Lectures on Supergeometry

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Contents

Disclaimer

The present text is in prefinal form and does not contain any new results. Its first typewritten draft was T. Covolo's Master Thesis, which she presented in 2009/2010. The latter, as well as the current version of our text are based upon a series of (post)doctoral seminars on Supergeometry, a reading and a lecture course, given by N. Poncin at the University of Luxembourg between 2006 and 2011. These lectures were themselves mainly based on [Var04], [Man02], and [DM99]. Since the work grew gradually over a number of years, some references might have been lost or forgotten; in this case, the authors would like to apologize and would be glad to add those references (in particular online encyclopedias such as nLab and Wikipedia were used).

Introduction

Two revolutionary physical theories appeared at the beginning of the 20th century, general relativity and quantum mechanics. Curved spacetime, introduced to deal with gravitation, had almost no impact on quantum physics, since at the atomic level gravitation and curvature may be neglected. However, gradually people realized that there should exist a unique model of spacetime, valid in the infinitely small as well as in the infinitely big context. The resulting need to unify quantum science and gravity led to the insight that radically new models of spacetime might be necessary. One of these models is **Superspace**.

Symmetry is a fundamental concept in Physics and Mathematics; **supersymmetry** (SUSY) was discovered at the beginning of the seventies. One of its predictions is that every elementary particle has a supersymmetric partner of opposite spin-parity: e.g. the SUSY partner of a boson is a fermion and vice versa. The Pauli exclusion principle entails that two fermions cannot occupy the same quantum state. This means that the Hilbert state space associated to a q-fermion system is the exterior power $\wedge^q \mathcal{H}_1$ of the Hilbert state space $\mathcal{H}_1$ of a single fermion. Similarly, the state space of a p-boson system is the symmetric power $S^p\mathcal{H}_0$ of the single boson space $\mathcal{H}_0$. Finally, the many particle space is

$$S\mathcal{H}_0 \otimes \wedge\mathcal{H}_1.$$

If we combine bosons and fermions in a unique framework, the base space is a superspace $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ and the quantum state space of a many boson and fermion system is the supersymmetric algebra of $\mathcal{H}$. This algebra is isomorphic to $S\mathcal{H}_0 \otimes \wedge\mathcal{H}_1$.

In the following, we provide a comprehensive introduction to **Superalgebra and Supergeometry**. In particular, we detail **differential and integral calculus** on supermanifolds.

The text is structured as follows.

The first two chapters contain all necessary preliminaries.

Chapter 1 deals with Superalgebra. The Deligne and Bernstein-Leites formalisms naturally appear from the construction of algebraic de Rham complexes. We introduce the superdeterminant or Berezinian and prove its multiplicativity.

The second chapter contains a brief introduction to Category and Sheaf Theory.

In Chapter 3, we define Kostant-Berezin-Leites supermanifolds, confining ourselves to the smooth category. Maximal ideals of the stalks of the sheaf of superfunctions are described, the smooth structure on the topological base space of a supermanifold is constructed, and the projection of the structure sheaf of a supermanifold onto the function sheaf of the base manifold is built. We then investigate extensively supermorphisms, and show that their local form is similar to that of classical morphisms – a property that makes Supergeometry a reasonable theory. Smoothness of the continuous base map of a supermorphism is proven. A number of proofs, in particular the construction and study of differential operators on supermanifolds, are based upon Hadamard's lemma, also known as polynomial approximation technique – which we explain in depth.

Special emphasis is put on differential calculus on supermanifolds and in particular on the construction of the super de Rham complex, which, just as integration on supermanifolds, is addressed only sparsely in the literature.

The last chapter finally provides a complete discussion of integration over supermanifolds. We first recall integration theory over classical manifolds. When passing to supermanifolds, the sheaf of top forms is to be replaced by the Berezinian sheaf, which we glue from trivial local line bundles and that we construct intrinsically as a quotient of super differential operators with coefficients in super differential forms. This allows defining the density bundle and hence integration over a supermanifold.

Chapter 1

Superalgebra

1.1 Super Vector Spaces

In the following we set $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \simeq \{0, 1\}$ and denote by k a field of characteristic 0, typically $\mathbb{R}$ or $\mathbb{C}$.

Definition 1. *A super vector space V over k is a $\mathbb{Z}_2$-graded vector space*

$$V = V_0 \oplus V_1,$$

where V_0 and V_1 are (ordinary) vector spaces over k. If V_0 and V_1 have dimension p and q, respectively, V is said to have dimension $p|q$.

The elements of V_0 (resp. of V_1) are called *even* (resp. *odd*) and elements in $V_0 \cup V_1$ are called *homogeneous*. The *parity* $p(v)$ is defined for all homogeneous elements v by $p(v) = 0$, for $v \in V_0$, and $p(v) = 1$, for $v \in V_1$.

Examples.

- *The ground field k can be viewed as a super k-vector space of dimension 1|0:*

$$k = k \oplus \{0\}.$$

- *Of course k^{p+q} is a k-vector space of dimension $p + q$. Let now $(e_i)_{1 \leq i \leq p+q}$ be the canonical basis and define e_i to be even (resp. odd) for $1 \leq i \leq p$ (resp. for $p + 1 \leq i \leq p + q$), whereas the elements of the field k are considered as even. Then, since any $v \in k^{p+q}$ reads*

$$v = \sum_{i=1}^{p} e_i v^i + \sum_{i=p+1}^{p+q} e_i v^i$$

and since the parity of a product is (always) the sum of the parities, we can view k^{p+q} as a super k-vector space of dimension $p|q$, which we denote by $k^{p|q}$.

Definition 2. *A super morphism between two super vector spaces V and W is a linear map $\ell : V \to W$, which preserves the grading, i.e. $\ell(V_i) \subset W_i$ ($i \in \{0,1\}$). The space consisting of all these morphisms is denoted by* $\mathrm{Hom}(V, W)$.

Super vector spaces and super morphisms form a *category* usually denoted by `SVect`.

Just as vectors, morphisms may have a parity. Indeed, besides the grading or parity preserving linear maps, we can consider as well the parity reversing linear maps, i.e. the operators $\ell : V \to W$, such that $\ell(V_i) \subset W_{i+1}$ ($i \in \{0,1\}$, $i+1$ denotes the sum in $\mathbb{Z}_2$). The linear maps ℓ that verify $\ell(V_i) \subset W_{i+\mathbf{0}}$ – the super morphisms – (resp. that verify $\ell(V_i) \subset W_{i+\mathbf{1}}$) are said to have parity $\mathbf{0}$ (resp. $\mathbf{1}$) and are called the *even morphisms* (resp. *odd morphisms*). The vector space of even (resp. odd) morphisms is denoted by $\mathbf{Hom}_0(V, W)$ – or, as above, by $\mathrm{Hom}(V, W)$ – (resp. $\mathbf{Hom}_1(V, W)$). The sum

$$\mathbf{Hom}(V, W) := \mathbf{Hom}_0(V, W) \oplus \mathbf{Hom}_1(V, W)$$

of these spaces of even and odd morphisms between the super vector spaces V and W is of course again a super vector space. Therefore **Hom** is (often) called the *internal* Hom. If $V = W$, we write, as usually, $\mathbf{End}(V)$ instead of $\mathbf{Hom}(V, V)$.

Example 1. *Let us come back to the super vector space $V = k^{p|q}$. Since it has a canonical basis, we can represent the (endo)morphisms in*

$$\mathbf{End}(k^{p|q}) = \mathbf{End}_0(k^{p|q}) \oplus \mathbf{End}_1(k^{p|q})$$

by matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are $p \times p, p \times q, q \times p$ and $q \times q$ matrices, respectively. It is easily seen that the even and odd (endo)morphisms are, respectively, of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ and } \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Note that these matrices have all their entries in the field k, i.e. all the entries are even.

1.2 Superalgebras

Definition 3. *A superalgebra $\mathcal{A}$ is a super vector space $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ endowed with an associative algebra structure with unit, which respects the parity, i.e.*

$$p(ab) = p(a) + p(b),$$

for any homogeneous $a, b \in \mathcal{A}$.

This means that, for all $i, j \in \{0, 1\}$,

$$\mathcal{A}_i \mathcal{A}_j \subset \mathcal{A}_{i+j}$$

and implies obviously that the unit is even.

A superalgebra is said to be *supercommutative*, if

$$ab = (-1)^{p(a)p(b)} ba, \tag{1.1}$$

for any homogeneous $a, b \in \mathcal{A}$. Hence, in a supercommutative superalgebra odd elements anticommute and are nilpotent, i.e. $ab = -ba$ and $a^2 = 0$, for any odd a and b.

Example 2. *Let M be a smooth manifold. The sum*

$$\Omega(M) = \Omega_0(M) \oplus \Omega_1(M)$$

of even and odd differential forms of M, where the parity is given by the cohomological degree, endowed with the exterior product, is a supercommutative superalgebra.

In the following, all (associative unital) superalgebras are assumed to be supercommutative.

Remark 1. Equation (1.1) is known as the *Koszul sign rule* **(Jean-Louis Koszul, born 3 January 1921, is a French mathematician), which underlies the whole supermathematics. It requires that whenever two homogeneous symbols a and b are exchanged, the sign $(-1)^{p(a)p(b)}$ must appear.**

In most formulas below, the involved symbols are implicitly assumed to be homogeneous.

1.3 Supermodules

Definition 4. *A super $\mathcal{A}$-module M is a module $M = M_0 \oplus M_1$ (direct sum in the category of abelian groups) over a superalgebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, such that the multiplication by scalars respects the parity, i.e. $p(am) = p(a) + p(m)$, for $a \in \mathcal{A}$ and $m \in M$.*

This definition means of course that $\mathcal{A}_i M_j \subset M_{i+j}, i, j \in \mathbb{Z}_2$.

Above we implicitly considered left $\mathcal{A}$-module structures; however, any left $\mathcal{A}$-module structure gives rise to a right $\mathcal{A}$-module structure, defined by: $ma := (-1)^{p(a)p(m)} am$.

Example 3. *Any superalgebra is a supermodule over itself.*

More generally, we consider in this text supermodules over superrings. A superring is a ring $R = R_0 \oplus R_1$ (direct sum in the category of abelian groups) such that $R_i R_j \subset R_{i+j}$. A super R-module is a module $M = M_0 \oplus M_1$ (direct sum in the category of abelian groups) such that $R_i M_j \subset M_{i+j}$.

As before, $\mathbf{Hom}(M, N) = \mathbf{Hom}_0(M, N) \oplus \mathbf{Hom}_1(M, N)$ denotes the set of all morphisms between two super $\mathcal{A}$-modules M and N. The homogeneous morphisms $t \in \mathbf{Hom}_i(M, N)$, $i \in \mathbb{Z}_2$, are the group morphisms $t : M \to N$ of parity $p(t) = i$ that are super $\mathcal{A}$-linear, i.e. that verify

$$t(am) = (-1)^{p(a)p(t)} a\, t(m),$$

for $a \in \mathcal{A}$ and $m \in M$. This $\mathcal{A}$-linearity condition is advantageously rephrased by means of the corresponding right module structure. Indeed, it is easily checked that it then reads

$$t(ma) = t(m)a. \tag{1.2}$$

Observe also that, since $k \simeq k \cdot 1_{\mathcal{A}} \subset \mathcal{A}_0 \subset \mathcal{A}$, where $1_{\mathcal{A}}$ denotes the identity element of $\mathcal{A}$, $\mathcal{A}$-linearity entails k-linearity.

The internal Hom set $\mathbf{Hom}(M, N)$ admits (of course) itself an $\mathcal{A}$-module structure: $(t + t')(m) := t(m) + t'(m)$, $(at)(m) := a(t(m))$. Note that $p(at) = p(a) + p(t)$; hence, $\mathbf{Hom}_0(M, N)$ is not an $\mathcal{A}$-module, but a vector space over k.

Definition 5. *A super $\mathcal{A}$-module morphism between two super $\mathcal{A}$-modules is an even morphism. The space $\mathbf{Hom}_0(M, N)$ of all the supermodule morphisms between M and N is often denoted simply by* $\mathrm{Hom}(M, N)$*.*

Super $\mathcal{A}$-modules and super $\mathcal{A}$-module morphisms for a category `SMod`.

Definition 6. *A free super $\mathcal{A}$-module of rank $p|q$ is a super $\mathcal{A}$-module that admits a basis $(e_i)_{1 \le i \le p+q}$, where e_i is even (resp. odd) for $1 \le i \le p$ (resp. for $p+1 \le i \le p+q$). In this case, the module is denoted by $\mathcal{A}^{p|q}$. Thus, we have*

$$\mathcal{A}^{p|q} = e_1 \mathcal{A} \oplus \cdots \oplus e_{p+q} \mathcal{A}.$$

As in the classical setting, the 'vectors' $m \in \mathcal{A}^{p|q}$ of a free super $\mathcal{A}$-module can be represented by the columns of their coordinates, and the morphisms $t \in \mathbf{Hom}(\mathcal{A}^{p|q}, \mathcal{A}^{r|s})$ between two free modules are represented by matrices. More precisely, to avoid signs, see Equation (1.2), we prefer right coordinates and write

$$m \simeq \mathfrak{a} = \begin{pmatrix} a^1 \\ \vdots \\ a^{p+q} \end{pmatrix},$$

if $m = \sum_i e_i a^i$. When m is even (resp. odd), the a^i are even (resp. odd), for $1 \le i \le p$, and odd (resp. even), for $p+1 \le i \le p+q$. For a morphism $t : M \to N$, we have

$$t(m) = t\left(\sum_j e_j a^j\right) = \sum_j t(e_j)a^j = \sum_i \sum_j e'_i T^i_j a^j;$$

here, (e_i) and (e'_i) denote bases of M and N, respectively. Thus t can be identified with a matrix:

$$t \simeq T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{M}(r|s \times p|q, \mathcal{A}).$$

If t is even (resp. odd), this matrix T is of the type

$$\begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix} \left(\text{resp.} \begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix} \right),$$

where "even" (resp. "odd") refers to matrices all entries of which are even (resp. odd). Eventually,

$$t(m) \simeq T\mathfrak{a},$$

i.e. the representative matrix acts from the left on columns of right coordinates. For the thus induced parity, the set of matrices $\mathrm{M}(r|s \times p|q, \mathcal{A})$ is a super $\mathcal{A}$-module that is naturally isomorphic to $\mathbf{Hom}(\mathcal{A}^{p|q}, \mathcal{A}^{r|s})$. *This means that the left multiplication of a matrix T by a 'scalar' a is obtained by multiplying any element of A, B by a and any element of C, D by $(-1)^{p(a)}a$.* Indeed, it is readily seen that $(at)(e_j) = (-1)^{p(a)p(e'_i)}\, e'_i\, aT^i_j$. The direct sum of the modules of matrices can be endowed with the usual matrix multiplication – which corresponds to the composition of morphisms.

1.4 Super tensor calculus

In this section we recall some facts from tensor algebra.

1.4.1 Tensor product over a commutative ring

The tensor product $M \otimes_R N$ of two modules M, N over a *commutative* ring R is defined as the quotient of the free R-module $R^{(M\times N)}$ generated by $M \times N$ – and thus made up by the combinations

$$\sum_{(x,y)\in M\times N} r_{(x,y)} e_{(x,y)},$$

where only a finite number of coefficients $r_{(x,y)} \in R$ are nonzero – by the R-submodule generated by the elements that "correspond to R-bilinearity", i.e. by the elements

$$-e_{(x+x',y)} + e_{(x,y)} + e_{(x',y)}, \; -e_{(x,y+y')} + e_{(x,y)} + e_{(x,y')},$$

$$-e_{(rx,y)} + re_{(x,y)},\ -e_{(x,ry)} + re_{(x,y)}.$$

This tensor product R-module together with the obvious R-bilinear map

$$\otimes : M \times N \ni (x,y) \mapsto x \otimes y = [e_{(x,y)}] \in M \otimes_R N$$

are universal.

1.4.2 Tensor product over a noncommutative ring

In the case of a *noncommutative* ring R, we consider a right R-module M and a left R-module N and define the tensor product $M \otimes_R N$ as a $\mathbb{Z}$-module, i.e. as an abelian group, and more precisely as the quotient of the free $\mathbb{Z}$-module $\mathbb{Z}^{(M\times N)}$ generated by $M\times N$, by the $\mathbb{Z}$-submodule generated by the elements that "correspond to the weakened bilinearity", i.e. by the elements

$$-e_{(x+x',y)} + e_{(x,y)} + e_{(x',y)},\ -e_{(x,y+y')} + e_{(x,y)} + e_{(x,y')},$$

$$-e_{(xr,y)} + e_{(x,ry)}.$$

The tensor product $\mathbb{Z}$-module $M \otimes_R N$ and the natural weakly bilinear map

$$\otimes : M \times N \ni (x,y) \mapsto x \otimes y = [e_{(x,y)}] \in M \otimes_R N$$

are universal. This means that the functor $- \otimes_R N$ from $\mathtt{Mod}_R$ to $\mathtt{AbGrp}$ is the left adjoint of the functor $\mathrm{Hom}_{\mathbb{Z}}(N,-)$, where the right module structure on $\mathrm{Hom}_{\mathbb{Z}}(N,P)$ is defined by $(fr)(n) = f(rn)$, i.e. we have

$$\mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N, P) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(N,P)),$$

functorially in M and P. In general it is not possible to define an R-module structure on $M \otimes_R N$ – investigate e.g. $r(mr' \otimes n)$.

Let us recall that an (S,R)-bimodule is an abelian group M equipped with a left S-module structure and a right R-module structure, which are compatible in the sense that $s(mr) = (sm)r$. If M is an (S,R)-bimodule, then $M \otimes_R N$ is a left S-module, with the obvious definition of the S-action (which does not lead as above to a contradiction in view of the compatibility of the left and right module structures). Similarly, if N is an (R,T)-bimodule, then $M \otimes_R N$ is a right T-module. If M and N have each bimodule structures as above, then the tensor product is an (S,T)-bimodule.

1.4.3 Tensor product over a supercommutative ring

Let us come to the tensor product of supermodules over *supercommutative rings.* If M and N denote supermodules over a supercommutative ring R, their tensor product as super R-modules is their tensor product as right and left modules over the noncommutative ring R. This $\mathbb{Z}$-module $M \otimes_R N$ is naturally $\mathbb{Z}_2$-graded:

$$M \otimes_R N = \bigoplus_{k\in\mathbb{Z}_2}(M \otimes_R N)_k = \bigoplus_{k\in\mathbb{Z}_2} \bigoplus_{i+j=k} \{\sum m \otimes n : m \in M_i, n \in N_j\}.$$

Since M (resp. N) is not only a right (resp. left) R-module, but even an (R, R)-bimodule, their tensor product is an (R, R)-bimodule as well, and the left and right module structures are related by the Koszul sign rule. Finally, the tensor product of two supermodules over a supercommutative ring R is itself a super R-module.

As usual, also the preceding tensor product can be characterized as a universal object. To formulate this fact, let us first consider homogeneous multilinear maps from super R-modules $S_1, \ldots, S_m$ to a super R-module T. We write the image of elements $(s_1, \ldots, s_m)$ by such a map ℓ in the form $s_1 \ldots s_i\, \ell\, s_{i+1} \ldots s_m$, $0 \le i \le m$. The interest in these notations originates for instance from the fact that many "products" are denoted by $s_1\, \ell\, s_2$. Each value of i actually leads to an a priori different concept of R-multilinear maps ℓ. Indeed, such a map is requested to be multiadditive and multilinear for multiplication by scalars in R, a condition that is defined by means of the Koszul sign rule applied to all commutating symbols, see [Man02]. We denote by $L^i = L^i(S_1 \times \ldots \times S_m, T)$ the set of i-R-multilinear maps. This set has a natural structure of super R-module: the $\mathbb{Z}_2$-grading and the group structure are obvious, the multiplication by scalars is defined by $s_1 \ldots s_i\,(r\ell)\, s_{i+1} \ldots s_m = s_1 \ldots (s_i r)\, \ell\, s_{i+1} \ldots s_m$. It can be shown that all the super R-modules L^i are isomorphic and thus isomorphic to $L = L^0$. As one might expect, we have the

Proposition 1. *Let R be a supercommutative ring and let $S_1, \ldots, S_m, T$ be super R-modules. The super R-module $L(S_1 \times \ldots \times S_m, T)$ of all super R-multilinear maps is isomorphic to the super R-module* $\mathbf{Hom}(S_1 \otimes_R \ldots \otimes_R S_m, T)$ *of all even and odd morphisms between the super R-modules $S_1 \otimes_R \ldots \otimes_R S_m$ and T.*

The tensor product of supermodules of a supercommutative ring R has properties similar to those of the usual tensor product [BBH91]. The tensor product of two morphisms of super R-modules $f : M \to M'$ and $g : N \to N'$ is defined by

$$f \otimes_R g : M \otimes_R N \ni m \otimes n \mapsto (-1)^{gm} f(m) \otimes g(n) \in M' \otimes_R N'.$$

The incorporation of the sign into the definition of $f \otimes_R g$ has the advantage that formulae in the supersetting and in the classical context are exactly the same as long as we write equations between maps – the difference appears only if we apply the maps to arguments.

1.4.4 Tensor product over an algebra over a (super)commutative ring

Other tensor products and universal properties are often needed.

Whereas algebras are vector spaces – modules over a commutative field – endowed with a multiplication, *R-algebras*, where R denotes a commutative ring, are R-modules together with a multiplication, or, better, not necessarily commutative rings A with an R-action. This action is given by a map $\varphi : R \to A$, a ring morphism, such that $\varphi(R) \subset Z(A)$, where $Z(A)$ is the center of A. The R-action is then defined by $r.a = \varphi(r)a$. Moreover, if M is a module over an R-algebra A, it is also a module over the commutative ring R: it suffices to set $r.m = \varphi(r)m$. The preceding center condition $\varphi(R) \subset Z(A)$, which also reads $\varphi(r)a = a\varphi(r)$, for all $a \in A$ and all $r \in R$, is for instance of importance if we consider the set of morphisms $\mathrm{Hom}_A(M, N)$, where M and N are modules over the R-algebra A. Let us be more precise. This set has an obvious group structure and, in view of the center condition (resp. the noncommutativity of A), even an R-module structure (resp. no A-module structure). Indeed, if for $f \in \mathrm{Hom}_A(M, N)$ we define the action of r on f by $(rf)(m) = r.(f(m))$, the map $rf : M \to N$ is actually an A-module morphism, since

$$(rf)(am) = r.(f(am)) = \varphi(r)(af(m)) = (\varphi(r)a)f(m)$$

$$= (a\varphi(r))f(m) = a(r.(f(m))) = a((rf)(m)).$$

A similar computation for r replaced by an element $a' \in A$ would not go through.

The same concept exists in superalgebra. If R is a supercommutative ring, a *super R-algebra* A is a superring A endowed with a superring morphism $\varphi : R \to A$ (remember that, if not otherwise specified, all morphisms are even), such that $\varphi(R) \subset Z(A)$, where $Z(A)$ denotes the supercenter of A. This means that $\varphi(r)a = (-1)^{ar}a\varphi(r)$.

