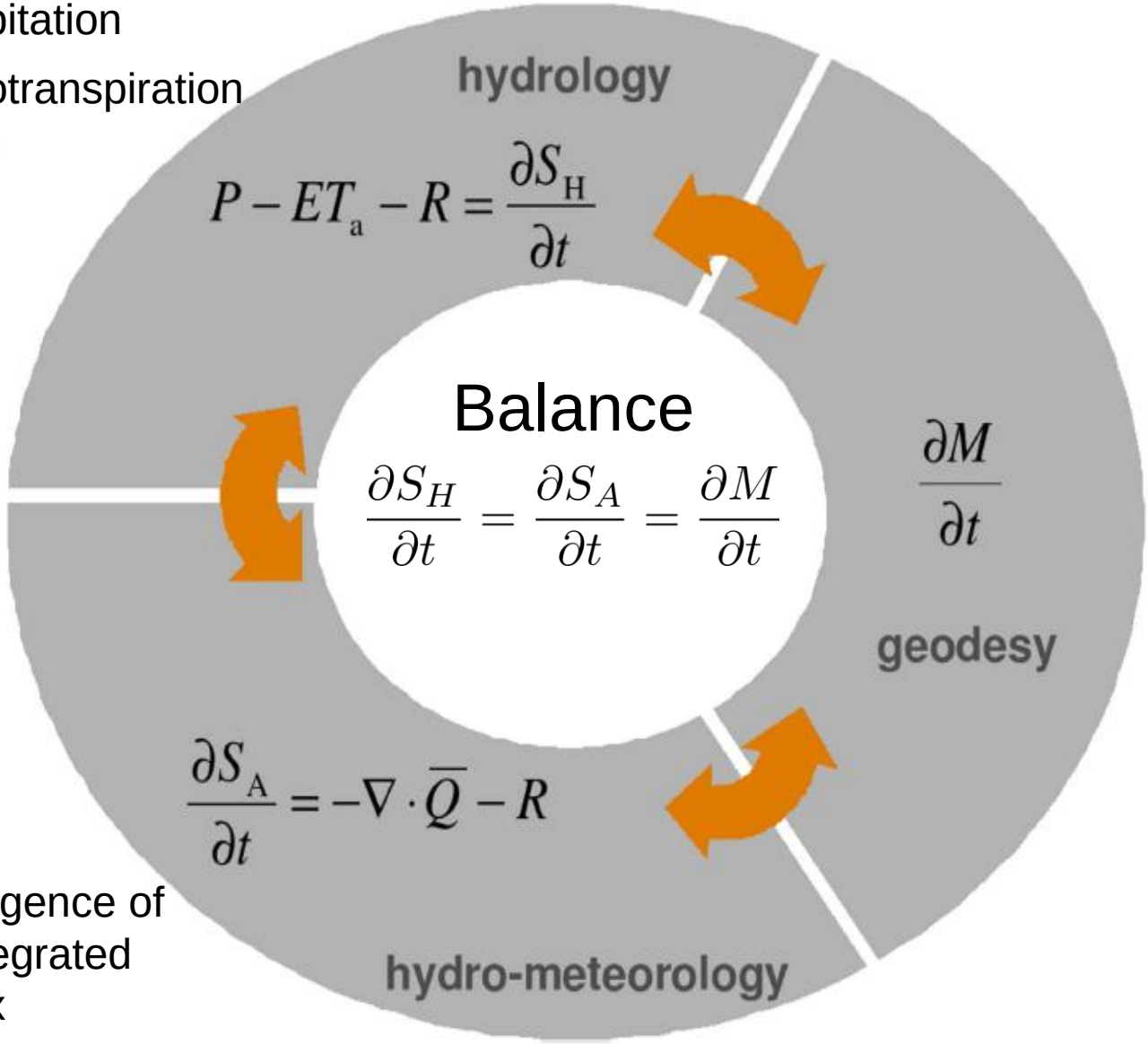
Mass change as a hydrological observable

Direct Water Balance - Project

P = precipitation

ET_a = evapotranspiration

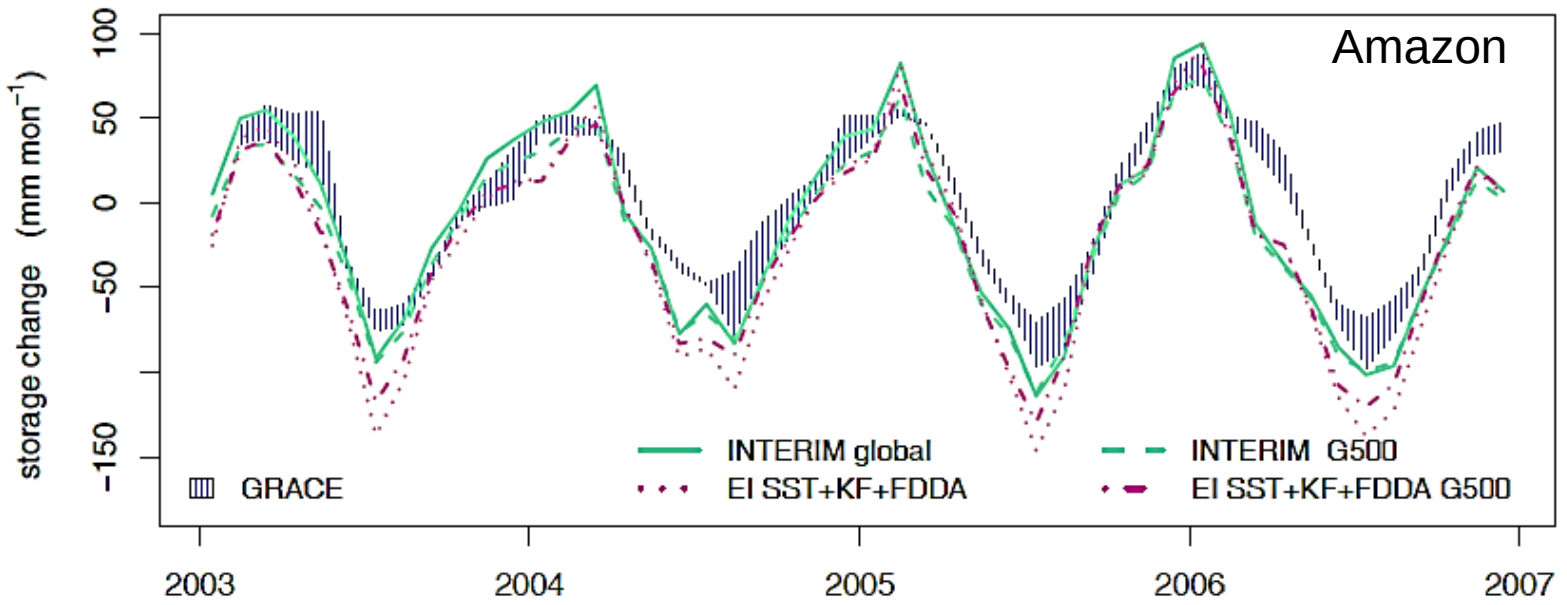
R = runoff



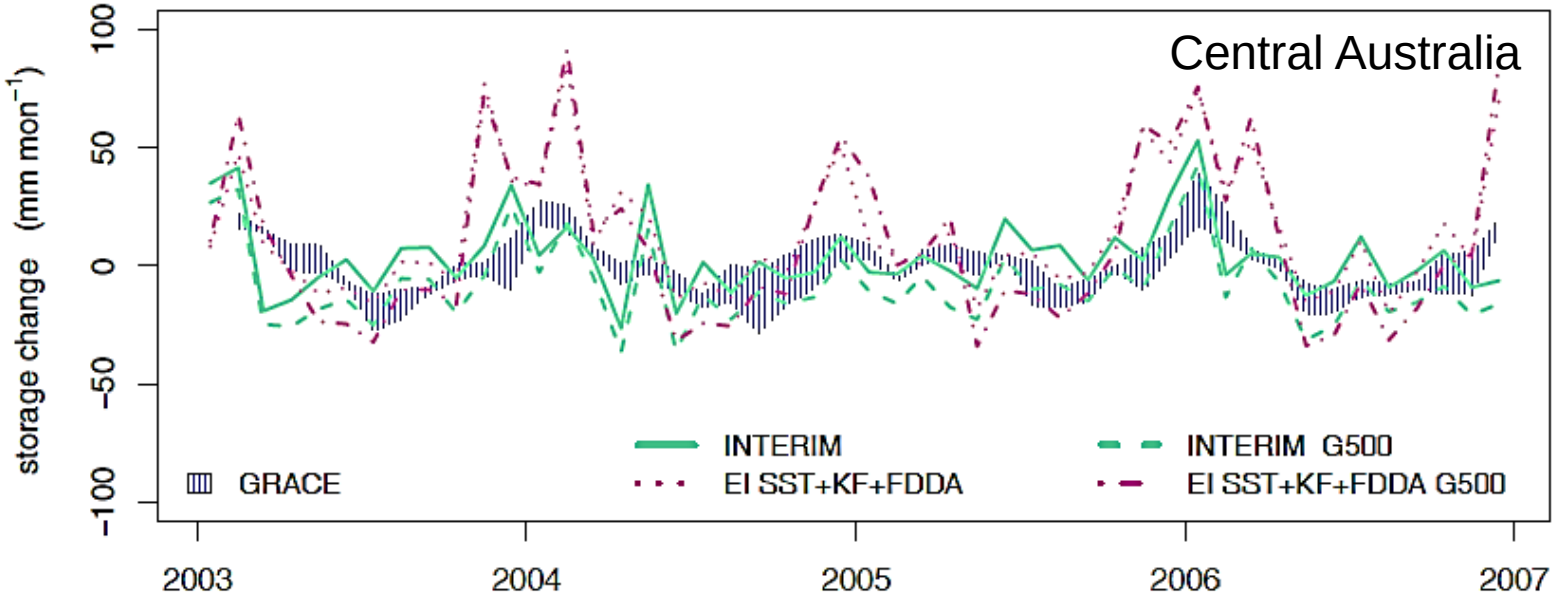
GRACE

$\nabla \cdot \bar{Q}$ = divergence of vertically integrated moisture flux

Examples of the comparisons



Fersch, KIT



Fersch, KIT

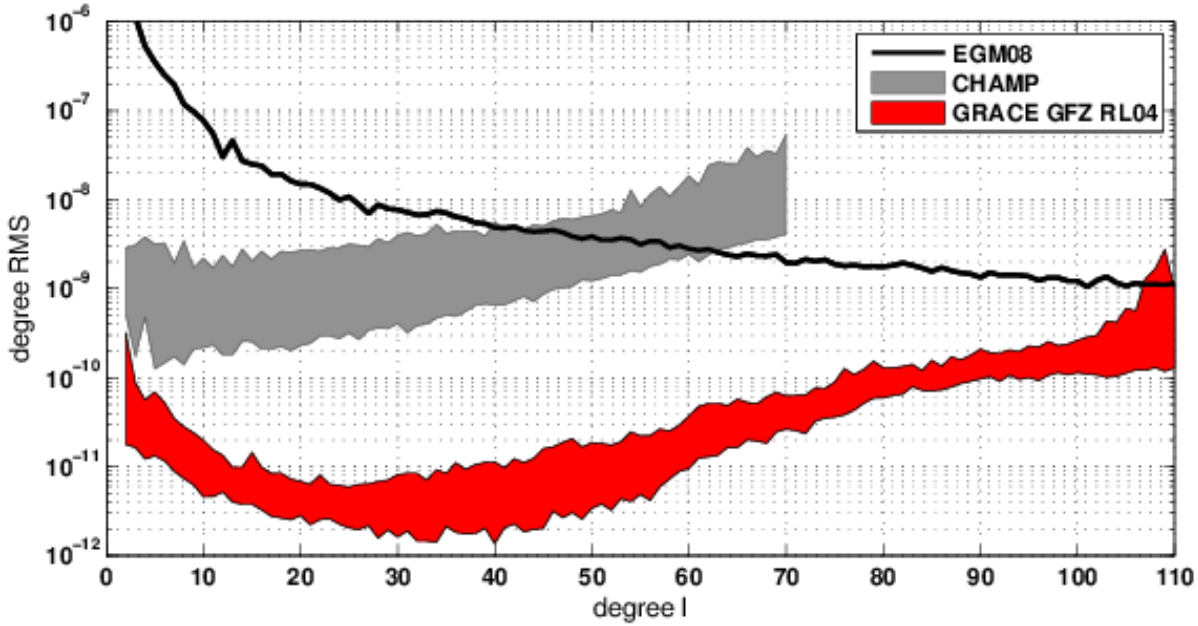
Some aspects in detail:

- CHAMP: time variability
- Spatial aliasing
- GRACE and GRACE Follow-on

PART I: CHAMP

Motivation

- Did we make use of the full potential of the CHAMP data?
- Single satellite scenario: sampling investigations
- Is the time variable gravity field really out of reach?



- What do we learn for future satellite missions?
- Can high-low SST serve as a transitory technology?

METHODOLOGY

Approach

- Acceleration approach:

$$\nabla V = \ddot{\vec{x}} - \vec{f}_{\mathrm{3rdBody}} - \vec{f}_{\mathrm{Tides}} - \vec{f}_{\mathrm{Rel}} - \vec{f}_{\mathrm{Grav}}$$

  - accelerations by numerical differentiation
  - three dimensional (pseudo-) observations
  - evaluation in the LNOF (local North-oriented frame)

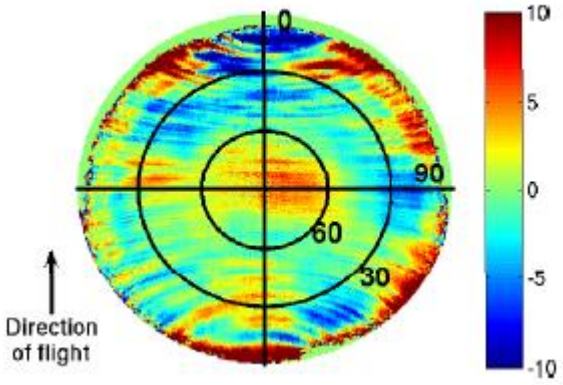
- Least-squares adjustment: $\vec{l} = A\vec{x} - \vec{\epsilon} \quad \Sigma_{ll} = \sigma_0^2 Q_{ll}$

- Covariance information:
  - unit matrix
  - epochwise covariance information (only correlations between coordinates)
  - empirical covariance information (only correlations between epochs)
  - scaled empirical covariance information (only correlations between epochs)
  - full error propagation (numerical stability ?)

Data sources

GPS positions: AIUB

- 10 s sampling
- estimated absolute antenna phase center model
- new IGS standards
- …



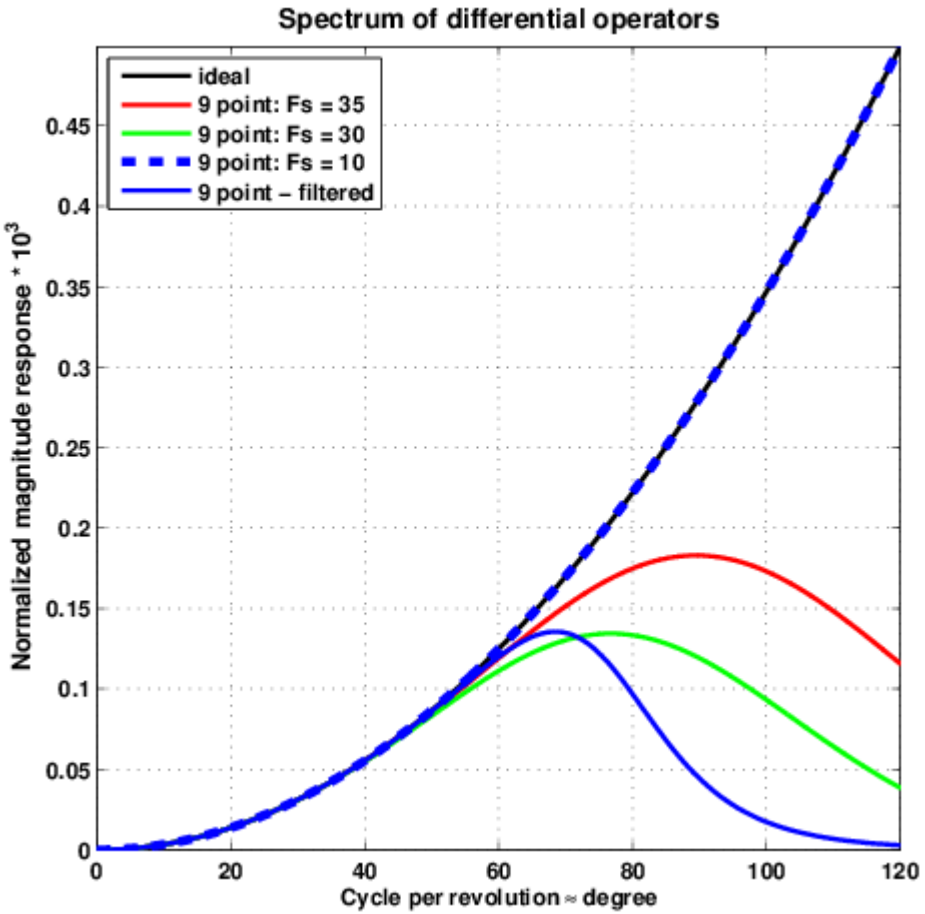
Prange 2010

Background models:

- JPL ephemeris DE405
- Solid Earth tides (IERS conventions)
- Solid Earth pole tides (IERS conventions)
- Ocean tides (FES 2004)
- Ocean pole tides (IERS conventions, Desai 2002)
- Atmospheric tides (N1-model, Biancale and Bode 2006)
- Relativistic corrections (IERS conventions)
- AOD1B-product (Flechtner 2008)

No accelerometer data needed!

Differentiation



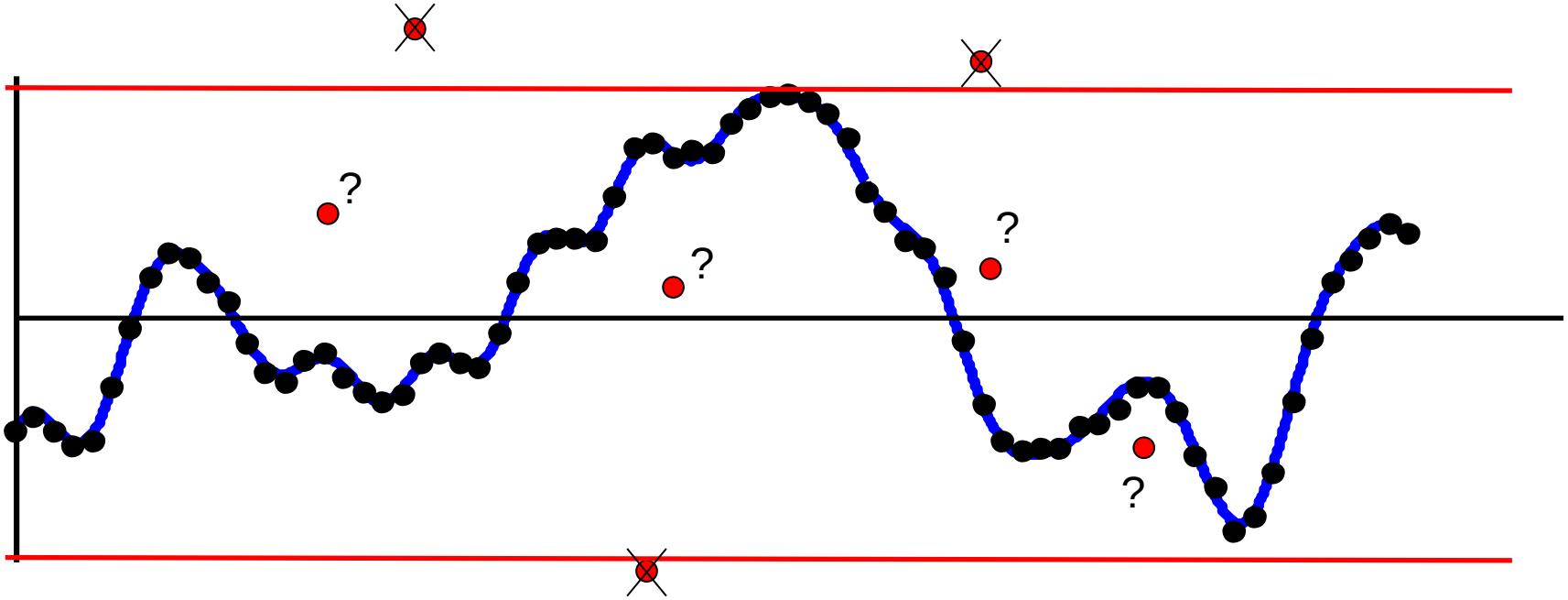
- Ideal differentiator: $|H(\omega)| = \omega^2$
- Numerical approximation by n-point central difference differentiator

$$\frac{\partial^j f_i}{\partial t^j} = \frac{1}{T} \sum_{k=-(n-1)/2}^{(n-1)/2} g_k^{(j)} f_{k+i}$$

- Filtering of high frequency noise
  - by lowpass filtering (warmup!)
  - inherently by the differentiator (varying the step size)

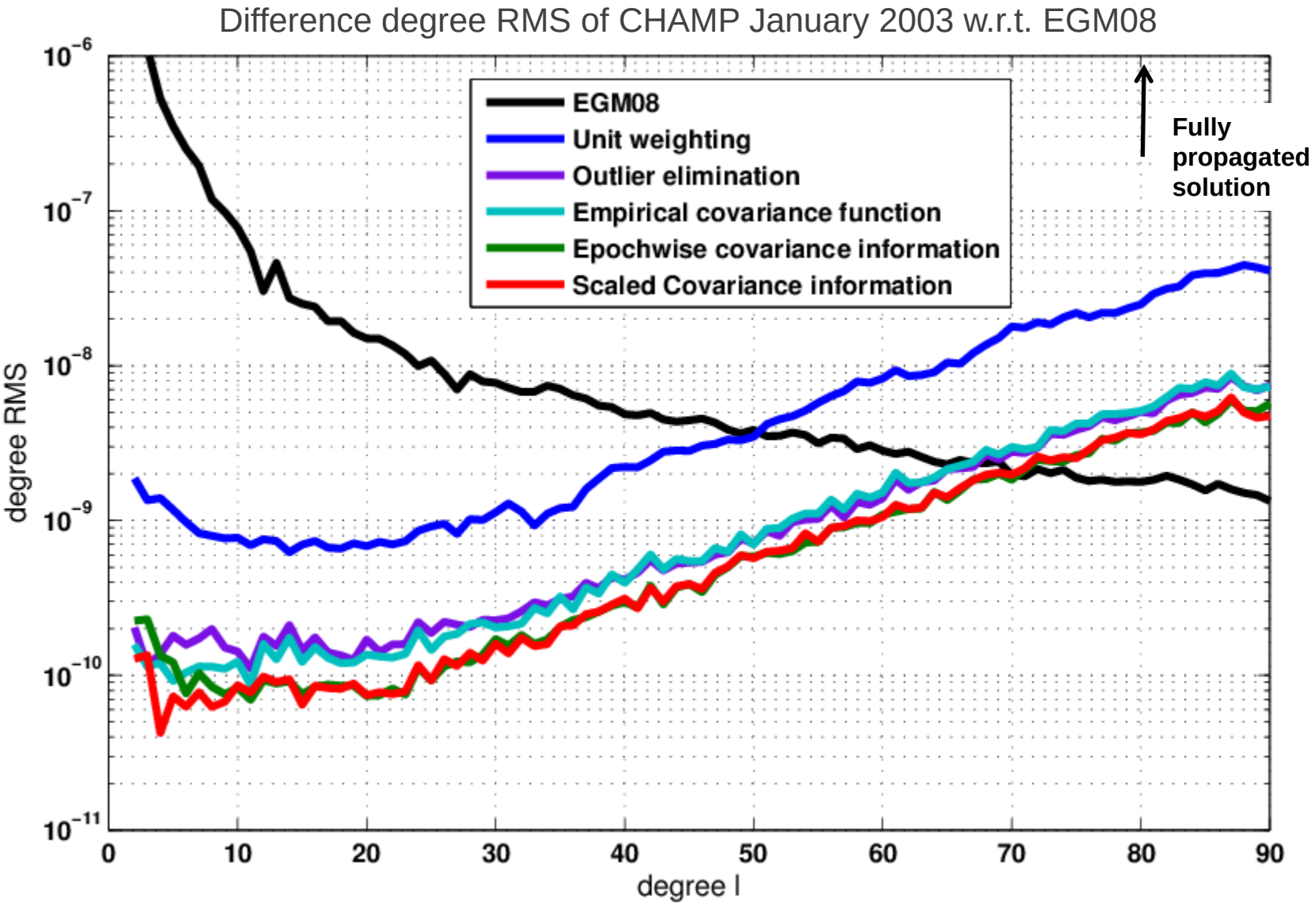
Outliers vs. poor observations

- Typically threshold based outlier detection based on residuals:



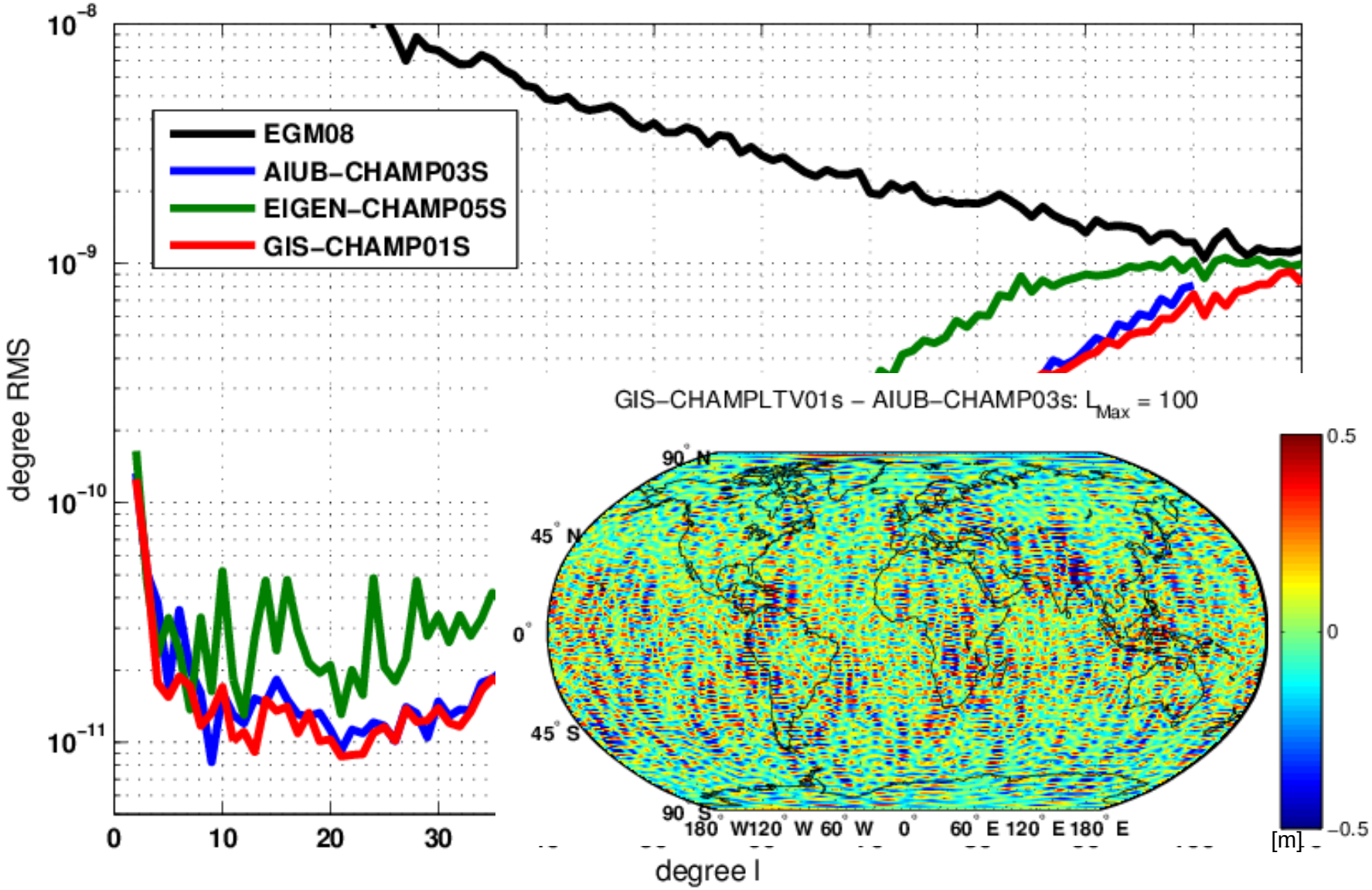
- Localizing outlier detection: moving windows (time and space domain)

Influence of outlier detection and covariance information



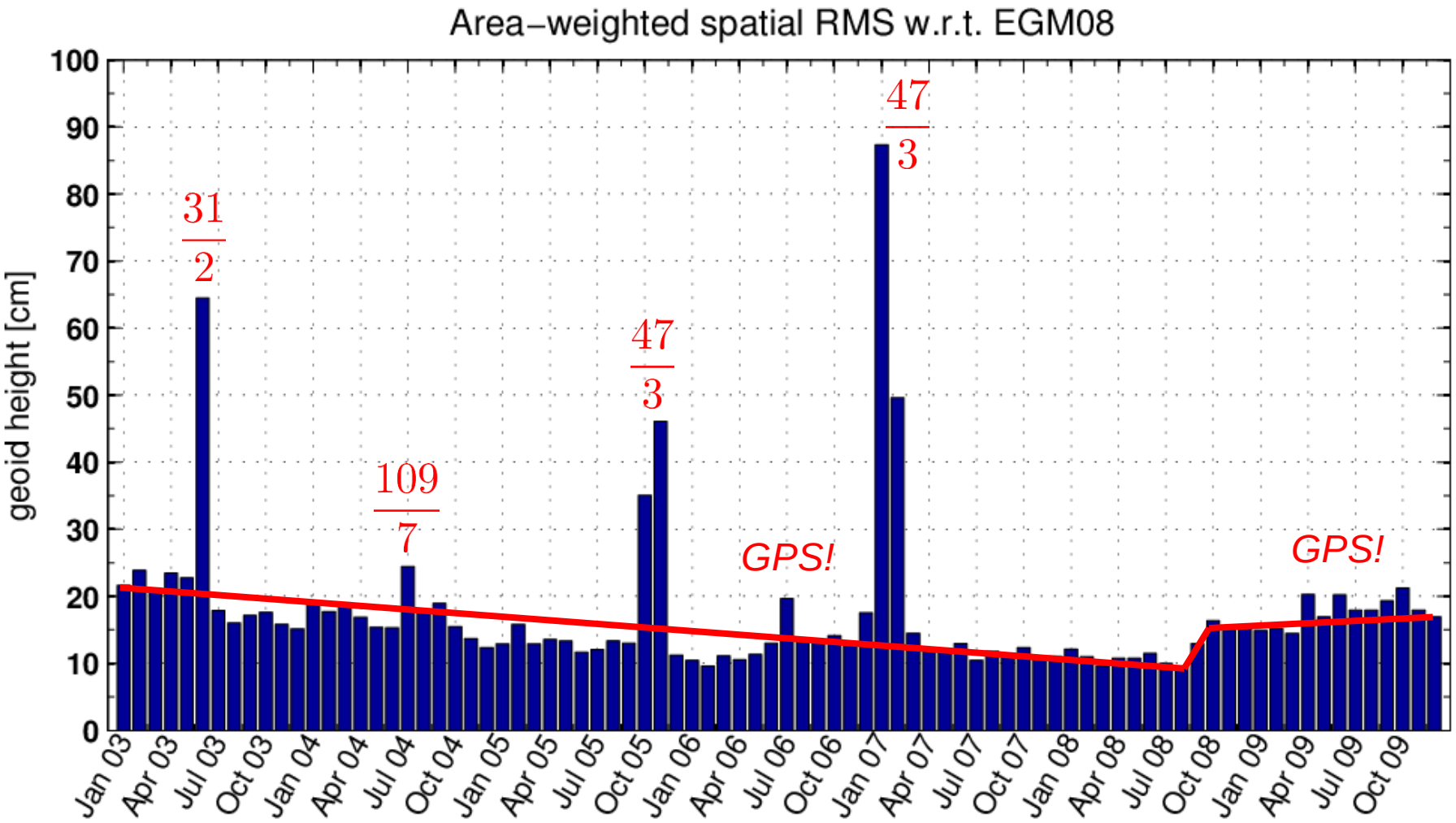
STATIC SOLUTION

CHAMP static gravity field

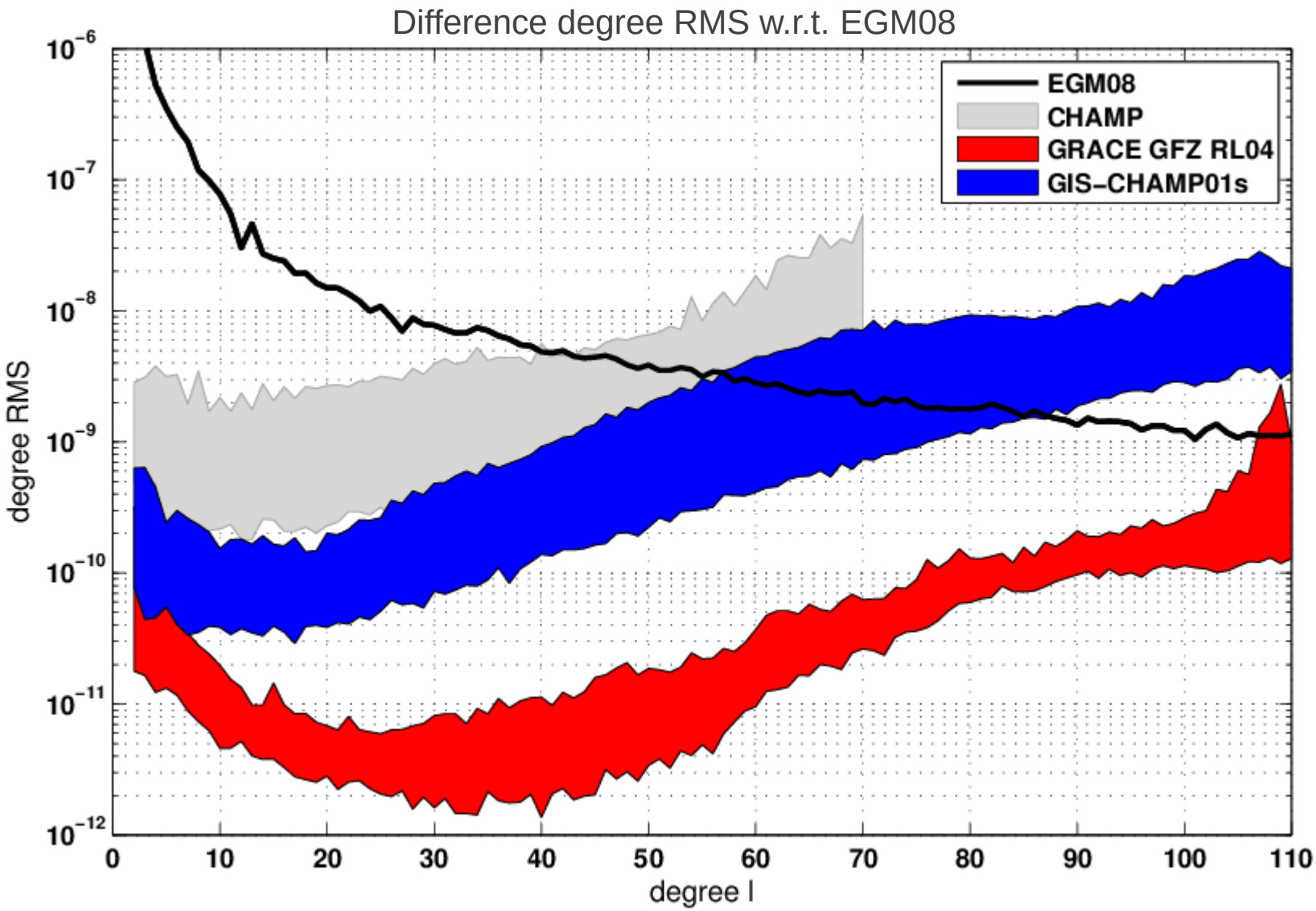


MONTHLY SOLUTIONS

CHAMP monthly gravity field solutions (1/4)

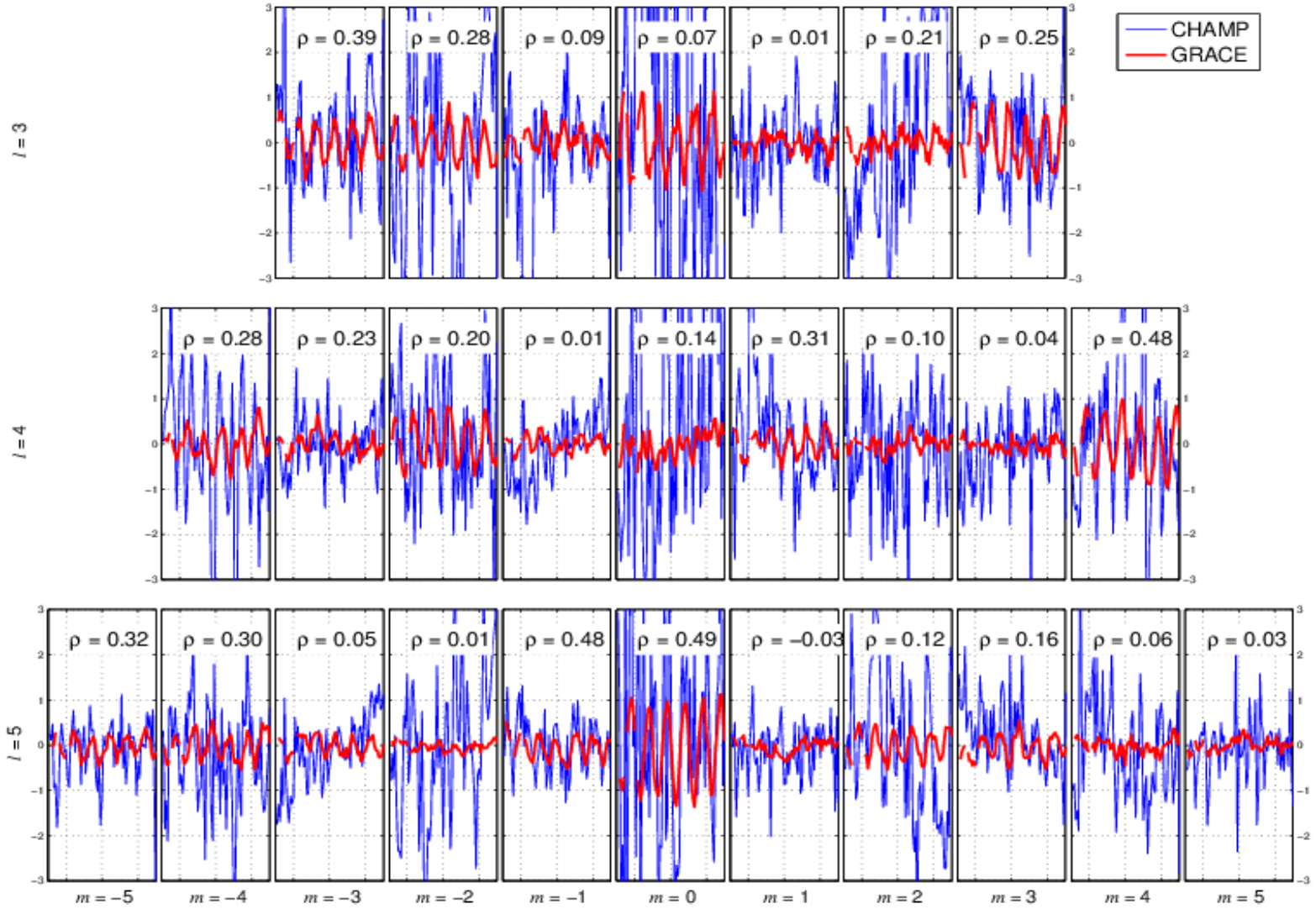


CHAMP monthly gravity field solutions (2/4)



CHAMP monthly gravity field solution (3/4)

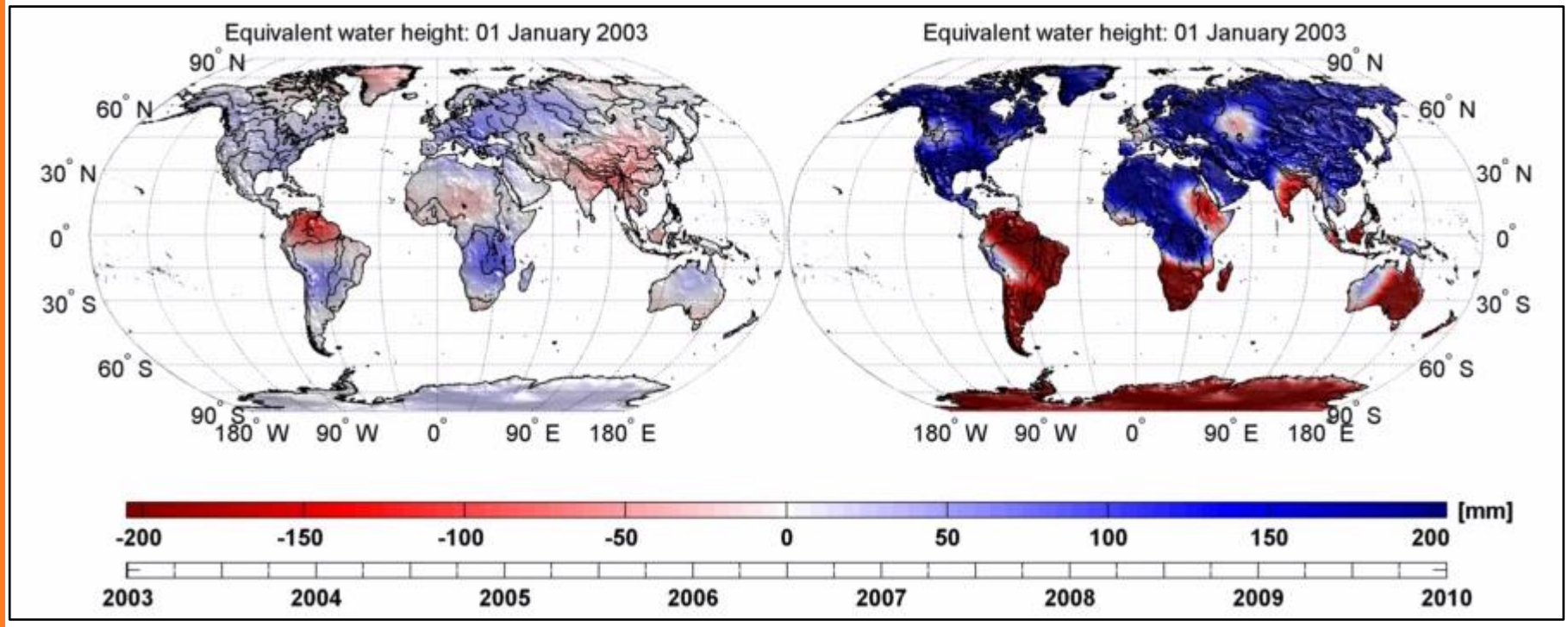
Time series of SH-coefficients: GRACE vs. CHAMP (annual) – scaled by 10^{10}



CHAMP monthly gravity field solutions (4/4)

GRACE GFZ Rel. 4

CHAMP



MODELING TIME VARIABILITY

Modeling time variability

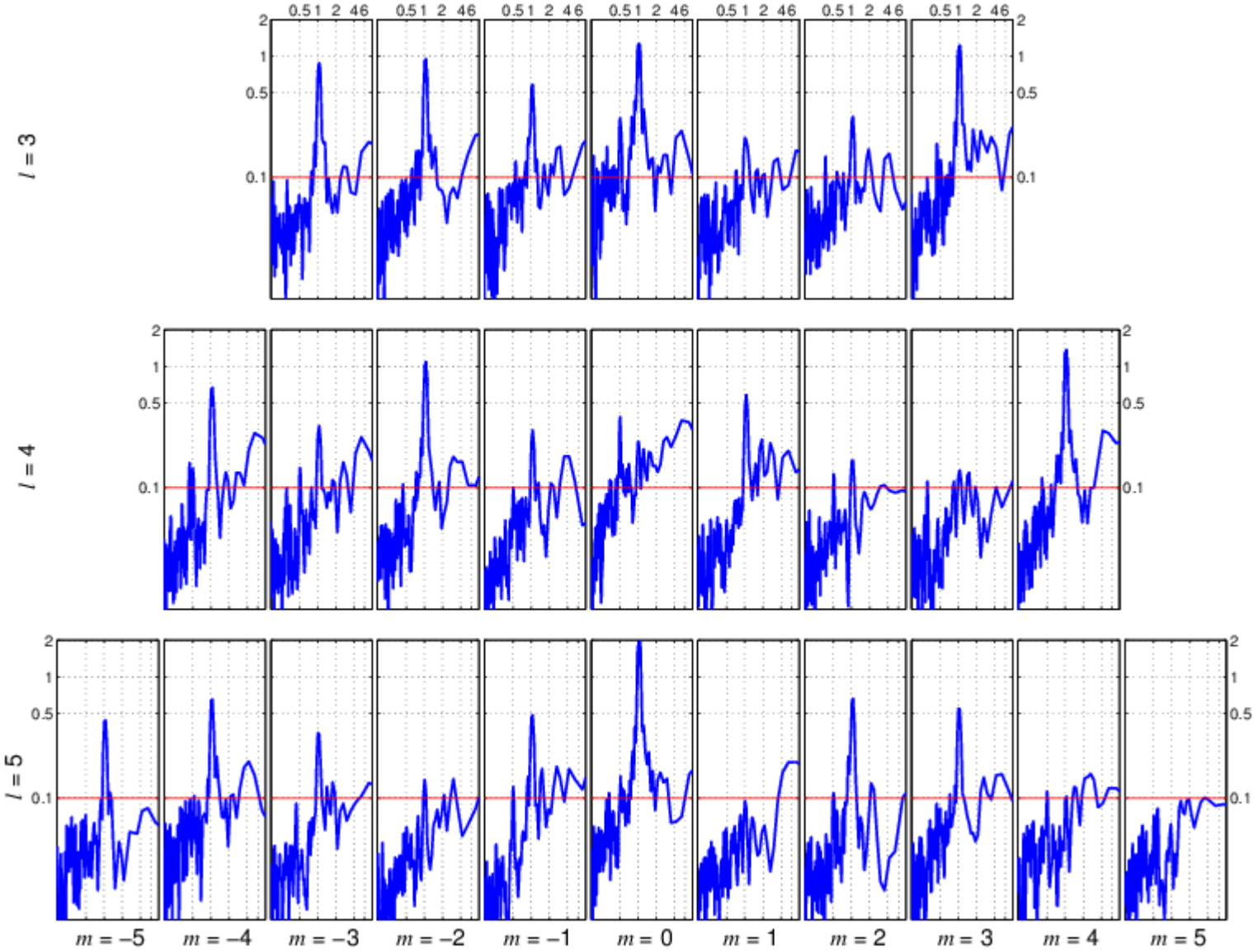
$$\bar{C}(t) = \bar{C}_0 + \dot{\bar{C}} \cdot t + \sum_{n=1}^{N} \bar{C}_n^{\cos} \cos(\omega_n t) + \bar{C}_n^{\sin} \sin(\omega_n t)$$

$$\bar{S}(t) = \bar{S}_0 + \dot{\bar{S}} \cdot t + \sum_{n=1}^{N} \bar{S}_n^{\cos} \cos(\omega_n t) + \bar{S}_n^{\sin} \sin(\omega_n t)$$

- Pro:
  - simple
  - considering spatial correlations between coefficients
- Con:
  - mean time-variable signal over the period of data
  - frequencies essentially unknown
  - variation of frequencies between coefficients
  - higher computational effort

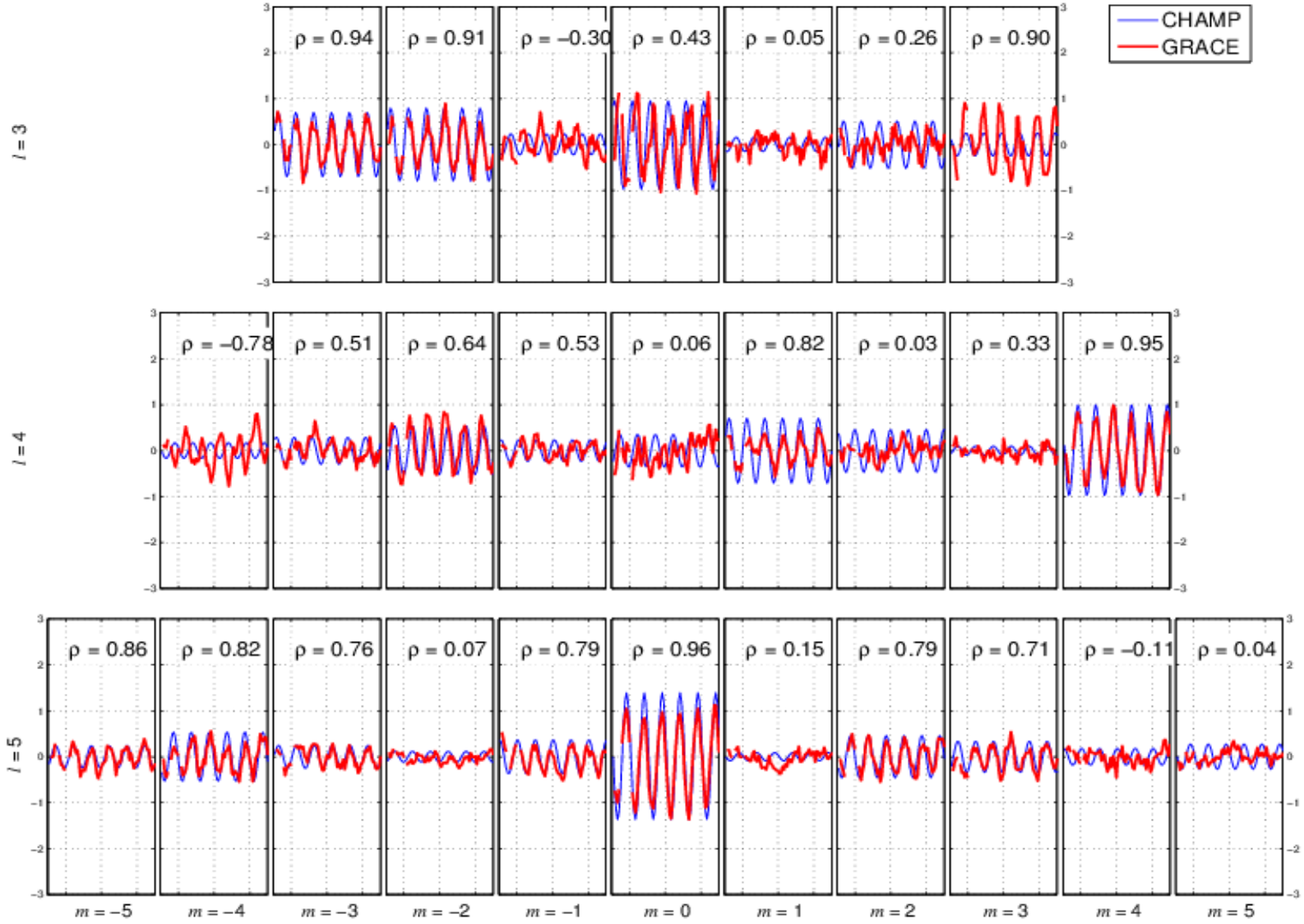
Modeling time variability

Spectrum of time series of SH-coefficients: GRACE GFZ RL 04 – scaled by 10^9



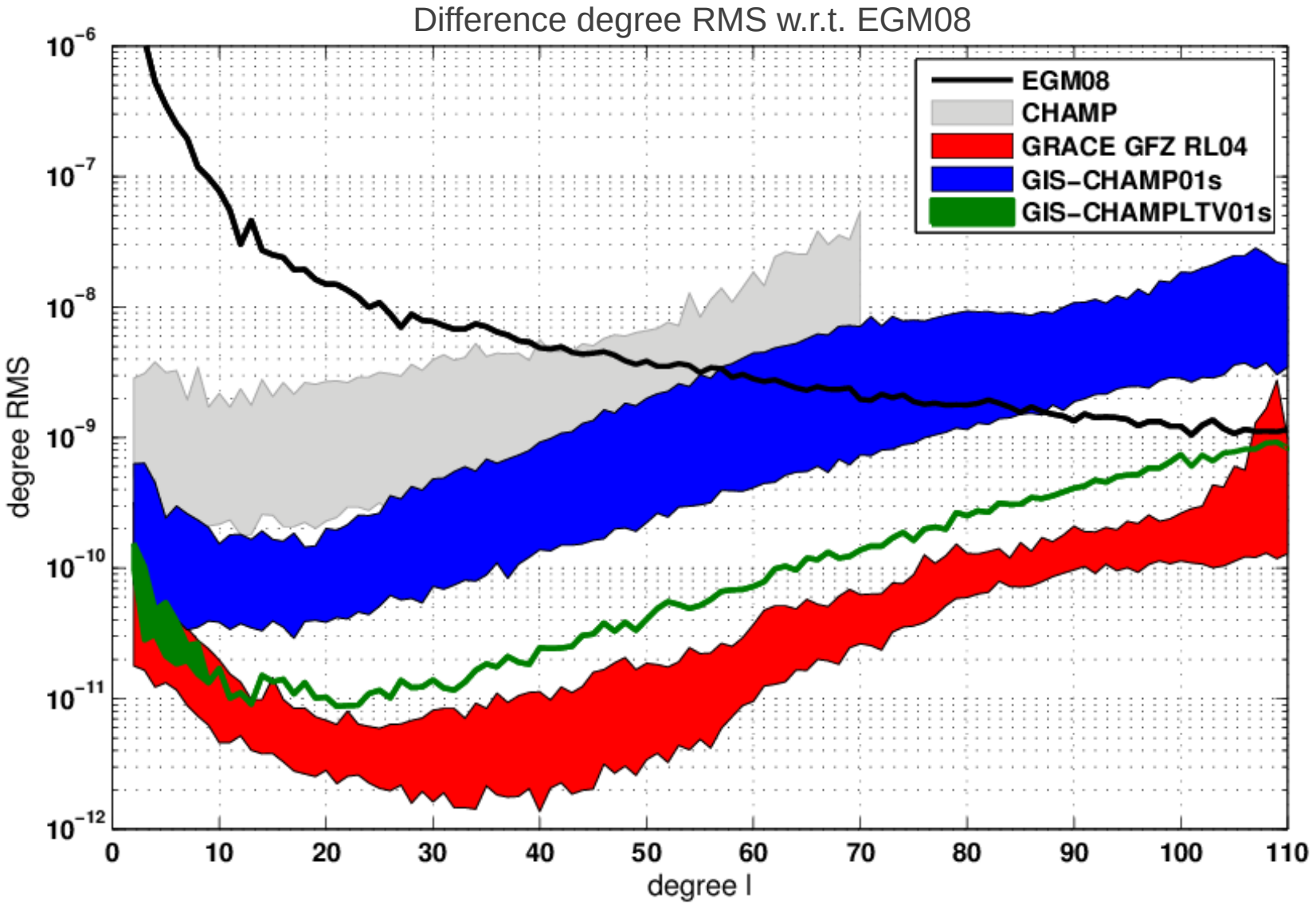
Modeled time-variable gravity field solution (1/4)

Time series of SH-coefficients: GRACE vs. CHAMP (annual) – scaled by 10^{10}



Modeled time-variable gravity field solution (2/4)

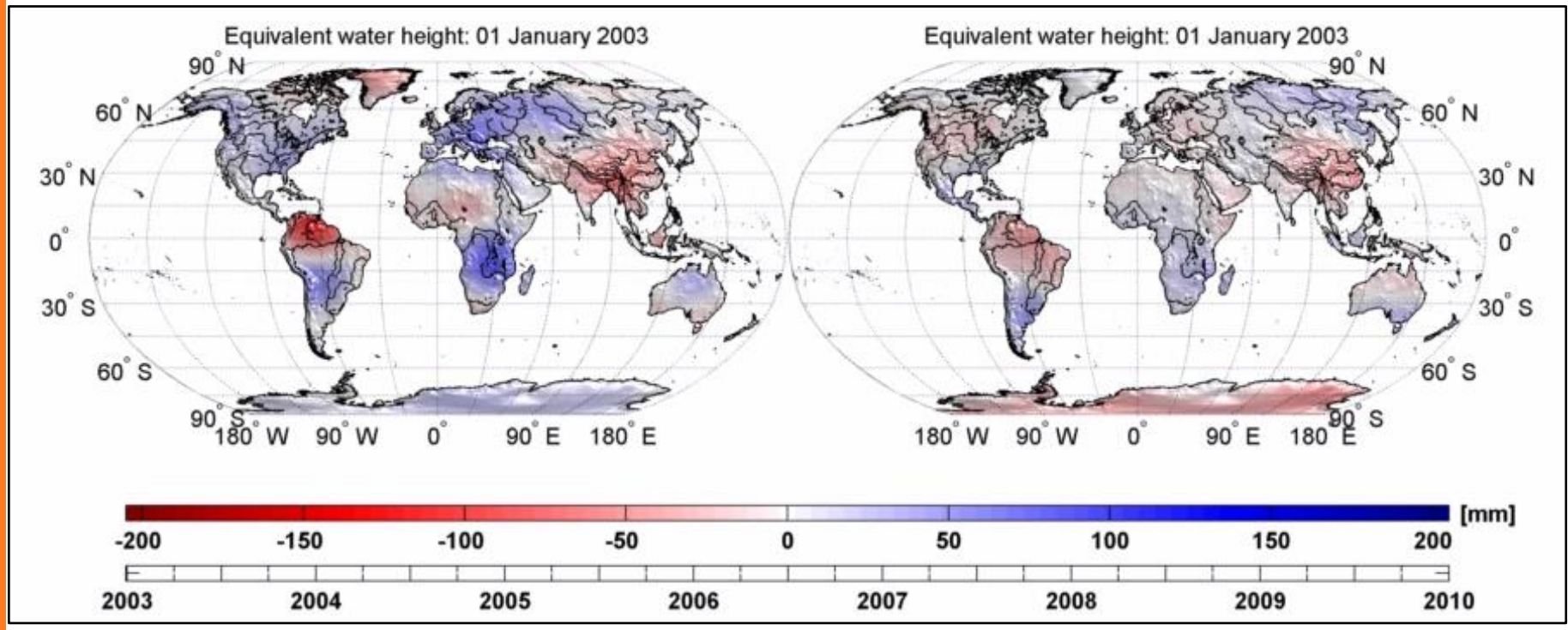
$$L_{Max} = 110$$
$$L_{Max}^{TV} = 8$$



Modeled time-variable gravity field solution (3/4)