The category of modules over an *R-algebra* A admits a concept of tensor product that takes into account both actions, the A- and the R-action. In the classical nongraded situation, if M and N denote a right and a left A-module respectively, these sets are modules over the, in general, noncommutative ring A, as well as over the commutative ring R. In view of what has been said before, their tensor product should therefore be an R-module. This tensor product, denoted by $M \otimes_A N$, is defined as the quotient of the free R-module $R^{(M\times N)}$ generated by $M \times N$, by the R-submodule generated by the elements that "correspond to the weakened A-bilinearity and to the usual R-bilinearity (in the following, we will speak of the (A, R)-bilinearity)", i.e. by the elements

$$-e_{(x+x',y)} + e_{(x,y)} + e_{(x',y)},\ -e_{(x,y+y')} + e_{(x,y)} + e_{(x,y')},$$

$$-e_{(xa,y)} + e_{(x,ay)}, -re_{(x,y)} + e_{(xr,y)}.$$

The tensor product R-module $M \otimes_A N$ and the (A, R)-bilinear map

$$\otimes : M \times N \ni (x, y) \mapsto x \otimes y = [e_{(x,y)}] \in M \otimes_A N$$

are universal. In other words, for any R-module P and any (A, R)-bilinear map $b : M \times N \to P$, there exists a unique R-module morphism $\tilde{b} : M \otimes_A N \to P$, such that $b = \tilde{b} \circ \otimes$.

1.5 Algebraic de Rham complexes, Deligne and Bernstein - Leites formalisms

The most representative simple examples of Koszul differentials are the coboundary and boundary operators on the graded vector space

$$S\mathbb{R}^{n\,*} \otimes \wedge^\bullet \mathbb{R}^n$$

of multivector fields of $\mathbb{R}^n$ with polynomial coefficients. If e_i (resp. x^i) denote the canonical basis (resp. coordinates) of $\mathbb{R}^n$, these operators are defined by

$$\partial = \sum_j \partial_{x^j} \otimes {}_{e_j}\wedge \quad \text{and} \quad \partial^* = \sum_j x^j \otimes i_{e_j},$$

where ${}_{e_j}\wedge$ (resp. i_{e_j}) is the left exterior product by e_j (resp. the interior product by e_j). The operator ∂ (resp. ∂^*) squares to 0, since the partial derivatives (resp. the multiplications by the coordinates) commute, whereas the exterior (resp. interior) products anticommute. Indeed, using first the commutativity of the derivative, then the anticommutativity of the exterior products, we get for instance

$$\partial^2(P\otimes v) = \sum_{ij} \partial_{x^i}\partial_{x^j} P\ e_i\wedge e_j\wedge v = \sum_{ji} \partial_{x^i}\partial_{x^j} P\ e_j\wedge e_i\wedge v = -\sum_{ij} \partial_{x^i}\partial_{x^j} P\ e_i\wedge e_j\wedge v = 0.$$

The abstract de Rham complexes will be constructed along the same lines. Consider a supercommutative R-algebra A over a supercommutative ring R and the free super R-module $G = R[\omega_1, \ldots, \omega_n]$ generated by formal generators ω_i. Let further $X_1, \ldots, X_n$ be homogeneous supercommutative superderivations of A, i.e. even or odd endomorphisms of the super R-module A, such that $X_i X_j = (-1)^{X_i X_j} X_j X_i$ and $X_i(aa') = (X_i a)a' + (-1)^{X_i a} a(X_i a')$. The cochain space is the super R-module

$$G \otimes_R A$$

and the coboundary operator is defined by

$$d = \sum_i \omega_i \otimes_R X_i.$$

Here ω_i denotes the super R-linear left multiplication by ω_i in G. Actually, we thus get a de Rham complex with even (resp. odd) differential, if we assume that, for any i, ω_i and X_i have the same parity (resp. opposite parity). Moreover, since d must be a square zero map, the ω_i must super anticommute (resp. supercommute). This assumption affects of course the free module $G = R[\omega_1, \ldots, \omega_n]$. To understand this specific commutation hypothesis, observe that

$$d^2(g \otimes a) = d\left(\sum_j (-1)^{X_j g}\,\omega_j g\; X_j a\right) = \sum_{ij} (-1)^{X_j g + X_i(\omega_j + g)}\,\omega_i\omega_j g\; X_i X_j a. \tag{1.3}$$

When exploiting the supercommutativity $X_iX_j = (-1)^{X_iX_j} X_jX_i$ and the, as understood in the introductory paragraph, obviously necessary commutativity $\omega_i\omega_j = \pm(-1)^{\omega_i\omega_j}\omega_j\omega_i$, we obtain

$$\begin{aligned} d^2(g \otimes a) &= \sum_{ji} (-1)^{X_i g + X_j(\omega_i + g)} (-1)^{X_iX_j}\,\omega_j\omega_i g\; X_iX_j a \\ &= \sum_{ij} (-1)^{X_i g + X_j(\omega_i + g)} (-1)^{X_iX_j} (\pm 1)(-1)^{\omega_i\omega_j}\,\omega_i\omega_j g\; X_iX_j a. \end{aligned} \tag{1.4}$$

If d is even (resp. odd), i.e. if $p(\omega_i) = p(X_i)$ (resp. $p(\omega_i) = p(X_i) + 1$), we have $(-1)^{X_i\omega_j} = (-1)^{\omega_i X_j}$ (resp. $(-1)^{X_i\omega_j} = (-1)^{(\omega_i+1)(X_j+1)}$) and $(-1)^{X_iX_j}(-1)^{\omega_i\omega_j} = 1$ (resp. $(-1)^{X_iX_j}(-1)^{\omega_i\omega_j} = (-1)^{\omega_i + X_j}$), and the Equations 1.3 and 1.4 differ by ± 1 (resp. ∓ 1), so that $d^2 = 0$, if we choose $\pm 1 = -1$ (resp. $\pm 1 = +1$), i.e. if we decide that the ω_i super anticommute (resp. supercommute).

As a special case, we obtain the local supergeometric de Rham complexes. Let $\mathcal{U}^{p|q}$ be a superdomain over $U \in \mathbb{R}^p$ with coordinates $(x^1, \ldots, \xi^q)$. Details about this concept and about notations used in the following can be found below. Set $R = \mathbb{R}$, $A = \mathcal{C}^\infty_{p|q}(U) = C^\infty(U)[\xi^1, \ldots, \xi^q]$, $G = \mathbb{R}[dx^1, \ldots, d\xi^q]$, $(X_1, \ldots, X_{p+q}) = (\partial_{x^1}, \ldots, \partial_{\xi^q})$, where the differentials of the coordinates are formal generators. Since we define

$$d = \sum_i \omega_i \otimes X_i = \sum_a dx^a \partial_{x^a} + \sum_\alpha d\xi^\alpha \partial_{\xi^\alpha},$$

we find in particular that the formal generators are the differentials of the coordinate functions.

Let us emphasize the next crucial observation. If we choose d to be even, the ω_i (have the same parity as the X_i and) must super anticommute, so that, if we denote the coordinates by a common symbol u^i, we must set

$$du^i du^j = \omega_i\omega_j = -(-1)^{u^iu^j}\omega_j\omega_i = -(-1)^{u^iu^j} du^j du^i, \tag{1.5}$$

whereas for an odd d, the ω_i (and the X_i have the opposite parity and the ω_i) must supercommute, so that we are forced to set

$$du^i du^j = \omega_i\omega_j = (-1)^{(u^i+1)(u^j+1)}\omega_j\omega_i = (-1)^{(u^i+1)(u^j+1)} du^j du^i. \tag{1.6}$$

Remark 2. *Equation (1.5) (resp. (1.6)) is known as the Deligne (resp. the Bernstein-Leites) sign convention for the wedge product.*

Naive local super differential forms (the simplest local concept of differential forms in Supergeometry) are defined as the elements of the cochain space

$$G \otimes_R A = \mathbb{R}[dx^1, \ldots, d\xi^q] \otimes C^\infty(U)[\xi^1, \ldots, \xi^q] =: \Omega_U^\bullet(U).$$

The freely generated space $\mathbb{R}[dx^1, \ldots, d\xi^q]$ and the space $\Omega_U^\bullet(U)$ depend on the choice of the just mentioned convention – Deligne or Bernstein-Leites. Further, we may choose the $\mathbb{N}$-grading of $\Omega_U^\bullet(U)$, also called *the cohomological grading*. The most naturel cohomological degree of super differential forms is defined by setting $\deg A = 0$ and $\deg du^i = 1$ (other conventions are possible, but will not be used in this text). Hence, a (naive) local super differential form of degree 1 is of the type

$$dx^a\, f_a(x,\xi) + d\xi^\alpha\, g_\alpha(x,\xi),$$

where sums are understood and where the coefficients are superfunctions. If we adopt the Deligne formalism, what we will always implicitly do if not otherwise stated, we have $dx^a\, dx^b = -dx^b\, dx^a$, $dx^a\, d\xi^\alpha = -d\xi^\alpha\, dx^a$, and $d\xi^\alpha\, d\xi^\beta = d\xi^\beta\, d\xi^\alpha$ (whereas in the Bernstein-Leites formalism, the commutation rules are $dx^a\, dx^b = -dx^b\, dx^a$, $dx^a\, d\xi^\alpha = d\xi^\alpha\, dx^a$, and $d\xi^\alpha\, d\xi^\beta = d\xi^\beta\, d\xi^\alpha$), so that a (naive) local super differential form of degree 2 is of the type

$$\sum_{a_1<a_2} dx^{a_1}\, dx^{a_2} f_{a_1a_2}(x,\xi) + \sum_{a,\alpha} dx^a\, d\xi^\alpha g_{a\alpha}(x,\xi) + \sum_{\alpha_1\le\alpha_2} d\xi^{\alpha_1}\, d\xi^{\alpha_2} h_{\alpha_1\alpha_2}(x,\xi).$$

Observe that the Deligne convention is just the superposition of the even situation and the usual Koszul sign rule. The Bernstein-Leites formulae ignore the even situation and are based upon the Koszul sign rule with weights defined as the sum of the parity and the cohomological degree.

1.6 Super symmetric and exterior algebras

Let us first mention some basic facts pertaining to the parity reversion (endo)functor of the category of super R-modules, where R is a supercommutative ring. If M is a (super) R-module, we define the R-module ΠM by $(\Pi M)_i := M_{i+1}$, $i \in \{0,1\}$,

$$\Pi m + \Pi m' := \Pi(m+m') \;\text{ and }\; r\Pi m := (-1)^r \Pi(rm).$$

It is easily seen that $\Pi(mr) = (\Pi m)r$. To make Π a functor, we define, for any (even) R-module morphism $\varphi : M \to N$, an R-module morphism $\varphi^\Pi : \Pi M \to \Pi N$ by

$$\varphi^\Pi(\Pi m) = \Pi(\varphi m).$$

Further, to any R-morphism $\varphi : M \to N$, we associate the R-mappings $\Pi\varphi : M \ni m \mapsto \Pi(\varphi m) \in \Pi N$ and $\varphi\Pi : \Pi M \ni \Pi m \mapsto \varphi m \in N$. Of course, $\varphi^{\Pi} = \Pi\varphi\Pi$.

Let M be a supermodule over a supercommutative ring R. Set $M^{\otimes n} = M \otimes_R \ldots \otimes_R M$ (n factors M, $n \geq 1$) and $M^{\otimes 0} = R$. The super R-module

$$T^{\bullet}M = \bigoplus_{n \geq 0} M^{\otimes n}$$

is the *tensor super R-algebra of the supermodule M*. The multiplication is the tensor product

$$\otimes_R : M^{\otimes n} \times M^{\otimes p} \to M^{\otimes n} \otimes_R M^{\otimes p} \simeq M^{\otimes (n+p)}.$$

The associative superalgebra TM (we omit $\bullet$) carries not only the previous $\mathbb{N}$-grading given by the number of factors in M, but also the natural $\mathbb{Z}_2$-grading induced by the $\mathbb{Z}_2$-grading of M. The associative super R-algebra TM and the R-morphism $i : M \to TM$ are universal, i.e. if A is an associative super R-algebra with a morphism $\ell : M \to A$, there exists a unique morphism of super R-algebras $\tilde{\ell} : TM \to A$ such that $\ell = \tilde{\ell} \circ i$.

To define the supersymmetric tensor algebra SM, we proceed as in the classical nongraded case. Let I_S be the ideal $I_S = (m \otimes n - (-1)^{mn} n \otimes m : m, n \in M)$. The *supersymmetric tensor R-algebra of the supermodule M* is then the quotient algebra

$$SM = TM/I_S.$$

It is universal for morphisms from M to supercommutative associative R-algebras. Assume now that R is a $\mathbb{Q}$-algebra, so that an action by the elements of R induces an action by the rational numbers. We can then define the supersymmetrizer

$$S : M^{\otimes n} \ni m_1 \otimes \ldots \otimes m_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \chi(\sigma) m_{\sigma(1)} \otimes \ldots \otimes m_{\sigma(n)} \in M^{\otimes n},$$

where $\chi(\sigma)$ is the Koszul sign associated with the permutation $m_1 \otimes \ldots \otimes m_n \mapsto m_{\sigma(1)} \otimes \ldots \otimes m_{\sigma(n)}$. As the ideal I_S is homogeneous, i.e. verifies

$$I_S = \bigoplus_{n \geq 0} I_S \cap M^{\otimes n},$$

the algebra SM is $\mathbb{N}$-graded:

$$S^{\bullet}M = \bigoplus_{n \geq 0} S^n M, \quad \text{where} \quad S^n M = M^{\otimes n} / (I_S \cap M^{\otimes n}).$$

Each R-module $S^n M$ can be injected into $M^{\otimes n}$ by $i : S^n M \ni [T] \mapsto S(T) \in M^{\otimes n}$, where S is the symmetrizer. The identification

$$S^n M \ni [T] \simeq S(T) \in M^{\otimes n}$$

allows sampling out the supersymmetric tensor algebra SM as a direct summand of TM. The product $\vee$ of supersymmetric tensors that is induced in the quotient by the product $\otimes$, i.e. $[T] \vee [U] = [T \otimes U]$, coincides with the product $\vee$ implemented by the symmetrizer, i.e. $S(T) \vee S(U) = S(S(T) \otimes S(U))$. It is clear that

$$[m] \vee [n] = [m \otimes n] = [(1/2)\left((m \otimes n + (-1)^{mn} n \otimes m) + (m \otimes n - (-1)^{mn} n \otimes m)\right)]$$

$$= [(1/2)(m \otimes n + (-1)^{mn} n \otimes m)] = (-1)^{mn}[n] \vee [m].$$

As for the super exterior algebra of a super R-module M, two different definitions exist, depending on the chosen formalism, Deligne or Bernstein-Leites. In the first case, we denote by I_A the ideal $I_A = (m \otimes n + (-1)^{mn} n \otimes m : m, n \in M)$ and define the *super exterior R-algebra of the supermodule M* as the quotient algebra

$$\wedge_{\mathrm{D}} M = TM/I_A.$$

We often omit subscript D. The induced product is denoted by $\wedge_{\mathrm{D}}$ or simply by $\wedge$. Again, the quotient algebra is graded:

$$\wedge_{\mathrm{D}} M = \bigoplus_{n \geq 0} \wedge^n_{\mathrm{D}} M.$$

In the second case, we define the exterior algebra by setting

$$\wedge_{\mathrm{BL}} M = S(\Pi M).$$

The product $\wedge_{\mathrm{BL}}$ in $\wedge_{\mathrm{BL}} M$ is defined by the product $\vee$ in $S(\Pi M)$. These definitions actually correspond to the respective sign conventions for the wedge product. Indeed,

$$[m] \wedge_{\mathrm{D}} [n] = [m \otimes n] = [(1/2)\left((m \otimes n - (-1)^{mn} n \otimes m) + (m \otimes n + (-1)^{mn} n \otimes m)\right)]$$

$$= [(1/2)(m \otimes n - (-1)^{mn} n \otimes m)] = -(-1)^{mn}[n] \wedge_{\mathrm{D}} [m],$$

and

$$[m] \wedge_{\mathrm{BL}} [n] = [\Pi m] \vee [\Pi n] = (-1)^{(m+1)(n+1)}[\Pi n] \vee [\Pi m] = (-1)^{(m+1)(n+1)}[n] \wedge_{\mathrm{BL}} [m].$$

The supermodules $\wedge_{\mathrm{D}} M$ and $\wedge_{\mathrm{BL}} M$ are isomorphic, but the algebra structures are of course not.

Let us still mention that if M is a free and finitely generated super R-module, we may use the identification $\wedge^n M^* \simeq \mathrm{Alt}(M^{\times n}, R)$, where the RHS denotes the R-module of alternating super R-multilinear forms on M.

1.7 Supertranspose

We now define the supertranspose of a matrix $T \in \mathrm{M}(r|s \times p|q, \mathcal{A})$, $T \simeq t \in \mathbf{Hom}(M, N)$, $M = \mathcal{A}^{p|q}$, $N = \mathcal{A}^{r|s}$, in the natural way, i.e. as the matrix of the transpose $t^* : N^* \to M^*$ of $t : M \to N$. The *dual super $\mathcal{A}$-module* of M is of course $M^* = \mathbf{Hom}(M, \mathcal{A})$, and the *dual morphism* t^* is naturally defined by

$$\langle t^*(n^*), m\rangle = (-1)^{p(t)p(n^*)}\langle n^*, t(m)\rangle,$$

for all $n^* \in N^*$ and $m \in M$; here $\langle -, -\rangle$ denotes the evaluation of the involved morphism on the corresponding source-module element. It follows that t and t^* have the same parity. To find the matrix of t^*, denote the basis of M (resp. N) by (e_i) (resp. (e'_i)) and define the dual basis (ε^i) of M^* (resp. (ε'^i) of N^*) as usually by $\varepsilon^i(e_j) = \delta^i_j \in \mathcal{A}_0$. The definition entails again that the elements of this dual basis and the corresponding elements of the inducing basis have the same parity. We then get

$$t^*(\varepsilon'^j)(e_i) = (-1)^{p(\varepsilon'^j)p(t)}\varepsilon'^j(t(e_i)) = (-1)^{p(\varepsilon'^j)p(t)}\varepsilon'^j(e'_k)t^k_{\ i} = (-1)^{p(\varepsilon'^j)p(t)}t^j_{\ i},$$

but also

$$t^*(\varepsilon'^j)(e_i) = (\varepsilon^k t^{*\ j}_{\ k})(e_i) = (-1)^{p(e_i)p(t^{*\ j}_{\ k})}\varepsilon^k(e_i)t^{*\ j}_{\ k} = (-1)^{p(e_i)p(t^{*\ j}_{\ i})}t^{*\ j}_{\ i},$$

where the left index is the row index. We thus get

$$t^{*\ j}_{\ i} = (-1)^{p(t^{*\ j}_{\ i})p(e_i)+p(\varepsilon'^j)p(t)}t^j_{\ i}.$$

Finally, the matrix representing t^*, i.e. the supertranspose of matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

representing t, is given by

$$T^{\mathrm{st}} = \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix}, \text{ for } T \text{ even},$$

and by

$$T^{\mathrm{st}} = \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix}, \text{ for } T \text{ odd}.$$

Moreover, the supertranspose has the following properties, which are easily verified.

Proposition 2. *The operation of supertransposition is*

1. *of period 4 (and not of period 2, like in the classical setting),*
2. *preserves the parity, i.e.* $p(T^{\mathrm{st}}) = p(T)$, *and*
3. $(ST)^{\mathrm{st}} = (-1)^{p(S)p(T)}T^{\mathrm{st}}S^{\mathrm{st}}$.

1.8 Supertrace

In the classical context, if M denotes a p-dimensional real vector space and if we fix a basis, we have $M^* \otimes M \simeq \mathrm{End}(M) \simeq \mathrm{M}(p \times p, \mathbb{R})$. The contraction is defined on $M^* \otimes M$ by

$$c : M^* \otimes M \ni \alpha \otimes m \mapsto \sum_i \alpha_i m^i \in \mathbb{R}.$$

Viewed as an endomorphism, $\alpha \otimes m$ reads $\alpha \otimes m : M \ni n \mapsto \alpha(n)\, m \in M$, and its matrix is $(\alpha_j m^i)_{i,j}$. Hence, the contraction is nothing but the trace

$$\mathrm{tr} : \mathrm{M}(p \times p, \mathbb{R}) \ni (\alpha_j m^i)_{i,j} \mapsto \sum_i \alpha_i m^i \in \mathbb{R}.$$

In the following, to simplify notations, we omit $p(-)$ and denote the objects and their parity by the same symbols. In the case of a free supermodule $M = \mathcal{A}^{p|q}$, the tensor $\alpha \otimes m$ coincides with the endomorphism $n \mapsto (-1)^{nm} \alpha(n)\, m$. Its matrix is

$$T = ((-1)^{e_j(e_i+m^i)+\alpha_j(e_i+e_j)} \alpha_j m^i)_{i,j}.$$

Indeed, when applying the endomorphism to a base vector e_j, we get

$$\begin{aligned}(-1)^{e_j m} \alpha(e_j) m &= (-1)^{e_j m} (\varepsilon^k \alpha_k)(e_j) m = (-1)^{e_j m + \alpha_k e_j} (\varepsilon^k(e_j)) \alpha_k m \\ &= (-1)^{e_j m + \alpha_j e_j} \alpha_j m = (-1)^{e_j(e_i+m^i)+\alpha_j e_j} \alpha_j e_i m^i \\ &= (-1)^{e_j(e_i+m^i)+\alpha_j(e_i+e_j)} e_i \alpha_j m^i.\end{aligned}$$

It now follows from the preceding classical observations, that the supertrace of matrix T should be defined by

$$\mathrm{str}(T) = \sum_i \pm_i (-1)^{e_i(e_i+m^i)+\alpha_i(e_i+e_i)} \alpha_i m^i = \sum_i \pm_i T^i_i, \tag{1.7}$$

where the sign $\pm_i$ has to be chosen such that the RHS coincides with the contraction of α by m, which is given in this supercontext by

$$\alpha(m) = \sum_i (-1)^{\alpha_i e_i} \alpha_i m^i.$$

This means that

$$\pm_i = (-1)^{e_i(e_i+\alpha_i+m^i)}. \tag{1.8}$$

It now follows from Equations (1.7) and (1.8) that we finally must choose the

Definition 7. *The supertrace of a square matrix*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{M}(p|q \times p|q, \mathcal{A})$$

is given by

$$\mathrm{str}(T) = \mathrm{tr}(A) - (-1)^{p(T)}\, \mathrm{tr}(D).$$

Indeed, the first term of the supertrace has to be $\mathrm{tr}(A)$, as e_i is even for $1 \leq i \leq p$, whereas for the second, e_i is odd, so that the sign is given by $p(\alpha_i m^i) = p(T)$.