GRACE GFZ Rel. 4

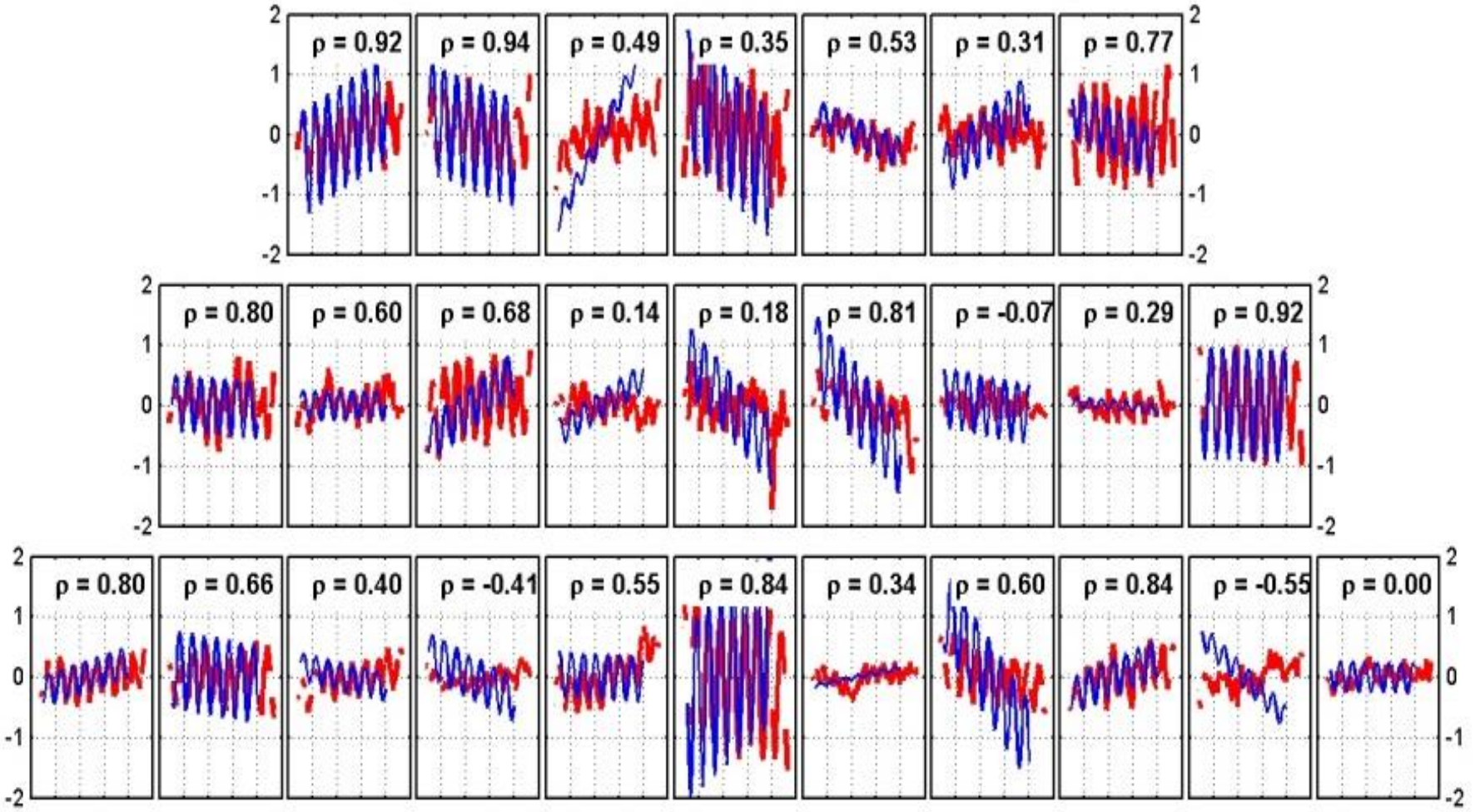
CHAMP



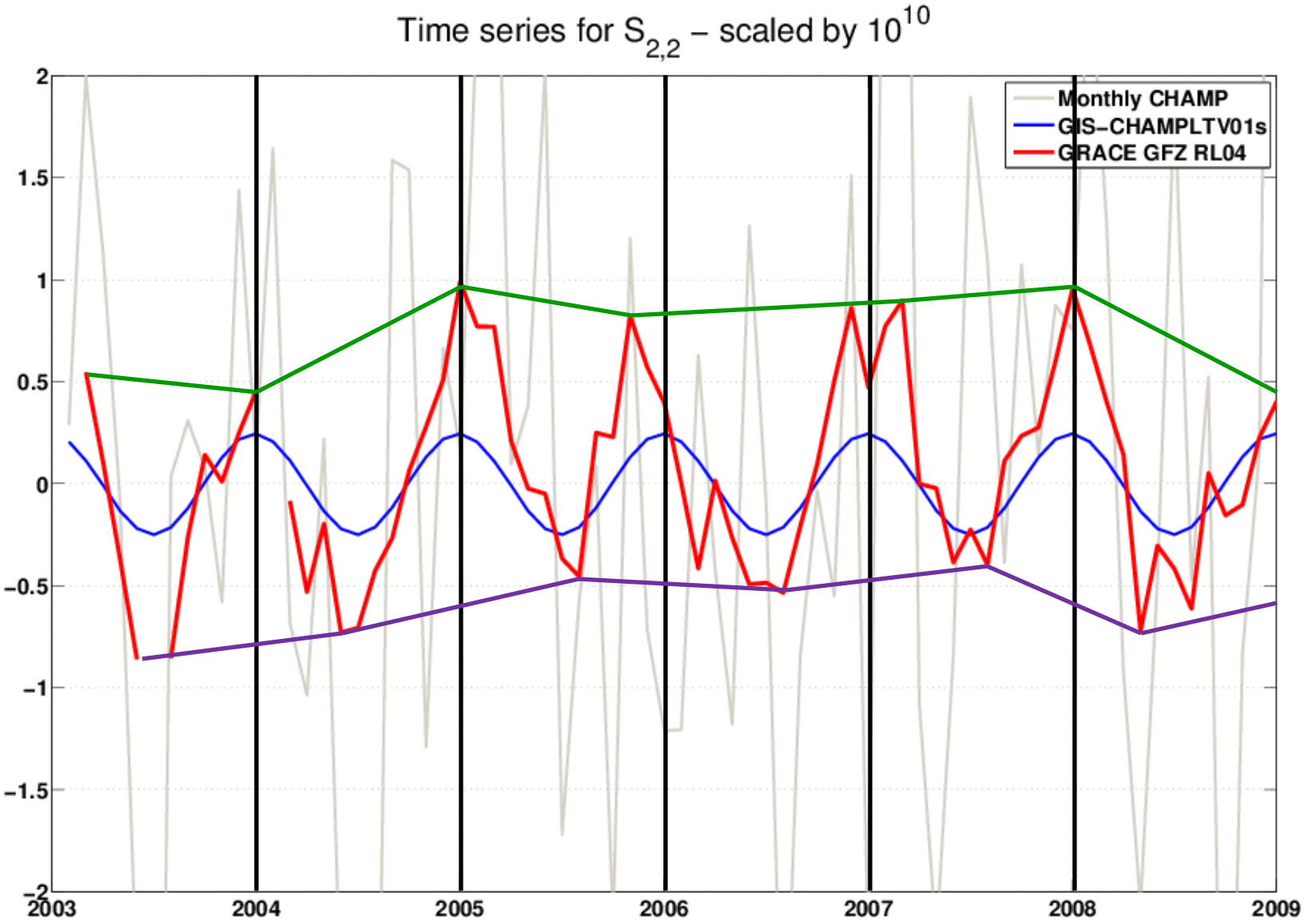
Modeled time-variable gravity field solution (4/4)

- additional trend estimation?

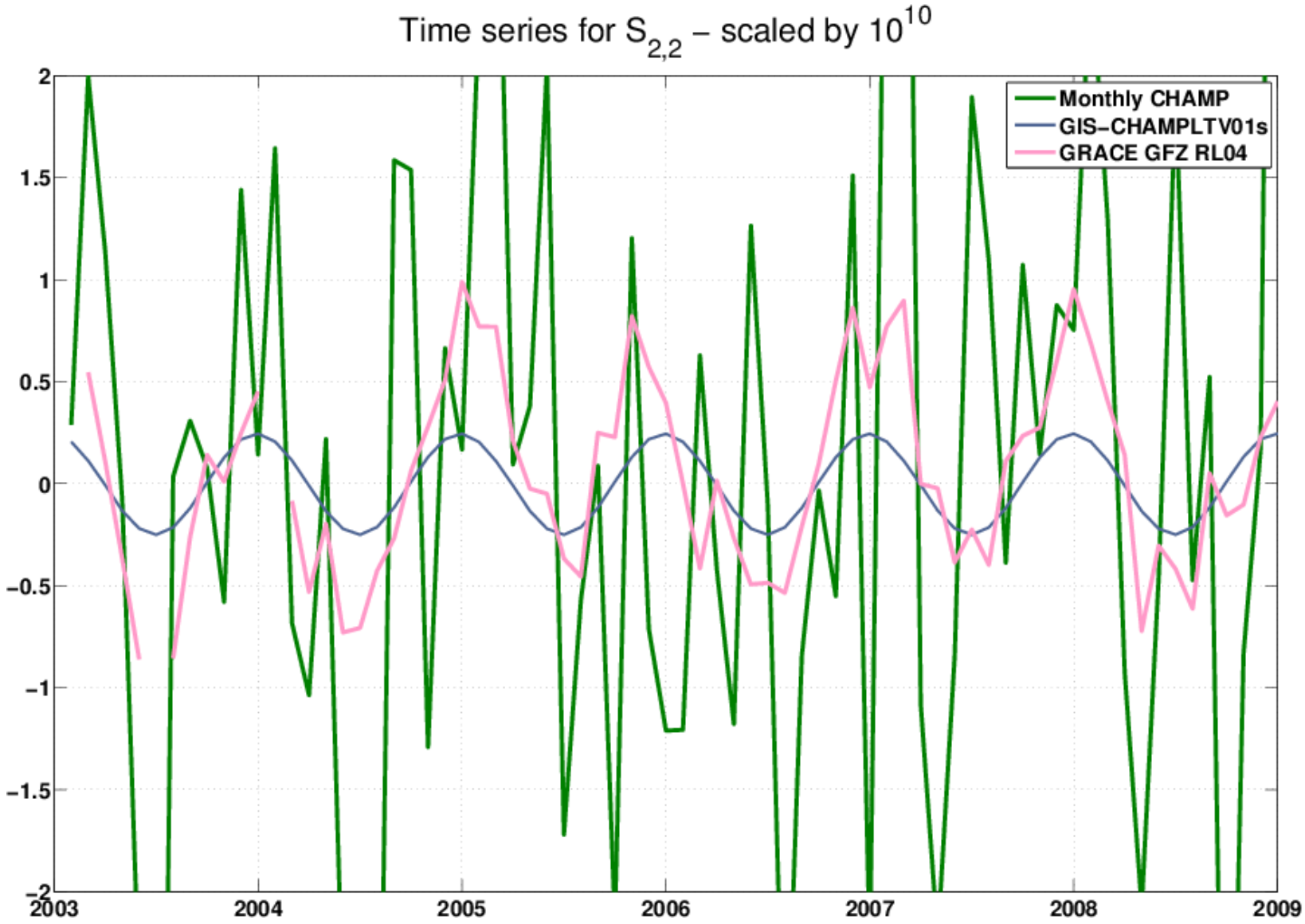
Time series of SH-coefficients: GRACE vs. CHAMP (annual + trend) – scaled by 10^{10}



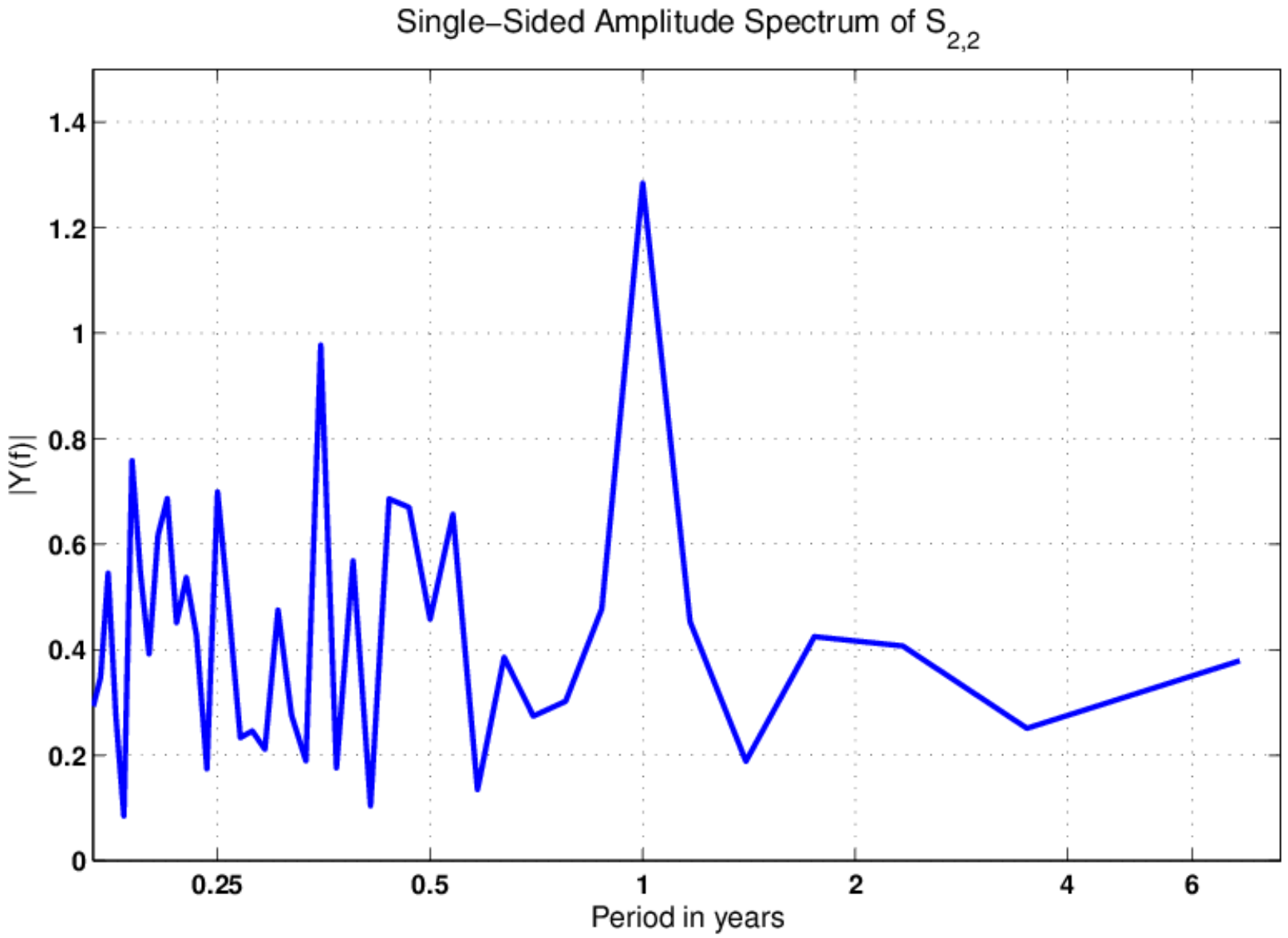
Time series of a coefficient



Time series of a coefficient

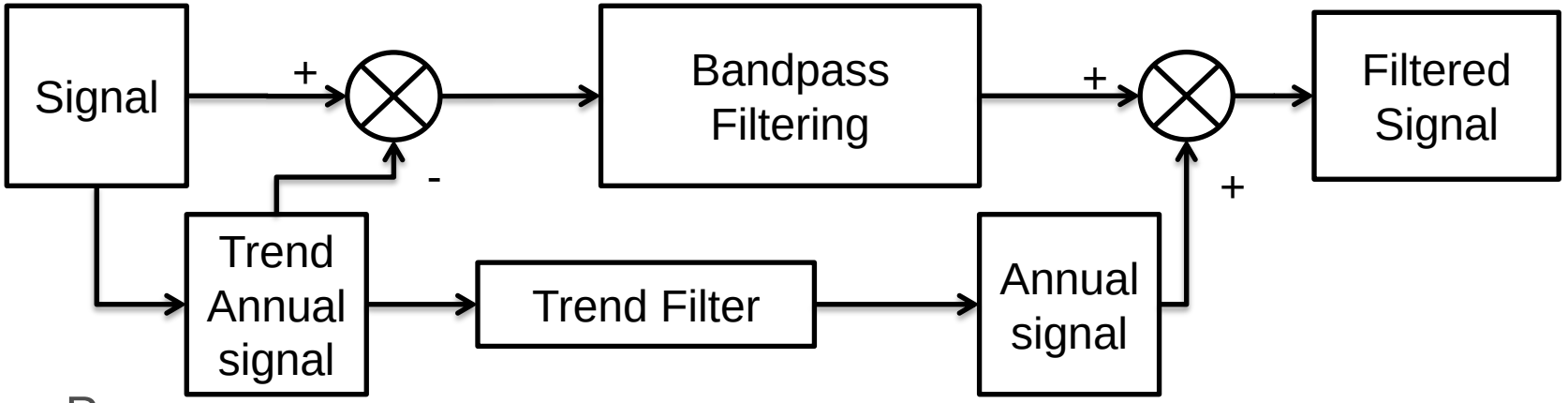


Spectrum of the time series



FILTERED MONTHLY SOLUTIONS

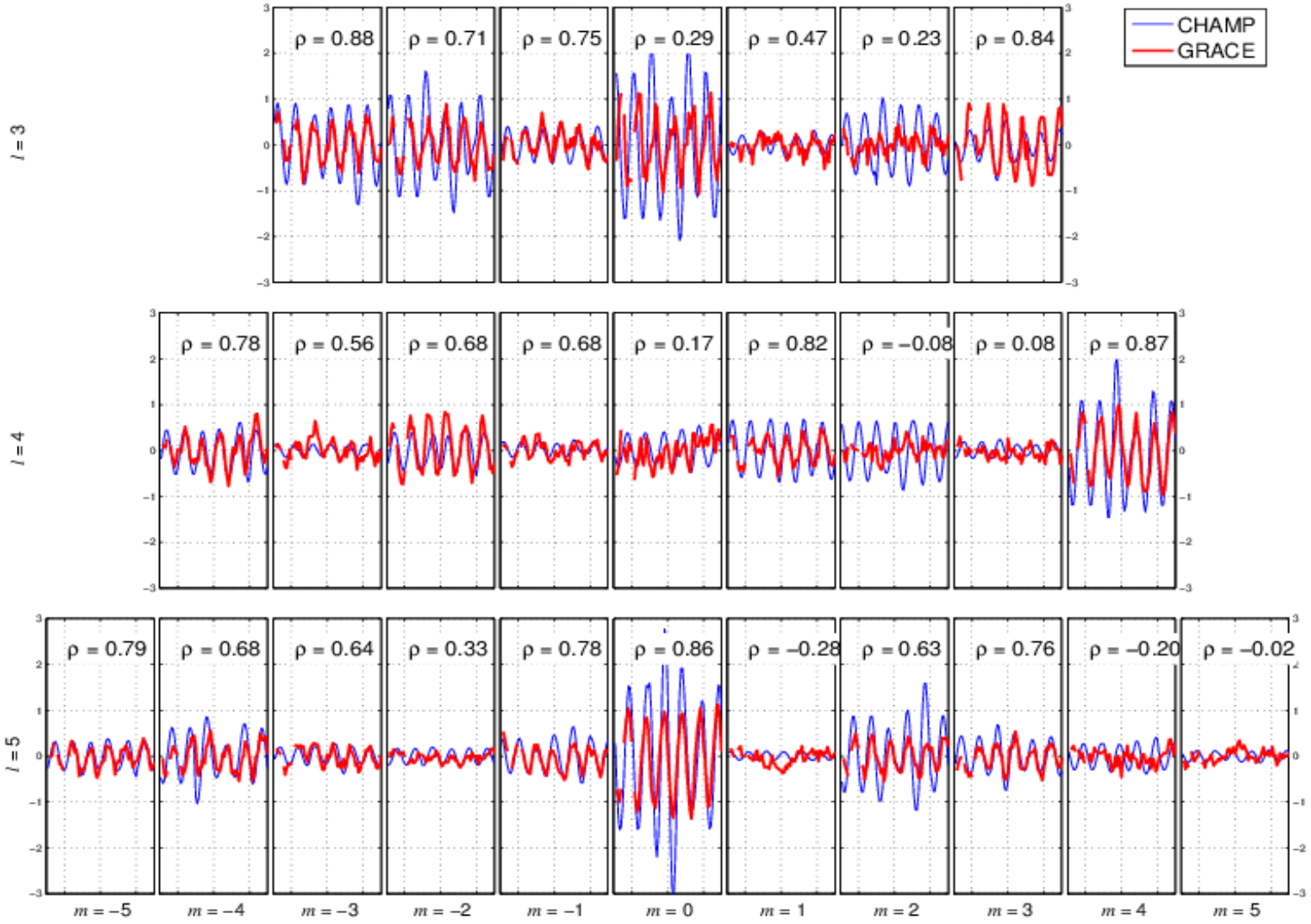
Filtering approach



- Pro:
  - variation of frequencies between coefficients possible (within passband)
  - applicable to all degrees and orders
  - filter design
- Con:
  - filter design
  - warmup
  - sophisticated outlier detection necessary
  - neglecting correlations between coefficients

Filtered monthly gravity field solutions (2/6)

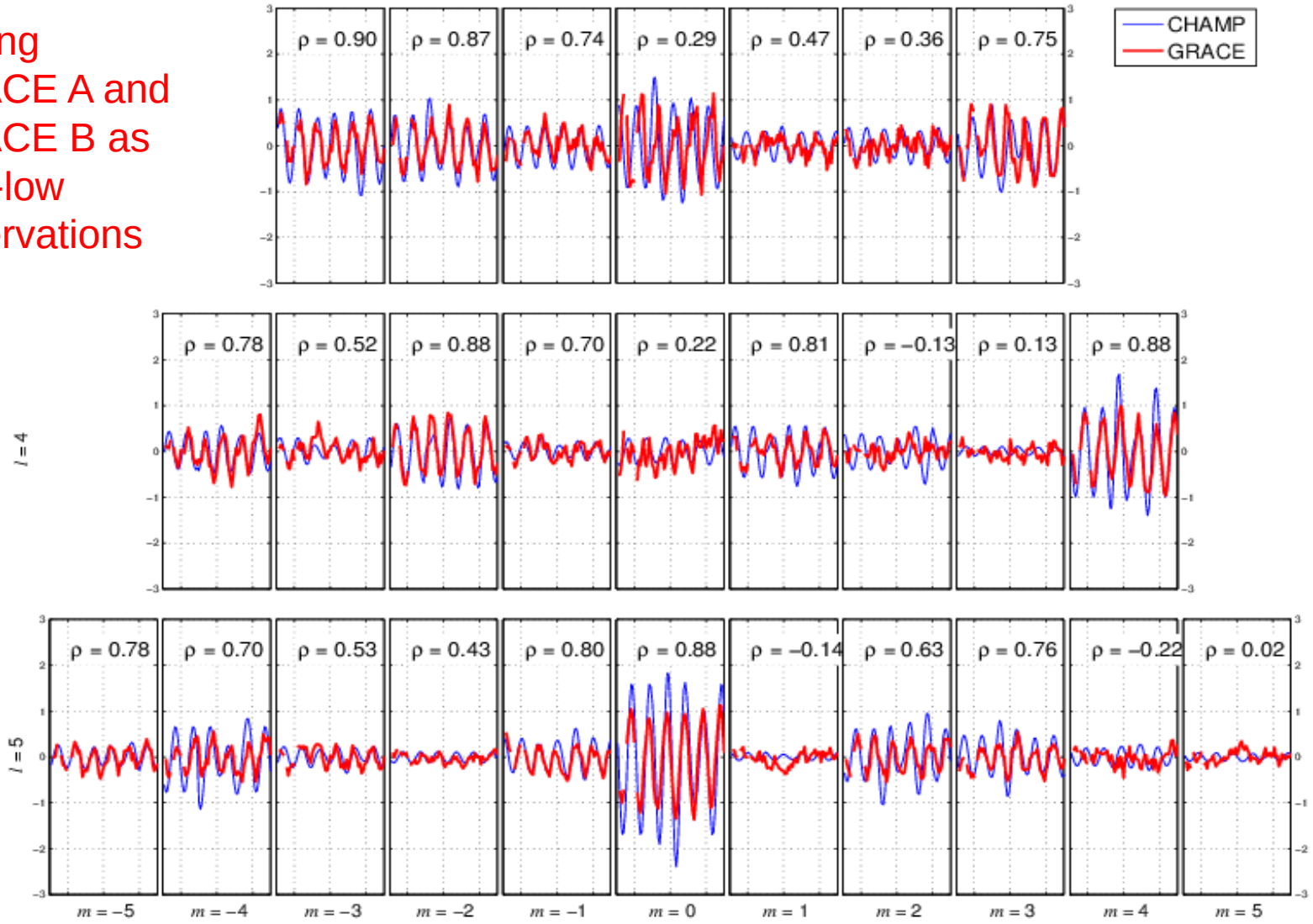
Time series of SH-coefficients: GRACE vs. CHAMP (filtered) – scaled by 10^{10}



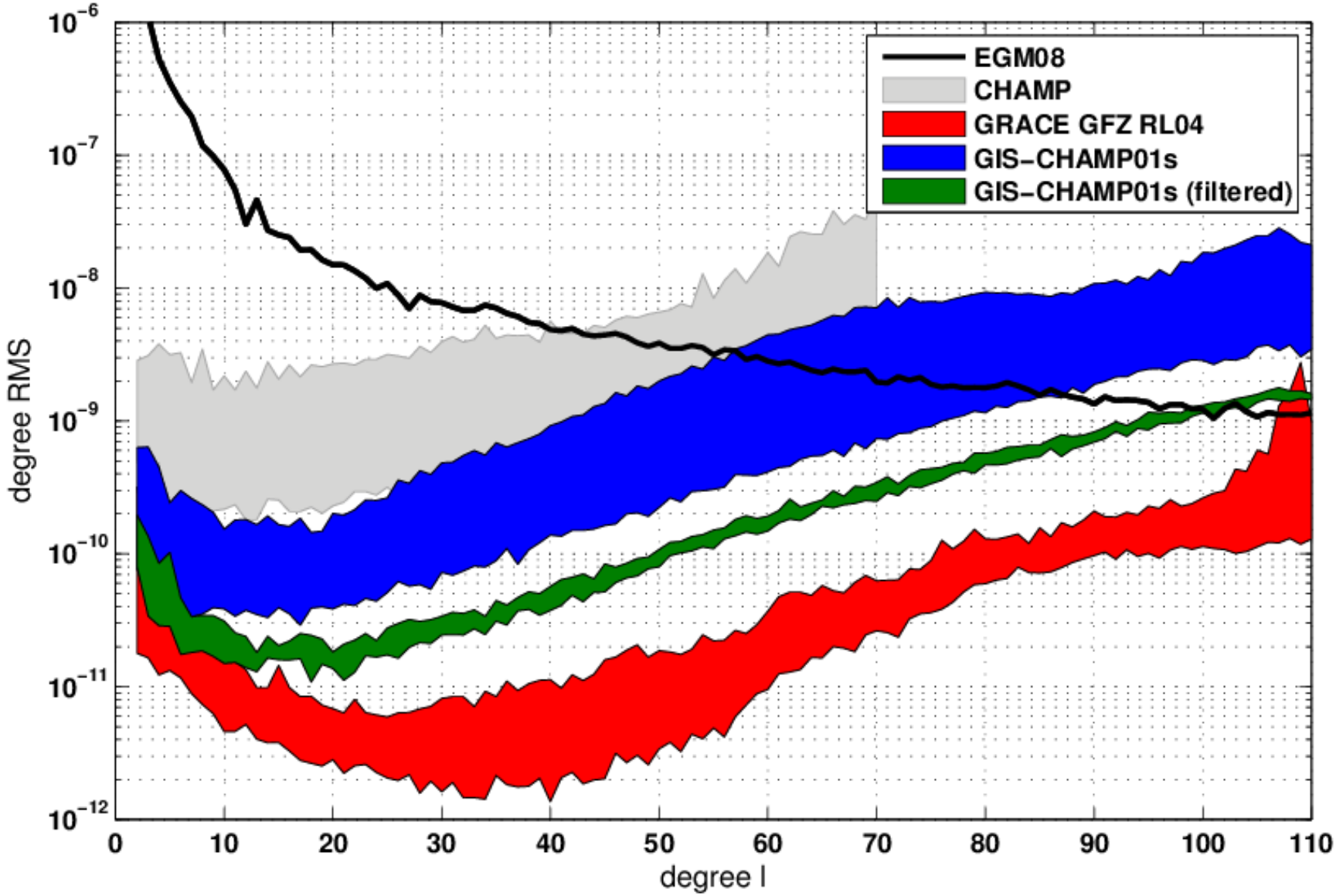
Filtered monthly gravity field solutions (3/6)

Time series of SH-coefficients: GRACE vs. High-Low (filtered) – scaled by 10^{10}

Adding GRACE A and GRACE B as high-low observations



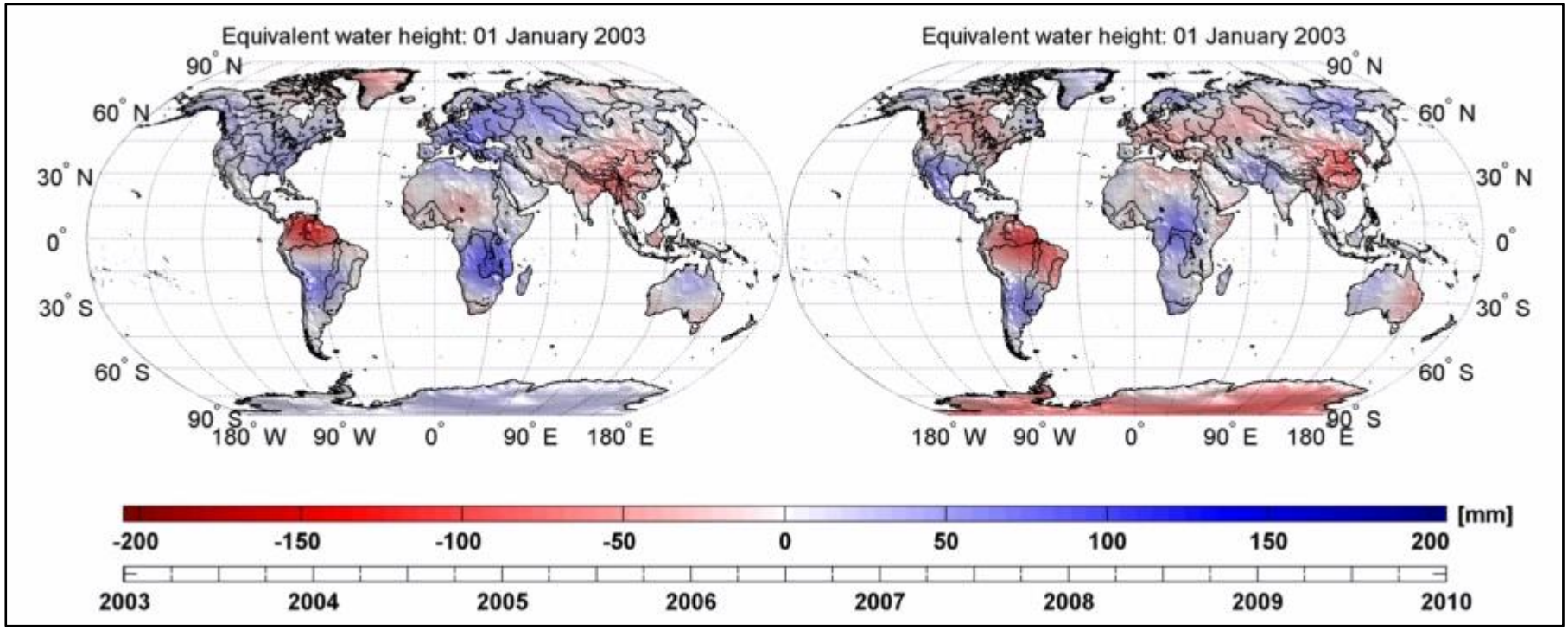
Filtered monthly gravity field solutions (1/6)



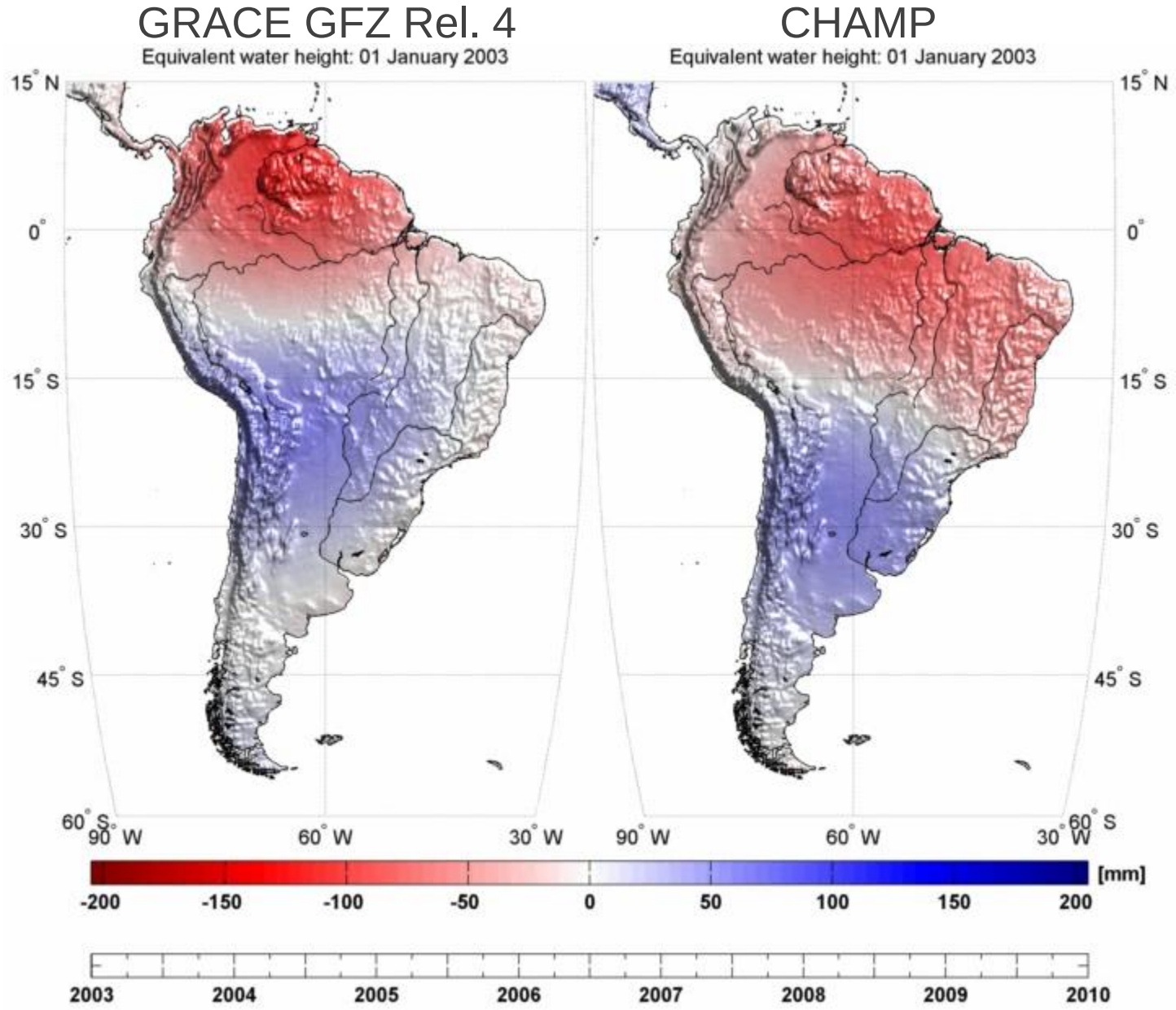
Filtered monthly gravity field solutions (4/6)

GRACE GFZ Rel. 4

CHAMP



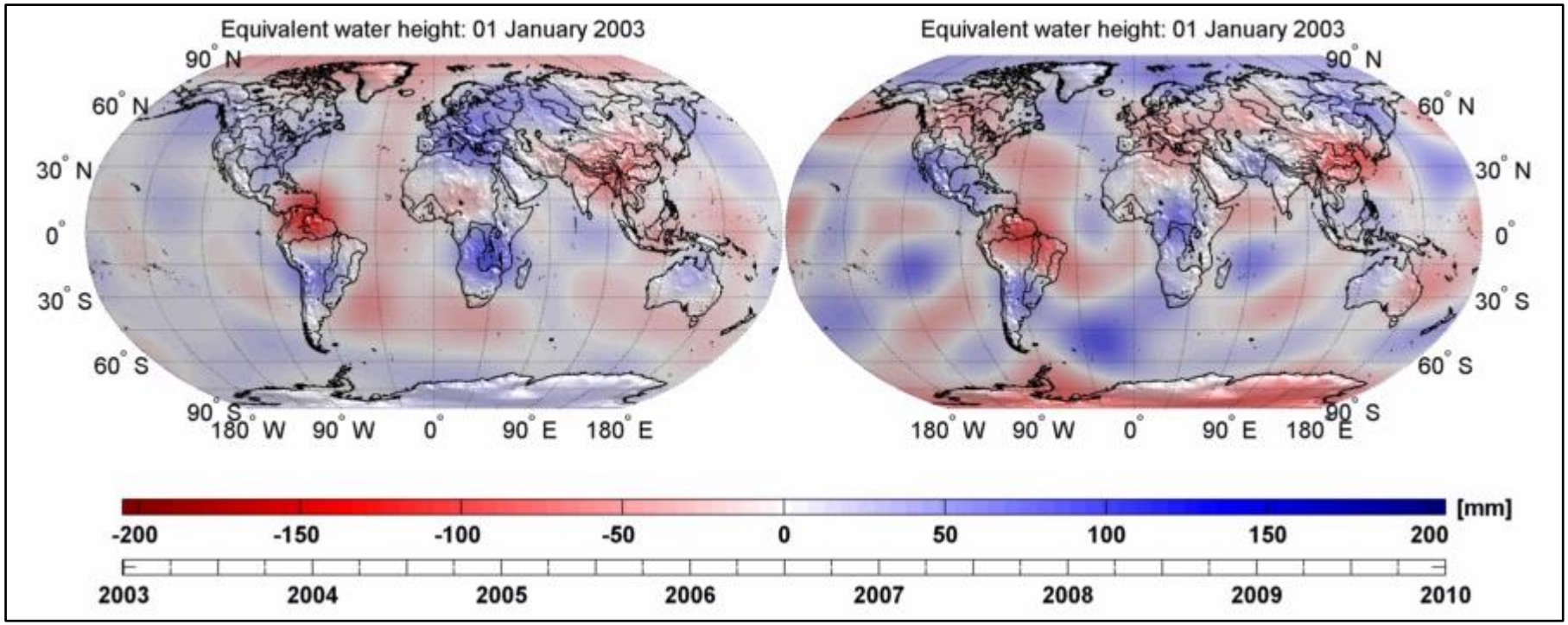
Filtered monthly gravity field solutions (5/6)



Filtered monthly gravity field solutions (6/6)

GRACE GFZ Rel. 4

CHAMP

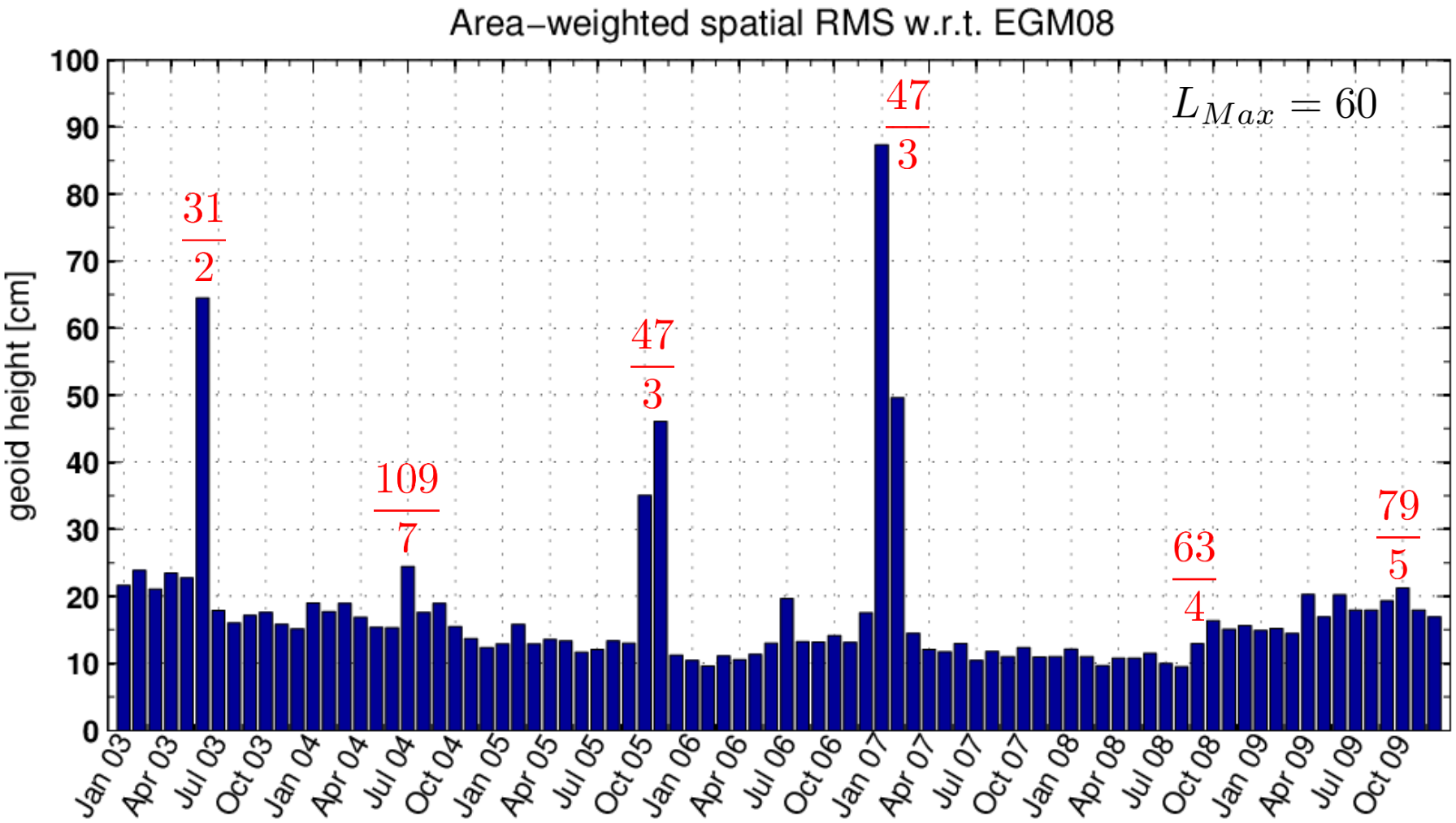


Summary Part I: CHAMP

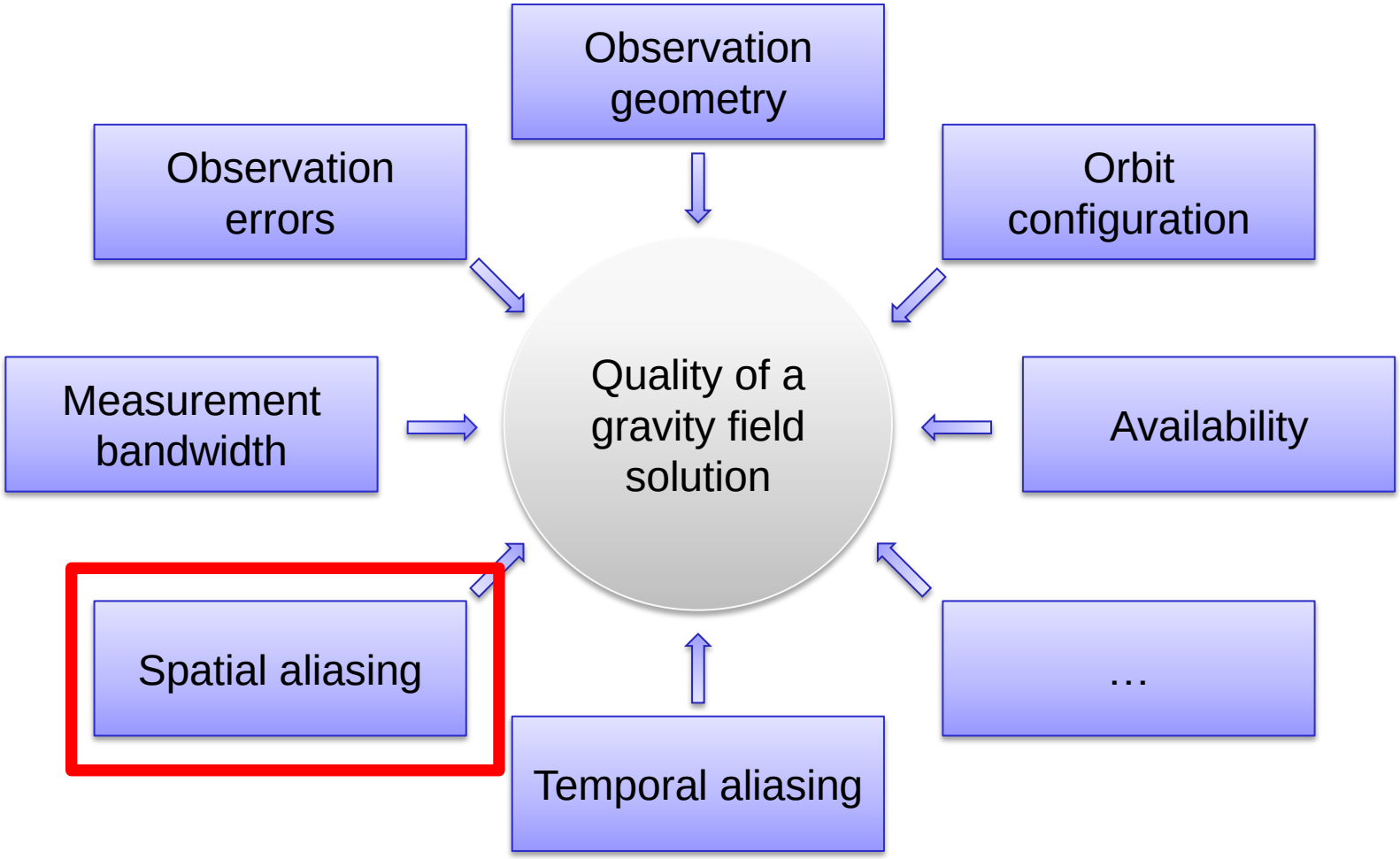
- Improved gravity field solution due AIUB - CHAMP position data including the time variability.
- Evaluation of the quality of time-variable solution necessary
- Processing refinements necessary (especially filtering)
- High-low SST is as a transitory technology till GFO (Swarm, Cosmic, Sentinel, …)
- Acceleration approach also suitable to satellite missions without accelerometer data
- Other approaches ?

PART II: SPATIAL ALIASING

CHAMP monthly gravity field solutions (1/4)



Influences on accuracy



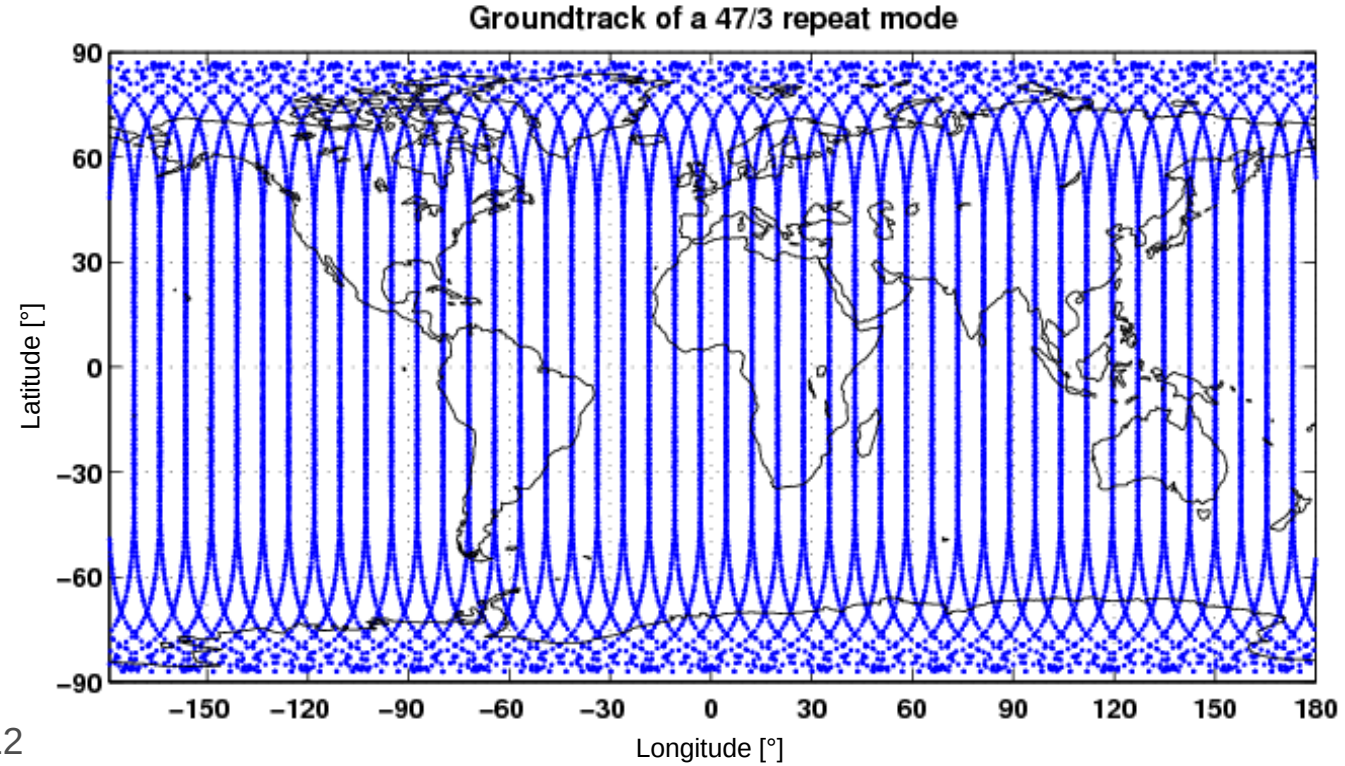
Colombo-Nyquist rule

$\frac{\beta}{\alpha}$ repeat orbit:

- β orbits
- α nodal days

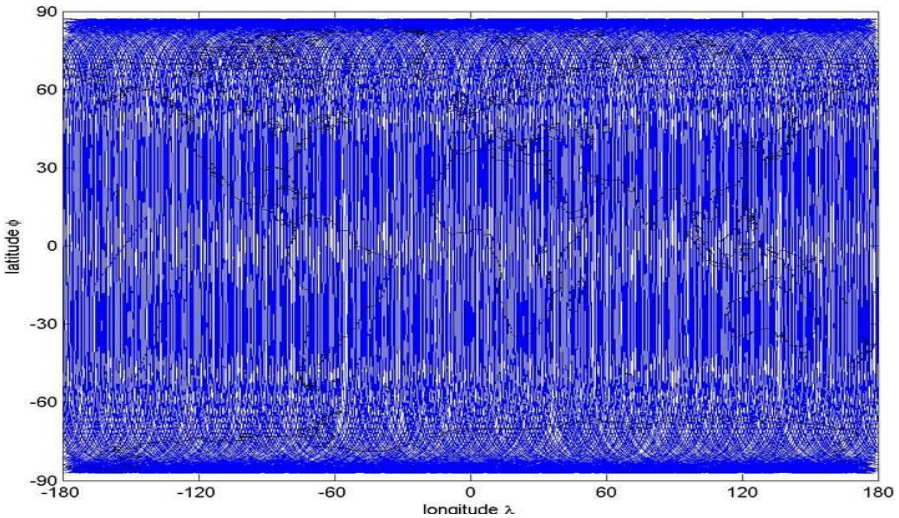
$\Rightarrow$ maximum spatial resolution: $L = \frac{\beta}{2}$

Example: β= 47, α = 3 $\Rightarrow$ L = 23

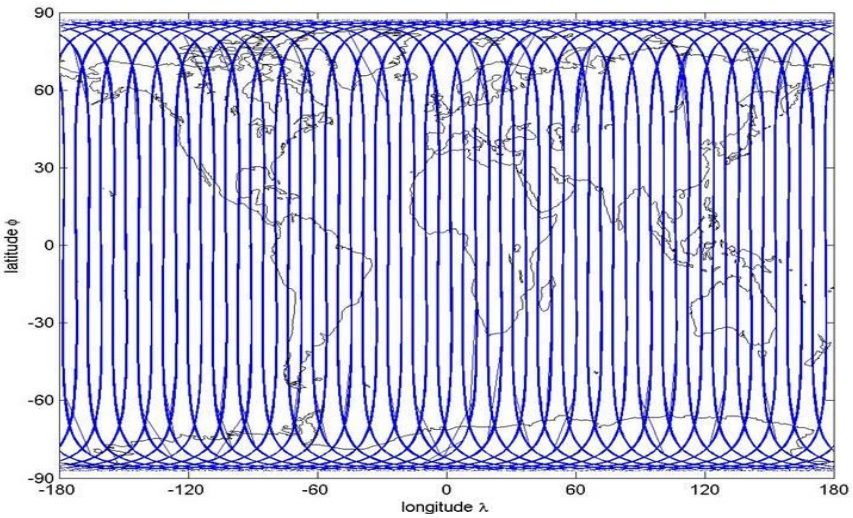


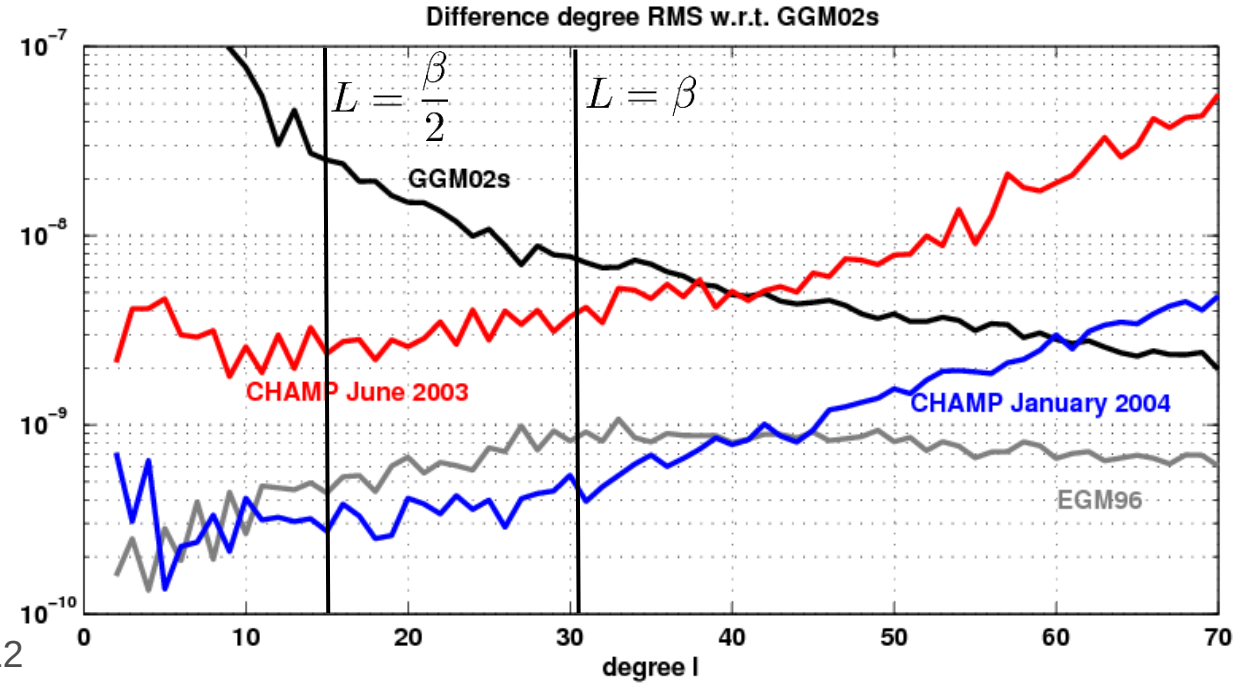
However…

January 2004



June 2003



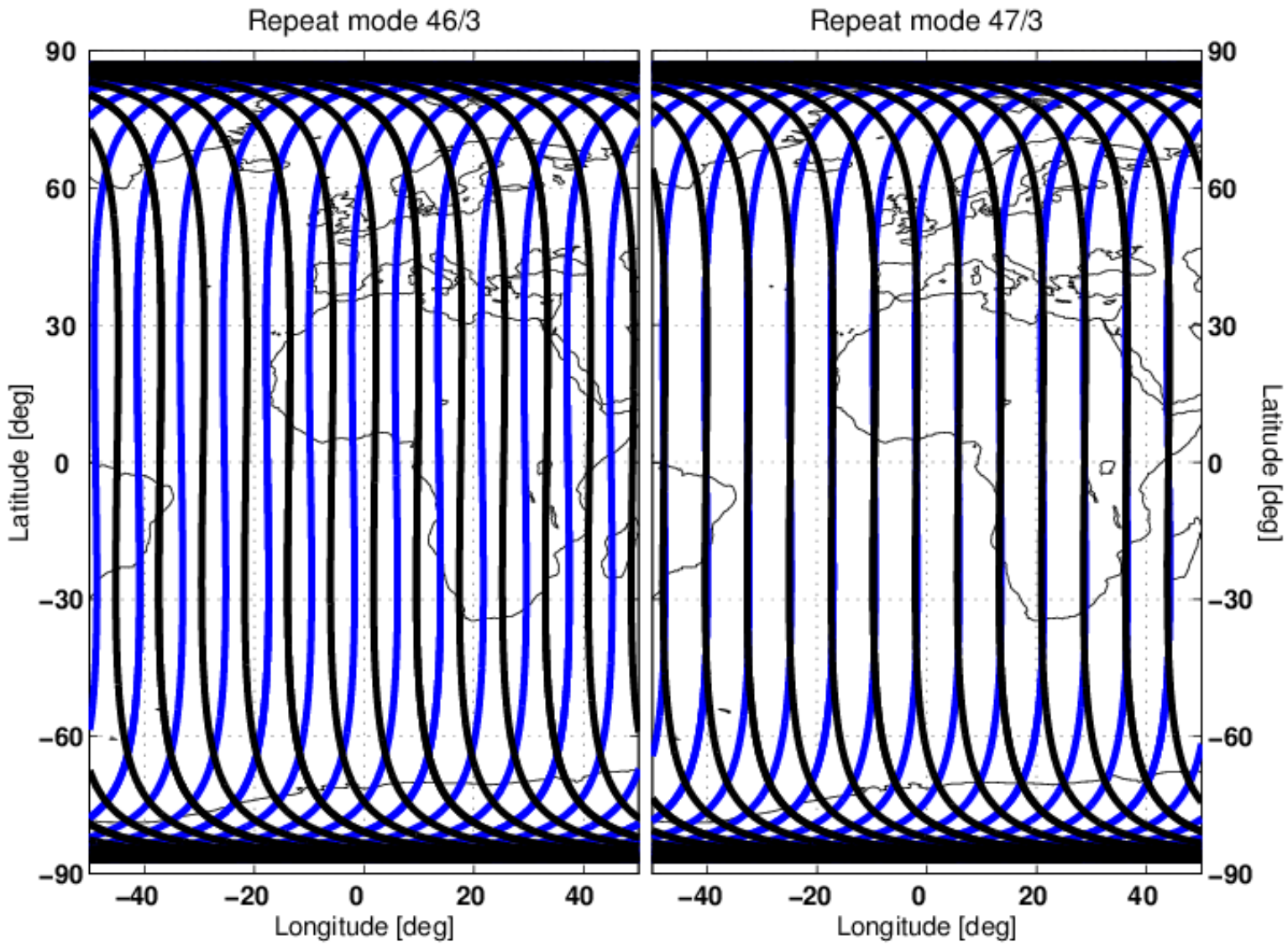


CHAMP in a 31/2 repeat orbit

GIS

NEW CRITERION?