Moreover, we have the

Proposition 3. *1.* $\mathrm{str}(STS^{-1}) = \mathrm{str}(T)$, *for any invertible even matrix S, which means that the supertrace can be defined for morphisms $t \in$* $\mathbf{End}(\mathrm{M})$*, where M is a free super $\mathcal{A}$-module,*

2. $\mathrm{str} : \mathrm{M}(p|q \times p|q, \mathcal{A}) \to \mathcal{A}$ *is a super $\mathcal{A}$-module morphism, i.e. in particular* $\mathrm{str}(aT) = a\,\mathrm{str}(T)$,

3. $\mathrm{str}(ST) = (-1)^{p(S)p(T)}\,\mathrm{str}(TS)$, *and*

4. $\mathrm{str}(T^{\mathrm{st}}) = \mathrm{str}(T)$.

1.9 Berezinian of an invertible even square matrix

The *Berezinian* or *superdeterminant* will play the role of the Jacobian and appear in the change of variables formula for integration on supermanifolds. Since a change of variables should be invertible and preserve the parity, it is natural to consider the Berezinian of invertible even matrices.

The Berezinian should verify two properties, which already hold true for the classical determinant, namely, it should be multiplicative and verify the superanalog of the classical formula $\det(e^X) = e^{\mathrm{tr}\,X}$, where X is a matrix.

Let us first consider an even square matrix

$$T = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathrm{M}_0(p|q \times p|q, k)$$

with entries in the ground field k. In this situation, we have $\mathrm{str}(T) = \mathrm{tr}(A) - \mathrm{tr}(D)$, $\det(e^A) = e^{\mathrm{tr}\,A}$, and similarly for D. Requiring that $\mathrm{Ber}(e^T) = e^{\mathrm{str}\,T}$ then immediately leads to the definition

$$\mathrm{Ber}(T) = \det(A)\det{}^{-1}(D). \tag{1.9}$$

Here, we already see that D has to be invertible; as aforementioned, it is natural to assume that T be invertible.

Lemma 1. *An even square matrix*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{M}_0(p|q \times p|q, \mathcal{A})$$

is invertible if and only if A and D are invertible matrices over $\mathcal{A}_0$, i.e. if $\det(A)$ *and* $\det(D)$ *are invertible in $\mathcal{A}_0$.*

Proof. To prove this lemma, we *divide the odd variables out.* Let J be the ideal in $\mathcal{A}$ generated by the odd elements. It is easily checked that $J = \mathcal{A}_1 \oplus \mathcal{A}_1^2$. Note that all elements of J are nilpotent, in particular $1 \notin J$, so J is a proper ideal and we define $\bar{\mathcal{A}} = \mathcal{A}/J \simeq \mathcal{A}_0/\mathcal{A}_1^2$. For a matrix L over $\mathcal{A}$, let $\bar{L}$ be the matrix over $\bar{\mathcal{A}}$ obtained by applying the projection $\mathcal{A} \to \bar{\mathcal{A}}$ to the entries of L.

First, we claim that L is invertible if and only if $\bar{L}$ is invertible over $\bar{\mathcal{A}}$. We will only consider right inverses, because the argument is the same for left inverses. If L is invertible, $\bar{L}$ is invertible as well, since, if $LM = I$, then $\bar{L}\bar{M} = I$. Conversely, suppose that $\bar{L}$ is invertible. Then there exists a matrix M over $\mathcal{A}$ such that $LM = I + X$, for some matrix X with entries only in J. To conclude that L is invertible, it just remains to prove that $I + X$ is invertible. It therefore suffices to show that X is nilpotent, i.e. $X^r = 0$, for some integer $r \geq 1$. Indeed, in this case $I + \sum_{k=1}^{r-1}(-1)^k X^k$ is the inverse of $I+X$. Note now that, since all entries of X are in J, there exist odd elements $o_1, \dots, o_N$, such that any entry of X is of the form $\sum_i a_i o_i$, with $a_i \in \mathcal{A}$. Iterated multiplication of X by itself thus leads to products of the o_i, and for $r = N + 1$ all these products $o_{i_1} \cdot \dots \cdot o_{i_r} = 0$, since at least one of the o_i appears twice. By this, the entries of X^r do all vanish and our claim is proven.

Coming back to the proof of the lemma, note that T is even, i.e. that A, D are even and B, C odd. Hence,

$$\bar{T} = \begin{pmatrix} \bar{A} & 0 \\ 0 & \bar{D} \end{pmatrix}$$

and $\bar{T}$ is invertible if and only if $\bar{A}$ and $\bar{D}$ are invertible. By the previous claim this means that T is invertible if and only if A and D are invertible. □

We are now prepared to find the appropriate definition of the Berezinian. It is easily checked that any invertible even matrix T admits the decomposition

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix} =: T_+ T_0 T_-. \quad (1.10)$$

Note now that if the Berezinian is multiplicative, which is a natural requirement, and if observation (1.9) extends to invertible even square matrices with entries in $\mathcal{A}$, we should choose the following

Definition 8. *Let* $\mathrm{GL}(p|q, \mathcal{A})$ *denote the group of all invertible even* $p|q \times p|q$ *matrices with entries in* $\mathcal{A}$*, i.e. the group* $\mathrm{Aut}(\mathcal{A}^{p|q})$ *of all (even) automorphisms of* $\mathcal{A}^{p|q}$*. For any*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(p|q, \mathcal{A}),$$

the Berezinian of T *is given by*

$$\mathrm{Ber}(T) = \det(A - BD^{-1}C) \det{}^{-1}(D).$$

Indeed, this actually implies that

$$\operatorname{Ber}\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} = 1 \textit{ and } \operatorname{Ber}\begin{pmatrix} I & 0 \\ C & I \end{pmatrix} = 1,$$

for odd B and C, and that Equation (1.9) holds still true for entries in $\mathcal{A}$. It is also important to notice that the involved classical determinants are computed for even matrices, and so make sense. Moreover, all the involved inverses exist and the Berezinian is an element of $\mathcal{A}_0$. Let us now prove that this definition entails the

Theorem 1. *The Berezinian is multiplicative, i.e. for any $X, Y \in \mathrm{GL}(p|q, \mathcal{A})$, we have*

$$\operatorname{Ber}(XY) = \operatorname{Ber}(X) \operatorname{Ber}(Y).$$

In particular, $\operatorname{Ber}(X)$ *is a unit of $\mathcal{A}_0$.*

Proof. Set

$$G = \{Y \in \mathrm{GL}(p|q, \mathcal{A}) : \operatorname{Ber}(XY) = \operatorname{Ber}(X) \operatorname{Ber}(Y), \forall X \in \mathrm{GL}(p|q, \mathcal{A})\}.$$

The proof consists of three parts.

(1) The set G is a subgroup of $\mathrm{GL}(p|q, \mathcal{A})$. Indeed, it is closed under products and inverses (and is nonempty in view of point (3) of this proof). Let us for instance show that if $Y \in G$, then $Y^{-1} \in G$. Since $1 = \operatorname{Ber}(Y^{-1}Y) = \operatorname{Ber}(Y^{-1}) \operatorname{Ber}(Y) = \operatorname{Ber}(Y) \operatorname{Ber}(Y^{-1})$ and since $\operatorname{Ber}(X) = \operatorname{Ber}(XY^{-1}Y) = \operatorname{Ber}(XY^{-1}) \operatorname{Ber}(Y)$, it follows that $\operatorname{Ber}(XY^{-1}) = \operatorname{Ber}(X) \operatorname{Ber}(Y^{-1})$.

(2) In view of decomposition (1.10), the invertible even matrices of the form

$$Y_- := \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}, Y_0 := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, Y_+ := \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$$

generate $\mathrm{GL}(p|q, \mathcal{A})$. Indeed, the central matrix T_0 in the RHS of this decomposition is invertible since all the other matrices T, T_+ and T_- are. As, obviously,

$$\begin{pmatrix} I & B' + B'' \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & B' \\ 0 & I \end{pmatrix} \begin{pmatrix} I & B'' \\ 0 & I \end{pmatrix},$$

we can even assume that matrix B in Y_+ is elementary, i.e. contains only one nonzero element.

(3) It now suffices to prove that G contains all invertible even matrices of the types Y_-, Y_0 and Y_+ (with an elementary B). Indeed, then G is a group and contains all the products of such matrices, thus all the elements of $\mathrm{GL}(p|q, \mathcal{A})$, so that finally $G = \mathrm{GL}(p|q, \mathcal{A})$, which completes the proof.

It is straightforwardly verified that the matrices Y_- and Y_0 are elements of G. This involves nothing more than the definition of the Berezinian. As for Y_+, we decompose X in the form $X = X_+X_0X_-$. Again, a direct computation immediately shows that the Berezinian is multiplicative for left multiplication by X_+ and X_0. Hence, it suffices to prove that $\mathrm{Ber}(X_-Y_+) = \mathrm{Ber}(X_-)\,\mathrm{Ber}(Y_+)$=1, where Y_+ contains an elementary B. If we omit the subscripts $\pm$ and write E instead of B (to remember that this matrix is elementary), we get

$$X = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix},\, Y = \begin{pmatrix} I & E \\ 0 & I \end{pmatrix},\, XY = \begin{pmatrix} I & E \\ C & I + CE \end{pmatrix},$$

and

$$\mathrm{Ber}(XY) = \det(I - E(I + CE)^{-1}C)\,\det{}^{-1}(I + CE).$$

Since E is odd, its only nonzero element β squares to 0 and all entries of matrices of the form EZ and ZE are divisible by β. Hence, any product of two entries of such matrices vanishes. In particular, $(CE)^2 = 0$, thus $I + CE$ is invertible with inverse $I - CE$ and $I - E(I + CE)^{-1}C = I - EC$. Since the structure of the classical determinant entails that, if L is a matrix with even entries such that any product of two of its entries is zero, then $\det(I + L) = 1 + \mathrm{tr}(L)$, we obtain

$$\det(I - E(I + CE)^{-1}C) = \det(I - EC) = 1 - \mathrm{tr}(EC)$$

and

$$\det{}^{-1}(I + CE) = (1 + \mathrm{tr}(CE))^{-1}.$$

As for the matrices C and E with odd entries, we have $\mathrm{tr}(EC) = -\mathrm{tr}(CE)$, we finally get

$$\mathrm{Ber}(XY) = (1 - \mathrm{tr}(EC))(1 + \mathrm{tr}(CE))^{-1} = (1 + \mathrm{tr}(CE))(1 + \mathrm{tr}(CE))^{-1} = 1,$$

which completes the proof of multiplicativity of the Berezinian. The fact that $\mathrm{Ber}(X)$ is a unit of $\mathcal{A}_0$ follows directly from the multiplicativity. □

The multiplicativity and the (obvious) decomposition

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix} \tag{1.11}$$

lead to the following

Corollary 1. *Let*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(p|q, \mathcal{A}).$$

Then

$$\mathrm{Ber}(T) = \det(A)\,\det{}^{-1}(D - CA^{-1}B).$$

The next proposition is a direct consequence of the preceding definition and results.

Proposition 4. *For any* $T \in \mathrm{GL}(p|q, \mathcal{A})$,

1. $\mathrm{Ber}(T^{-1}) = \mathrm{Ber}^{-1}(T)$,
2. *The Berezinian does not depend on the chosen basis, i.e.* $\mathrm{Ber}(STS^{-1}) = \mathrm{Ber}(T)$, *for any invertible even matrix* S,
3. $\mathrm{Ber}(T^{\mathrm{st}}) = \mathrm{Ber}(T)$,
4. $\mathrm{Ber}(e^T) = e^{\mathrm{str}(T)}$.

1.10 Berezinian of a free supermodule

The Berezinian of a free supermodule is the superversion of the Determinant of a vector space. If S is a free module of rank n over a commutative ring R, we set

$$\mathrm{Det}\, S := \wedge^n S.$$

To any basis (e_i) of S, corresponds a basis $e_1 \wedge \ldots \wedge e_n$ of $\mathrm{Det}\, S$, such that if (e'_i) is another basis of S with $e'_i = B^k_i e_k$, then

$$e'_1 \wedge \ldots \wedge e'_n = \mathrm{Det}\, B\; e_1 \wedge \ldots \wedge e_n.$$

When trying to extend this concept to the supercase, we note that the exterior product of odd vectors commutes, so that there is no top exterior power for an odd module. It is easily understood that the Berezinian of a free supermodule $S = S_0 \oplus S_1$ of rank $p|q$ over a supercommutative ring $R = R_0 \oplus R_1$ (in which 2 is invertible) is isomorphic to

$$\mathrm{Ber}\, S :\simeq \wedge^p S_0 \otimes \vee^q S_1^*.$$

If $(e_1, \ldots, e_{p+q})$ is a standard basis of S and $(\varepsilon^1, \ldots, \varepsilon^{p+q})$ denotes the dual basis, the vector

$$[e_1, \ldots, e_{p+q}] \simeq e_1 \wedge \ldots \wedge e_p \otimes \varepsilon^{p+1} \vee \ldots \vee \varepsilon^{p+q}$$

is a basis of $\mathrm{Ber}\, S$ (of cohomological degree p). It can be shown that if $(e'_1, \ldots, e'_{p+q})$ is a second basis of S related to the first by $e'_i = e_k B^k_i$, we have

$$[e'_1, \ldots, e'_{p+q}] = [e_1, \ldots, e_{p+q}]\; \mathrm{Ber}\, B. \tag{1.12}$$

It is clear that $\mathrm{Ber}\, S$ is a free super R-module of rank $1|0$, if q is even, and of rank $0|1$, if q is odd. Hence, since a change of basis $B : S \to S$ and its Berezinian $\mathrm{Ber}\, B : \mathrm{Ber}\, S \to \mathrm{Ber}\, S$ are even automorphisms, the Berezinian Ber is an endofunctor of the category of free super R-modules of finite rank and corresponding even automorphisms.

For a precise treatment of these questions – even in the $\mathbb{Z}_2^n$-graded situation – we refer the reader to [Cov12].

Chapter 2

Sheaf Theory

2.1 Categories and functors

We first recall some basic category theoretical concepts.

Many examples of categories are well known. The category $\mathtt{Set}$ of sets and maps between them, the category $\mathtt{Vect}$ of vector spaces and linear maps, $\mathtt{Top}$ of topological spaces and continuous maps... An abstract category is made up roughly by a class of objects that need not be sets (and are thought of as points), and by a class of morphisms that thus need not be maps assigning a unique target element to each source element (and are thought of as arrows between points). The definition of a category extends the basic properties of the preceding concrete categories to this abstract setting. In this lecture course,

Definition 9. *A* **category** $\mathtt{C}$ *consists of a set* $\mathrm{Ob}(\mathtt{C})$ *(or simply* $\mathtt{C}$*) of objects, and, for each objects* $A, B \in \mathtt{C}$*, of a set* $\mathrm{Hom}_{\mathtt{C}}(A, B)$ *(or simply* $\mathrm{Hom}(A, B)$*) of morphisms* $f : A \to B$ *from* A *to* B*. Moreover, for any objects* $A, B, C \in \mathtt{C}$*, there exists a composition map* $\circ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \ni (f, g) \to g \circ f \in \mathrm{Hom}(A, C)$*. This operation is required to be associative and to have identities, i.e. for each object* $A \in \mathtt{C}$ *there exists an identity morphism* $1_A \in \mathrm{Hom}(A, A)$ *such that if* $f \in \mathrm{Hom}(A, B)$ *and* $g \in \mathrm{Hom}(C, A)$*, we have* $f \circ 1_A = f$ *and* $1_A \circ g = g$ *(this entails that the identities are actually unique).*

The **opposite category** $\mathtt{C}^{\mathtt{op}}$ of $\mathtt{C}$ is the category defined by the same objects $\mathrm{Ob}(\mathtt{C}^{\mathtt{op}}) = \mathrm{Ob}(\mathtt{C})$ and by the "reversed morphisms" $\mathrm{Hom}_{\mathtt{C}^{\mathtt{op}}}(A, B) := \mathrm{Hom}_{\mathtt{C}}(B, A)$.

Remark 3. *For simplicity, we will skip in these notes set-theoretical problems related with cardinality and universes.*

Categories form themselves a "metacategory". Morphisms between categories, which respect of course the categorical structure, are called functors.

Definition 10. *Let* $\mathtt{C}$ *and* $\mathtt{C}'$ *be two categories. A* **functor** $F : \mathtt{C} \to \mathtt{C}'$ *is a made up by a map* $F : \mathrm{Ob}(\mathtt{C}) \ni A \mapsto F(A) \in \mathrm{Ob}(\mathtt{C}')$ *and, for any* $A, B \in \mathrm{Ob}(\mathtt{C})$*, by a map* $F : \mathrm{Hom}_{\mathtt{C}}(A, B) \ni f \mapsto F(f) \in \mathrm{Hom}_{\mathtt{C}'}(F(A), F(B))$*. In addition, these assignments have to respect the categorical structure, i.e. the composition and the identities. More precisely, we ask that* $F(g \circ f) = F(g) \circ F(f)$ *and that* $F(1_A) = 1_{F(A)}$*, for all composable* f *and* g *and for all* A.

This definition is in fact the definition of a **covariant functor**. A **contravariant functor** is defined in the same way, except that it "reverses the arrows", *i.e.* $F : \mathrm{Hom}_{\mathtt{C}}(A, B) \ni f \mapsto F(f) \in \mathrm{Hom}_{\mathtt{C}'}(F(B), F(A))$. In other words, a contravariant functor from $\mathtt{C}$ to $\mathtt{C}'$ is a covariant functor from $\mathtt{C}^{\mathtt{op}}$ to $\mathtt{C}'$.

The Hom "bifunctor" provides an example of a covariant and of a contravariant functor. Let $\mathtt{C}$ be a category and X an arbitrary fixed object of $\mathtt{C}$. Then

$$\begin{array}{cccc} \mathrm{Hom}(X, -): & \mathtt{C} & \to & \mathtt{Set} \\ & A & \mapsto & \mathrm{Hom}(X, A) \\ & (f : A \to B) & \mapsto & (f \circ - : \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B)) \end{array}$$

is a covariant functor from $\mathtt{C}$ to $\mathtt{Set}$, and

$$\begin{array}{cccc} \mathrm{Hom}(-, X): & \mathtt{C} & \to & \mathtt{Set} \\ & A & \mapsto & \mathrm{Hom}(A, X) \\ & (f : A \to B) & \mapsto & (- \circ f : \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)) \end{array}$$

is a contravariant functor.

In Physics, an object $\bullet$ may be viewed as a particle and a morphism $\bullet \rightsquigarrow \bullet$ as a static string. A motion of strings

$$\begin{array}{c} \bullet \rightsquigarrow \bullet \\ \Downarrow \\ \bullet \rightsquigarrow \bullet \end{array}$$

then corresponds to a morphism $\alpha : f \Rightarrow g$ between two morphisms $f : A \to B$ and $g : A \to B$. In fact such morphisms α (called 2-morphisms) between morphisms f and g (called 1-morphisms) extend the categorical structure from a category (also called 1-category) to a 2-category. More concretely, if the objects under consideration are categories, we have morphisms $F : \mathtt{C} \to \mathtt{C}'$ between categories – 1-morphisms, just defined under the name of functors – and we have morphisms between functors $\alpha : F \Rightarrow G$ – 2-morphisms, which will be called natural transformations.

Definition 11. *Let* $F : \mathtt{C} \to \mathtt{C}'$ *and* $G : \mathtt{C} \to \mathtt{C}'$ *be two functors between the same categories* $\mathtt{C}$ *and* $\mathtt{C}'$*. A* **natural transformation** $\alpha : F \Rightarrow G$ *assigns to any object* $A \in \mathtt{C}$

of the source category a unique morphism $\alpha_A : F(A) \to G(A)$ of the target category $\mathtt{C}'$*, such that the following diagram commutes for every $A, B \in \mathtt{C}$ and $f \in \mathrm{Hom}(A, B)$:*

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \alpha_A}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_B} \\ G(A) & \xrightarrow[G(f)]{} & G(B) \end{array}$$

2.2 Presheaves and sheaves

The prototype of a sheaf is the sheaf of algebras of continuous functions over a topological space X – which assigns to any open subset U of X the algebra $\mathcal{C}^0(U)$. These continuous functions can be restricted and glued in the standard way. Another example is the sheaf of modules of sections of a vector bundle E over a smooth manifold X. It associates to any open subset U of X the $\mathcal{C}^\infty(U)$-module $\Gamma(U, E)$ of sections of E above U. Again the standard restriction and gluing properties hold. The definition of a presheaf and of a sheaf extend these model situations to a more abstract and general level.

Definition 12. *A* **presheaf** $\mathcal{F}$ *over a topological space X with values in a category* $\mathtt{C}$ *assigns to any open subset $U \subset X$ an object $\mathcal{F}(U)$ in* $\mathtt{C}$*. Moreover, for any inclusion $V \subset U$ of open subsets of X, there exists a morphism $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ of* $\mathtt{C}$*, called restriction morphism, which verifies two coherence properties:*

1. $\rho^U_U = 1_{\mathcal{F}(U)}$*, for any open subset $U \subset X$,*
2. $\rho^V_W \circ \rho^U_V = \rho^U_W$*, for all open subsets $W \subset V \subset U \subset X$.*

Actually the open subsets of X form a category $\mathtt{Op}_X$ with the inclusion maps i as morphisms, *i.e.*, for any open subsets U, V of X,

$$\mathrm{Hom}(V, U) = \begin{cases} \{i : V \hookrightarrow U\}, & \text{if } V \subset U, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The preceding definition of a presheaf can now be rephrased as follows:

Definition 13. *A* ***presheaf*** $\mathcal{F}$ *over X with values in* $\mathtt{C}$ *is a contravariant functor from the category* $\mathtt{Op}_X$ *to the category* $\mathtt{C}$.

Let us now assume for simplicity that $\mathtt{C} = \mathtt{Set}$. The elements of the set $\mathcal{F}(U)$, U open in X, are called the **sections** of $\mathcal{F}$ over U. We often write $\Gamma(U, \mathcal{F})$ instead of $\mathcal{F}(U)$.

A presheaf is a sheaf, if the usual identity and gluing properties known from the above prototypical examples are verified:

Definition 14. *A* ***sheaf*** $\mathcal{F}$ *over a topological space* X *with values in* `Set` *is a* `Set`*-valued presheaf over* X*, which verifies the following two additional requirements: For any open subset* $U \subset X$ *and for any open cover* $\{U_i\}_{i\in I}$ *of* U*,*

1. **Local identity***: if* $s, t \in \mathcal{F}(U)$ *and if* $\rho^U_{U_i} s = \rho^U_{U_i} t$*, for all* $i \in I$*, then* $s = t$*,*
2. **Gluing property***: if* $\{s_i\}_{i\in I}$ *is a family of sections* $s_i \in \mathcal{F}(U_i)$*, such that*

$$\rho^{U_i}_{U_i\cap U_j} s_i = \rho^{U_j}_{U_i\cap U_j} s_j, \quad \forall i, j \in I,$$

then there exists a section $s \in \mathcal{F}(U)$*, such that* $\rho^U_{U_i} s = s_i$*, for all* $i \in I$ *(the local identity property implies that the global section* s *is actually unique).*

There exist presheaves that are not sheaves. An example are continuous bounded functions over the real line. These, together with the usual restriction maps, form a presheaf. However, they do not define a sheaf, since, when gluing such functions, we may get a global unbounded function.

A sheaf $\mathcal{F}$ over X is denoted by $(X, \mathcal{F})$, if we wish to emphasize the underlying space. Of course, a sheaf can be valued in categories of sets endowed with additional structure – as for instance the category of abelian groups, the category of supercommutative algebras... The category `C`, in which the sheaf takes values, defines the **type** of the sheaf. If `C` is the category `Gr` of groups, the sheaf is a sheaf of groups. For our purposes, sheaves of $\mathcal{O}$-modules, where $\mathcal{O}$ is a sheaf of rings, and locally free sheaves of $\mathcal{O}$-modules are of particular importance.

Definition 15. *Let* $\mathcal{O}$ *be a sheaf of rings over a topological space* X*. A sheaf of* $\mathcal{O}$*-modules over* X *is a sheaf* $\mathcal{F}$ *over* X*, such that, for each open subset* $U \subset X$*, the set* $\mathcal{F}(U)$ *is a module over the ring* $\mathcal{O}(U)$*. If in addition, for every point* $x \in X$ *there exists an open neighborhood* $U \subset X$*, such that* $\mathcal{F}(U)$ *is a free* $\mathcal{O}(U)$*-module, then* $\mathcal{F}$ *is a* ***locally free sheaf of*** $\mathcal{O}$***-modules****.*

2.3 Morphisms

Presheaves and sheaves of the same type over a given topological space form categories. Morphisms of presheaves and morphisms of sheaves are defined identically. As presheaves are just special contravariant functors, morphisms of presheaves are particular cases of natural transformations, so that the next definition is obvious.

Definition 16. *Consider two* `C`*-valued presheaves* $\mathcal{F}$ *and* $\mathcal{G}$ *defined over the same topological space* X*. A* ***morphism*** *of presheaves* $\phi : \mathcal{F} \Rightarrow \mathcal{G}$ *is a family of* `C`*-morphisms*

$$\phi_U : \mathcal{F}(U) \to \mathcal{G}(U),$$

indexed by $U \in \mathtt{Op}_X$, which commute with the restrictions, i.e., if ρ and r denote the restriction morphisms of $\mathcal{F}$ and $\mathcal{G}$, respectively, the following diagram commutes, for any open subsets $V \subset U \subset X$:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho^U_V"] \arrow[d, "\phi_U"'] & \mathcal{F}(V) \arrow[d, "\phi_V"'] \\ \mathcal{G}(U) \arrow[r, "r^U_V"'] & \mathcal{G}(V) \end{tikzcd}$$

The fact that morphisms of presheaves and of sheaves are the same means that the category $\mathtt{Sh}(X, \mathtt{C})$ of sheaves is a full subcategory of the category $\mathtt{PreSh}(X, \mathtt{C})$ of presheaves:

Definition 17. *A category $\mathtt{C}'$ is a* **subcategory** *of a category $\mathtt{C}$, if $\mathrm{Ob}(\mathtt{C}') \subset \mathrm{Ob}(\mathtt{C})$, if, for any objects $A, B \in \mathtt{C}'$, $\mathrm{Hom}_{\mathtt{C}'}(A, B) \subset \mathrm{Hom}_{\mathtt{C}}(A, B)$, and if the composition and identities in $\mathtt{C}'$ are induced by those of $\mathtt{C}$. If $\mathrm{Hom}_{\mathtt{C}'}(A, B) = \mathrm{Hom}_{\mathtt{C}}(A, B)$, for all $A, B \in \mathtt{C}'$, then $\mathtt{C}'$ is a* **full subcategory** *of $\mathtt{C}$.*

For sheaves of the same type over different topological spaces, the preceding definition of morphisms is to be extended. We need this generalized definition later, in the case of sheaves of rings.

Definition 18. *A* ***morphism*** *of sheaves $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ of rings over two topological spaces X and Y,*

$$\Phi : (X, \mathcal{F}) \Rightarrow (Y, \mathcal{G}),$$

(to simplify, we often substitute $\to$ for $\Rightarrow$) is made up by a continuous map

$$\phi : X \to Y$$

and a family of ring morphisms ϕ^*_V, indexed by $V \in \mathtt{Op}_Y$, called pullback morphisms,

$$\phi^*_V : \mathcal{G}(V) \to \mathcal{F}(\phi^{-1}(V)),$$

which commute with the restrictions, i.e. if we denote by ρ and r the restriction morphisms of $(X, \mathcal{F})$ and $(Y, \mathcal{G})$, respectively, the following diagram commutes, for all open subsets $W \subset V \subset Y$:

$$\begin{tikzcd} \mathcal{G}(V) \arrow[r, "r^V_W"'] \arrow[d, "\phi^*_V"'] & \mathcal{G}(W) \arrow[d, "\phi^*_W"] \\ \mathcal{F}(\phi^{-1}(V)) \arrow[r, "\rho^{\phi^{-1}(V)}_{\phi^{-1}(W)}"'] & \mathcal{F}(\phi^{-1}(W)) \end{tikzcd}$$

2.4 Germs and stalks

If X is a topological space, the concept of germ at a point $x \in X$ of concrete functions $f : X \to \mathbb{R}$ is clear: it is the class $[f]_x$ of all functions f defined around x and such that any two of them coincide in some neighborhood of x.

This notion can be extended to the sections of a presheaf $\mathcal{F}$ of rings over X. Let $x \in X$ and consider the sections over the neighborhoods U of x, or, better, take the disjoint union $\sqcup_{U \ni x} \mathcal{F}(U)$. To define the class $[s]_x$, $s \in \mathcal{F}(U)$, we need an equivalence relation in this union. A section $s \in \mathcal{F}(U)$, $U \ni x$, is equivalent to a section $t \in \mathcal{F}(V)$, $V \ni x$, and we write $s \sim t$, if and only if there exists a neighborhood $W \subset U \cap V$ of x such that

$$\rho_W^U s = \rho_W^V t. \tag{2.1}$$

It is quite obvious that the quotient set $\mathcal{F}_x := \sqcup_{U \ni x} \mathcal{F}(U) / \sim$ inherits a ring structure.

Definition 19. *Let $\mathcal{F}$ be a presheaf of rings over a topological space X. For any $x \in X$, the ring*

$$\mathcal{F}_x := \sqcup_{U \ni x} \mathcal{F}(U) / \sim,$$

where $\sim$ denotes the equivalence (2.1), is called the **stalk** *of $\mathcal{F}$ at x. Its elements $[s]_x$ are the* **germs** *of sections at x.*

As morphisms $\phi : \mathcal{F} \to \mathcal{G}$ of presheaves over X commute with restrictions, they induce morphisms

$$\phi_x : \mathcal{F}_x \ni [s]_x \mapsto [\phi s]_x \in \mathcal{G}_x,$$

$x \in X$, between the stalks of these presheaves. A similar result is valid for morphisms of presheaves over different spaces.

The preceding quotient is an example of an **inductive or direct limit**. An inductive limit is a limit $\varinjlim X_i$ of rings, modules, algebras... X_i, $i \in I$, and is itself a ring, module, algebra...

More precisely, start with a family $(X_i)_{i \in I}$ of algebraic objects of one of the preceding types, indexed by a directed set $(I, \leq)$. Consider further a collection of morphisms $f_{ji} : X_i \to X_j$, $i \leq j$ (previously the restriction morphisms), which satisfy the compatibility conditions

$$f_{ii} = 1_{X_i}, \quad f_{kj} \circ f_{ji} = f_{ki}, \tag{2.2}$$

for all $i \leq j \leq k$. We say that $x_i \in X_i \subset \sqcup_i X_i$ and $x_j \in X_j \subset \sqcup_i X_i$ are equivalent, and write $x_i \sim x_j$, if and only if there is k, $i \leq k$, $j \leq k$, such that

$$f_{ki}(x_i) = f_{kj}(x_j). \tag{2.3}$$

It follows from the compatibility relations that $\sim$ is an equivalence and we denote by $X = \sqcup_i X_i / \sim$ the quotient set. The quotient maps $\pi_i : X_i \to X$ allow defining on

X the same algebraic structure as on the X_i. Indeed, to define in X for instance a product $\pi_i x_i \cdot \pi_j x_j$, it suffices to note that $x_i \sim f_{ki} x_i$ (which means that $\pi_i = \pi_k f_{ki}$) and $x_j \sim f_{kj} x_j$, for any k bigger than i and j, and to set

$$\pi_i x_i \cdot \pi_j x_j = (\pi_k f_{ki} x_i) \cdot (\pi_k f_{kj} x_j) = \pi_k \left(f_{ki} x_i \cdot f_{kj} x_j \right), \tag{2.4}$$

which actually leads to a well-defined product. Of course, the π_k are then morphisms for the considered algebraic structure.

Definition 20. *Consider a category of sets with some algebraic structure. A* **direct system** *of objects and morphisms is a system (X_i, f_{ji}) that satisfies the compatibility conditions (2.2). The algebraic object $X = \sqcup_i X_i / \sim$, see (2.3) and (2.4), and the corresponding morphisms $\pi_i : X_i \to X$, such that $\pi_i = \pi_k f_{ki}$, then form the* **direct limit** *of the system (X_i, f_{ji}) and we write $X = \varinjlim X_i$.*

Actually the direct limit of a direct system is a (the) solution of the universal problem that is obvious from the preceding definition. This limit can be defined for a direct system in a category `C`, precisely by means of this universal problem. However, in an arbitrary category `C` direct limits may not exist.

2.5 Sheafification

We already mentioned that there exist presheaves that are not sheaves. However, to any presheaf of sets we can associate a sheaf that has the same stalks.

Proposition 5. *Let $\mathcal{F}$ be a `Set`-valued presheaf over X. There exists a sheaf $\mathcal{F}^\sharp$ and a presheaf morphism $\varphi : \mathcal{F} \to \mathcal{F}^\sharp$, such that for any other sheaf F and presheaf morphism $f : \mathcal{F} \to F$, there is a unique sheaf morphism $\psi : \mathcal{F}^\sharp \to F$, such that $f = \psi\varphi$.*

Definition 21. *The solution $(\mathcal{F}^\sharp, \varphi)$ of the preceding universal problem (which is of course unique up to unique isomorphism) is called the* **sheafification** *of the presheaf $\mathcal{F}$.*

Proof of Proposition 5. We first replace abstract sections $s \in \mathcal{F}(U)$, U open in X, by concrete sections

$$\sigma : U \ni y \mapsto \sigma(y) \in \mathcal{F}_y \subset \sqcup_{x \in U} \mathcal{F}_x, \tag{2.5}$$

so that gluing problems disappear. Observe that any section $s \in \mathcal{F}(U)$ induces a section

$$\tilde{s} : U \ni y \mapsto [s]_y \in \mathcal{F}_y \subset \sqcup_{x \in U} \mathcal{F}_x.$$

Conversely, for any section σ and any $y \in U$, we have $\sigma(y) = [s]_y$, where s is a section of $\mathcal{F}$ defined in a neighborhood of y. However, this s may vary from point to point. To get a sheaf $\mathcal{F}^\sharp$ with all desired properties, we define $\mathcal{F}^\sharp(U)$ as the set of all the maps (2.5) with the additional requirement that they be locally implemented by a section of $\mathcal{F}$. More precisely, we ask that for any $y \in U$, there exists an open neighborhood $V \subset U$ of y and a section $s_V \in \mathcal{F}(V)$ such that $\sigma = \tilde{s}_V$ in V. This condition is weak enough to provide a sheaf and it is strong enough to get the best possible one. The map φ is of course given by $\varphi_U : \mathcal{F}(U) \ni s \mapsto \tilde{s} \in \mathcal{F}^\sharp(U)$, and any presheaf morphism $f : \mathcal{F} \to F$ valued in a sheaf F factors uniquely through $\mathcal{F}^\sharp$. Indeed, it suffices to define $\psi_U(\sigma) \in F(U)$ locally in any subset $V \subset U$ in which $\sigma = \tilde{s}_V$ by $\psi_V(\sigma) = f s_V$. □

Remark 4. *It is easily seen that the presheaf $\mathcal{F}$ and its sheafification $\mathcal{F}^\sharp$ have the same stalks.*

Eventually, the universal problem formulation of the sheafification can be equivalently restated as a left adjoint functor problem for the forgetful functor.

2.6 Exact sequences of sheaves

Complexes and exact (in particular, short exact) sequences are frequently met in Mathematics. E.g., in Linear Algebra, if $\ell : E \to F$ is a linear map between two real finite-dimensional vector spaces, the short sequence

$$0 \to \ker \ell \xrightarrow{i} E \xrightarrow{\ell} E/\ker \ell \simeq \operatorname{im} \ell \to 0 \tag{2.6}$$

of vector spaces and linear maps, where i is the injection, is a complex and it is exact.

Definition 22. *A sequence of vector spaces and linear maps*

$$\ldots \to E \xrightarrow{f} F \xrightarrow{g} G \to \ldots$$

is a **complex** *(resp. is* **exact***) if and only if, at every spot, the image of the incoming map is included in the kernel (resp. coincides with the kernel) of the outgoing map. In particular, a short sequence*

$$0 \to E \xrightarrow{f} F \xrightarrow{g} G \to 0$$

is a complex if and only if $\operatorname{im} f \subset \ker g$ *(resp. if and only if f is injective,* $\operatorname{im} f = \ker g$, *and g is surjective).*

Let us mention that in the last sequence the left (resp. right) arrow $0 \to E$ (resp. $G \to 0$) represents the unique linear map from $\{0\}$ to E (resp. from G to $\{0\}$) $(\star)$.

We now provide some information about Category Theory. The reader may skip these observations and view Propositions 6 and 7 as definitions.

If we do not work in the category $\mathtt{Vect}$ of vector spaces, but in an abstract category $\mathtt{C}$, the morphisms are not necessarily maps between sets of elements, but just arrows between abstract objects. Therefore,

Remark 5. *The usual concepts of injective morphism (or monomorphism), surjective morphism (or epimorphism), kernel, image, exact sequence..., which are based upon the notion of element, must be extended to this more general abstract setting.*

Definition 23. *Let $\mathtt{C}$ be a category. A morphism $f : A \to B$ in $\mathtt{C}$ is a* **monomorphism** *(resp.* **epimorphism***), if and only if it is left (resp. right) cancelable, i.e. if and only if, for any $\mathtt{C}$-morphisms g_1 and g_2, $f \circ g_1 = f \circ g_2$ (resp. $g_1 \circ f = g_2 \circ f$) entails $g_1 = g_2$.*

The above remark ($\star$) leads to the next extension.

Definition 24. *A category $\mathtt{C}$ admits a* **zero** *object 0, if, for any object A in $\mathtt{C}$, there exists a unique $\mathtt{C}$-morphism $f : 0 \to A$ and a unique $\mathtt{C}$-morphism $g : A \to 0$.*

A category does not necessarily have a zero object, but if such an object exists, it is unique (up to unique isomorphism). If a zero object 0 exists, there are, for any objects A, B, unique morphisms $f : A \to 0$ and $g : 0 \to B$, we write $0_{AB} := g \circ f : A \to B$, and we call 0_{AB} a zero morphism from A to B. These zero morphisms verify, for any $h : B \to C$ and any $k : A \to B$, $h \circ 0_{AB} = 0_{AC} = 0_{BC} \circ k$.

Definition 25. *Let $\mathtt{C}$ be a category with a zero object. For any $\mathtt{C}$-morphism $f : A \to B$, a* **kernel** *of f is a morphism $k : \ker f \to A$ (whose source is denoted by $\ker f$), such that $f \circ k = 0_{\ker f, B}$, and which is universal in the obvious sense. Similarly, a* **cokernel** *of f is a morphism $c : B \to \operatorname{coker} f$ (whose target is denoted by $\operatorname{coker} f$), such that $c \circ f = 0_{A, \operatorname{coker} f}$, and which is universal.*

Not every morphism in every category needs have a kernel or cokernel, but if such a morphism exists it is unique and it is necessarily a monomorphism or epimorphism, respectively. In the concrete case $\mathtt{C} = \mathtt{Vect}$, the cokernel of a linear map $f : A \to B$ is given by $B/\operatorname{im} f$ (together with the projection c onto this quotient).

The notions of image of a morphism and of exact sequence cannot be defined in any category – the category has to be abelian. The prototype of an (abstract) abelian category is the (concrete) category $\mathtt{AbGrp}$ of abelian groups. However, it is time consuming to detail abelian categories in whole generality, so we confine ourselves to the

Remark 6. *In an abelian category any morphism admits a kernel, a cokernel and an image (defined as the kernel of its cokernel), so that exact sequences or exact complexes can be defined as usual by the requirement that any "incoming image" must coincide with the corresponding "outgoing kernel".*

In the following, we investigate sheafs of abelian groups over a topological space X, e.g. sheaves of modules over a ring.

Remark 7. *The category of sheaves of modules over a topological space is abelian, and exact complexes of sheaves of modules are defined.*

Let us recall that the main aspect of sheaves is that they can be glued from local pieces. Hence, it is not surprising that a number of properties involving sheaves are equivalent to the same properties at the level of stalks. Starting from the above general definitions of monomorphisms, epimorphisms and exact complexes in the category of sheaves, one can prove the following propositions that corroborate this remark.

Proposition 6. *A morphism of sheaves of modules over a topological space X, $\varphi : \mathcal{F} \to \mathcal{G}$, is a monomorphism (resp. epimorphism), if and only if the induced module morphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective), for all $x \in X$.*

Proposition 7. *A short sequence of morphisms of sheaves of modules over X is* ***exact****, if and only if the induced sequence of module morphisms between the corresponding stalks at x is exact, for all $x \in X$.*

The preceding properties do not exactly correspond to the same properties for morphisms of modules of sections.

Proposition 8. *A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves of modules over X is*

1. *a monomorphism, if and only if the morphisms $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ between modules of sections are injective, for each open subset $U \subset X$,*

2. *an epimorphism, if and only if the morphisms $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ between modules of sections are "weakly surjective", for any $U \subset X$, i.e. have the property that for any $t \in \mathcal{G}(U)$, there are an open cover $U = \cup_\alpha U_\alpha$ and sections $s_\alpha \in \mathcal{F}(U_\alpha)$, such that $\varphi_{U_\alpha} s_\alpha = t|_{U_\alpha}$.*

Proposition 9. *A short sequence of morphisms of sheaves of modules over X*

$$0 \longrightarrow \mathcal{F} \xrightarrow{i} \mathcal{G} \xrightarrow{p} \mathcal{H} \longrightarrow 0$$

is exact, if and only if, for any open subset $U \subset X$, the sequence of morphisms of modules of sections

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{i_U} \mathcal{G}(U) \xrightarrow{p_U} \mathcal{H}(U)$$

is exact, and additionally the map p_U satisfies the preceding weak surjectivity property.

Sketch of proof. To study simultaneously the three spots of the sequences of sheaves, stalks, and modules of sections, change notations and let $\alpha : \mathcal{A} \to \mathcal{B}$ (resp. $\beta : \mathcal{B} \to \mathcal{C}$) be one of the three incoming morphisms (resp. the corresponding outgoing morphism) of sheaves. Denote by the same symbols α and β the induced morphisms between stalks or modules of sections. Let us focus on the 'top-down' implication, *i.e.* assume that the sequences of sheaves and stalks are exact and investigate whether at the level of modules of sections over $U \subset X$, we have as well, *e.g.* $\ker \beta \subset \operatorname{im} \alpha$.

Let $t \in \mathcal{B}(U) \cap \ker \beta$. Then, for any $x \in U$, we have $[t]_x \in \mathcal{B}_x \cap \ker \beta = \mathcal{B}_x \cap \operatorname{im} \alpha$, so that there exists $[s]_x \in \mathcal{A}_x$, such that $\alpha[s]_x = [t]_x$. It follows that there is a neighborhood $U_x \subset U$ of x, such that $s \in \mathcal{A}(U_x)$ and $\alpha s = t|_{U_x}$.

If the considered spot is 'Spot 1', the result means that $0 = \alpha s = t|_{U_x}$, so that $t = 0$, which means that the sequence of modules is exact at this spot and that i_U is injective.

If it is 'Spot 2', the map $\alpha = i_U$ is injective. If we write $s_x := s \in \mathcal{A}(U_x)$, we have

$$\alpha(s_x|_{U_x \cap U_y}) = t|_{U_x \cap U_y} = \alpha(s_y|_{U_x \cap U_y}),$$

and the sections $s_x \in \mathcal{A}(U_x)$, $x \in U$, can be glued and provide a section $S \in \mathcal{A}(U)$, such that $S|_{U_x} = s_x$. It is clear that $t = \alpha S \in \mathcal{B}(U) \cap \operatorname{im} \alpha$, so that we get as well exactness at the central spot (at least we just proved the inclusion $\mathcal{B}(U) \cap \ker \beta \subset \mathcal{B}(U) \cap \operatorname{im} \alpha$).

In case of 'Spot 3', the above result means precisely that p_U is weakly surjective, but no gluing as for 'Spot 2' is possible, as α is no longer injective. $\square$

The next remark is intended for readers who are already familiar with Sheaf Theory.

The preceding proposition shows in particular that the global section functor, which sends a sheaf to its module of global sections, is only a left exact covariant functor. If the source category has enough injectives - which is for instance the case for the category of abelian groups - this functor admits right derived functors and sheaf cohomology can be defined. There are particular classes of sheaves, for example flabby sheaves, which are acyclic, *i.e.* all their higher sheaf cohomology groups vanish, and for which the global section functor is exact, so that a short exact sequence of sheaves provides a short exact sequence of global sections.

2.7 Quotient sheaf

In this section, we consider sheaves, for example of abelian groups, over a topological space X.

Definition 26. *A* **subsheaf** *of a sheaf $\mathcal{G}$ is a sheaf $\mathcal{F}$ over the same topological space, such that any $\mathcal{F}(U)$ is a subobject of the corresponding $\mathcal{G}(U)$ and any restriction morphism of $\mathcal{F}$ is induced by the corresponding restriction map of $\mathcal{G}$.*

To define the notion of quotient sheaf $\mathcal{G}/_{\mathcal{F}}$ of a sheaf $\mathcal{G}$ by a subsheaf $\mathcal{F}$, we first construct a presheaf

$$\operatorname{pre}\mathcal{G}/_{\mathcal{F}} : U \mapsto (\operatorname{pre}\mathcal{G}/_{\mathcal{F}})\,(U) := \mathcal{G}(U)/\mathcal{F}(U).$$

As this assignment is not necessarily a sheaf,

Definition 27. *The* ***quotient sheaf*** $\mathcal{G}/_{\mathcal{F}}$ *of a sheaf* $\mathcal{G}$ *by a subsheaf* $\mathcal{F}$ *is the sheafification of the presheaf* $\operatorname{pre}\mathcal{G}/_{\mathcal{F}}$.