Dependency on the parity (1/2)



Dependency on the parity (2/2)

Ascending equator crossings: $\lambda_p^a = \lambda_0^a + \frac{2\pi}{\beta} p \quad \text{with} \;\; p = 0 \ldots \beta - 1$

First descending equator crossings: $\lambda_0^d = \lambda_0^a + \pi - \frac{1}{2} T_u \omega_E$

$$\lambda_0^d = \lambda_0^a + \pi - \pi \frac{\alpha}{\beta}$$

Condition: $\lambda_p^a - \lambda_0^d = 0$

$$\lambda_0^a + \frac{2\pi}{\beta} p - \lambda_0^a - \pi + \pi \frac{\alpha}{\beta} = 0$$

$$p = \frac{1}{2} (\beta - \alpha)$$

Solution only for $\beta - \alpha$ even, since all elements are integer.

Introducing the concept of unique equator crossings:

$$\chi = \begin{cases} 2\beta, & \text{for } \beta - \alpha \text{ odd} \\ \beta, & \text{for } \beta - \alpha \text{ even} \end{cases}$$

Time series criterion

- Spatial representation:

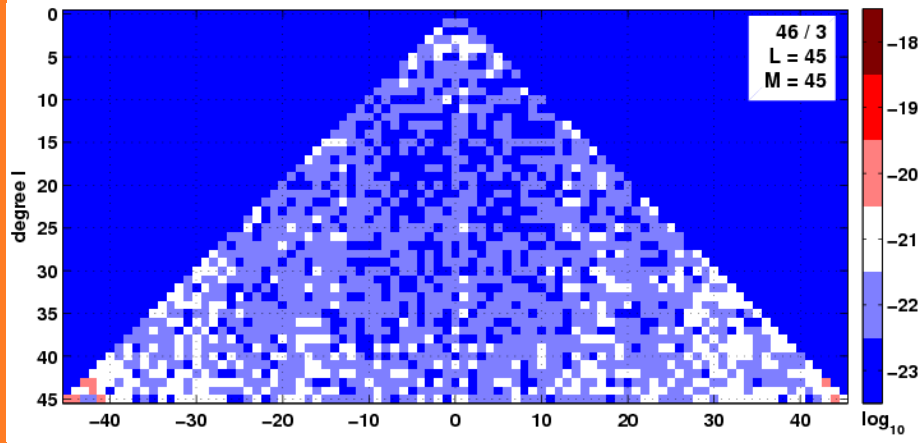
$$V(r,\phi,\lambda) = \frac{GM}{R}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^{l+1}\sum_{m=-l}^{l}\bar{K}_{lm}\bar{P}_{lm}(\sin\phi)\,e^{im\lambda}$$

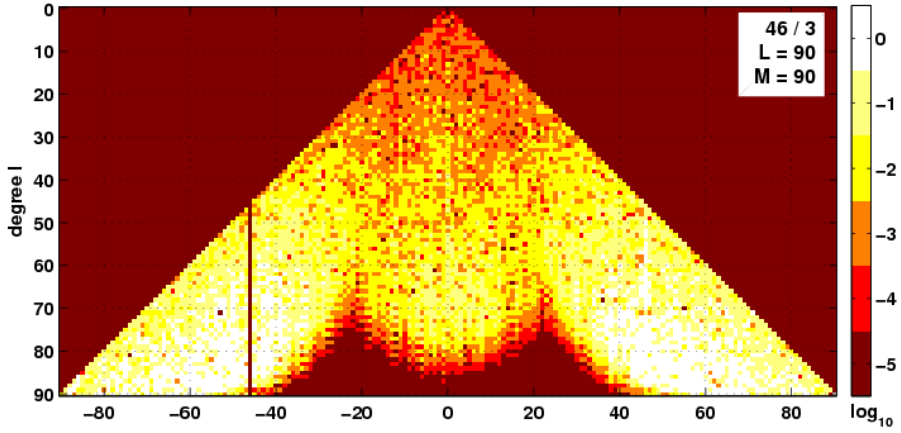
GIS

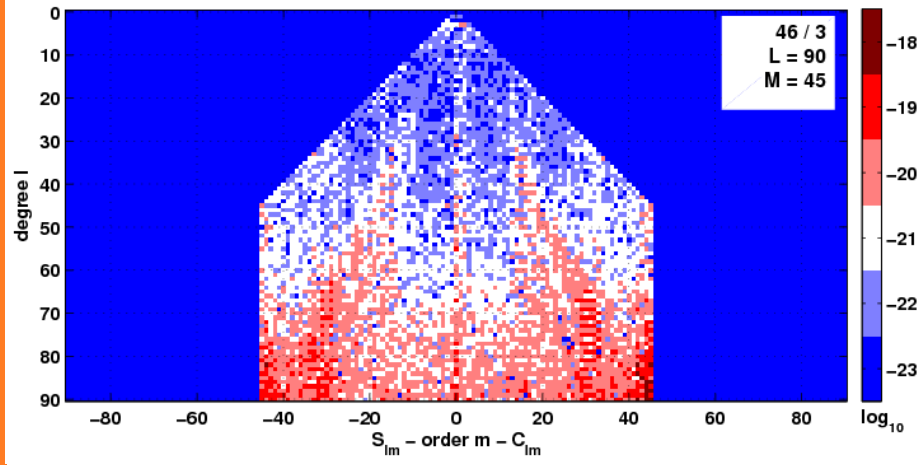
SIMULATION STUDY

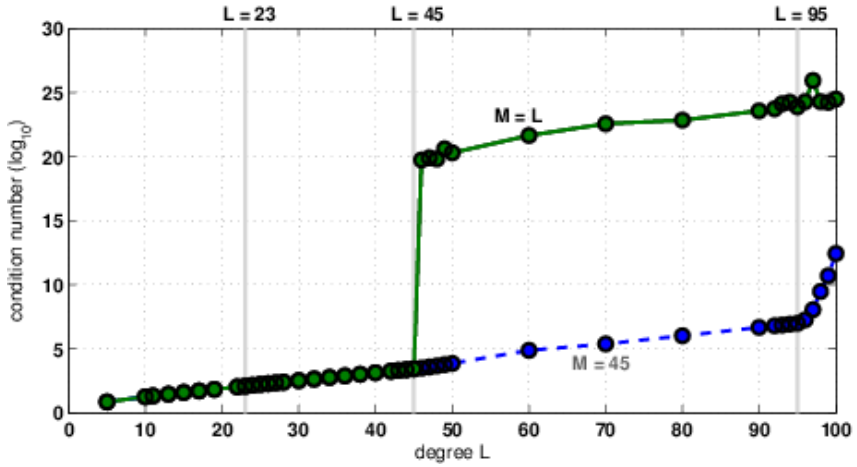
Simulation study: odd parity 46/3

$$M < \beta$$



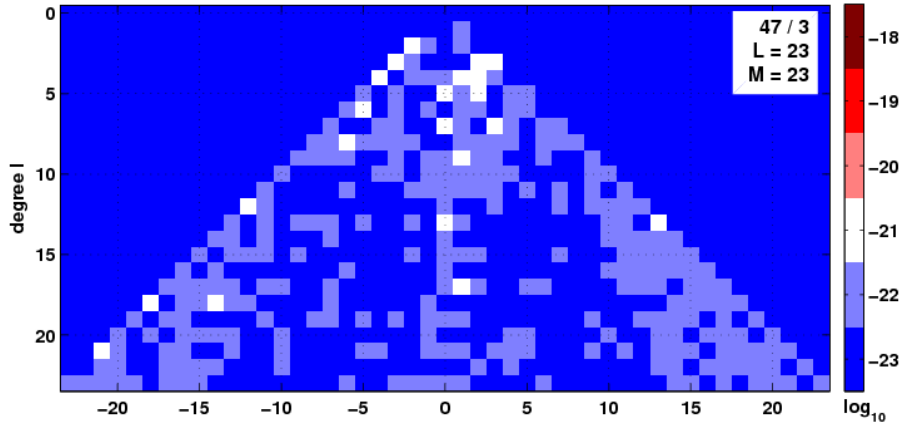


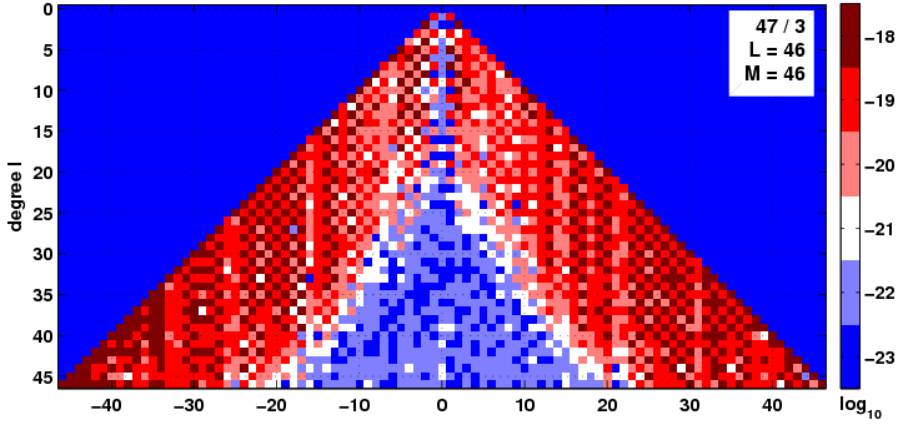


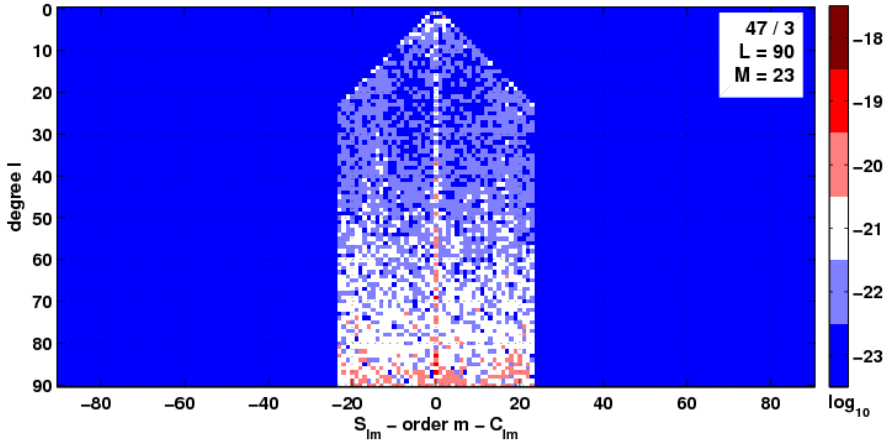


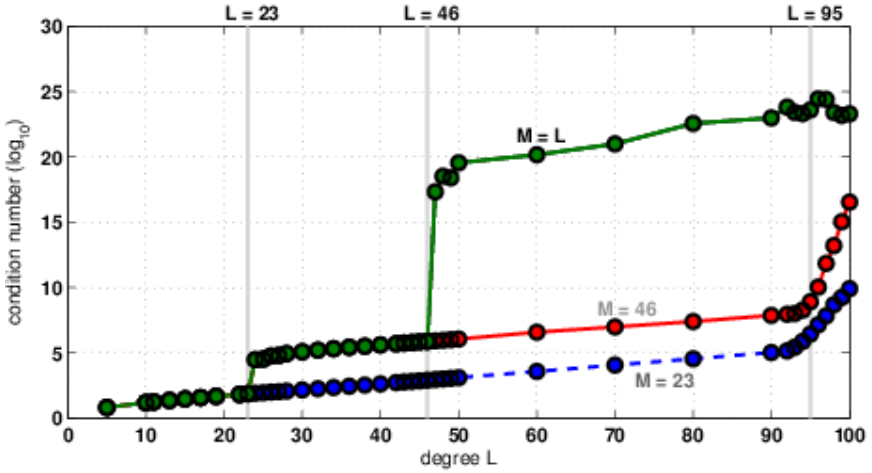
Simulation study: even parity 47/3

$$M < \frac{\beta}{2}$$

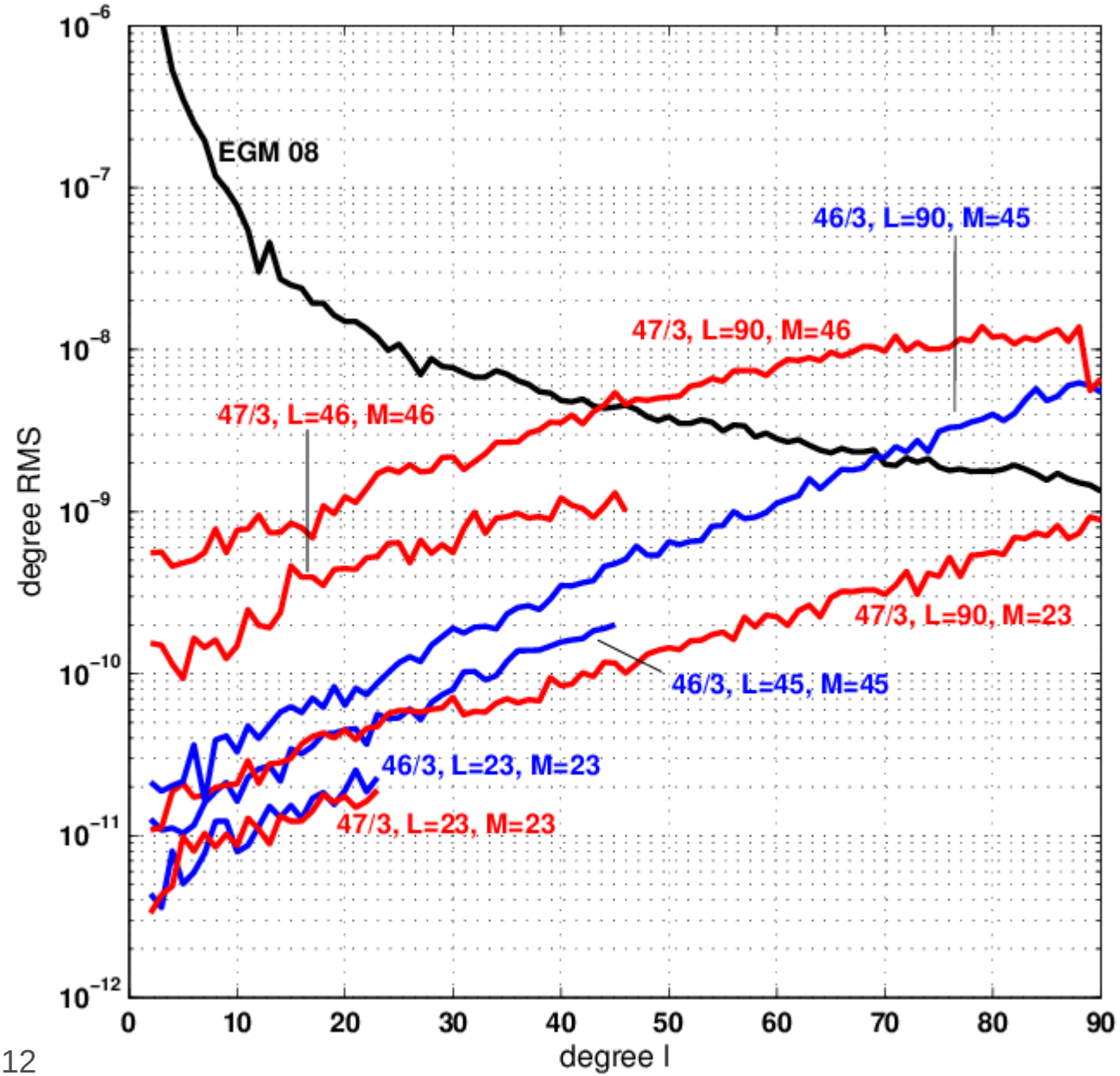








Noise impact

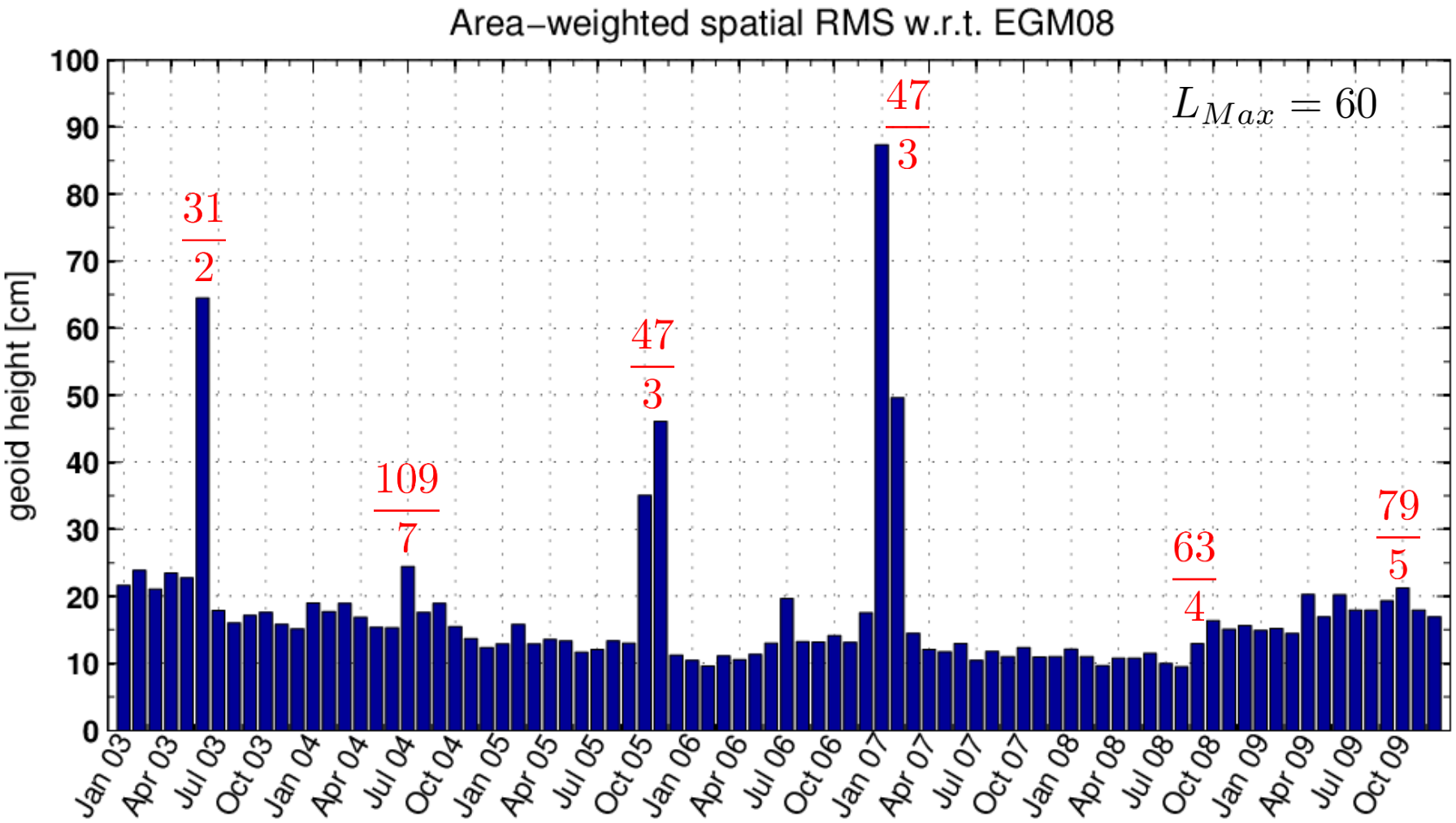


IMPACT ON CHAMP AND GRACE

CHAMP repeat cycles (1/2)

β	α	Time	Parity	h [km]	χ	M	Old: $L =$
139	9	January 2002	Even	409	139	69	69
31	2	May 2002	Odd	393	62	30	15
		October 2002					
		June 2003					
109	7	July 2004	Even	372	109	54	54
47	3	November 2005	Even	345	47	23	23
		January 2007					
63	4	September 2008	Odd	321	126	62	31
79	5	October 2009	Even	307	79	39	39

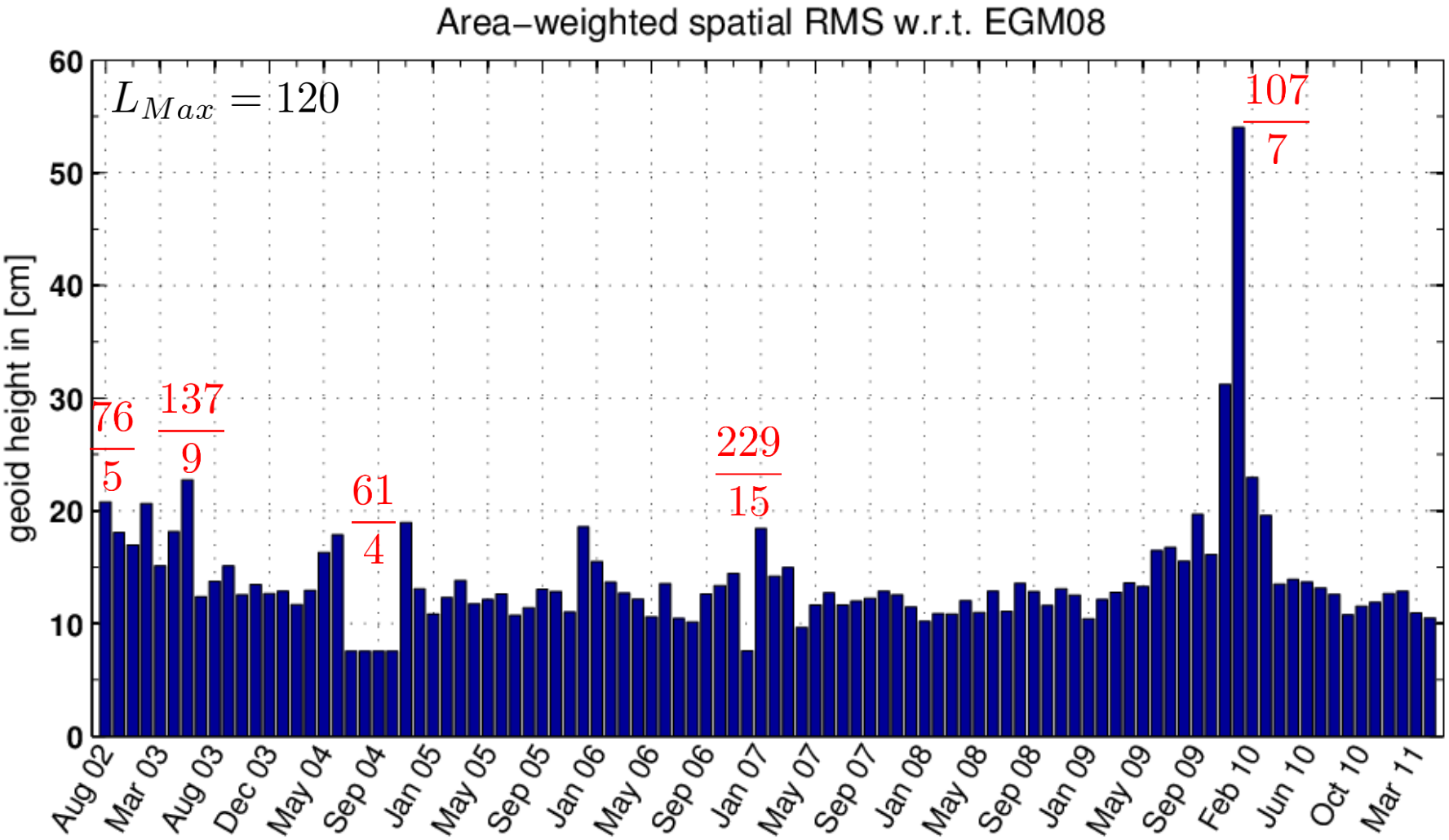
CHAMP repeat cycles (2/2)



GRACE repeat cycles (1/2)

β	α	Time	Parity	h [km]	χ	M	Old: $L =$
76	5	September 2002	Odd	485	152	75	37
137	9	April 2003	Even	478	137	68	68
61	4	September 2004	Odd	470	122	60	30
229	15	January 2006	Even	465	229	114	114
107	7	December 2009	Even	459	107	53	53
199	13		Even	453	199	99	99
46	3		Odd	445	92	45	22
169	11		Even	436	169	84	84
…	…	…	…	…	…	…	

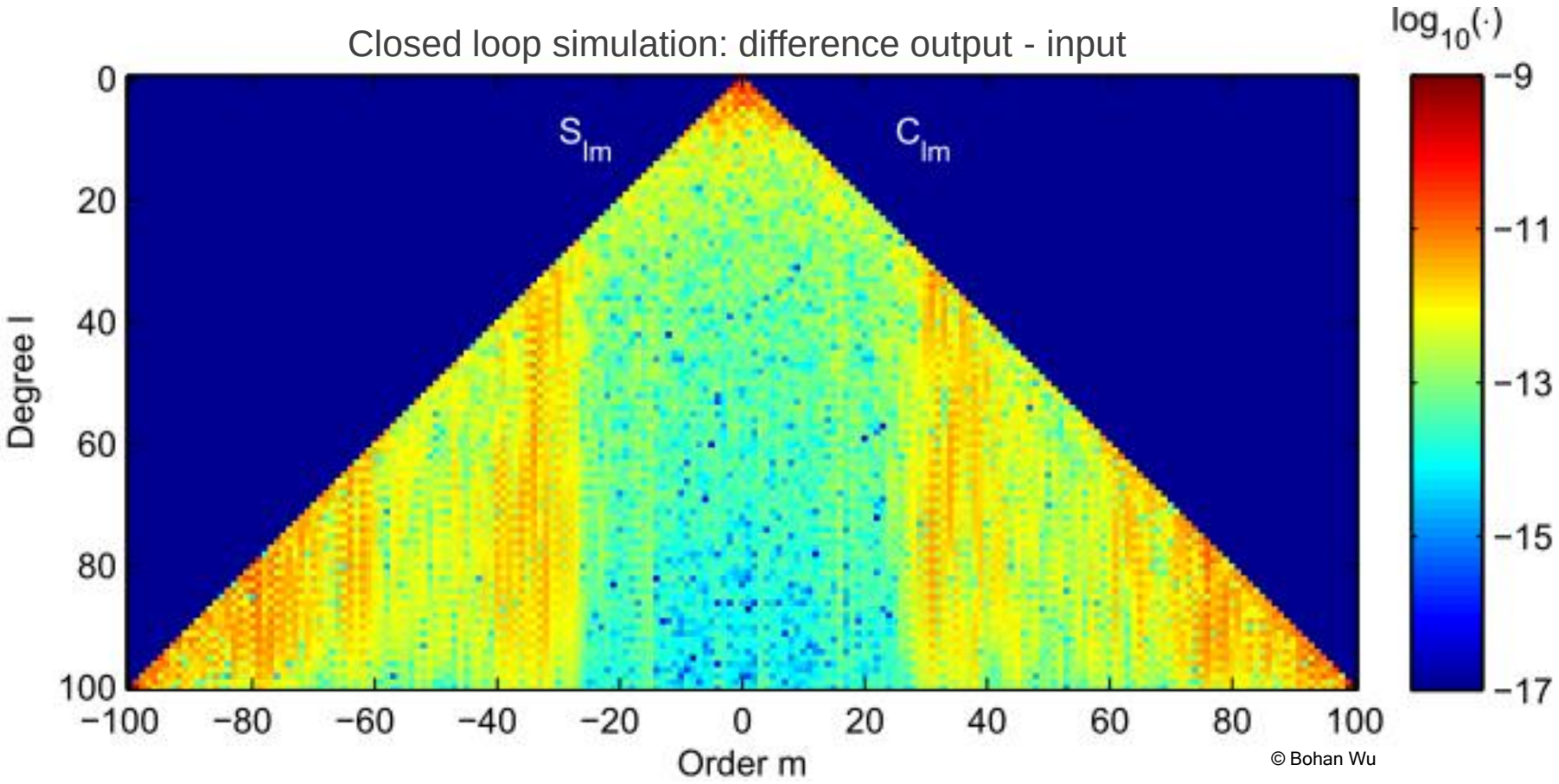
GRACE repeat cycles (2/2)



TEMPORAL ALIASING

Temporal aliasing?