Proposition 10. *If* $\mathcal{F}$ *is a subsheaf of a sheaf* $\mathcal{G}$, *the short sequence of sheaves*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{G}/_{\mathcal{F}} \to 0$$

is exact.

Proof. Since

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{G}(U)/\mathcal{F}(U) \to 0$$

is exact for every open subset $U \subset X$, see e.g. Equation (2.6), the sequence

$$0 \to \mathcal{F}_x \to \mathcal{G}_x \to (\operatorname{pre}\mathcal{G}/_{\mathcal{F}})_x \to 0$$

is exact, for any $x \in X$, so that the sequence

$$0 \to \mathcal{F}_x \to \mathcal{G}_x \to (\mathcal{G}/_{\mathcal{F}})_x \to 0$$

is exact as well, as the last stalks of the two last sequences are isomorphic. Hence, the result. $\square$

Chapter 3

Supermanifolds

The categorical formulation of Supergeometry establishes a link between the Kostant-Berezin-Leites (KBL) and the deWitt-Rogers (WR) approaches to Supergeometry (the latter being valid provided that all constructions are functorial with respect to a change of the underlying Grassmannian). In the following, we confine ourselves to the KBL viewpoint.

3.1 The category of local ringed spaces

3.1.1 Local ringed spaces

Definition 28. *A* ***ringed space*** *is a pair* $(M, \mathcal{O})$ *made up by a topological space* M *and a sheaf* $\mathcal{O}$ *of commutative rings with unit. If in addition, for every point* x *of* M*, the stalk* $\mathcal{O}_x$ *of* $\mathcal{O}$ *at* x *is a local ring, i.e. a ring that admits a unique maximal ideal, then* $(M, \mathcal{O})$ *is a* ***local ringed space*** (*or locally ringed space*) (*LRS*).

These maximal ideals can be described in a relatively simple way:

Proposition 11. *If* $(M, \mathcal{O})$ *is a LRS,* x *a point of* M*, and* $\mathfrak{m}_x$ *denotes the maximal ideal of* $\mathcal{O}_x$*, the difference* $\mathcal{O}_x \setminus \mathfrak{m}_x$ *is the set of invertible elements of* $\mathcal{O}_x$*.*

Proof. If $f \in O_x$ is invertible and $f \in \mathfrak{m}_x$, we have $fg = 1$, so that $1 \in \mathfrak{m}_x$ and $O_x = \mathfrak{m}_x$, which is a contradiction. If $f \notin \mathfrak{m}_x$, consider the ideal $I_f = \{fg : g \in O_x\}$ generated by f. If it is proper, it is contained in a maximal ideal (due to Zorn's lemma), hence in $\mathfrak{m}_x$, since this is the unique maximal ideal. Thus $f \in \mathfrak{m}_x$ – a contradiction. It follows that $I_f = O_x \ni 1$, thus that there is $g \in \mathcal{O}_x$ such that $fg = 1$, and eventually that f is invertible. $\square$

3.1.2 Morphisms of local ringed spaces

Morphisms of local ringed spaces are of course maps between ringed spaces that respect the whole ringed space structure, the topological structure, the sheaf structure, as well as locality:

Definition 29. *A* ***morphism*** *of LRS is a map Φ between two ringed spaces $(M, \mathcal{O})$ and $(N, \mathcal{R})$, made up by*

1. *a continuous base map*
$$\phi \;:\; M \;\to\; N,$$
2. *a family, indexed by the open subsets $V \subset N$, of ring morphisms, called pullbacks, of the form*
$$\phi_V^* \;:\; \mathcal{R}(V) \;\to\; \mathcal{O}\left(\phi^{-1}(V)\right),$$
which
   - *commute with the restriction maps of the sheaves $\mathcal{O}$ and $\mathcal{R}$,*
   - *and naturally induce, for every $x \in M$, a ring morphism*
$$\phi_x^* : \mathcal{R}_{\phi(x)} \to \mathcal{O}_x$$
that respects the maximal ideals, i.e. verifies
$$\phi_x^*(\mathfrak{m}_{\phi(x)}) \subset \mathfrak{m}_x. \tag{3.1}$$

Local ringed spaces form a category with obvious identity morphisms, the composition $\circ$ being given for $\Phi = (\phi, \phi^*) \in \mathrm{Hom}((M, \mathcal{O}), (N, \mathcal{R}))$ and $\Psi = (\psi, \psi^*) \in \mathrm{Hom}((N, \mathcal{R}), (P, \mathcal{S}))$ by

$$\Psi \circ \Phi = (\psi, \psi^*) \circ (\phi, \phi^*) = (\psi \circ \phi, \phi^* \circ \psi^*).$$

Example 4. *The pair $(\mathbb{R}^n, \mathcal{C}^\infty)$ is a local ringed space. Here $\mathcal{C}^\infty$ is the sheaf of rings, whose sections are the rings of real-valued smooth functions defined on the open subsets of $\mathbb{R}^n$ and whose restriction maps are the usual restrictions of functions. The unique maximal ideal of the stalk at $x \in \mathbb{R}^n$ is known to be*

$$\mathfrak{m}_x = \left\{[f]_x \in \mathcal{C}_x^\infty | f(x) = 0\right\}. \tag{3.2}$$

3.1.3 Local ringed spaces of functions

The preceding example is part of a special class of LRS, for which the rings of sections are commutative $\mathbb{K}$-algebras with unit made up by functions valued in the commutative field $\mathbb{K}$. We refer to such LRS as *LRS of functions*. In this case, the unique maximal ideals are always of the form (3.2) and the pullback morphisms ψ_V^* are necessarily those implemented by the base map ψ.

Proposition 12. *Let $(M, \mathcal{O})$ and $(N, \mathcal{R})$ be two LRS of functions, and let $x \in M$. Then,*

1. *the unique maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_x$ is given by*
$$\mathfrak{m}_x = \{[f]_x \in \mathcal{O}_x | f(x) = 0\},$$
2. *the only possible morphisms (ψ, ψ^*) between $(M, \mathcal{O})$ and $(N, \mathcal{R})$ are those whose pullbacks are defined, for any $V \subset N$ and any $g \in \mathcal{R}(V)$, by*
$$\psi_V^*(g) := g \circ \psi.$$

Proof. 1. For any $x \in M$, the set $I_x := \{[f]_x \in \mathcal{O}_x | f(x) = 0\}$ is an ideal of $\mathcal{O}_x$. Since it is the kernel of the evaluation map at x, it has codimension 1.

Since any proper ideal is contained in a maximal one, we get $I_x \subset \mathfrak{m}_x$. As $\mathfrak{m}_x$ is itself proper, it has at least codimension 1. Hence, $\mathfrak{m}_x = I_x$.

2. Let (ψ, ψ^*) be a morphism of LRS of functions from $(M, \mathcal{O})$ to $(N, \mathcal{R})$. Assume that, for some open subset $V \subset N$, the pullback ψ_V^* is not canonically induced by ψ, *i.e.* that, for some $g \in \mathcal{R}(V)$ and some $x \in \psi^{-1}(V)$, we have
$$\psi_V^*(g)\,(x) \neq (g \circ \psi)\,(x). \tag{3.3}$$
Since, for any $k \in \mathbb{K} \hookrightarrow \mathcal{R}(V)$,
$$\psi_V^*(g + k)\,(x) = \psi_V^*(g)\,(x) + k \neq (g \circ \psi)\,(x) + k = ((g + k) \circ \psi)\;(x),$$
there exists $h \in \mathcal{R}(V)$ (take $h = g + k$ for some $k \in \mathbb{K}$) such that
$$\psi_V^*(h)\,(x) = 0, \tag{3.4}$$
$$(h \circ \psi)\,(x) \neq 0. \tag{3.5}$$
In view of Proposition 11 and the preceding description of maximal ideals, Equation (3.5) implies that $[h]_{\psi(x)} \in \mathcal{R}_{\psi(x)}$ is invertible, *i.e.* that h is invertible in some neighborhood $W \subset V$ of $\psi(x)$, and thus that $\psi_W^*(h)$ is invertible in $\mathcal{O}(\psi^{-1}(W))$. On the other hand, Equation (3.4) entails that $[\psi_V^*(h)]_x \in \mathfrak{m}_x$, so that $\psi_W^*(h)$ is not invertible – a contradiction.

□

Corollary 2. *For morphisms of LRS of functions, the stalk condition (3.1) is automatically satisfied.*

3.2 Definition of supermanifolds

3.2.1 Algebraic approach to manifolds

It is well-known that the differential structure of a manifold is encrypted in the associative algebra structure of its space of functions – in the sense that if the algebras of functions of two manifolds are isomorphic, the manifolds are diffeomorphic. Furthermore, if $(M, \mathcal{O})$ is a ringed space (not necessarily a LRS of functions), which is locally isomorphic to the LRS $(\mathbb{R}^n, \mathcal{C}^\infty)$, one can endow M with a differential structure of dimension n (such that the resulting sheaf of smooth functions $(M, \mathcal{C}^\infty)$ is isomorphic to the sheaf $(M, \mathcal{O})$). When substituting the sheaf $(\mathbb{R}^n, \mathcal{A})$ of real analytic functions (resp. the sheaf of holomorphic functions $(\mathbb{C}^n, \mathcal{H})$) for the **local model** $(\mathbb{R}^n, \mathcal{C}^\infty)$, we obtain similarly real analytic manifolds (resp. complex manifolds). We will define supermanifolds along the same lines.

3.2.2 Super ringed spaces

Superrings were defined in the chapter on Superalgebra.

Definition 30. *A superring $R = R_0 \oplus R_1$ is called **local** if it admits a unique maximal homogeneous ideal I, i.e. a unique maximal ideal of the form*

$$I = I_0 \oplus I_1 = (I \cap R_0) \oplus (I \cap R_1).$$

Definition 31. *A **super ringed space** is a pair $(M, \mathcal{O})$ made up by a topological space M and a sheaf $\mathcal{O}$ of supercommutative superrings with unit. If in addition, for any $x \in M$, the stalk $\mathcal{O}_x$ of $\mathcal{O}$ at x is a local superring, we say that $(M, \mathcal{O})$ is a **local super ringed space** (LSRS).*

Note that the restriction morphisms of a sheaf of superrings (resp. the pullback morphisms of a morphism of local super ringed spaces) respect the $\mathbb{Z}_2$-grading, since they are morphisms of superrings. Local super ringed spaces form a category.

3.2.3 Superdomains

The LSRS on which supermanifolds will be locally modelled are sometimes called superdomains.

Definition 32. *A smooth* **superdomain** *of dimension $p|q$ is a LSRS of the type $\mathcal{U}^{p|q} := (U, \mathcal{C}^\infty_{p|q})$, where U is an open subset of $\mathbb{R}^p$ and where $\mathcal{C}^\infty_{p|q}$ is the sheaf of supercommutative associative $\mathbb{R}$-algebras with unit, defined, for any open subset $V \subset U$, by*

$$\mathcal{C}^\infty_{p|q}(V) = \mathcal{C}^\infty(V)\left[\xi^1, \ldots, \xi^q\right]. \tag{3.6}$$

The RHS is the Grassmann algebra over smooth functions of V generated by q odd (anticommuting) symbols ξ^i.

More precisely, if $x = (x^1, \ldots, x^p)$ are the canonical commuting (even) coordinates of $\mathbb{R}^p$, the "superfunctions" $f \in \mathcal{C}^\infty_{p|q}(V)$ read

$$f(x, \xi) = \sum_\alpha f_\alpha(x)\xi^\alpha = \sum_{k=0}^{q} \sum_{\alpha^1 < \ldots < \alpha^k} f_{\alpha^1 \ldots \alpha^k}(x)\, \xi^{\alpha^1} \ldots \xi^{\alpha^k}, \tag{3.7}$$

with all coefficients f_α in $\mathcal{C}^\infty(V)$.

As the x^i and the smooth functions of these variables are even and the ξ^i are odd, the terms of a superfunction have a canonical parity or $\mathbb{Z}_2$-degree. Furthermore, just as in the prototype of a Grassmann algebra, i.e. in the algebra of differential forms, these terms have a canonical cohomological or $\mathbb{N}$-degree as well.

If we choose $U = \mathbb{R}^p$, we get the smooth superdomain or the superspace $\mathbb{R}^{p|q}$ – the local model for smooth supermanifolds of dimension $p|q$. One defines similarly the local models of real analytic (resp. complex) supermanifolds, just replacing $\mathcal{C}^\infty(V)$ by $\mathcal{A}(V)$ (resp. $\mathcal{H}(V)$).

Let us briefly comment on locality of the super ringed space $\mathcal{U}^{p|q} = (U, \mathcal{C}^\infty_{p|q})$ and detail the unique maximal homogeneous ideals.

Proposition 13. *The unique maximal homogeneous ideal of a stalk $\mathcal{C}^\infty_{p|q,x}$ is given by*

$$\mathfrak{m}_x = \{[f]_x : f_0(x) = 0\}\,.$$

Proof. Let $p_0(V) : \mathcal{C}^\infty_{p|q}(V) \to \mathcal{C}^\infty(V)$ be the projection onto the term of cohomological degree 0 and let $J(V)$ be its kernel. Consider the short exact sequence of algebras

$$0 \to J_x \to \mathcal{C}^\infty_{p|q,x} \to \mathcal{C}^\infty_x \to 0$$

and take a maximal homogeneous ideal $\mathfrak{m}_x$ of $\mathcal{C}^\infty_{p|q,x}$. In the sequel, we omit subscript x and prefer the simplified notation f to the germ notation $[f]_x$.

It is clear that $J \subset \mathfrak{m}$. Indeed, the $(q+1)$-th power of each $f \in J$ vanishes, so that

$$J \subset \sqrt{\mathfrak{m}} = \left\{f \in \mathcal{C}^\infty_{p|q} \mid \exists\, n \in \mathbb{N}^* : f^n \in \mathfrak{m}\right\}.$$

However, since $\mathfrak{m}$ is proper, $1 \notin \sqrt{\mathfrak{m}}$, so $\sqrt{\mathfrak{m}}$ is a proper homogeneous ideal of $\mathcal{C}^\infty_{p|q}$ that contains $\mathfrak{m}$, and $J \subset \sqrt{\mathfrak{m}} = \mathfrak{m}$. Hence, any maximal homogeneous ideal of $\mathcal{C}^\infty_{p|q}$ is of the form

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 = I \oplus J,$$

where $I = p_0(\mathfrak{m})$ is an ideal of $\mathcal{C}^\infty$. This ideal I is maximal in $\mathcal{C}^\infty$, otherwise $\mathfrak{m}$ is not maximal. It follows now from the description, see Proposition 12, of the unique maximal ideal of $\mathcal{C}^\infty = \mathcal{C}^\infty_x$ that any maximal homogeneous ideal $\mathfrak{m}_x$ of $\mathcal{C}^\infty_{p|q\,,x}$ coincides necessarily with

$$\mathfrak{m}_x = \{[f_0]_x : f_0(x) = 0\} \oplus J_x = \{[f]_x : f_0(x) = 0\}.$$

It suffices now to check that this $\mathfrak{m}_x$ is a maximal homogeneous ideal in the stalk $\mathcal{C}^\infty_{p|q\,,x}$. □

3.2.4 Supermanifolds

Definition 33. *A supermanifold $\mathcal{M}$ of dimension $p|q$ is a SRS $(M, \mathcal{O})$ – over a second countable Hausdorff topological space M – that is locally isomorphic to a model LSRS. In the smooth (resp. real analytic, holomorphic) category, the local model is the superspace $(\mathbb{R}^p, \mathcal{C}^\infty_{p|q})$ (resp. $(\mathbb{R}^p, \mathcal{A}_{p|q})$, $(\mathbb{C}^p, \mathcal{H}_{p|q})$).*

In this text, if not differently specified, we implicitly consider smooth supermanifolds. Further, we treat rather SRS whose structure sheaves are sheaves of supercommutative associative unital $\mathbb{R}$-algebras.

Remark 8. *As the local model is a LSRS, supermanifolds are automatically LSRS as well.*

The prototype of a supermanifold is the SRS (M, Ω) of differential forms over a classical smooth manifold. More generally, if E is a smooth vector bundle over M, the SRS $(M, \Gamma(\wedge E^*))$ is a supermanifold of dimension $n|k$, where $n = \dim M$ and $k = \operatorname{rank} E$. This supermanifold is usually denoted by $E[1]$ or ΠE and viewed as the total space of the vector bundle E with shifted parity in the fibers. Thus the functions of this total space are, in local even base coordinates x and odd fiber coordinates ξ, exactly the same as the sections of $\Gamma(\wedge E^*)$ (note that dual base vectors $\varepsilon^i \in E^*_m$ can be viewed as $\varepsilon^i(\xi) = \xi^i$ coordinates of the vectors $\xi \in E_m$). The importance of the example $\Pi E = (M, \Gamma(\wedge E^*))$ relies on the fact that any smooth supermanifold is of this type [Bat79]. More precisely, for any smooth supermanifold $\mathcal{M}$ over a manifold M, there exists a vector bundle E over M, such that $\mathcal{M}$ is diffeomorphic to ΠE. This isomorphism is not canonical and cannot be used in the real analytic or complex category. However, it provides an $\mathbb{N}$-grading of the algebra of functions of $\mathcal{M}$. Nevertheless, the study of (smooth) supermanifolds cannot be reduced to the mere study of vector bundles. Indeed, the category of supermanifolds has more morphisms than that of vector bundles, see below.

3.3 Projection of superfunctions onto classical functions

Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold of dimension $p|q$. Locally, classical functions are embedded into superfunctions and the latter project onto the first. The same holds true globally, if we choose an isomorphism $\mathcal{M} \simeq \Pi E$. However, a global projection of $\mathcal{O}(M)$ onto $\mathcal{C}^\infty(M)$ can be constructed canonically, i.e. independently of the choice of an isomorphism (of course, we have to construct a smooth manifold structure on the base M).

If U denotes a classical smooth manifold, a function $f \in \mathcal{C}^\infty(U)$ is invertible in $\mathcal{C}^\infty(U)$ if and only if $f(x) \neq 0$, for all $x \in U$. Hence,

Remark 9. *For any classical function $f \in \mathcal{C}^\infty(U)$, the value $f(x) \in \mathbb{R}$, $x \in U$, can be characterized as the unique $k \in \mathbb{R}$ such that the function $f - k$ is not invertible, in any neighborhood of x.*

Observe that superfunctions $f \in \mathcal{C}^\infty_{p|q}(U)$, U open in $\mathbb{R}^p$, cannot be evaluated at a "point" (x, ξ), since the ξ are formal parameters. The evaluation of f at a point $x \in U$ leads to a value $f(x, \xi) \in \mathbb{R}[\xi^1, \ldots, \xi^q]$.

Proposition 14. *Let $G(R) = R[\xi^1, \ldots, \xi^q]$ be the Grassmann algebra generated over a unital commutative ring R by anticommuting generators ξ^i. An element $s \in G(R)$,*

$$s = s_0 + \sum_a s_a\, \xi^a + \sum_{a,b} s_{ab}\, \xi^a \xi^b + \ldots,$$

is invertible in $G(R)$ if and only if s_0 is invertible in R.

Proof. The map

$$\begin{array}{rccl} \varepsilon: & G(R) & \to & R \\ & s & \mapsto & s_0 \end{array}$$

is clearly a surjective homomorphism of rings. Thus, if $s \in G(R)$ is invertible, then $\varepsilon(s) = s_0 \in R$ is also invertible (if the inverse of s is s^{-1}, the inverse of $s_0 = \varepsilon(s)$ is $\varepsilon(s^{-1})$).

Conversely, suppose s_0 is invertible in R an let s_0^{-1} be its inverse. Replacing s by $s_0^{-1}s$, we can assume that $s_0 = 1$. Thus, $s = 1 - t$, with t in the ideal J of $G(R)$ generated by the nilpotent elements $\xi^1, \ldots, \xi^q$. In particular, $t^{q+1} = 0$ and thus s is invertible with inverse $s^{-1} = 1 + \sum_{m=1}^{q} t^m$. $\square$

This result entails in the trivial geometric case $R = \mathcal{C}^\infty(U)$, $G(R) = \mathcal{C}^\infty_{p|q}(U)$, U open in $\mathbb{R}^p$, that $f = \sum_\alpha f_\alpha(x)\xi^\alpha \in \mathcal{C}^\infty_{p|q}(U)$ is invertible if and only if $f_0 \in \mathcal{C}^\infty(U)$ is invertible. When combining the latter equivalence with the above characterization of the value $f_0(x)$, $x \in U$, we get the following

Lemma 2. *Let U be open in $\mathbb{R}^p$. Then, for any $x \in U$ and any $f \in \mathcal{C}^\infty_{p|q}(U)$, there exists a unique $k \in \mathbb{R}$ such that $f - k$ is not invertible, in any neighborhood of x in U.*

Since Lemma 2 is a local SRS-property, the local SRS-isomorphism between the sheaf $\mathcal{C}^\infty_{p|q}$ and the structure sheaf $\mathcal{O}$ of $\mathcal{M}$ entails that the same property holds true in $\mathcal{O}$. For any open subset U of M, any $f \in \mathcal{O}(U)$ and any $x \in U$, the unique $k \in \mathbb{R}$ such that $f - k$ is not invertible, in any neighborhood of x, is denoted by $\tilde{f}(x)$ or $\varepsilon_U(f)(x)$. If x runs through U, we obtain a function $\varepsilon_U(f) : U \to \mathbb{R}$, and if f runs through $\mathcal{O}(U)$, we get a map $\varepsilon_U : \mathcal{O}(U) \to \mathcal{F}(U)$, where $\mathcal{F}(U) = \operatorname{im}\varepsilon_U$ is the algebra of these functions. Actually ε_U is a surjective algebra morphism and the short sequence of algebras

$$0 \ \to \ \mathcal{J}(U) \ \to \ \mathcal{O}(U) \ \to \mathcal{F}(U) \ \to \ 0,$$

where $\mathcal{J}(U) = \ker\varepsilon_U$, is exact. In particular, $\mathcal{F}(U) \simeq \mathcal{O}(U)/\mathcal{J}(U)$ and these presheaves define sheaves $\mathfrak{F} \simeq \mathcal{O}/\mathcal{J}$. It follows that we have a short exact sequence of sheaves

$$0 \ \to \ \mathcal{J} \ \to \ \mathcal{O} \ \to \mathfrak{F} \ \to \ 0.$$

Observe now that $\mathcal{O}$ is locally isomorphic to $\mathcal{C}^\infty_{p|q}$, that $\mathfrak{F}$ is locally isomorphic to $\mathcal{C}^\infty_{\mathbb{R}^p}$, and that $\mathfrak{F}$ thus provides a differentiable manifold structure on the topological space M such that $\mathcal{C}^\infty_M \simeq \mathfrak{F}$. Eventually, we have the

Proposition 15. *For any supermanifold $\mathcal{M} = (M, \mathcal{O})$, there exists a short exact sequence of sheaves*

$$0 \to \mathcal{J} \to \mathcal{O} \xrightarrow{\varepsilon} \mathcal{C}^\infty \to 0,$$

where $\mathcal{J} = \ker\varepsilon$ and where $\mathcal{C}^\infty$ is the function sheaf of a smooth manifold structure on the base M of $\mathcal{M}$.

The projection $\varepsilon : \mathcal{O} \to \mathcal{C}^\infty$, of the structure sheaf of $\mathcal{M}$ onto the structure sheaf of M, provides an embedding $i : M \to \mathcal{M}$ of the classical base manifold M into the supermanifold $\mathcal{M}$. We can thus view a supermanifold as a “classical manifold surrounded by a cloud of odd stuff”.

Let us conclude with the

Remark 10. *If $\mathcal{M} = (M, \mathcal{O})$ is a supermanifold and $m \in M$, the unique maximal homogeneous ideal $\mathfrak{m}_m$ of $\mathcal{O}_m$ is*

$$\mathfrak{m}_m = \{[f]_m : \varepsilon(f)(m) = 0\}. \tag{3.8}$$

3.4 Morphisms of supermanifolds

3.4.1 First properties of supermorphisms

Supermanifolds form a full subcategory of the category of LSRS. In other words, a morphism of supermanifolds

$$\Psi : \mathcal{M} = (M, \mathcal{O}) \quad \to \quad \mathcal{N} = (N, \mathcal{R})$$

is just a morphism of the corresponding LSRS. Hence, $\Psi = (\psi, \psi^*)$ is made up by a continuous map between the underlying topological spaces

$$\psi : M \quad \to \quad N,$$

together with a family of pullbacks

$$\psi^*_V : \mathcal{R}(V) \quad \to \quad \mathcal{O}\left(\psi^{-1}(V)\right),$$

indexed by the open subsets $V \subset N$, which are morphisms of associative unital $\mathbb{R}$-algebras, commuting with the restriction maps and preserving the parity. Moreover, the algebra morphisms ψ_x, $x \in M$, between stalks, naturally induced by these pullbacks, respect the maximal homogeneous ideal, i.e.

$$\psi^*_m(\mathfrak{m}_{\psi(m)}) \subset \mathfrak{m}_m, \quad \forall m \in M.$$

Morphisms of supermanifolds commute with the projections onto the base:

Proposition 16. *Let*

$$\Psi = (\psi, \psi^*) : \mathcal{M} = (M, \mathcal{O}) \quad \to \quad \mathcal{N} = (N, \mathcal{R})$$

be a morphism of supermanifolds, $V \subset N$ an open subset, and $U = \psi^{-1}(V)$. Then,

$$\varepsilon_U \circ \psi^* = \psi^* \circ \varepsilon_V. \tag{3.9}$$

Proof. The proposition claims that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{R}(V) & \xrightarrow{\psi^*} & \mathcal{O}(U) \\ \varepsilon_V \downarrow & & \downarrow \varepsilon_U \\ \mathcal{C}^\infty_N(V) & \xrightarrow{\psi^*} & \mathcal{C}^\infty_M(U) \end{array},$$

where the pullback of classical functions is given, for $f \in \mathcal{C}^\infty_N(V)$, by $\psi^*(f) = f \circ \psi$.

Let $t \in \mathcal{R}(V)$ and $m \in U$. If we set $s = \psi^*(t) \in \mathcal{O}(U)$, we have to show that

$$\varepsilon_U(s)(m) = \varepsilon_V(t)(\psi(m)).$$

The RHS of this equation is by definition the unique real number k such that $t - k$ is not invertible in any neighborhood of $\psi(m)$. It suffices thus to prove that the LHS has this property. Suppose that $t - \varepsilon_U(s)(m)$ is invertible in some neighborhood of $\psi(m)$. Then, since ψ^* is an associative unital $\mathbb{R}$-algebra morphism,

$$\psi^* \left(t - \varepsilon_U(s)(m)\right) = \psi^*(t) - \varepsilon_U(s)(m)\psi^*(1) = s - \varepsilon_U(s)(m)$$

is invertible in some neighborhood of m – a contradiction. $\square$

Remark 11. *The next result states in particular that morphisms of supermanifolds automatically respect the maximal ideal, see Equation (3.8) – so that this requirement is actually redundant in the definition of morphisms of supermanifolds.*

Corollary 3. *With slightly simplified notations, we get*

1. $\psi^* \left(\ker \varepsilon_V\right) \subset \ker \varepsilon_{\psi^{-1}(V)}$, *for any open subset* $V \subset N$,
2. $\psi^* \left(\mathfrak{m}_{\psi(m)}\right) \subset \mathfrak{m}_m$, *for any point* $m \in M$,
3. $\psi^* \left(\mathfrak{m}^k_{\psi(m)}\right) \subset \mathfrak{m}^k_m$, *for any* $k \in \mathbb{N}^*$ *and any* $m \in M$.

The reader may have guessed the

Corollary 4. *The base map* $\psi : M \to N$ *of a morphism of supermanifolds is smooth between the smooth base manifolds.*

Proof. Let $m \in M$, let $(V, y = (y^1, \ldots, y^n))$ be a chart of N around $\psi(m)$, and set $U = \psi^{-1}(V)$. For any $g \in \mathcal{C}^\infty(V)$, there is $t \in \mathcal{R}(V)$ (just restrict V), such that

$$g \circ \psi = \varepsilon_V(t) \circ \psi = \psi^* \left(\varepsilon_V(t)\right) = \varepsilon_U \left(\psi^*(t)\right) \in \mathcal{C}^\infty(U).$$

In particular, for $g = y^i$, we get

$$\psi^i = y^i \circ \psi \; \in \mathcal{C}^\infty(U),$$

so that $\psi \in \mathcal{C}^\infty(U, N)$ and, since U is a neighborhood of an arbitrary point $m \in M$, $\psi \in \mathcal{C}^\infty(M, N)$. $\square$

3.4.2 Fundamental Theorem of Supermorphisms

Let $\Psi = (\psi, \psi^*) : \mathcal{M} = (M, \mathcal{O}) \to \mathcal{V}^{r|s} = (V, \mathcal{C}^\infty_{r|s})$ be a morphism of supermanifolds and let (y, η) be a global system of coordinates of the superdomain $\mathcal{V}^{r|s}$. Then the functions

$$\begin{aligned} s^i &= \psi^* y^i, \quad \text{for } 1 \le i \le r, \\ \sigma^a &= \psi^* \eta^a, \quad \text{for } 1 \le a \le s, \end{aligned}$$

satisfy

1. $s^i \in \mathcal{O}_0(M)$ and $\sigma^a \in \mathcal{O}_1(M)$, for all i and all a,
2. $(\varepsilon s^1, \ldots, \varepsilon s^r)\,(M) \subset V$.

Actually these pullbacks of the super coordinate functions completely determine the supermorphism:

Theorem 2 (Fundamental Theorem of Supermorphisms)**.** *If $\mathcal{M} = (M, \mathcal{O})$ is a supermanifold, $V \subset \mathbb{R}^r$ an open subset, and if (s, σ) is an $(r+s)$-tuple of functions in $\mathcal{O}(M)$ that satisfy the preceding conditions 1. and 2., there exists a unique morphism of supermanifolds $\Psi = (\psi, \psi^*) : \mathcal{M} \to \mathcal{V}^{r|s}$ such that*

$$s^i = \psi^* y^i \quad \text{and} \quad \sigma^a = \psi^* \eta^a, \tag{3.10}$$

where (y, η) are the coordinates of the superdomain.

The proof of this theorem requires some preparatory work.

3.4.3 Local form of supermorphisms

A morphism of classical smooth manifolds $\psi \in \mathcal{C}^\infty(M, N)$ reads, locally in coordinates $x = (x^1, \ldots, x^m)$ of M (resp. $y = (y^1, \ldots, y^n)$ of N), $\psi(x) = y$ or $y^i = \psi^i(x)$. Using notations from Physics, we can write as well

$$y^i = y^i(x),$$

which is actually the most used local form. Since $\psi^* y^i = y^i(\psi(x)) = y^i(x)$, the comparison with the preceding form shows that a change of interpretation allows ignoring the pullback map ψ^*.

A supermorphism $(\psi, \psi^*) : \mathcal{M} \to \mathcal{N}$ between supermanifolds of dimension $p|q$ and $r|s$ is locally an algebra morphism

$$\psi^* : \mathcal{C}^\infty(V)[\eta^1, \ldots, \eta^s] \to \mathcal{C}^\infty(U)[\xi^1, \ldots, \xi^q]$$

over ψ ($V \subset \mathbb{R}^r$, $U = \psi^{-1}(V) \subset \mathbb{R}^p$), which commutes with restrictions and preserves the parity, and which is completely determined by the pullbacks $\psi^* y^i = s^i(x,\xi)$ and $\psi^* \eta^a = \sigma^a(x,\xi)$, where y and x are coordinates in V and U, respectively. In other words, when omitting the pullback map ψ^* and committing the usual notational abuse, we obtain that a supermorphism locally reads

$$\begin{aligned} y^i &= y^i(x,\xi) \text{ (even)}, \\ \eta^a &= \eta^a(x,\xi) \text{ (odd)}. \end{aligned}$$

Hence,

Remark 12. *The local form of supermorphisms is completely similar to that of classical ones.*

Moreover,

Remark 13. *It is now clear that the category of supermanifolds admits more morphisms than the category of vector bundles. Indeed, whereas a supermorphism locally reads*

$$\begin{aligned} y^i &= y^i_0(x) + y^i_{\alpha^1\alpha^2}(x)\xi^{\alpha^1}\xi^{\alpha^2} + \ldots, \\ \eta^a &= \eta^a_\alpha(x)\xi^\alpha + \eta^a_{\alpha^1\alpha^2\alpha^3}(x)\xi^{\alpha^1}\xi^{\alpha^2}\xi^{\alpha^3} + \ldots, \end{aligned}$$

where summations are understood and restricted to increasing sequences, a morphism of vector bundles is locally of the type

$$\begin{aligned} y^i &= y^i(x), \\ \eta^a &= \eta^a_\alpha(x)\xi^\alpha. \end{aligned}$$

3.4.4 Instructive intuitive example

Consider the local supermorphism defined by

$$\begin{aligned} y &= y(x,\xi) = x + \xi^1\xi^2 \text{ (even)}, \\ \eta &= \eta(x,\xi) = f(x)\xi^1 + g(x)\xi^2 \text{ (odd)}. \end{aligned}$$

To pull e.g. the superfunction $\sin y$ back to a superfunction of the variables (x,ξ), we use a formal Taylor expansion

$$\sin y = \sin(x + \xi^1\xi^2) = \sin x + (\cos x)\ \xi^1\xi^2.$$

Observe first that this expansion is finite due to nilpotency, and second that the classical function $\sin y$ is transformed into a nonclassical superfunction.

Remark 14. *We know that for any supermanifold the choice of a local isomorphism provides a local embedding of classical functions into superfunctions, but this embedding is not intrinsic. Indeed, the second preceding remark shows that another local isomorphism will lead to another embedding.*

The proof of the Fundamental Theorem will provide a rigorous justification of the used Taylor expansion.

3.4.5 Polynomial Approximation Technique

Since ψ^* is an algebra morphism, the data $\psi^* y^i = s^i$ and $\psi^* \eta^a = \sigma^a$ uniquely determine at least the pullback $\psi^* P$ of any polynomial section $P \in \mathcal{C}^\infty_{r|s}(V)$, *i.e.* of all the sections of the form

$$\sum_\alpha P_\alpha(y) \prod_a (\eta^a)^{\alpha^a},$$

where the

$$P_\alpha(y) = \sum_\beta P_{\alpha\beta} \prod_i (y^i)^{\beta^i}$$

are polynomial functions in $\mathcal{C}^\infty(V)$. Hence the question whether there might exist a kind of appropriate polynomial approximation for sections $f \in \mathcal{C}^\infty_{r|s}(V)$ that allows extending the previous observation from polynomial sections P to arbitrary sections f. This approximation will be given below.

When taking an interest in germs $[f]_m \in \mathcal{O}_m$, $m \in M$, of a supermanifold $(M, \mathcal{O})$ of dimension $p|q$, we can of course choose a centered super coordinate system (x, ξ) around m and work in a superdomain $\mathcal{U}^{p|q}$ associated with a convex open subset $U \subset \mathbb{R}^p$, in which $m \simeq x = 0$. In view of the description (3.8) of the unique maximal homogeneous ideal of $\mathcal{O}_m$, $\mathfrak{m}_m = \{[f]_m : \varepsilon(f)(m) = 0\}$, the Taylor expansion around $m \simeq x = 0$ of the local form of $\varepsilon(f)$ allows seeing that the germs of the maximal ideal are "functions" of the type

$$\mathfrak{m}_m = \{[f]_0 : f(x,\xi) = 0(x) + \sum_a s_a(x)\xi^a + \sum_{a<b} s_{ab}(x)\xi^a\xi^b + \ldots\},$$

where $0(x)$ are terms of degree 1 at least in x. It follows that $\mathfrak{m}^k_m$, $1 \leq k \leq q+1$ is made up by the "functions"

$$\mathfrak{m}^k_m = \{[f]_0 : f(x,\xi) = \sum_{\ell=0}^{k-1} \sum_{a_1<\ldots<a_\ell} 0_{a_1\ldots a_\ell}(x^{k-\ell})\, \xi^{a_1}\ldots\xi^{a_\ell}$$

$$+ \sum_{\ell=k}^{q} \sum_{a_1<\ldots<a_\ell} f_{a_1\ldots a_\ell}(x)\, \xi^{a_1}\ldots\xi^{a_\ell}\}. \tag{3.11}$$

In particular,

$$\mathfrak{m}^{q+1}_m = \{[f]_0 : f(x,\xi) = 0(x^{q+1}) + \sum_a 0_a(x^q)\xi^a + \ldots + 0(x)\xi^1\ldots\xi^q\}.$$

Hence, the

Corollary 5. *If* $[f]_{m'} \in \mathfrak{m}^{q+1}_{m'}$, *for any* m' *close to* m, *then* $[f]_m = 0$.

We can think of two germs at m, which differ by a germ in $\mathfrak{m}_m^k$, as being "the closer to each other", the bigger k.

The next result is basic.

Theorem 3 (Polynomial Approximation)**.** *Let $(M, \mathcal{O})$ be a supermanifold, $m \in M$ an arbitrary point, and $f \in \mathcal{O}(U)$ any section of $\mathcal{O}$ defined in a neighborhood U of m. For any fixed degree of approximation $k \in \mathbb{N}^*$, there exists a polynomial $P = P(x, \xi)$ (depending on m, f, and k), where (x, ξ) are super coordinates centered at m, such that*

$$[f]_m - [P]_m \in \mathfrak{m}_m^k.$$

Proof. Using the Taylor expansion of the $f_\alpha(x)$ at $m \simeq x = 0$, we get

$$f = \sum_\alpha f_\alpha(x)\xi^\alpha = \sum_\alpha P_\alpha(x)\xi^\alpha + \sum_\alpha 0_\alpha(x^k)\xi^\alpha,$$

where the first sum of the RHS is the searched polynomial $P = P(x, \xi)$ and where the second sum belongs to $\mathfrak{m}_m^k$. □

We are now prepared to prove the Fundamental Theorem of Supermorphisms.

Proof of Theorem 2. If the searched supermorphism exists, it is necessarily unique. Indeed, let (ψ_1, ψ_1^*) and (ψ_2, ψ_2^*) be two morphisms defined by the same (s, σ). As mentioned above, the pullbacks $\psi_1^*(P)$ and $\psi_2^*(P)$ coincide on polynomial sections $P \in \mathcal{C}^\infty_{r|s}(V)$. Further, the base maps ψ_1 and ψ_2 coincide. Indeed, if we denote by (y, η) supercoordinates in $\mathcal{V}^{r|s}$, the commutation of the pullback maps with the projections onto the base entails that, for all i,

$$\psi_1^i = y^i \circ \psi_1 = \varepsilon\psi_1^* y^i = \varepsilon s^i = \varepsilon\psi_2^* y^i = y^i \circ \psi_2 = \psi_2^i.$$

The Polynomial Approximation allows now showing that the previous results imply that $\psi_1^*(f) = \psi_2^*(f)$, for any section $f \in \mathcal{C}^\infty_{r|s}(V)$, and so that $(\psi_1, \psi_1^*) = (\psi_2, \psi_2^*)$. Indeed, let $f \in \mathcal{C}^\infty_{r|s}(V)$ and let $m \in M$. Set $n := \psi_1(m) = \psi_2(m) \in V$. By Theorem 3 there exists a polynomial $P = P(y, \eta)$ such that $[f]_n - [P]_n \in \mathfrak{m}_n^{q+1}$. Applying ψ_i^*, $i = 1, 2$, we obtain

$$[\psi_i^*(f)]_m - [\psi_i^*(P)]_m \quad \in \mathfrak{m}_m^{q+1}, \quad i = 1, 2,$$

in view of Corollary 3. However, this implies that

$$[\psi_1^*(f)]_m - [\psi_1^*(P)]_m - [\psi_2^*(f)]_m + [\psi_2^*(P)]_m = [\psi_1^*(f) - \psi_2^*(f)]_m \quad \in \mathfrak{m}_m^{q+1},$$

for all $m \in M$. By Corollary 5, we finally get $\psi_1^*(f) = \psi_2^*(f)$, and as the same argument goes through for smaller sections $f \in \mathcal{C}^\infty_{r|s}(W)$, $W \subset V$, $\psi_1^* = \psi_2^*$.

We now prove the existence of a morphism (ψ, ψ^*) that pulls the coordinate functions (y, η) back to the (s, σ).

Since the pullbacks ψ^* must commute with the restrictions ε, we necessarily have $\psi^*(\varepsilon(y^i)) = \varepsilon(\psi^*(y^i))$, i.e. $y^i \circ \psi = \varepsilon s^i$, so that we set $\psi = (\varepsilon s^1, \ldots, \varepsilon s^r) \in \mathcal{C}^\infty(M, V)$.

Let $V \subset U$ be open. To construct the algebra morphism

$$\psi_V^* : \mathcal{C}^\infty_{r|s}(V) \to \mathcal{O}\left(\psi^{-1}(V)\right),$$

which respects the parity and commutes with the restriction maps, we cover the open subset $\psi^{-1}(V) \subset M$ by chart domains $(W, (x, \xi))$ and construct, for any $f \in \mathcal{C}^\infty_{r|s,\,k}(V)$ of parity k, pullbacks $\psi_W^*(f) \in \mathcal{O}_k(W)$. Let $f = f(y, \eta) = \sum_\alpha f_\alpha(y)\, \eta^\alpha$ be a superfunction of parity k defined over V and set

$$\psi_W^*(f) = \sum_\alpha f_\alpha(\psi^* y)\, (\psi^* \eta)^\alpha = \sum_\alpha f_\alpha(s)\, \sigma^\alpha, \tag{3.12}$$

where $s = (s^1, \ldots, s^r) \in (\mathcal{O}_0(W))^{\times r}$ and $\sigma^\alpha = (\sigma^1)^{\alpha^1} \ldots (\sigma^s)^{\alpha^s} \in \mathcal{O}_k(W)$. Here s and σ^α are viewed as restricted to W. Define now the RHS of Equation (3.12) as in Example 3.4.4 by means of a formal Taylor expansion. When setting

$$s^i = \sum_\beta s^i_\beta(x)\, \xi^\beta =: \varepsilon s^i + n^i,$$

where n^i denotes the nilpotent part, and expanding

$$f_\alpha(s) = f_\alpha(s^1, \ldots, s^r) = f_\alpha((\varepsilon s^1, \ldots, \varepsilon s^r) + (n^1, \ldots, n^r)) =: f_\alpha(\varepsilon s + n),$$

we obtain

$$f_\alpha(s) = \sum_\beta \frac{1}{\beta!}\, (\partial_y^\beta f_\alpha)(\varepsilon s)\, n^\beta,$$

so that

$$\psi_W^*(f) = \sum_\alpha \sum_\beta \frac{1}{\beta!} (\partial_y^\beta f_\alpha)(\varepsilon s)\, n^\beta\, \sigma^\alpha. \tag{3.13}$$

This definition makes sense. First, the sum is finite due to nilpotency. As $n^i \in \mathcal{O}_0(W)$, we have $n^\beta = (n^1)^{\beta^1} \ldots (n^r)^{\beta^r} \in \mathcal{O}_0(W)$. Since $\varepsilon s = \psi \in \mathcal{C}^\infty(M, U)$, its restriction used in (3.13), $\varepsilon s : W \subset \psi^{-1}(V) \to V$, is smooth as well, and, as $f_\alpha \in \mathcal{C}^\infty(V)$, the RHS of (3.13) is an element of $\mathcal{O}_k(W)$. Since the ψ_W^* pull the coordinate functions (y, η) back to the restrictions of the (s, σ), they coincide over overlaps $W \cap W'$ and their values $\psi_W^*(f)$ can be glued to provide sections $\psi_V^*(f) \in \mathcal{O}_k(\psi^{-1}(V))$.

Whereas commutation of the pullbacks with the restrictions is quite easily checked, the algebra morphism property is essentially a consequence of Leibniz' rule for partial derivatives. $\square$

It is instructive to verify the algebra morphism property for (ψ, ψ^*) defined by

$$y = y(x, \xi) = x + \xi^1 \xi^2,$$
$$\eta = \eta(x, \xi) = f(x)\xi^1 + g(x)\xi^2,$$

and the product $\sin y \cos y$. We get

$$\psi^* \sin y = \sin x + (\cos x)\xi^1 \xi^2,$$

$$\psi^* \cos y = \cos x - (\sin x)\xi^1 \xi^2,$$

and

$$\psi^*(\sin y \cos y) = \sin x \cos x + (\cos^2 x - \sin^2 x)\xi^1 \xi^2.$$

Chapter 4

Differential Calculus on Supermanifolds

In this section, we consider as usual a supermanifold $\mathcal{M} = (M, \mathcal{O})$ of dimension $p|q$.

4.1 Tangent sheaf and super vector fields

Definition 34. *A homogeneous* **superderivation** *of parity i of the super $\mathbb{R}$-algebra $\mathcal{O}(U)$, U open in M, is an $\mathbb{R}$-linear map $D \in \mathbf{End}_i\,\mathcal{O}(U)$ of weight i, which verifies the graded Leibniz rule*

$$D(st) = (Ds)t + (-1)^{ij} s(Dt),$$

for all $s \in \mathcal{O}_j(U)$ and all $t \in \mathcal{O}(U)$. We denote by $\mathrm{Der}_i\,\mathcal{O}(U)$ the set of all derivations of parity i of $\mathcal{O}(U)$.