- Closed loop simulation: GRACE
- Difference of ocean tide models (EOT10a –FES2004) as the only noise source

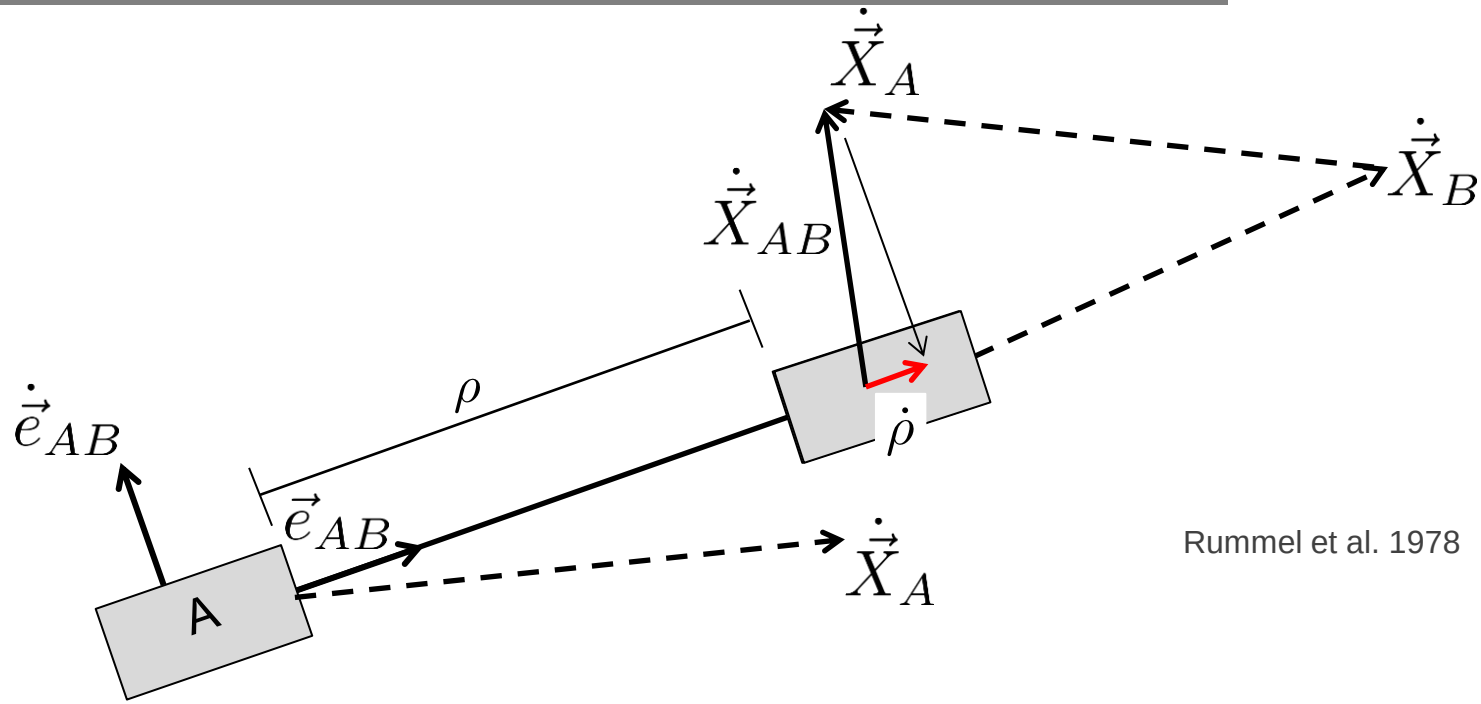


Summary Spatial Aliasing

- Refinement of Colombo-Nyquist rule necessary
- Nyquist criterion separately applicable to longitude and latitude direction due to near polar (limit?) orbits in gravity field missions
- Limitation practically on the maximum order *M*
- New rule of thumb describes only the influence due to the spatial aliasing of the static field.
- Signal higher than maximum order *M* ?
- Dependency on the observation quantity?
- Criterion ?

PART III:
GRACE and GRACE Follow-On

Geometry of the GRACE system



Rummel et al. 1978

Differentiation

Integration

$$\rho = \vec{X}_{AB} \cdot \vec{e}_{AB}$$

$$\dot{\rho} = \dot{\vec{X}}_{AB} \cdot \vec{e}_{AB}$$

$$\ddot{\rho} = \ddot{\vec{X}}_{AB} \cdot \vec{e}_{AB} + \dot{\vec{X}}_{AB} \cdot \dot{\vec{e}}_{AB}$$

$$= \nabla V_{AB}$$

SOLUTION STRATEGIES

Solution strategies

Variational equations

- $\rho, \dot{\rho}$ — *Classical* (Reigber 1989, Tapley 2004)
- $\rho, \dot{\rho}, \Delta\rho$ — *Celestial mechanics approach* (Beutler et al. 2010, Jäggi 2007)
- $\rho, \dot{\rho}$ — Short-arc method (Mayer-Gürr 2006)
- …

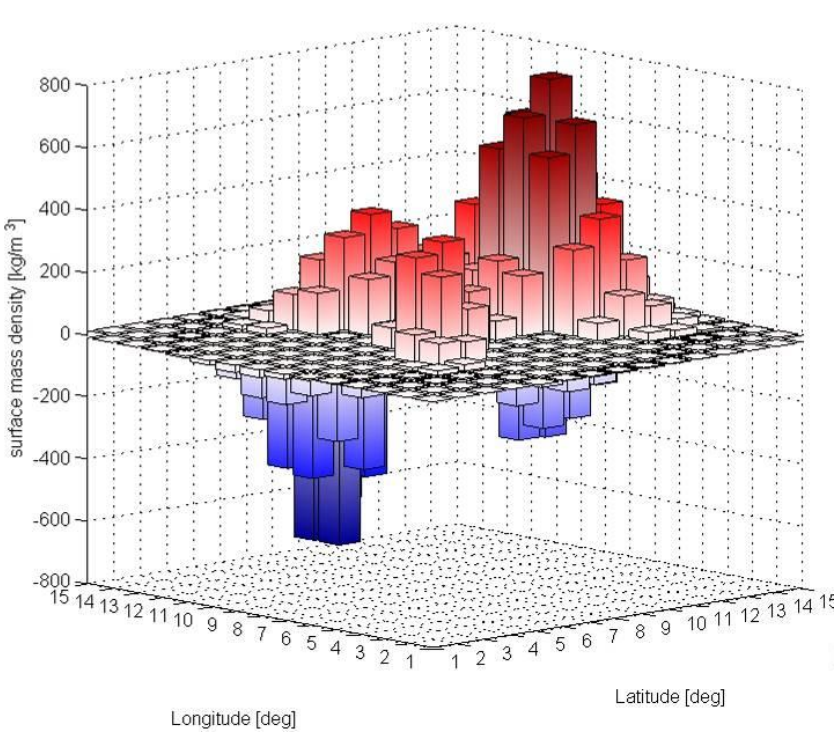
In-situ observations

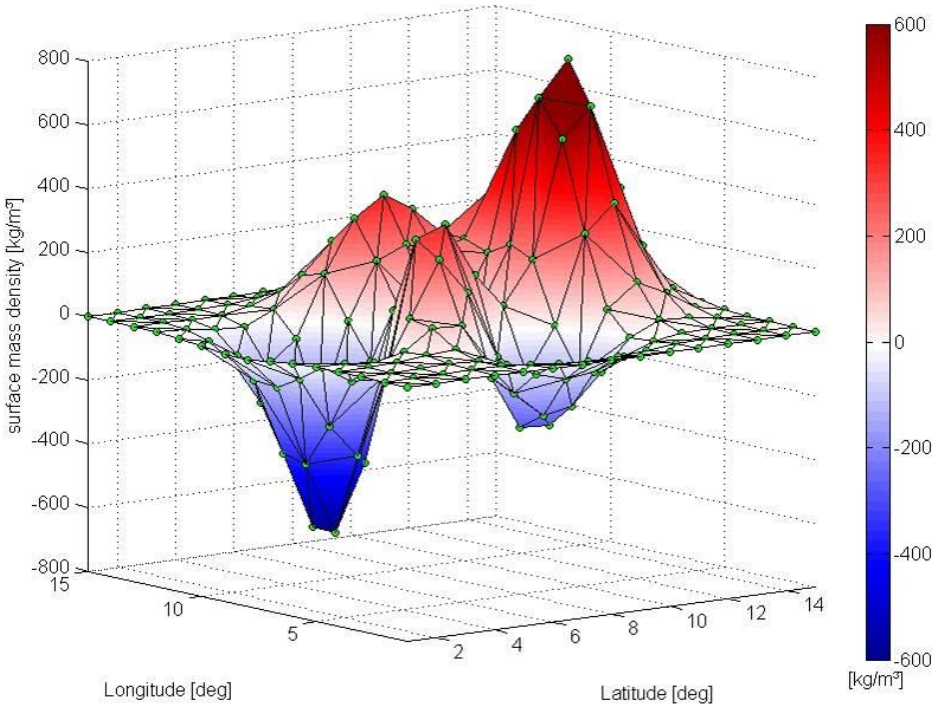
- Energy Integral (Han 2003, Ramillien et al. 2010) — $\dot{\rho}$
- Differential gravimetry (Liu 2010) — $\ddot{\rho}$
- LoS Gradiometry (Keller and Sharifi 2005) — $\frac{\ddot{\rho}}{\rho}$
- …

Motivation for in-situ observations: local modeling

- Boundary element method

$$V = \int_{\Omega} \frac{\sigma\left(\vec{X}_Q\right)}{\|\vec{X} - \vec{X}_Q\|} d\vec{X}_Q = \sum_{i=1}^{N} \sigma_i \int_{\Omega_i} \frac{\Phi_i}{\|\vec{X} - \vec{X}_Q\|} d\Omega_i$$





Motivation for in-situ observations: local modeling

Water problems can only be solved on a basin scale.

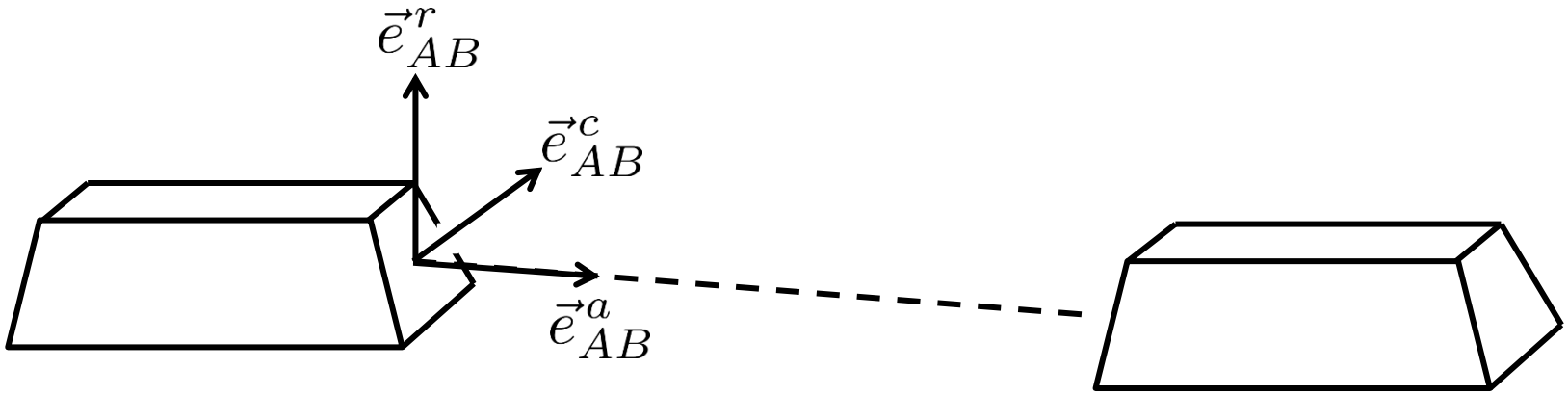
(Robert Kandel – Water from Heaven)

July - September, 1989

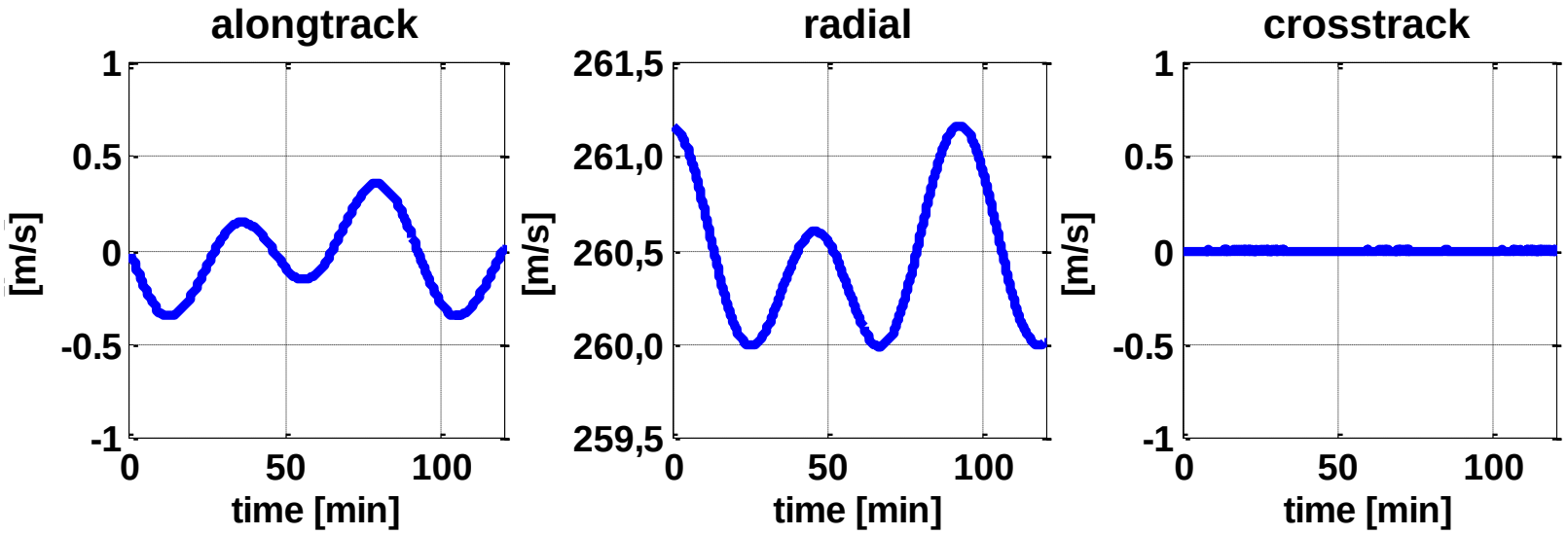
October 5, 2008

IN-SITU OBSERVATIONS: DIFFERENTIAL GRAVITMETRY APPROACH

Instantaneous relative reference frame



Velocity



In-situ observation

Range observables:

$$\vec{X}_{AB} = \rho \cdot \vec{e}^{\,a}_{AB}$$

$$\dot{\vec{X}}_{AB} = \dot{\rho} \cdot \vec{e}^{\,a}_{AB} + \rho \cdot \dot{\vec{e}}^{\,a}_{AB}$$

$$\ddot{\vec{X}}_{AB} = \ddot{\rho} \cdot \vec{e}^{\,a}_{AB} + 2 \cdot \dot{\rho} \cdot \dot{\vec{e}}^{\,a}_{AB} + \rho \cdot \ddot{\vec{e}}^{\,a}_{AB}$$

Multiplication with unit vectors:

GRACE

$$\ddot{\vec{X}}_{AB} \cdot \vec{e}^{\,a}_{AB} = \ddot{\rho} \quad + \quad 0 \quad + \quad \rho \cdot \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB}$$

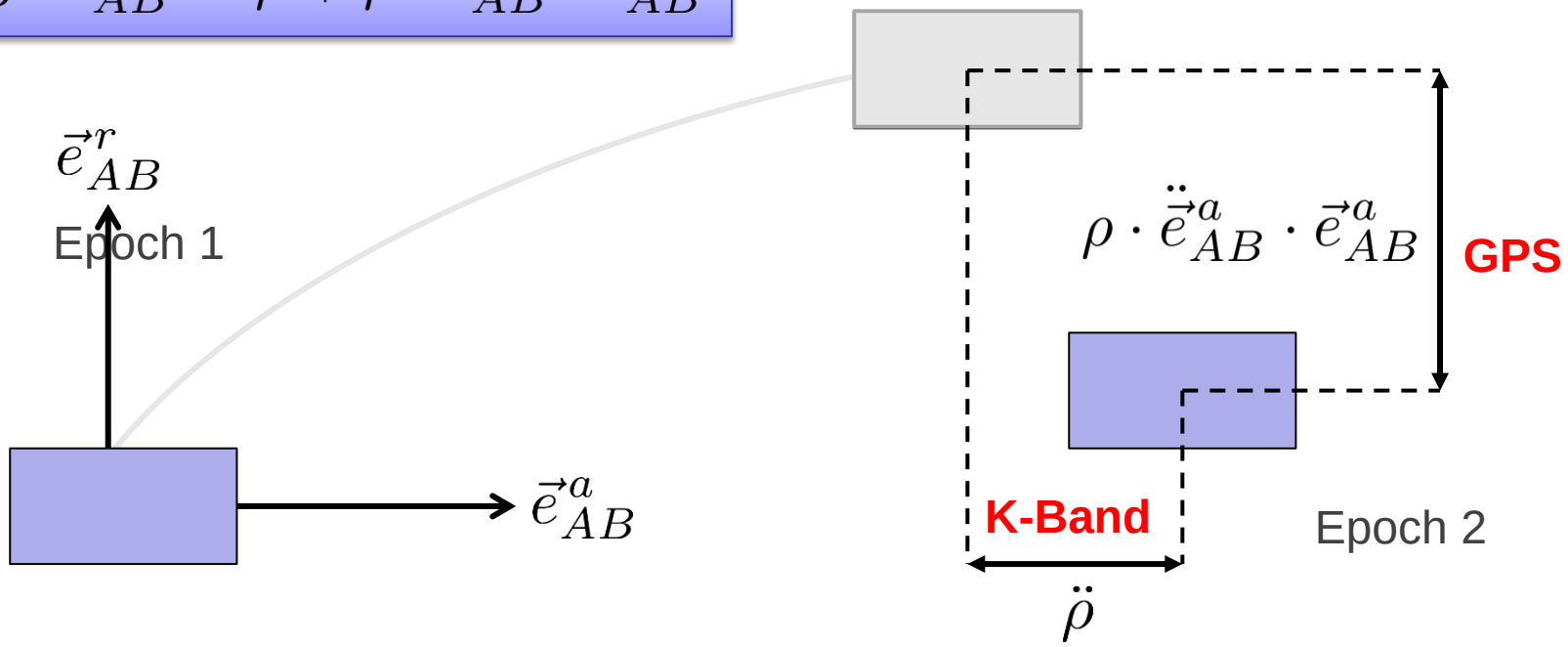
$$\ddot{\vec{X}}_{AB} \cdot \vec{e}^{\,r}_{AB} = 0 \quad + \quad 2 \cdot \dot{\rho} \cdot \|\dot{\vec{e}}^{\,a}_{AB}\| \quad + \quad \rho \cdot \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,r}_{AB}$$

$$\ddot{\vec{X}}_{AB} \cdot \vec{e}^{\,c}_{AB} = 0 \quad + \quad 0 \quad + \quad \rho \cdot \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,c}_{AB}$$

Relative motion

absolute motion neglected!

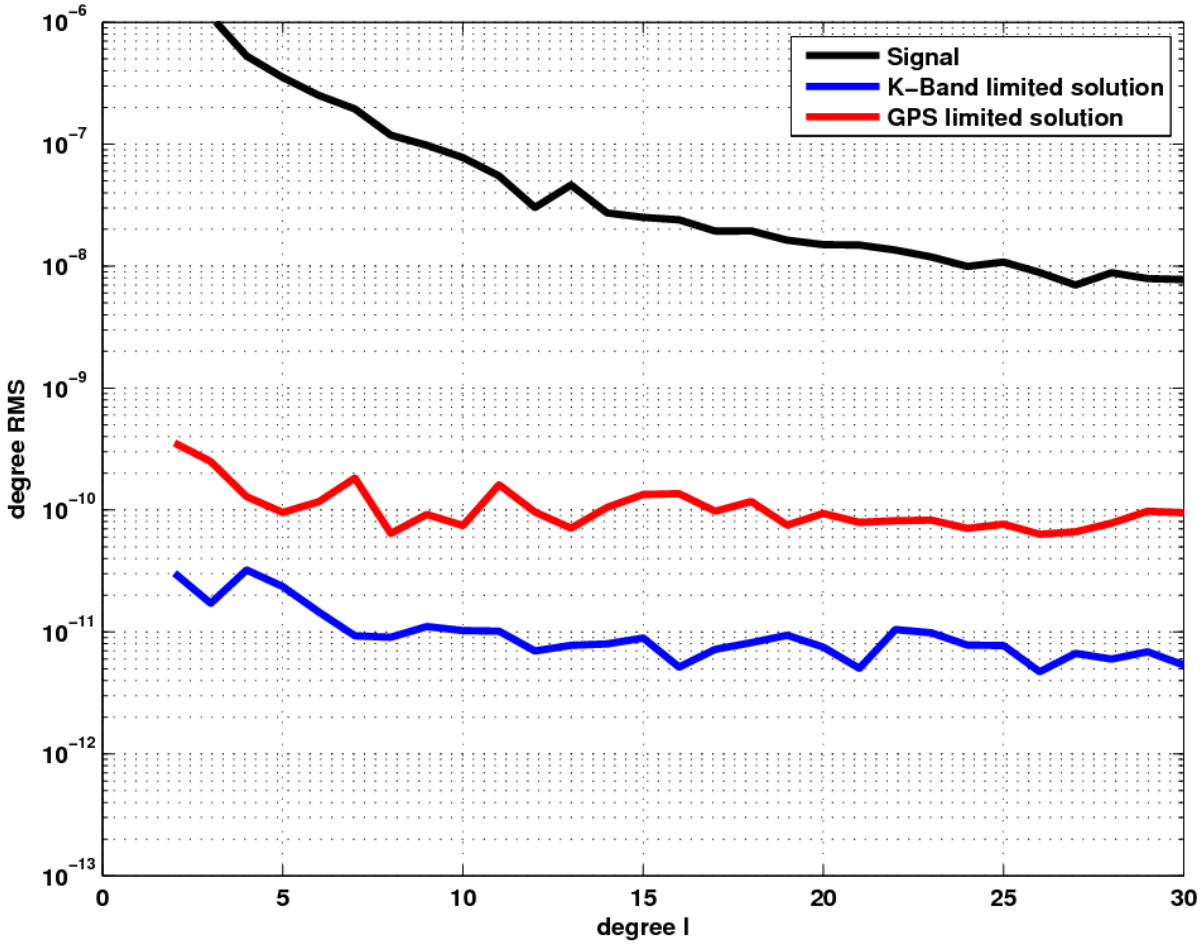
$$\ddot{\vec{X}}_{AB} \cdot \vec{e}^{\,a}_{AB} = \ddot{\rho} + \rho \cdot \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB}$$



$$\begin{aligned}
\rho \cdot \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}_{AB} &= \vec{X}_{AB} \cdot \ddot{\vec{e}}^{\,a}_{AB} &&= -\dot{\vec{X}}_{AB} \cdot \dot{\vec{e}}^{\,a}_{AB} \\
&= -\dot{\rho} \cdot \|\dot{\vec{e}}^{\,a}_{AB}\|^2 &&= -\frac{1}{\rho}\left(\dot{\vec{X}}_{AB} \cdot \dot{\vec{X}}_{AB} - \dot{\rho}^2\right) \\
&= -\rho \cdot \omega_c^2
\end{aligned}$$

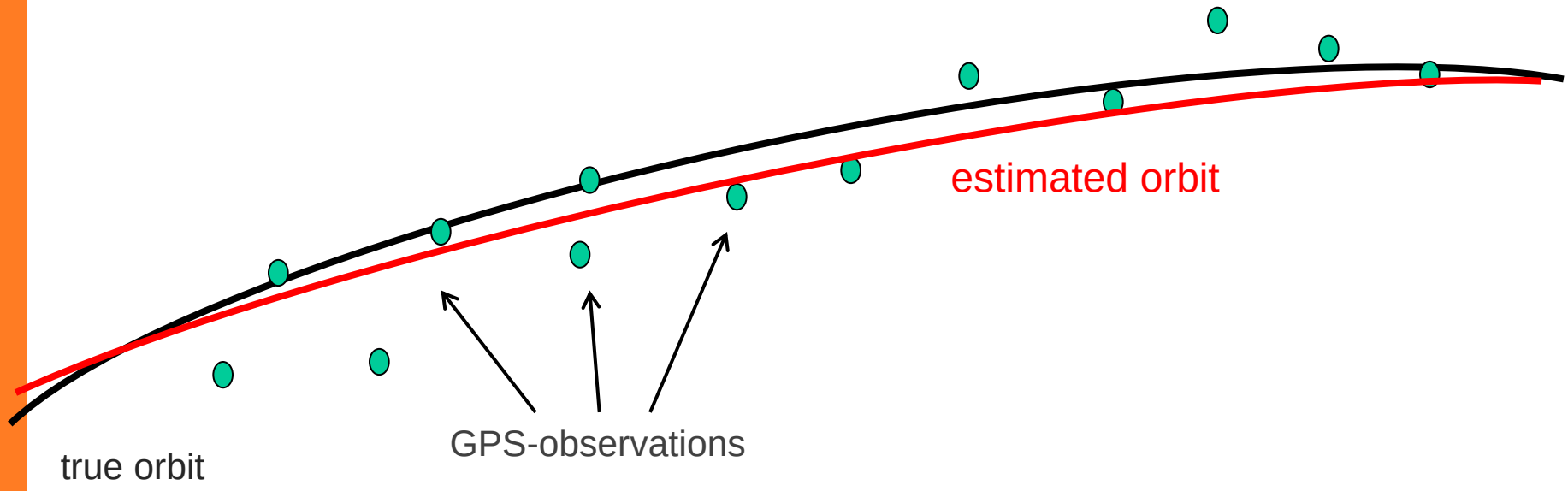
Limitation

Combination of highly precise K-Band observations with comparably low accurate GPS relative velocity