The sets $\mathrm{Der}_i\,\mathcal{O}(U)$ are clearly $\mathbb{R}$-vector spaces, so that the set

$$\mathrm{Der}\,\mathcal{O}(U) := \mathrm{Der}_0\,\mathcal{O}(U) \oplus \mathrm{Der}_1\,\mathcal{O}(U)$$

of all superderivations of $\mathcal{O}(U)$ is a super vector space over $\mathbb{R}$. Moreover, the group $\mathrm{Der}\,\mathcal{O}(U)$ has a super $\mathcal{O}(U)$-module structure, defined, for $s \in \mathcal{O}_i(U)$ and $D \in \mathrm{Der}_j\,\mathcal{O}(U)$, by $(sD)(t) := s(Dt)$, $t \in \mathcal{O}(U)$, so that $sD \in \mathrm{Der}_{i+j}\,\mathcal{O}(U)$.

The super vector space $\mathrm{Der}\,\mathcal{O}(U)$ has also a super Lie algebra structure, defined, for any $D \in \mathrm{Der}_i\,\mathcal{O}(U)$ and $D' \in \mathrm{Der}_j\,\mathcal{O}(U)$, by

$$[D, D'] := D \circ D' - (-1)^{i \cdot j} D' \circ D \in \mathrm{Der}_{i+j}\,\mathcal{O}(U).$$

To simplify notations, the restriction $\rho^U_V s$ of a section $s \in \mathcal{O}(U)$ to an open subset $V \subset U$ will be denoted in the following by $s|_V$.

Proposition 17. *Any derivation $D \in \operatorname{Der} \mathcal{O}(U)$ is a local operator and, for any open subset $V \subset U$, there exists $D|_V \in \operatorname{Der} \mathcal{O}(V)$ – of the same parity as D, if D is homogeneous – such that*

$$D|_V s|_V = (Ds)|_V,$$

for any $s \in \mathcal{O}(U)$.

The proof of this proposition uses two additional concepts.

Definition 35. *The* **support** *of $s \in \mathcal{O}(U)$ is the closed subset* $\operatorname{supp} s = U \setminus \Omega$, *where*

$$\Omega = \{m \in U : \exists \textit{ a neighborhood } V \subset U \textit{ of } m \textit{ such that } s|_V = 0\}.$$

Definition 36. *A* **super bump function** *around $m \in M$ is a section $\gamma \in \mathcal{O}_0(M)$ with support* $\operatorname{supp} \gamma \subset U$ *contained in a neighborhood U of m, and restriction $\gamma|_V = 1$ in a neighborhood $V \subset U$ of m.*

For a proof of existence of super bump functions, we refer the reader e.g. to [Lei80].

On a smooth manifold, a function f defined in the neighborhood of a point m can be extended, by multiplication by a bump function around m, to a global function, which coincides with f in a neighborhood of m and whose support is contained in $\operatorname{supp} f$. The following lemma extends this result [Lei80].

Lemma 3. *For any point $m \in U$ and any section $s \in \mathcal{O}_i(U)$, there is a global section $S \in \mathcal{O}_i(M)$ and a neighborhood $V \subset U$ of m such that $S|_V = s|_V$ and* $\operatorname{supp} S \subset \operatorname{supp} s$.

Proof of Proposition 17. We first prove that any derivation $D \in \operatorname{Der} \mathcal{O}(U)$ is a local operator, *i.e.* that, for $V \subset U$ and $s \in \mathcal{O}(U)$, we have $(Ds)|_V = 0$, if $s|_V = 0$. Let $m \in V$ and let $W \subset V \subset U$ be a neighborhood of m. There exists a bump function $\gamma \in \mathcal{O}_0(U)$, which restricts to 1 in W and whose support is included in V. Hence, $\gamma s \in \mathcal{O}(U)$, which vanishes inside V and also in $U \setminus \operatorname{supp} \gamma$. Since $\{V, U \setminus \operatorname{supp} \gamma\}$ is an open cover of U, we get $\gamma s = 0$. Hence,

$$0 = (D(\gamma s))\,|_W = ((D\gamma)s + \gamma(Ds))\,|_W = (D\gamma)|_W s|_W + \gamma|_W (Ds)|_W = (Ds)|_W,$$

where W is a neighborhood of an arbitrary $m \in V$. Finally, $(Ds)|_V = 0$.

As for the second part of the proposition, consider $D \in \operatorname{Der}_i \mathcal{O}(U)$ and $s \in \mathcal{O}_j(V)$. For any $m \in V$, there exists a section $S \in \mathcal{O}_j(U)$ such that $S|_W = s|_W$, for some neighborhood $W \subset V$ of m. The sections $(DS)|_W \in \mathcal{O}_{j+i}(W)$ and $(DS')|_{W'} \in \mathcal{O}_{j+i}(W')$, defined in neighborhoods W of m and W' of m', coincide in the overlap $W \cap W'$, in view of the just proved locality property, and thus these sections define a unique section $D|_V s \in \mathcal{O}_{j+i}(V)$, such that $(D|_V s)|_W = (DS)|_W$. The map

$$D|_V : \quad \mathcal{O}_j(V) \to \mathcal{O}_{j+i}(V)$$

is obviously $\mathbb{R}$-linear, has parity i, and verifies $D|_V s|_V = (Ds)|_V$, for any $s \in \mathcal{O}(U)$. In fact, we have $D|_V \in \operatorname{Der}_i \mathcal{O}(V)$, as, for any two sections $s, t \in \mathcal{O}(V)$, s of parity j,

$$\begin{aligned}
(D|_V(st))\,|_W &= (D(ST))\,|_W \\
&= (DS)|_W T|_W + (-1)^{ij} S|_W (DT)|_W \\
&= (D|_V s)|_W t|_W + (-1)^{ij} s|_W (D|_V t)|_W \\
&= \big((D|_V s)t + (-1)^{ij} s (D|_V t)\big)\,|_W,
\end{aligned}$$

so that

$$D|_V(st) = (D|_V s)t + (-1)^{ij} s(D|_V t).$$

□

The restriction maps $\rho^U_V : \operatorname{Der}\mathcal{O}(U) \ni D \mapsto D|_V \in \operatorname{Der}\mathcal{O}(V)$ of Proposition 17 make the assignment

$$\operatorname{Der}\mathcal{O} : U \mapsto (\operatorname{Der}\mathcal{O})(U) = \operatorname{Der}\mathcal{O}(U)$$

of a super $\mathcal{O}(U)$-module (resp. super Lie algebra) to any open subset of the base M of a supermanifold $\mathcal{M} = (M, \mathcal{O})$, a presheaf and even a sheaf of super $\mathcal{O}$-modules (resp. super Lie algebras).

Definition 37. *Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold. The sheaf* $\operatorname{Der}\mathcal{O}$ *of derivations of the structure sheaf $\mathcal{O}$ is called the* **tangent sheaf** $T\mathcal{M}$ *of $\mathcal{M}$. The $\mathcal{O}(M)$-module of global sections* $\operatorname{Der}\mathcal{O}(M)$ *is the module of* **super vector fields** *of $\mathcal{M}$.*

Super vector fields admit a local description similar to that of classical vector fields. Indeed, let $\mathcal{M}$ be of dimension $p|q$ and let $(x, \xi) = (x^1, \ldots, x^p, \xi^1, \ldots, \xi^q)$ be a system of local coordinates in $U \subset M$. We then define p derivations (resp. q derivations)

$$\partial_{x^i} \in \operatorname{Der}_0 \mathcal{O}(U) \quad (\text{resp. } \partial_{\xi^a} \in \operatorname{Der}_1 \mathcal{O}(U))$$

by setting, for any $s = \sum_\alpha s_\alpha(x)\xi^\alpha \in \mathcal{O}(U)$,

$$\partial_{x^i} s = \sum_\alpha (\partial_{x^i} s_\alpha(x))\xi^\alpha \quad (\text{resp. } \partial_{\xi^a} s = \sum_\alpha s_\alpha(x)\,(\partial_{\xi^a}\xi^\alpha) \quad \text{and} \quad \partial_{\xi^a}\xi^\alpha = \delta^\alpha_a).$$

Proposition 18. *If (x, ξ) is a coordinate system in U, the tuple $(\partial_x, \partial_\xi)$ is a basis of the $\mathcal{O}(U)$-module $(T\mathcal{M})(U)$, i.e. any $X \in (T\mathcal{M})(U)$ admits a unique decomposition*

$$X = \sum_{1 \le i \le p} \mathcal{X}^i \, \partial_{x^i} + \sum_{1 \le a \le q} \mathscr{X}^a \, \partial_{\xi^a}, \tag{4.1}$$

where $\mathcal{X}^i, \mathscr{X}^a \in \mathcal{O}(U)$. Hence, $T\mathcal{M} = Der\, \mathcal{O}$ is a locally free sheaf of super $\mathcal{O}$-modules. Further, it is a sheaf of super Lie algebras for the super commutator bracket of derivations.

Proof. Freeness is easily obtained. Indeed, if X admits a decomposition (4.1), the coefficients $\mathcal{X}^i$ and $\mathscr{X}^a$ are necessarily $\mathcal{X}^i = Xx^i$, $\mathscr{X}^a = X\xi^a$.

As for existence, let $Y \in \mathrm{Der}\, \mathcal{O}(U)$ be the difference of the LHS and the RHS of (4.1), with the just determined coefficients. Obviously, $Yx^i = 0$ and $Y\xi^a = 0$, so that $YP = 0$, for any polynomial section $P \in \mathcal{O}(U)$. It suffices to show that $Ys = 0$, for an arbitrary section $s \in \mathcal{O}(U)$. This result follows from the Polynomial Approximation Technique. Indeed, for any $m \in U$, there is a polynomial section P such that

$$[s]_m - [P]_m \in \mathfrak{m}_m^{q+1} \, .$$

Hence,

$$[Ys]_m = [Ys]_m - [YP]_m = Y\left([s]_m - [P]_m\right) \in Y\mathfrak{m}_m^{q+1} \, ,$$

where Y is the derivation of $\mathcal{O}_m$ induced by the derivation Y of $\mathcal{O}(U)$, $U \ni m$. We can of course assume that $m \simeq 0$ in the local coordinates. Recall now the description (3.11) asserting that of the elements of the powers $\mathfrak{m}_0^{q+k}$ of the maximal homogeneous ideal $\mathfrak{m}_0$ are classes of sections of the form

$$\begin{aligned} f &= 0(x^{q+k}) + \sum_a 0(x^{q+k-1})\xi^a + \ldots + 0(x^k)\, \xi^1 \ldots \xi^q \\ &= \sum \varepsilon(x)\, x^{i^1} \ldots x^{i^{q+k-\ell}} \xi^{a^1} \ldots \xi^{a^\ell}, \end{aligned}$$

with self-explaining notations. Since Y is a derivation that vanishes on polynomial sections, we get

$$\begin{gathered} Yf = \sum (Y\varepsilon(x))\, x^{i^1} \ldots x^{i^{q+k-\ell}} \xi^{a^1} \ldots \xi^{a^\ell} = \\ \sum (s_\alpha(x)\, x^{i^1})\, x^{i^2} \ldots x^{i^{q+k-\ell}} \xi^\alpha \xi^{a^1} \ldots \xi^{a^\ell} \in \mathfrak{m}_0^{q+k}, \end{gathered}$$

so that $Y\mathfrak{m}_0^{q+1} \subset \mathfrak{m}_0^{q+1}$. In view of Corollary 5, this implies that $Ys = 0$. □

It is known that locally free sheaves of $\mathcal{C}^\infty$-modules over a manifold M are 1-to-1 with vector bundles over M. In supergeometry,

Definition 38. *A* **super vector bundle** *over a supermanifold $\mathcal{M} = (M, \mathcal{O})$ is a locally free sheaf of $\mathcal{O}$-modules over M.*

Example 5. *The tangent sheaf $T\mathcal{M}$ of $\mathcal{M}$ is a super vector bundle over $\mathcal{M}$.*

4.2 Tangent space

If M is a smooth manifold and $m \in M$, the map

$$L : T_m M \ni X_m \mapsto L_{X_m} \in \operatorname{Der}_m \mathcal{C}^\infty(M),$$

where

$$L_{X_m} : \mathcal{C}^\infty(M) \ni f \mapsto (d_m f)(X_m) \in \mathbb{R},$$

is a vector space isomorphism. More precisely, L_{X_m} is a derivation at m that is actually defined on the stalk $\mathcal{C}^\infty_m$, i.e. it is an $\mathbb{R}$-linear map

$$L_{X_m} : \mathcal{C}^\infty_m \to \mathbb{R}$$

such that

$$L_{X_m}(fg) = (L_{X_m} f)\, g(m) + f(m)(L_{X_m} g).$$

Analogously,

Definition 39. *If $\mathcal{M} = (M, \mathcal{O})$ is a supermanifold and $m \in M$, a homogeneous* **super tangent vector** *of parity i, at m to $\mathcal{M}$, is a derivation of parity i at m of $\mathcal{O}_m$, i.e. an $\mathbb{R}$-linear map*

$$X_m : \quad \mathcal{O}_m \to \mathbb{R}$$

of parity i (if X_m has parity 0 (resp. 1), the elements of $\mathcal{O}_{m,1}$ (resp. $\mathcal{O}_{m,0}$) are mapped to 0), such that, for any $s \in \mathcal{O}_{m,j}$ and any $t \in \mathcal{O}_m$,

$$X_m(st) = (X_m s)\,(\varepsilon t)(m) + (-1)^{ij}\,(\varepsilon s)(m)\,(X_m t),$$

where $\varepsilon : \mathcal{O}_m \to \mathcal{C}^\infty_m$ is the morphism induced by the projection $\varepsilon : \mathcal{O} \to \mathcal{C}^\infty$.

The super vector space $\operatorname{Der}_m \mathcal{O}_m$ of all derivations at m of $\mathcal{O}_m$, i.e. of all super tangent vectors at m to $\mathcal{M}$, is called the **super tangent space** *of $\mathcal{M}$ at m and is denoted by $T_m\mathcal{M}$.*

Proposition 19. *Any vector field $X \in \operatorname{Der} \mathcal{O}(U) = (T\mathcal{M})(U)$, which is defined in a neighborhood U of a point $m \in M$, induces a tangent vector $X_m \in \operatorname{Der}_m \mathcal{O}_m = T_m\mathcal{M}$ at m – of the same parity as X, if X is homogeneous.*

Proof. Let $X : \mathcal{O}_m \to \mathcal{O}_m$ be the derivation (of parity i) induced by $X \in \operatorname{Der} \mathcal{O}(U)$ (if X has parity i), let $\varepsilon : \mathcal{O}_m \to \mathcal{C}^\infty_m$ be the algebra morphism (of parity 0) induced by the projection map, and let $\operatorname{ev}_m : \mathcal{C}^\infty_m \to \mathbb{R}$ be the evaluation morphism (of parity 0) at m. Then, $X_m := \operatorname{ev}_m \circ \varepsilon \circ X$ is, as immediately checked, a derivation at m of $\mathcal{O}_m$ (of the same parity as X). □

Just as $T\mathcal{M}$ is a locally free sheaf of $\mathcal{O}$-modules with local frame $(\partial_{x^i}, \partial_{\xi^a})$ implemented by local coordinates (x^i, ξ^a), we have:

Proposition 20. *For every $m \in M$, the tangent space $T_m\mathcal{M}$ is a super vector space over $\mathbb{R}$ with basis $\partial_{x^i,m} \in T_{m,0}\,\mathcal{M}$, $\partial_{\xi^a,m} \in T_{m,1}\,\mathcal{M}$ induced by the local vector fields $\partial_{x^i} \in (T\mathcal{M})_0(U)$, $\partial_{\xi^a} \in (T\mathcal{M})_1(U)$. Hence, the tangent space $T_m\mathcal{M}$ has the same dimension as the supermanifold $\mathcal{M}$.*

Proof. The proof is along the same lines as that of the local freeness of $T\mathcal{M}$. □

The next proposition compares the stalks $(T\mathcal{M})_m$ of the tangent sheaf and the corresponding tangent spaces $T_m\mathcal{M}$.

Proposition 21. *For any $m \in M$,*

$$T_m\mathcal{M} \simeq (T\mathcal{M})_m/\mathfrak{m}_m(T\mathcal{M})_m,$$

where $\mathfrak{m}_m$ is the maximal ideal.

Proof. The result follows from the proof of Proposition 19, as well as from the local forms of the involved objects. □

4.3 Tangent map of a supermorphism, modified Jacobian

Let $\psi \in \mathcal{C}^\infty(M, N)$ be a morphism of smooth manifolds. The tangent map $T_m\psi$, $m \in M$, is the linear map

$$T_m\psi : T_mM \ni X_m \to X_m \circ \psi^* \in T_{\psi(m)}N,$$

where $\psi^* : \mathcal{C}^\infty_{\psi(m)} \to \mathcal{C}^\infty_m$. This result, which is quite obvious in view of the interpretation of a tangent space as space of derivations at the corresponding point, is easily checked.

Definition 40. *Let $\Psi = (\psi, \psi^*) : \mathcal{M} \to \mathcal{N}$ be a morphism of supermanifolds. The* **tangent map** *$T_m\Psi$, $m \in M$, of Ψ at m is the super vector space morphism defined by*

$$\begin{array}{rccl} T_m\Psi : & T_m\mathcal{M} & \to & T_{\psi(m)}\mathcal{N} \\ & X_m & \mapsto & X_m \circ \psi^*, \end{array}$$

where ψ^* is the pullback morphism between stalks.

Let $\mathcal{M} = (M, \mathcal{O})$ and $\mathcal{N} = (N, \mathcal{R})$. Since $(T_m\Psi)(X_m)$ is the composite of the algebra morphism (of parity 0) $\psi^* : \mathcal{R}_{\psi(m)} \to \mathcal{O}_m$ and the derivation $X_m : \mathcal{O}_m \to \mathbb{R}$, it is a derivation at $\psi(m)$, i.e. an element of the target space (with same parity as X_m). Hence, $T_m\Psi$ is actually a super vector space morphism.

The usual theorem that governs tangent maps of composite maps holds true in Supergeometry.

Proposition 22. *Let $\Psi = (\psi, \psi^*) : \mathcal{M} \to \mathcal{N}$ and $\Phi = (\phi, \phi^*) : \mathcal{N} \to \mathcal{P}$ be morphisms of supermanifolds. Their composite $\Phi \circ \Psi = (\phi \circ \psi, \psi^* \circ \phi^*) : \mathcal{M} \to \mathcal{P}$ is a morphism of supermanifolds whose tangent map at $m \in M$ is given by*

$$T_m(\Phi \circ \Psi) = T_{\psi(m)}\Phi \circ T_m\Psi.$$

Proof. Obvious. □

In classical geometry, the preceding proposition is the global version of the chain rule. Similarly,

Proposition 23. *Let $(\psi, \psi^*) : (M, \mathcal{O}) \to (N, \mathcal{R})$ be a supermorphism. If $V \subset N$ is a domain with coordinates $v = (y, \eta)$ and $\psi^{-1}(V)$ a domain with coordinates $u = (x, \xi)$, we have, for any $t \in \mathcal{R}(V)$,*

$$\partial_{u^{\mathfrak{a}}}(\psi^* t) = \sum_{\mathfrak{b}} \partial_{u^{\mathfrak{a}}}(\psi^* v^{\mathfrak{b}})\, \psi^*(\partial_{v^{\mathfrak{b}}} t). \tag{4.2}$$

If we ignore the pullbacks, see above, this result is the usual chain rule with exchanged order of factors in the RHS.

Proof. The LHS and RHS of (4.2) are composites from $\mathcal{R}(V)$ to $\mathcal{O}(U)$, $U = \psi^{-1}(V)$, of derivations and algebra morphisms (actually the RHS is a combination with coefficients in $\mathcal{O}(U)$ of such composites). It follows that both are derivations from $\mathcal{R}(V)$ to $\mathcal{O}(U)$. As they coincide on coordinate functions and thus on polynomial sections, the Polynomial Approximation Method shows that they coincide everywhere. □

Let $\Psi = (\psi, \psi^*) : \mathcal{M} = (M, \mathcal{O}) \to \mathcal{N} = (N, \mathcal{R})$ again be a supermorphism from a supermanifold of dimension $p|q$ to a supermanifold of dimension $r|s$, and let $m \in M$. We take now an interest in the representative matrix of the super vector space morphism $T_m\Psi : T_m\mathcal{M} \to T_{\psi(m)}\mathcal{N}$ in the bases

$$\partial_{u^{\mathfrak{a}}, m} = (\partial_{x^i, m}, \partial_{\xi^a, m})$$

of $T_m\mathcal{M}$ and

$$\partial_{v^{\mathfrak{b}}, \psi(m)} = (\partial_{y^j, \psi(m)}, \partial_{\eta^b, \psi(m)})$$

of $T_{\psi(m)}\mathcal{N}$ induced by local coordinates in a neighborhood of m and $\psi(m)$, respectively. Since the searched matrix is a diagonal block matrix $(r+s)\times(p+q)$ with entries in $\mathbb{R}$ and as Ψ has a local form $v=v(u)$ that respects the parity, we may ask whether this matrix is, as in the classical case, the Jacobian matrix

$$\partial_u v|_m = \begin{pmatrix} \partial_x y & \partial_\xi y \\ \partial_x \eta & \partial_\xi \eta \end{pmatrix}|_m =$$

$$\begin{pmatrix} \varepsilon(\partial_x y)\,(m) & \varepsilon(\partial_\xi y)\,(m) \\ \varepsilon(\partial_x \eta)\,(m) & \varepsilon(\partial_\xi \eta)\,(m) \end{pmatrix} = \begin{pmatrix} \varepsilon(\partial_x y)\,(m) & 0 \\ 0 & \varepsilon(\partial_\xi \eta)\,(m) \end{pmatrix}.$$

To answer this question, we compute $(T_m\Psi)(\partial_{u^{\mathfrak{a}},m}) = \partial_{u^{\mathfrak{a}},m}\circ\psi^* \in \mathrm{Der}_{\psi(m)}\mathcal{R}_{\psi(m)}$, or, better, we compute its value at $[t]=[t]_{\psi(m)}\in\mathcal{R}_{\psi(m)}$ and decompose the result in the target basis $\partial_{v^{\mathfrak{b}},\psi(m)}$. Remember that for any vector field X the induced tangent vector X_m is defined by

$$X_m[s]_m = \mathrm{ev}_m[\varepsilon X s]_m = \mathrm{ev}_m\,\varepsilon X s,$$

so that in the case of $\partial_{u^{\mathfrak{a}},m}\,\psi^*[t]_{\psi(m)} = \partial_{u^{\mathfrak{a}},m}[\psi^* t]_m$ it suffices to compute $\partial_{u^{\mathfrak{a}}}(\psi^* t)$ and to apply $\mathrm{ev}_m\circ\varepsilon$ afterwards. Since

$$\partial_{u^{\mathfrak{a}}}(\psi^* t) = \sum_{\mathfrak{b}} \partial_{u^{\mathfrak{a}}}(\psi^* v^{\mathfrak{b}})\ \psi^*(\partial_{v^{\mathfrak{b}}} t),$$

we get

$$\partial_{u^{\mathfrak{a}},m}\,\psi^*[t]_{\psi(m)} = \partial_{u^{\mathfrak{a}},m}[\psi^* t]_m = \sum_{\mathfrak{b}} \varepsilon(\partial_{u^{\mathfrak{a}}} v^{\mathfrak{b}})(m)\ \mathrm{ev}_m\,\varepsilon\,\psi^*\partial_{v^{\mathfrak{b}}} t,$$

and, as

$$\mathrm{ev}_m\,\varepsilon\,\psi^*\partial_{v^{\mathfrak{b}}} t = \varepsilon(\partial_{v^{\mathfrak{b}}} t)(\psi(m)) = \partial_{v^{\mathfrak{b}},\psi(m)}[t]_{\psi(m)},$$

we really obtain the Jacobian matrix $\partial_u v|_m$.