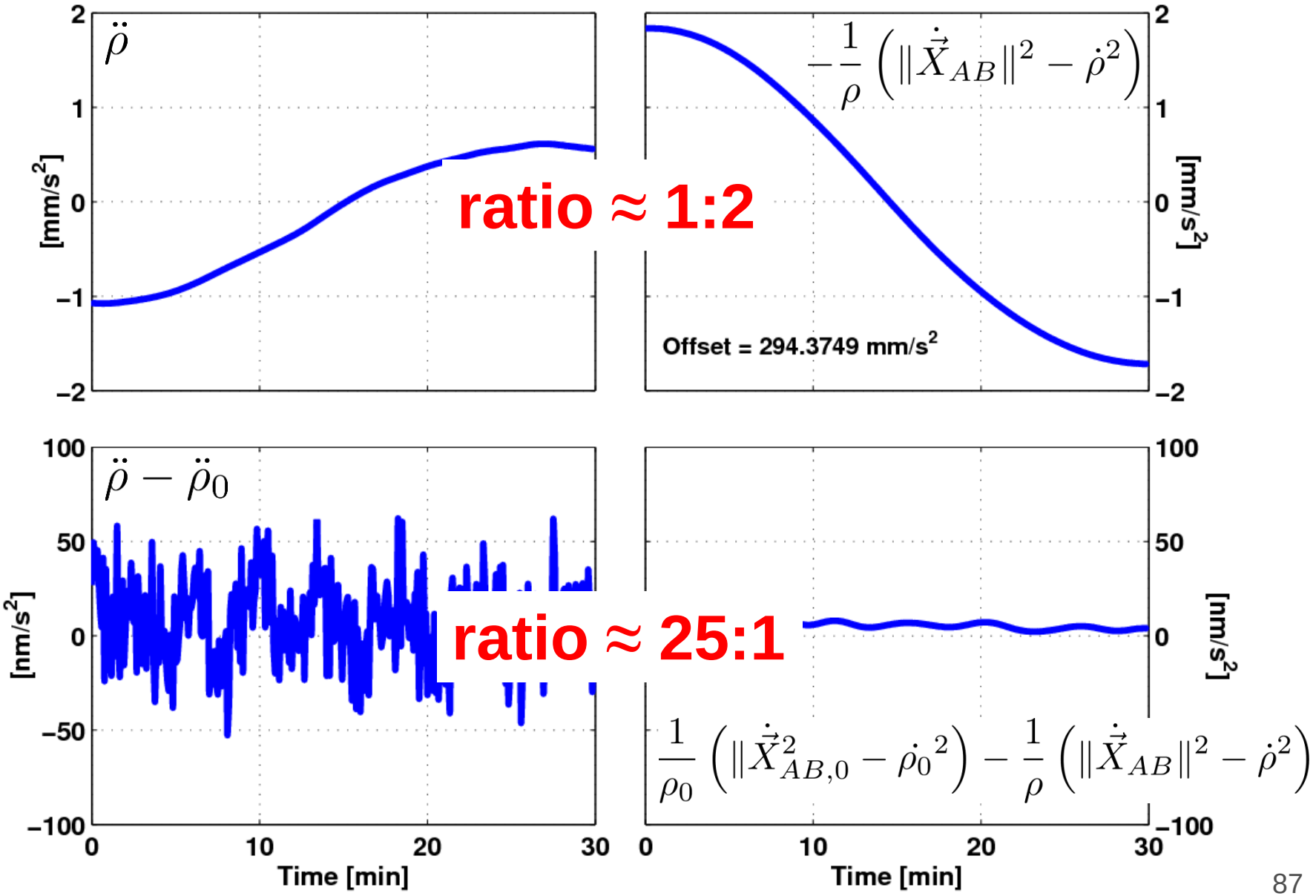
Residual quantities

- Orbit fitting using the homogeneous solution of the variational equation with a known a priori gravity field

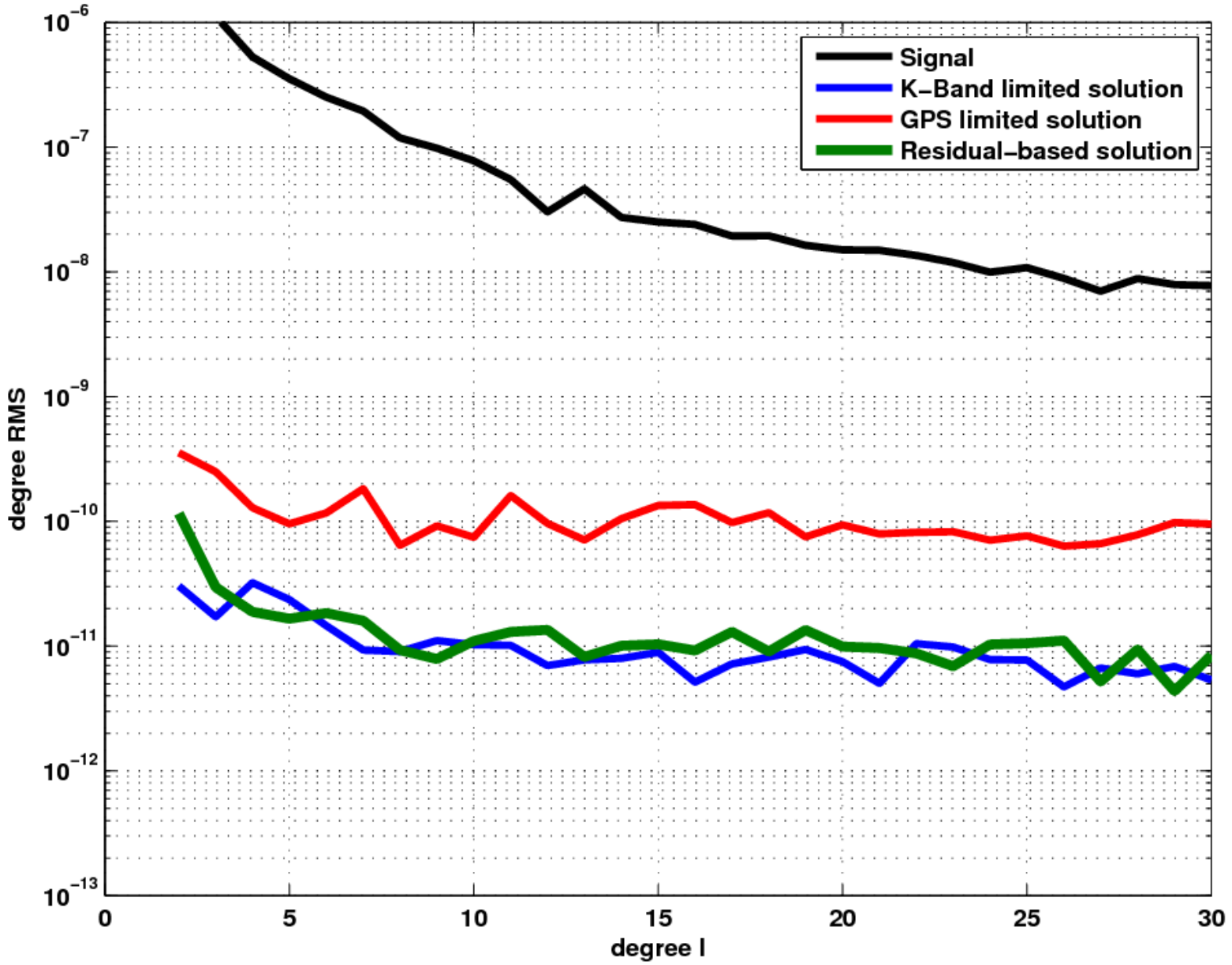


- Avoiding the estimation of empirical parameters by using short arcs

Residual quantities



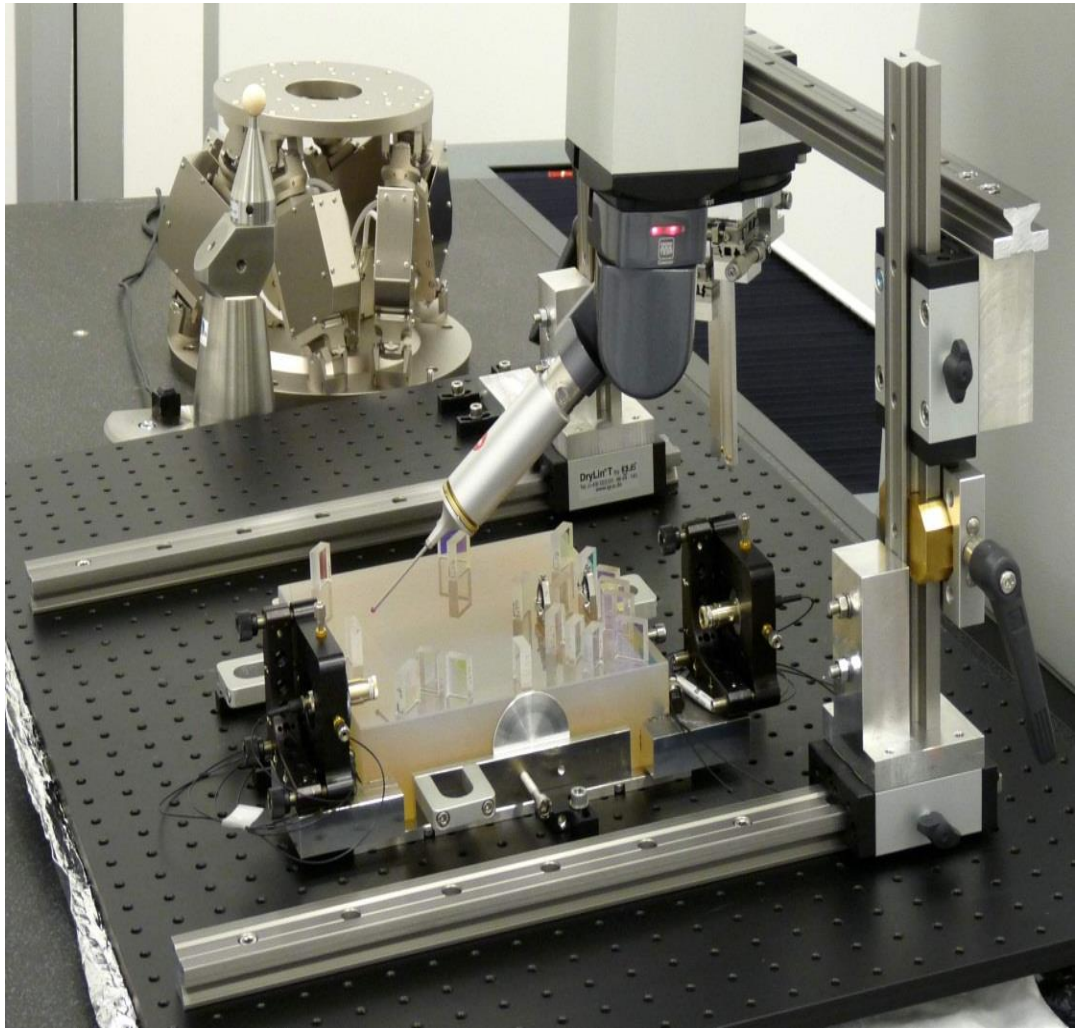
Approximated solution



GRACE FOLLOW-ON

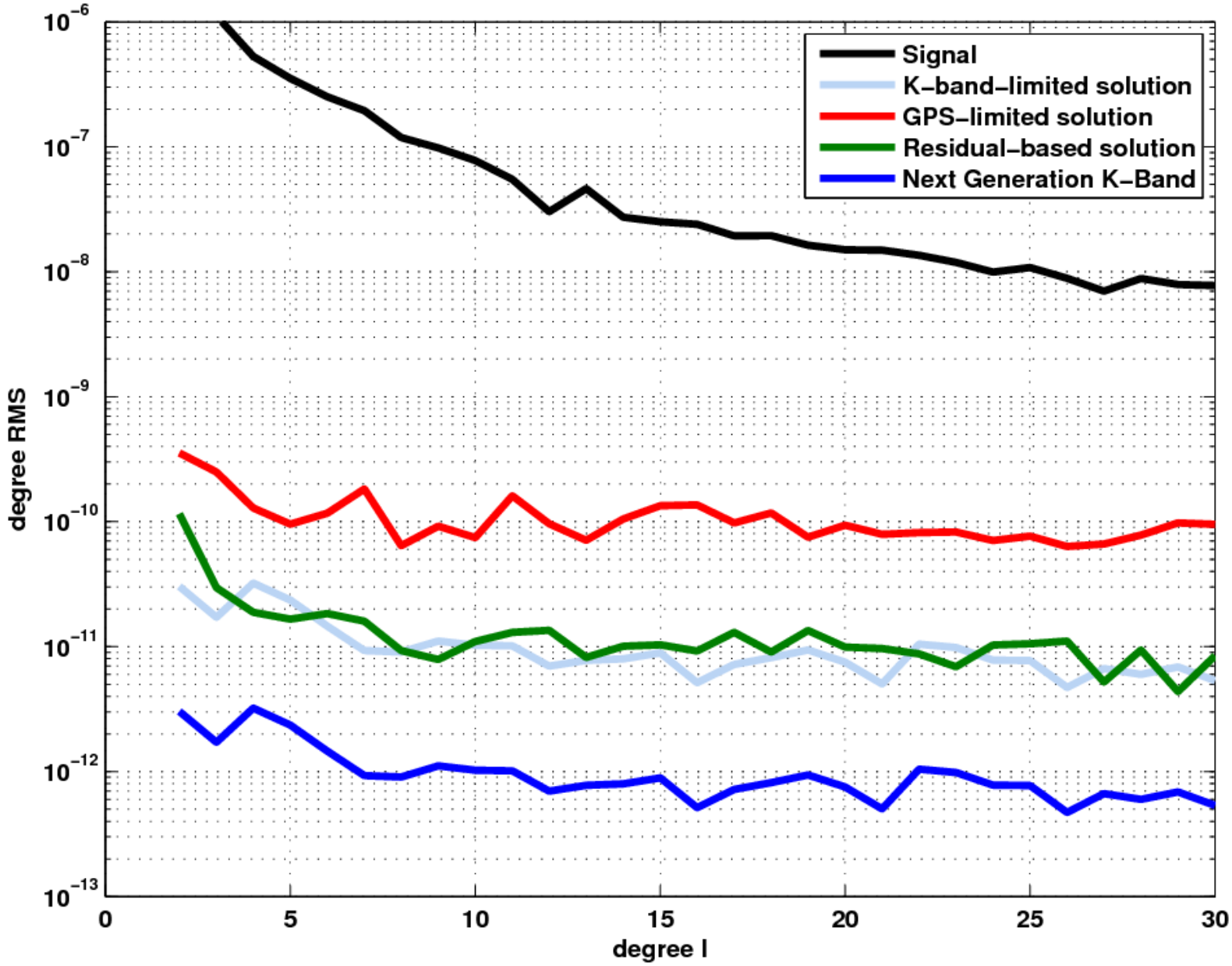
Next generation GRACE

- New type of inter-satellite distance measurement based on laser interferometry
- Noise reduction by factor 10 expected (factor 1000 possible)



M. Dehne, Quest

Solution for next generation GRACE



Variational equations for velocity term

- Reduction to residual quantity insufficient
- Modeling the velocity term by variational equations:

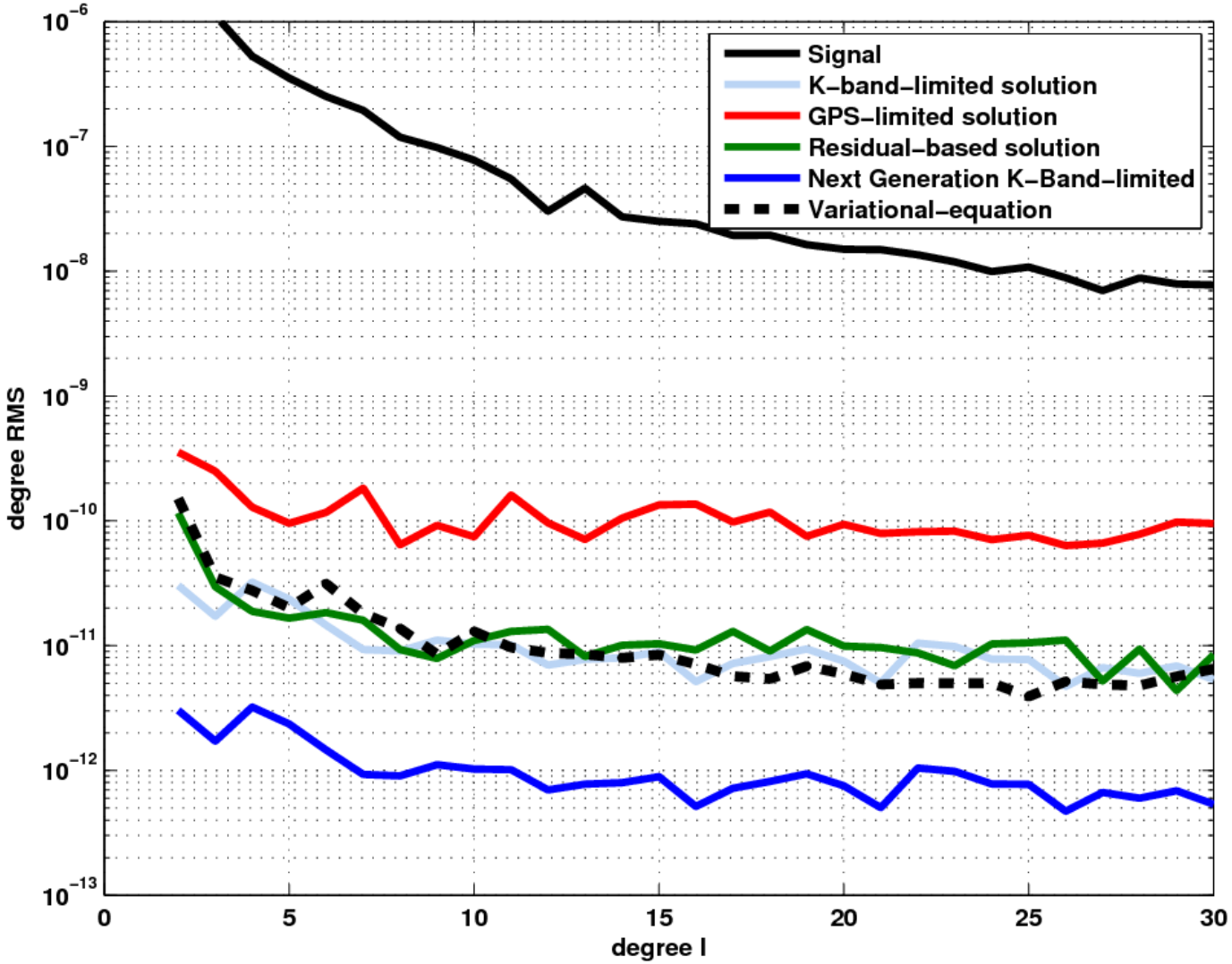
$$f = -\frac{1}{\rho}\left(\|\dot{\vec{X}}_{AB}\|^2 - \dot{\rho}^2\right)$$

$$\frac{\partial f}{\partial p_i} = \frac{\partial f}{\partial x_A}\frac{\partial x_A}{\partial p_i} + \frac{\partial f}{\partial x_B}\frac{\partial x_B}{\partial p_i} + \ldots\ldots\ldots + \frac{\partial f}{\partial \dot{z}_B}\frac{\partial \dot{z}_B}{\partial p_i}$$

- Application of the method of the variations of the constants
- Alternatively: application of the Hill equations

Results

- only minor improvements
- possibly the orbit fit to the GPS positions as a limiting factor



ICCT

Applicability of current GRACE solution strategies to the next generation of inter-satellite range observations

Chair: Matthias Weigelt

Co-Chair: Adrian Jäggi

Objectives

The objectives of the study group are to:

- *investigate each solution strategy, identify approximations and linearizations and test them for their permissibility to the next generation of inter-satellite range observations,*
- *i*
- *q*
  force modeling,
- *investigate the interaction with global and local modeling,*
- *extend the applicability to planetary satellite mission, e.g. GRAIL*
- *establish a platform for the discussion and in-depth understanding of each approach and provide documentation.*

It is **not** the idea,
to find the best approach!

Example: sensitivity to non-gravitational error sources

- Calibration of the accelerometer:
  - Calibration model for GRACE:

$$\tilde{\vec{f}} = (1 + S_1)\,\vec{f} + \vec{b}$$

  - Full calibration model for an accelerometer

$$\tilde{\vec{f}} = (1 + S_1)\,\vec{f} + S_2\vec{f}^{\,2} + N\vec{f} + \vec{b}$$

with: $\vec{b}$ bias

S_1 linear scale matrix

S_2 non-linear scale matrix

N misalignment matrix

- Calibration with GPS-positioning!

Interaction between approaches and modeling

In collaboration with JSG 0.3:

Comparison of methodologies for regional gravity field modeling

Chair: M. Schmidt, Co-Chair: Ch. Gerlach

Users interest in **regional mass variations**, e.g. floodings:

Requirements are:

- **enhanced** spatial **resolution**,
- **arbitrary regions** of interests (river basins, etc.),
- **combinations of measurement techniques** (space gravity missions, airborne, terrestrial, altimetry, other remote sensing, etc.).

Methodology

- Simulation

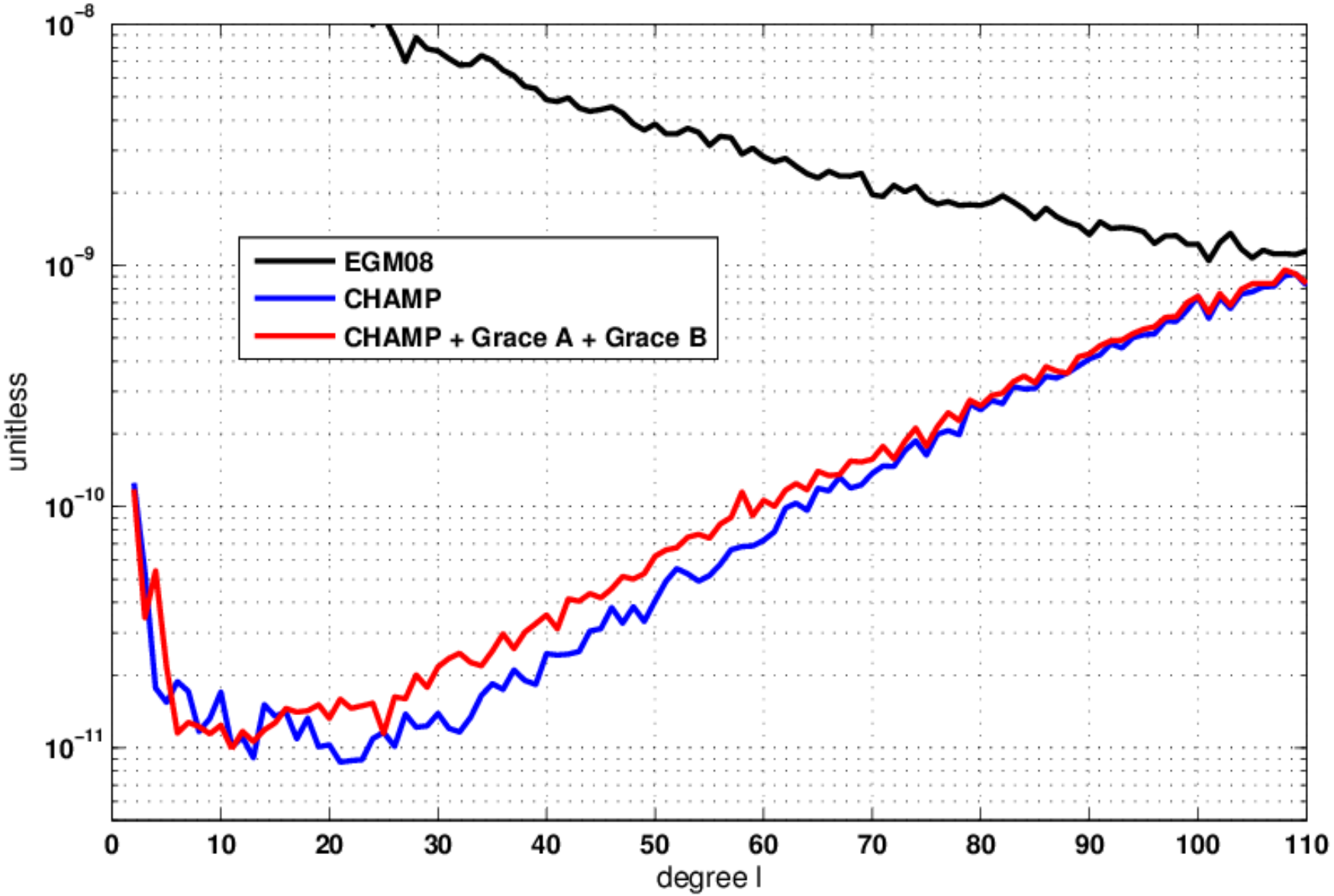
Conclusions

Are we prepared for future satellite missions?

- *Yes, but we should also make use of the time.*
  - e.g. applicability of approaches to GFO
- *Satellite systems (…) need to be better understood.*
  - e.g. spatial aliasing
- *Thinking outside of the box*
  - e.g. high-low SST as transitory technology

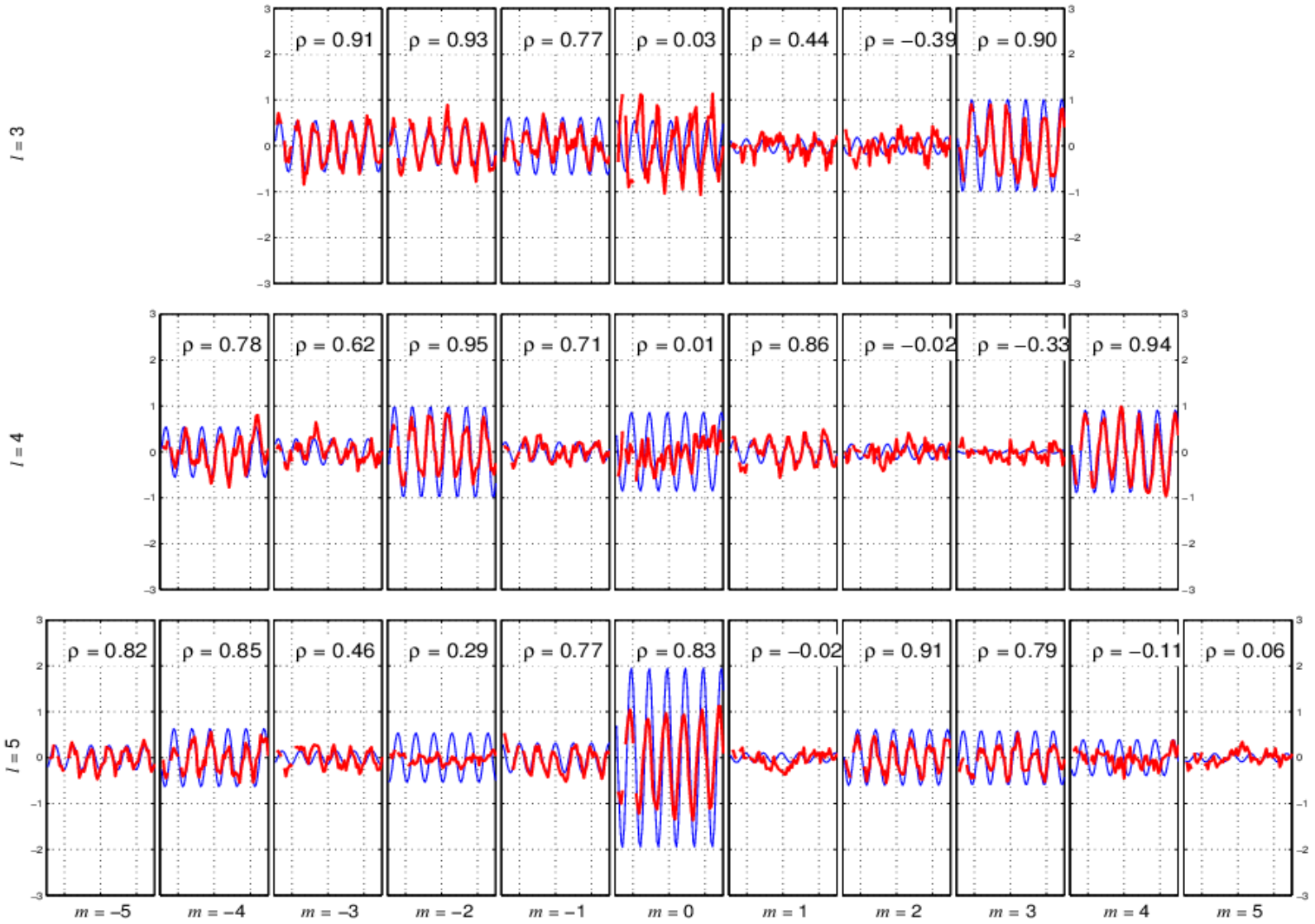
BACKUP

CHAMP + GRACE A + GRACE B



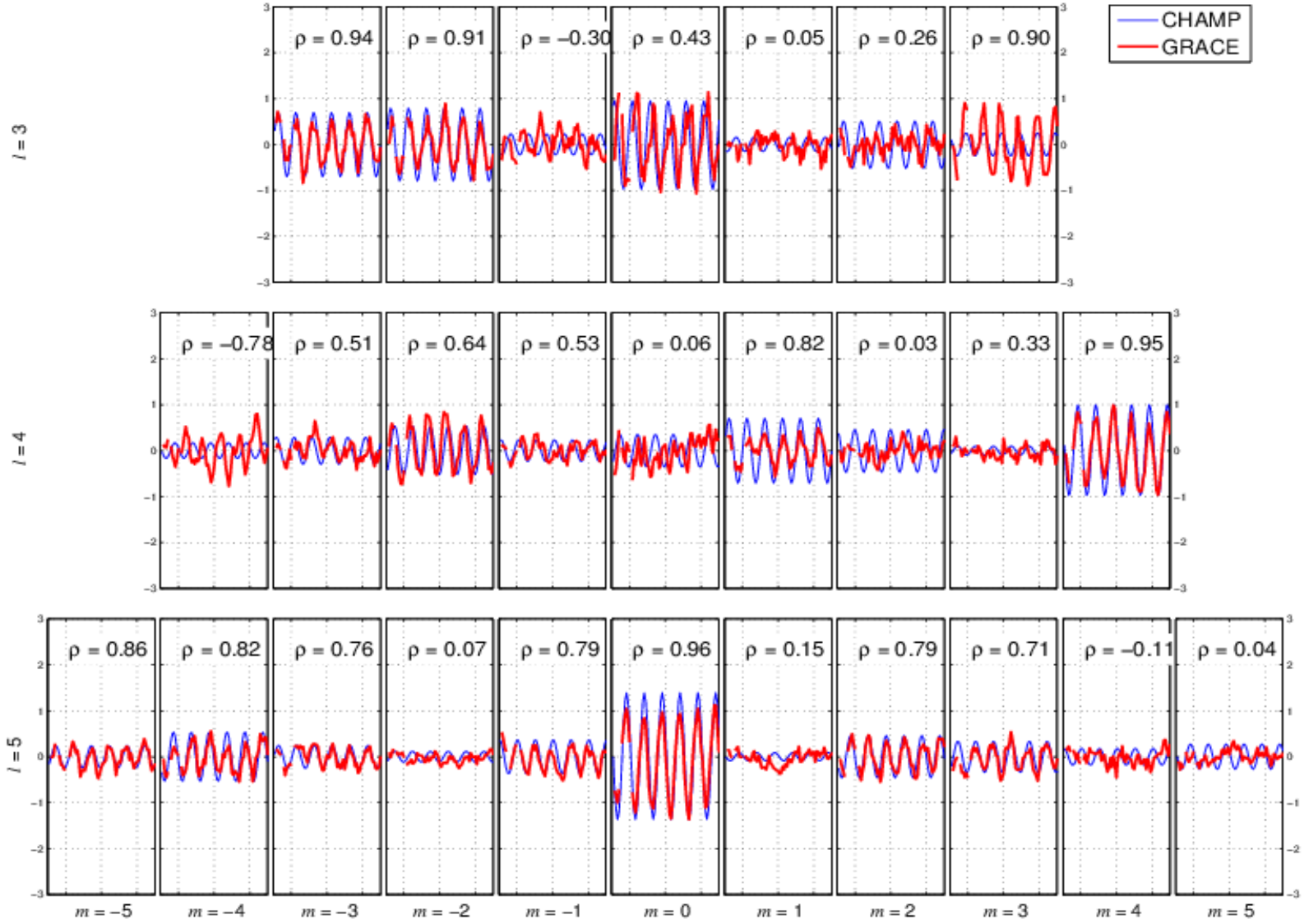
CHAMP + GRACE A + GRACE B

Time series of SH-coefficients: GRACE vs. CH+GA+GB (annual) – scaled by 10^{10}

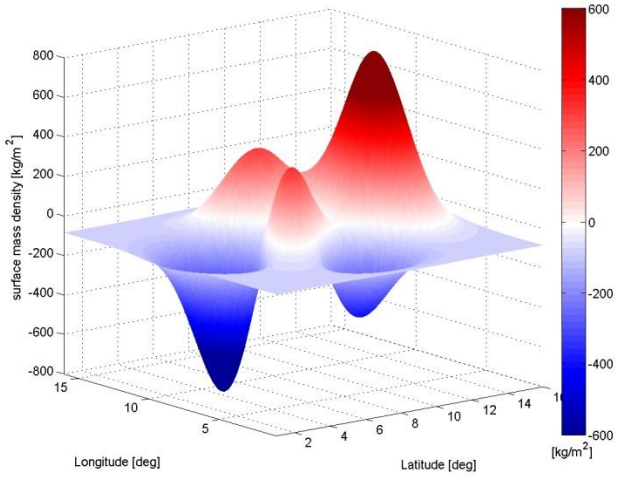


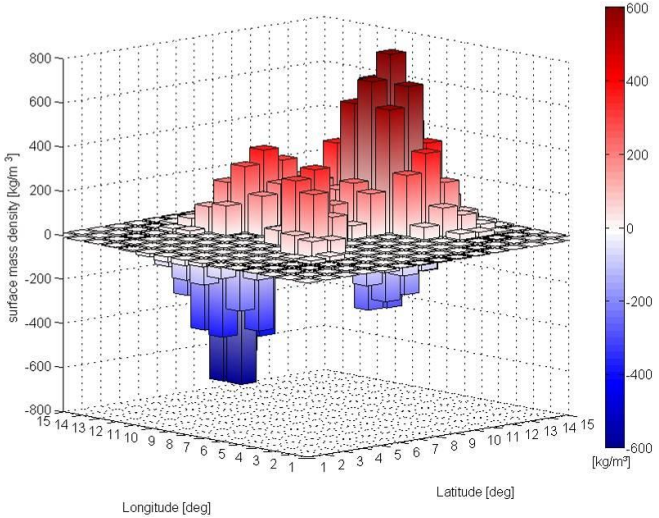
CHAMP

Time series of SH-coefficients: GRACE vs. CHAMP (annual) – scaled by 10^{10}

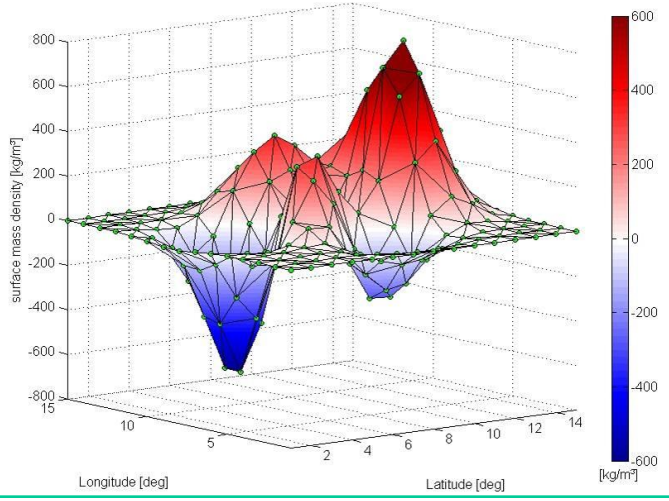


Decomposition of the surface into elements with finite extend (boundary elements)





e.g. blocks



e.g. triangles

Example: Consider the single layer potential

$$V = \int_{\Omega} \frac{\sigma\left(\vec{X}_Q\right)}{\|\vec{X} - \vec{X}_Q\|} d\vec{X}_Q$$

Separation of the surface into elements: $\Omega = \sum_{i=1}^{N} \Omega_i$

Assumption:

$$\vec{X} = \sum_{k=1}^{K} \Phi_{i,k}^{\vec{X}} \vec{X}_{i,k}$$

$$\sigma_i\left(\vec{X}_Q\right) = \sum_{k=1}^{K} \Phi_{i,k}^{\sigma} \sigma_{i,k}\left(\vec{X}_{i,k}\right)$$

The boundary elements are called ***isoparametric*** *if* $\Phi_{i,k}^{\vec{X}} = \Phi_{i,k}^{\sigma}$

Example: Consider the single layer potential

Including transformation to the normal triangle and spherical integration

$$V = \frac{GR^2}{4\pi} \sum_{i=1}^{N} \sum_{k=1}^{K} \sigma_{i,k} \int_{-1}^{1} \int_{-1}^{-\xi} \frac{J_i(\xi,\eta) \cdot \Phi_{i,k}(\xi,\eta) \cdot \cos\phi(\xi,\eta)}{\| \vec{X}(r,\phi,\lambda) - \vec{X}(R,\phi(\xi,\eta),\lambda(\xi,\eta)) \|} d\eta d\xi$$

Integration by Gaussian quadrature

$$\int_{-1}^{1} \int_{-1}^{-\xi} \ldots d\eta d\xi \Rightarrow \sum_{l=1}^{L} \sum_{m=1}^{L} w_l w_m \ldots \qquad \text{with } w_l w_m = 0 \text{ for } m > l$$

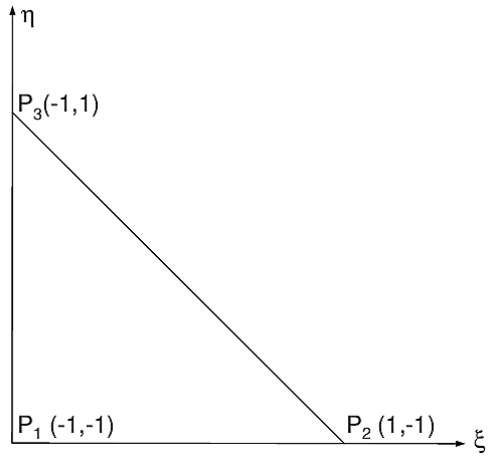
Integration is exact for a polynomial of order 2L.

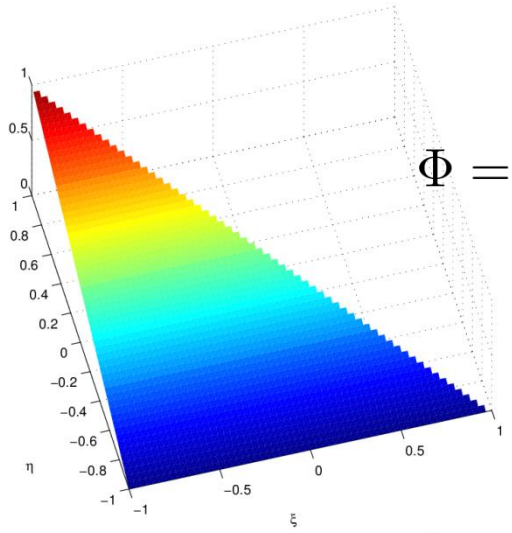
Base functions are

- *strictly space-limited (i.e. band-unlimited)*
- *compact*
- *continuous but not differentiable at the edges*
- *singular if the point of interest lies inside or on the corner of the element*
  - *Weak singularity for potential* $\left(\frac{1}{r}\right)$
  - *Strong singularity for gradient* $\left(\frac{1}{r^3}\right)$

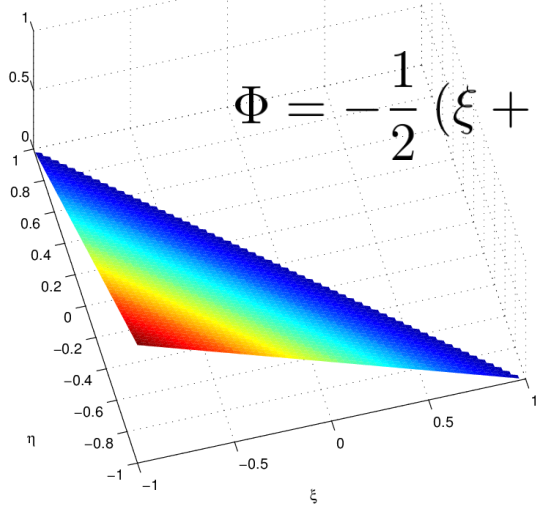
Base function

Example: linear triangle

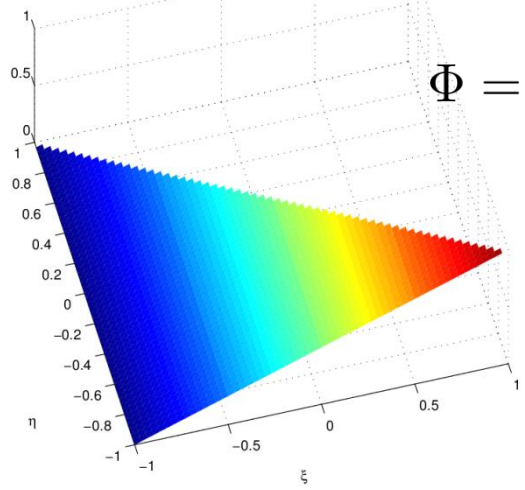




$$\Phi = \frac{1}{2}(\eta + 1)$$



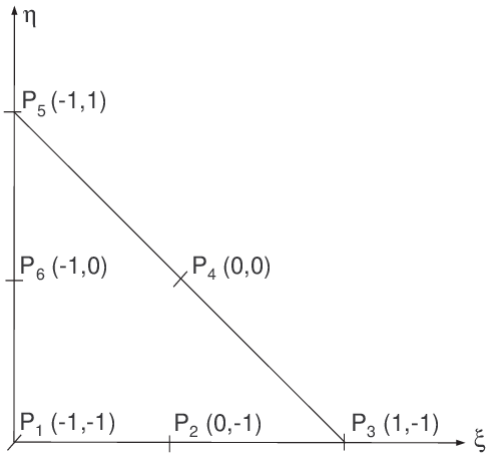
$$\Phi = -\frac{1}{2}(\xi + \eta)$$

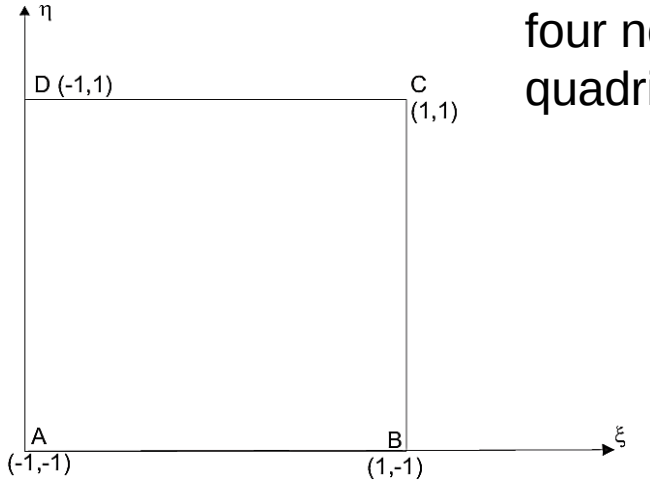


$$\Phi = \frac{1}{2}(\xi + 1)$$

Other base functions

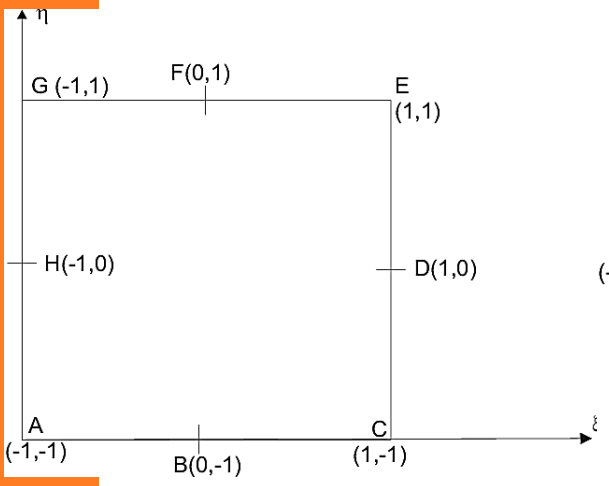
six node triangle



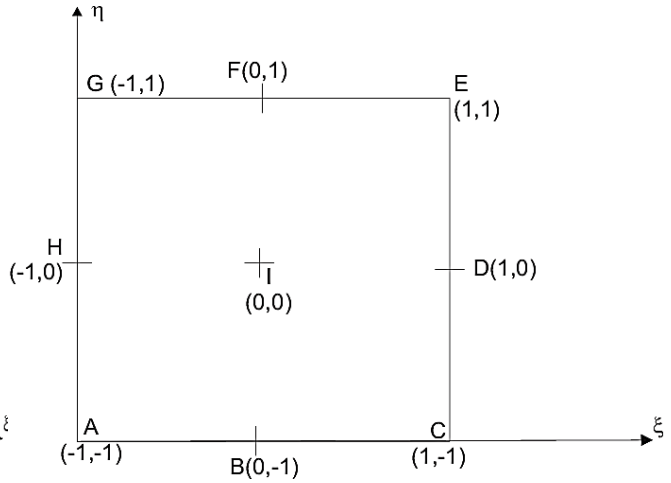


four node quadrilateral

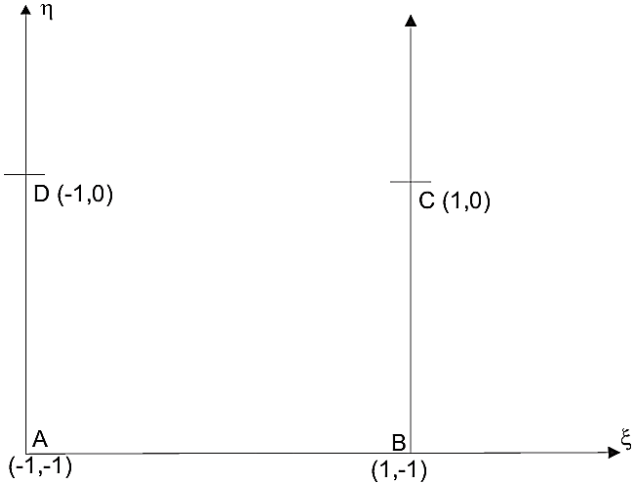
eight node quadrilateral:



nine node quadrilateral:



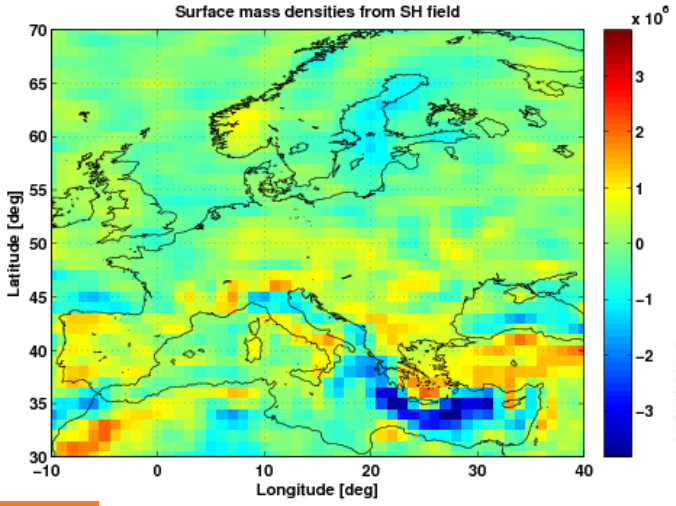
infinite quadrilateral:

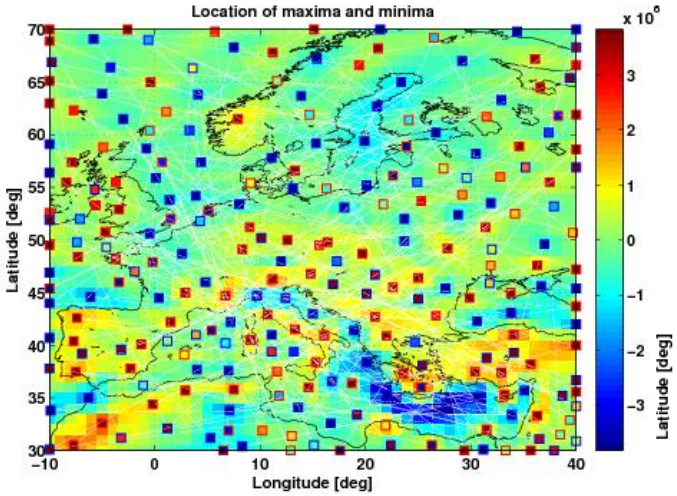


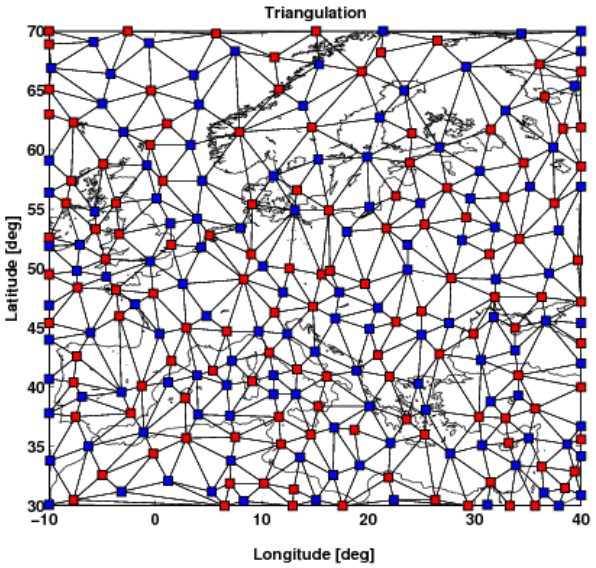
Point Grids

Point grid: <u>any</u> (as long as a proper tessellation is possible)

Currently based on the maxima and minima of an a priori field



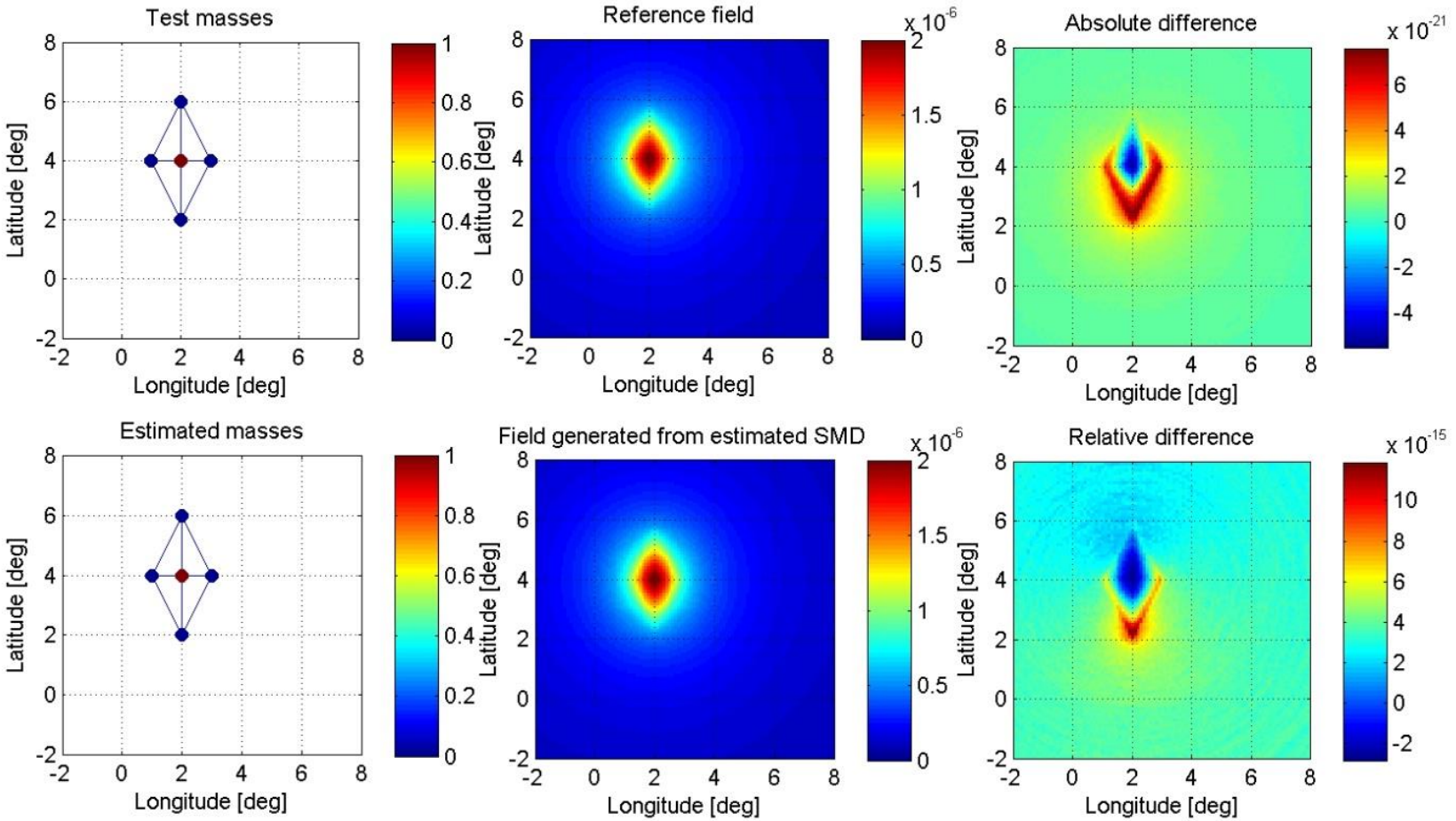




Solution approach

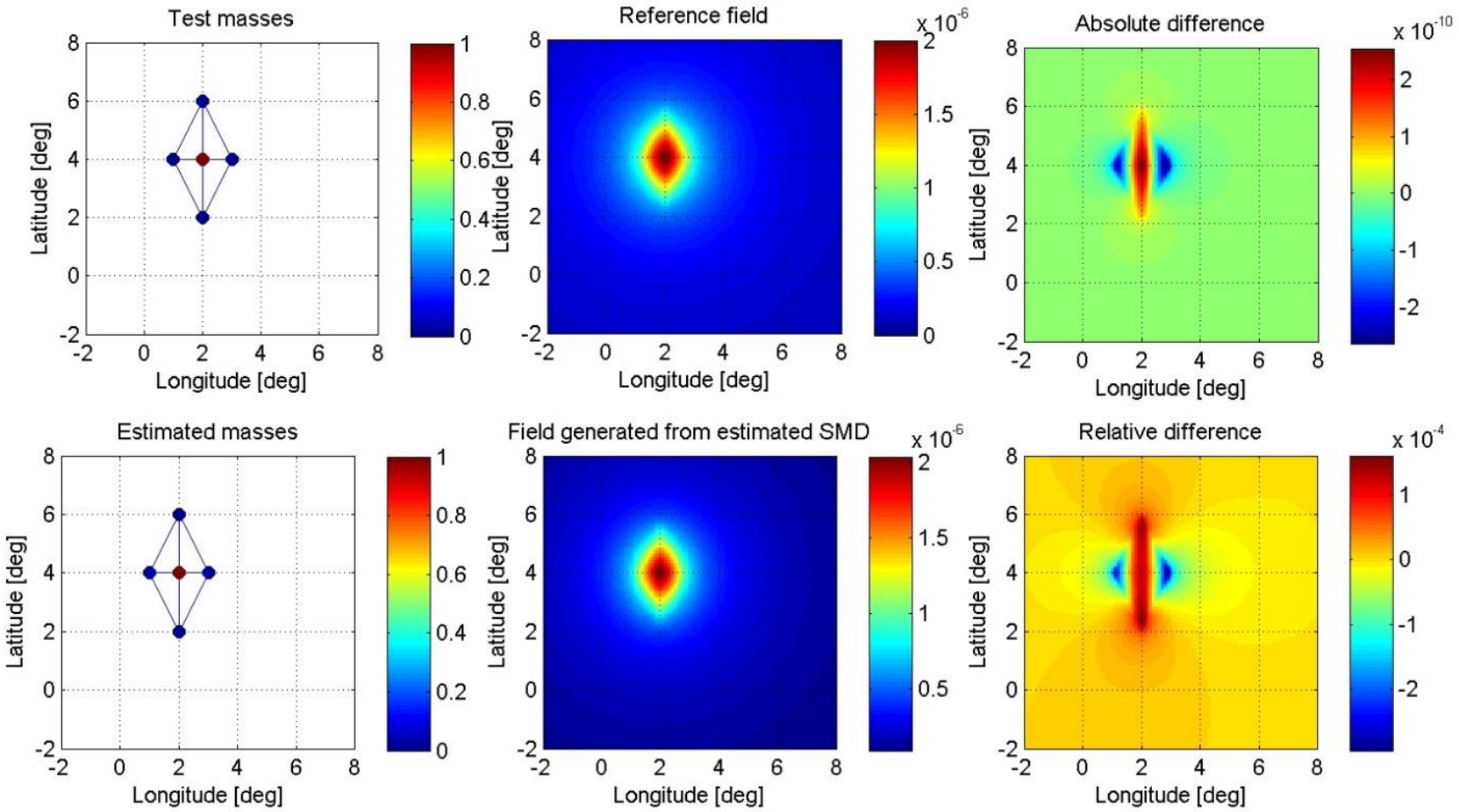
- *Search for maxima and minima of an a priori field*
- *Triangulation by Delaunay tessellation*
- *Least-squares adjustment*
  - *brute-force*
  - *assembly of the normal matrix (singularity!)*
  - *full consideration of the stochastic information*
- *No regularization*
  - *objective: avoid regularization by proper grid*
  - *iterative search for vertices*

Closed loop simulation: noisefree and h=0km



Simulation study

Closed loop simulation: noise=1% and h=400km



Simulation study