Consider now two morphisms of supermanifolds $\Psi:\mathcal{M}\to\mathcal{N}$ and $\Phi:\mathcal{N}\to\mathcal{P}$ and assume that Ψ, Φ, and $\Phi\circ\Psi$ locally read $v=v(u)$, $w=w(v)$, and $w=w(u)$, respectively. It is natural to hope that the matrix counterpart of $T_m(\Phi\circ\Psi) = T_{\psi(m)}\Phi\circ T_m\Psi$ be $\partial_u w|_m = \partial_v w|_{\psi(m)}\partial_u v|_m$. However,

$$\partial_{u^{\mathfrak{b}}} w^{\mathfrak{a}} = \sum_{\mathfrak{c}} \partial_{u^{\mathfrak{b}}} v^{\mathfrak{c}} \partial_{v^{\mathfrak{c}}} w^{\mathfrak{a}} = \sum_{\mathfrak{c}} (-1)^{(u^{\mathfrak{b}}+v^{\mathfrak{c}})(v^{\mathfrak{c}}+w^{\mathfrak{a}})} \partial_{v^{\mathfrak{c}}} w^{\mathfrak{a}} \partial_{u^{\mathfrak{b}}} v^{\mathfrak{c}}. \tag{4.3}$$

To absorb the redundant sign, we define

Definition 41. *The* **modified super Jacobian matrix** *of a local supermorphism Ψ from a superdomain $\mathcal{U}^{p|q}$ to a superdomain $\mathcal{V}^{r|s}$, given by $y=y(x,\xi),\eta=\eta(x,\xi)$, is the even $(r+s)\times(p+q)$ supermatrix*

$$J\Psi = \begin{pmatrix} \partial_x y & -\partial_\xi y \\ \partial_x \eta & \partial_\xi \eta \end{pmatrix}.$$

Proposition 24. *Let $\Psi = (\psi, \psi^*) : \mathcal{M} \to \mathcal{N}$ be a supermorphism, let $m \in M$, and let (x, ξ) and (y, η) be local coordinates in a neighborhood of m and of $\psi(m)$, respectively. In the canonical induced bases of the tangent spaces, the representative matrix of the super vector space morphism $T_m\Psi : T_m\mathcal{M} \to T_{\psi(m)}\mathcal{N}$ is the modified super Jacobian $J\Psi|_m = \varepsilon(J\Psi)(m)$ computed from the local form $y = y(x, \xi), \eta = \eta(x, \xi)$ of Ψ. Moreover, the matrix form of*

$$T_m(\Phi \circ \Psi) = T_{\psi(m)}\Phi \circ T_m\Psi$$

is

$$J(\Phi \circ \Psi)|_m = J\Phi|_{\psi(m)} \cdot J\Psi|_m.$$

Proof. If we denote $y = y(x, \xi), \eta = \eta(x, \xi)$ more compactly by $v = v(u)$, the matrix of $T_m\Psi$ is

$$\partial_u v|_m = \begin{pmatrix} \partial_x y & \partial_\xi y \\ \partial_x \eta & \partial_\xi \eta \end{pmatrix} |_m = \begin{pmatrix} \varepsilon(\partial_x y)\,(m) & 0 \\ 0 & \varepsilon(\partial_\xi \eta)\,(m) \end{pmatrix} = J\Psi|_m,$$

since the difference between the Jacobian and the modified Jacobian matrices disappears when we project and evaluate.

As for the second statement, note that the entries of the modified Jacobian matrix $J\Psi$ are

$$(v^{\mathfrak{a}})'_{u^{\mathfrak{b}}} := (-1)^{(v^{\mathfrak{a}}+1)u^{\mathfrak{b}}} \partial_{u^{\mathfrak{b}}} v^{\mathfrak{a}}.$$

Equation (4.3) then gives

$$(w^{\mathfrak{a}})'_{u^{\mathfrak{b}}} =$$

$$\sum_{\mathfrak{c}} (-1)^{(w^{\mathfrak{a}}+1)u^{\mathfrak{b}}} (-1)^{(u^{\mathfrak{b}}+v^{\mathfrak{c}})(v^{\mathfrak{c}}+w^{\mathfrak{a}})} (-1)^{(w^{\mathfrak{a}}+1)v^{\mathfrak{c}}} (-1)^{(v^{\mathfrak{c}}+1)u^{\mathfrak{b}}} (w^{\mathfrak{a}})'_{v^{\mathfrak{c}}} (v^{\mathfrak{c}})'_{u^{\mathfrak{b}}} =$$

$$\sum_{\mathfrak{c}} (w^{\mathfrak{a}})'_{v^{\mathfrak{c}}} (v^{\mathfrak{c}})'_{u^{\mathfrak{b}}},$$

which proves the claim. $\square$

4.4 Universal derivations

The Kähler differential is the algebraic counterpart of the de Rham differential of functions of a manifold.

Definition 42. *A* **Kähler differential** *or* **universal derivation** *of a commutative algebra A with unit over a commutative ring R is a pair $(\Omega^1_{A/R}, d)$ made up by an A-module $\Omega^1_{A/R}$ and an R-linear derivation $d : A \to \Omega^1_{A/R}$, which are universal in the sense that for any A-module B and any R-linear derivation $\delta : A \to B$, there exists a unique A-module morphism $\varphi : \Omega^1_{A/R} \to B$, such that $\delta = \varphi \circ d$. The A-module $\Omega^1_{A/R}$ is then called module of Kähler differentials of the R-algebra A.*

Let us first provide three models of the module of Kähler differentials and of the corresponding derivations.

Theorem 4. *For any unital commutative algebra A over a commutative ring R, the A-module $\Omega^1_{A/R}$ of Kähler differentials exists (it is therefore unique up to unique isomorphism) and it admits the models constructed as follows:*

- *The A-module $\Omega^1_{A/R}$ can be defined as the free A-module generated by the symbols df, $f \in A$, modulo the relations $d(r.1) = 0$ (1 denotes the unit of A), for all $r \in R$, and $d(f+g) = df + dg$, $d(fg) = df\, g + f\, dg$, for all $f, g \in A$.*
- *Let $A \otimes_R A$ be the usual R-module, which is here even an A-algebra. The kernel $I = \ker \mu$ of the algebra morphism $\mu : A \otimes_R A \ni f \otimes g \to fg \in A$ is an ideal and so is $I^2 \subset I$. The pair $(\Omega^1_{A/R}, d)$ can then be defined as the A-module $\Omega^1_{A/R} = I/I^2$ together with the R-linear derivation $d : A \ni f \mapsto [f \otimes 1 - 1 \otimes f] \in \Omega^1_{A/R}$.*
- *If $R = \mathbb{K}$ is a commutative field and A a local unital commutative $\mathbb{K}$-algebra, i.e. an algebra with a unique maximal ideal $\mathfrak{m}$, and if there exists a short split exact sequence $0 \to \mathfrak{m} \to A \xrightarrow{p} \mathbb{K} \to 0$, then the A-module $\Omega^1_{A/\mathbb{K}} = \mathfrak{m}/\mathfrak{m}^2$ and the $\mathbb{K}$-linear derivation $d : A \ni f \mapsto [f - p(f).1_A] \in \Omega^1_{A/\mathbb{K}}$ form a model for Kähler differentials.*

It is easily checked that the maps d valued in I/I^2 or $\mathfrak{m}/\mathfrak{m}^2$ are actually derivations. For a proof of the theorem we refer the reader to [Pfl00], [Mat80], [Wei95].

Theorem 5. *Let now A be an R-algebra as above and denote by $\Omega^1_{A/R}$ the corresponding A-module of Kähler differentials. The exterior algebra*

$$\Omega^\bullet_{A/R} := \wedge^\bullet \Omega^1_{A/R} = \bigoplus_{n \geq 0} \wedge^n \Omega^1_{A/R} =: \bigoplus_{n \geq 0} \Omega^n_{A/R}$$

is then an associative A-algebra and the Kähler derivation $d : A \to \Omega^1_{A/R}$ admits a unique well-defined R-linear extension $d : \Omega_{A/R} \to \Omega_{A/R}$ (we omit $\bullet$) as degree 1 derivation for the wedge product, which squares to 0 [Pfl00], [Mat80], [Wei95].

Proof. Let us provide some explanations concerning the results of Theorems 4 and 5.

The first claim of Theorem 4 is obvious. Indeed, it suffices to construct the morphism φ on the generators df and to check that it descends to the quotient.

As for the second claim, note first that the A-module structure of $A \otimes_R A$ is given by the action on the first factor. Observe further that I is generated over A by the differences $1 \otimes f - f \otimes 1$, $f \in A$, since, if $\sum_i f_i \otimes g_i \in I$, we have $\sum_i f_i g_i = 0$, and thus

$$\sum_i f_i(1 \otimes g_i - g_i \otimes 1) = \sum_i (f_i \otimes g_i - f_i g_i \otimes 1) = \sum_i f_i \otimes g_i.$$

It follows that I^2 is generated by the products $(1\otimes f - f\otimes 1)(1\otimes g - g\otimes 1)$, $f, g \in A$.

We now show that the A-module I/I^2 and the differential

$$\overline{d} : A \ni f \mapsto [1\otimes f - f\otimes 1] \in I/I^2$$

form a model of $(\Omega^1_{A/R}, d)$. It is easily checked that $\overline{d}$ is an R-linear derivation. Indeed,

$$\overline{d}(fg) = [1\otimes fg - fg\otimes 1] =$$

$$[g(1\otimes f - f\otimes 1) + f(1\otimes g - g\otimes 1) + (1\otimes f - f\otimes 1)(1\otimes g - g\otimes 1] = g\overline{d}f + f\overline{d}g\ .$$

Hence, there is a unique A-module morphism $\varphi : \Omega^1_{A/R} \ni df \mapsto \overline{d}f \in I/I^2$. This morphism φ is clearly surjective, and, in view of the preceding observation, it is easily seen that it is also injective.

As for the unique (R-linear) extension (as square 0 degree 1 derivation) announced in Theorem 5, let us mention that $d : \Omega^n_{A/R} \to \Omega^{n+1}_{A/R}$ is characterized by

$$d(fdg_1 \wedge \ldots \wedge dg_n) = df \wedge dg_1 \wedge \ldots \wedge dg_n\ .$$

When using the model of Kähler 1-forms given in Item 1 of Theorem 4, we can show that d is actually well-defined (it is e.g. easily checked that $d(fd(gh)) = d(fgdh + fhdg)$). $\square$

An analogous concept can be defined in Superalgebra.

Definition 43. *A* **universal superderivation** *of a supercommutative algebra A with unit over a supercommutative ring R is a pair $(\Omega^1_{\mathrm{ev}}, d_{\mathrm{ev}})$ (resp. $(\Omega^1_{\mathrm{odd}}, d_{\mathrm{odd}})$) made up by a super A-module and an even (resp. odd) R-linear superderivation $d_{\mathrm{ev}} : A \to \Omega^1_{\mathrm{ev}}$ (resp. $d_{\mathrm{odd}} : A \to \Omega^1_{\mathrm{odd}}$) that are universal: for any super A-module B and any R-linear superderivation of same parity $\delta : A \to B$, there exists a unique (even) A-module morphism $\varphi : \Omega^1_{\mathrm{ev}} \to B$ (resp. $\varphi : \Omega^1_{\mathrm{odd}} \to B$) such that $\delta = \varphi \circ d$.*

To construct a universal superderivation, it suffices, as in the classical context, to consider the free super A-module Ω^1_i, $i \in \{\mathrm{ev}, \mathrm{odd}\} \simeq \{0, 1\}$, generated by the symbols $\{d_i f : f \in A_j, j \in \{0,1\}\} \subset \Omega^1_{i,i+j}$ modulo the relations $d_i(r.1) = 0$, for any $r \in R$ (here 1 denotes the unit of A), and $d_i(f+g) = d_i f + d_i g$, $d_i(fg) = d_i f\, g + (-1)^{if} f\, d_i g$, for any $f, g \in A$. It is then straightforwardly checked that (Ω^1_i, d_i) is the universal superderivation of A over R.

Observe the following relationship between even and odd universal superderivations.

Proposition 25. *If $(\Omega^1_{\mathrm{ev}}, d_{\mathrm{ev}})$ is the even universal superderivation of A over R, then $(\Pi\Omega^1_{\mathrm{ev}}, \Pi d_{\mathrm{ev}})$, where Π is the parity reversion functor, is the odd universal superderivation of A over R. A similar result holds of course true if we apply Π to $(\Omega^1_{\mathrm{odd}}, d_{\mathrm{odd}})$.*

Proof. We detail only part 1 of the proposition. To simplify notations, let (Ω^1, d) be the even universal superderivation of A over R. Then, $\Pi\Omega^1$ is an A-module and $\Pi d : A \to \Pi\Omega^1$ is an odd R-linear superderivation. Consider now an arbitrary A-module B and an odd R-linear superderivation $\delta : A \to B$. Since $\Pi\delta : A \to \Pi B$ is an even R-linear superderivation, there exists a unique A-module morphism $\varphi : \Omega^1 \to \Pi B$ such that $\Pi\delta = \varphi \circ d$. But then $\varphi^\Pi : \Pi\Omega^1 \to B$ is an A-morphism and $\delta = \varphi^\Pi \circ \Pi d$. $\square$

4.5 Super 1-forms and cotangent sheaf

Definition 44. *The* **cotangent sheaf** *of a supermanifold $\mathcal{M} = (M, \mathcal{O})$ is the dual of its tangent sheaf, i.e. it is the sheaf of super $\mathcal{O}$-modules*

$$\Omega^1\mathcal{M} := T^*\mathcal{M} := \mathbf{Hom}_{\mathcal{O}\text{- mod}}(T\mathcal{M}, \mathcal{O}).$$

The sections of $\Omega^1\mathcal{M}$ are called **super differential 1-forms**.

We now define the differential of superfunctions confining ourselves to the Deligne formalism.

Definition 45. *For any open subset $U \subset M$ and any $i \in \{0, 1\}$, we define the* **differential of a superfunction** *$f \in \mathcal{O}_i(U)$,*

$$d_U f \in (\Omega^1\mathcal{M})_i(U) = \mathbf{Hom}_i((T\mathcal{M})(U), \mathcal{O}(U)),$$

by

$$(d_U f)(X) = (-1)^{ij} X f \in \mathcal{O}(U),$$

for all $X \in (T\mathcal{M})_j(U) = \operatorname{Der}_j \mathcal{O}(U)$.

It is clear that $d_U f$ has the same parity as f (hence, that d_U is even) and is actually an $\mathcal{O}(U)$-module morphism. Indeed, $d_U f$ is clearly additive and

$$(d_U f)(gX) = (-1)^{f(g+X)}(gX)(f) = (-1)^{fg} g(d_U f)(X),$$

for any function g and any vector field X.

Moreover, the even $d_U : \mathcal{O}(U) \to (\Omega^1\mathcal{M})(U)$ is obviously $\mathbb{R}$-linear and it is straightforwardly checked that it is a superderivation. In fact,

Remark 15. *The pair $((\Omega^1\mathcal{M})(U), d_U)$ is the universal even superderivation of $\mathcal{O}(U)$ [DM99].*

Remark 16. *Let us draw the attention of the reader to the fact that, in the case of smooth algebras $A = \mathcal{O}(U)$ (or $A = \mathcal{C}^\infty(U)$, if we work in the classical nongraded context), the ring-theoretic Kähler forms considered above do not lead to the ordinary concept of smooth 1-forms. However, when viewing A as a $\mathcal{C}^\infty$-ring, the Kähler differential and forms, in the sense of the Fermat theory of $\mathcal{C}^\infty$-rings, do produce the correct notions. In other words, the differential d_U of superfunctions is the universal derivation of $\mathcal{O}(U)$ in the sense of the Fermat theory of $\mathcal{C}^\infty$-ring. For details on this topic, we refer the reader to [CR12].*

As in the classical setting, d_U admits a unique well-defined ($\mathbb{R}$-linear) extension to the super exterior algebra $(\Omega^\bullet\mathcal{M})(U) := \wedge^\bullet(\Omega^1\mathcal{M})(U)$ of $(\Omega^1\mathcal{M})(U)$ (as a square 0 degree 1 even derivation).

Proposition 26. *The sheaf $\Omega^1\mathcal{M} = T^*\mathcal{M}$ of $\mathcal{O}$-modules is locally free, i.e. it is a super vector bundle – the* **cotangent bundle***. More precisely, if (x, ξ) are supercoordinates over $U \subset M$, the $\mathcal{O}(U)$-module $(\Omega^1\mathcal{M})(U)$ admits the basis $(dx^1, \ldots, d\xi^q)$, where d is the above defined differential $d_U : \mathcal{O}(U) \to (\Omega^1\mathcal{M})(U)$. Any super differential 1-form ω over the coordinate patch U thus reads uniquely in the form*

$$\omega = dx^i f_i(x, \xi) + d\xi^a g_a(x, \xi),$$

and the components $f_i \in \mathcal{O}(U)$ (resp. $g_a \in \mathcal{O}(U)$) are given by the evaluations $\omega(\partial_{x^i})$ (resp. $(-1)^\omega\omega(\partial_{\xi^a})$) of ω on the basic vector fields $(\partial_{x^1}, \ldots, \partial_{\xi^q})$. Eventually, the operator $d = d_U$ is given by

$$d_U = dx^i\partial_{x^i} + d\xi^a\partial_{\xi^a}.$$

Note that these results are in accordance with those obtained via a different approach in Section 1.5.

Proof. Since U is a domain of local coordinates (x, ξ), $(T\mathcal{M})(U) = \operatorname{Der}\mathcal{O}(U)$ is a free $\mathcal{O}(U)$-module with basis $(\partial_{x^1}, \ldots, \partial_{\xi^q})$. Therefore, the dual module $(T^*\mathcal{M})(U) = (\Omega^1\mathcal{M})(U)$ admits the dual basis, defined as usual, see Section 1.7, and is thus free. It follows from the above definition of the differential $d = d_U : \mathcal{O}(U) \to (\Omega^1\mathcal{M})(U)$, that

$$\begin{aligned} dx^i\left(\partial_{x^j}\right) &= \partial_{x^j}x^i = \delta^i_j, \\ dx^i\left(\partial_{\xi^b}\right) &= \partial_{\xi^b}x^i = 0, \\ d\xi^a\left(\partial_{x^j}\right) &= \partial_{x^j}\xi^a = 0, \\ d\xi^a\left(\partial_{\xi^b}\right) &= -\partial_{\xi^b}\xi^a = -\delta^a_b, \end{aligned}$$

so that the differentials $(dx^1, \ldots, d\xi^q)$ of the supercoordinate functions form a basis as well. The expressions $f_i = \omega(\partial_{x^i})$ and $g_a = (-1)^\omega\omega(\partial_{\xi^a})$ of the corresponding components of an arbitrary 1-form ω are readily checked. As a particular case, we get the components of an exact 1-form $d_U f$, $f \in \mathcal{O}(U)$, and the announced form of d_U in a coordinate chart. $\square$

4.6 Super differential forms

Definition 46. *Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold and let $U \subset M$ be open. Define the set*

$$(\Omega\mathcal{M})(U) := \wedge(\Omega^1\mathcal{M})(U) = \bigoplus_{k=0}^{\infty} \wedge^k(\Omega^1\mathcal{M})(U) =: \bigoplus_{k=0}^{\infty}(\Omega^k\mathcal{M})(U)$$

of **super differential forms** *over U, as the Deligne super exterior $\mathcal{O}(U)$-algebra of the super $\mathcal{O}(U)$-module $(\Omega^1\mathcal{M})(U)$.*

In view of its definition, see Section 1.6, the wedge or exterior product $\wedge$ – the associative algebra structure of $(\Omega\mathcal{M})(U)$ – respects both, the $(\Omega^1\mathcal{M})(U)$–induced $\mathbb{Z}_2$- and the cohomological $\mathbb{N}$-grading. Let us recall the behavior of the $\mathcal{O}(U)$-module structure of $(\Omega\mathcal{M})(U)$ with respect to the exterior product: if $f \in \mathcal{O}_i(U)$, $\omega \in (\Omega_j\mathcal{M})(U)$ and $\omega' \in (\Omega_k\mathcal{M})(U)$ have the parities i, j, k, we get

$$f(\omega \wedge \omega') = (f\omega) \wedge \omega' = (-1)^{ij}(\omega f) \wedge \omega' = (-1)^{ij}\omega \wedge (f\omega')$$

$$= (-1)^{ij+ik}\omega \wedge (\omega' f) = (-1)^{i(j+k)}(\omega \wedge \omega')f.$$

Further, since $(\Omega^1\mathcal{M})(U)$ is generated over $\mathcal{O}(U)$ by the df, $f \in \mathcal{O}(U)$, – we omit subscript U – the $\mathcal{O}(U)$-modules $(\Omega^k\mathcal{M})(U)$ are generated over $\mathcal{O}(U)$ by the $df_1 \wedge \ldots \wedge df_k$, $f_i \in \mathcal{O}(U)$. It is then easily checked that the used Deligne wedge product of super differential forms is graded-super symmetric, i.e. that, if $\omega \in (\Omega^k\mathcal{M})(U)$ and $\omega' \in (\Omega^\ell\mathcal{M})(U)$ are homogeneous of parity $\tilde{\omega}$ and $\tilde{\omega}'$, we have

$$\omega \wedge \omega' = (-1)^{k\ell+\tilde{\omega}\tilde{\omega}'}\omega' \wedge \omega,$$

which corresponds to the superposition of the classical $\mathbb{N}$-graded formalism and the Koszul sign rule.

Remark 17. *Deligne super differential forms are an example of a $\mathbb{Z}_2^2$-graded algebra and are one of the motivations to extend Supergeometry to higher gradings [COP12].*

Proposition 27. *The derivation $d_U : \mathcal{O}(U) \to (\Omega^1\mathcal{M})(U)$ extends uniquely as degree 1 even derivation of $((\Omega\mathcal{M})(U), \wedge)$ that squares to 0. In other words, there exists a unique well-defined map $d_U : (\Omega\mathcal{M})(U) \to (\Omega\mathcal{M})(U)$ that has weight 1 with respect to the cohomological degree ($\mathbb{N}$-degree) and weight 0 with respect to the parity ($\mathbb{Z}_2$-degree),* *coincides with $d_U : \mathcal{O}(U) \to (\Omega^1\mathcal{M})(U)$ on $(\Omega^0\mathcal{M})(U) = \mathcal{O}(U)$, verifies $d_U^2 = 0$, and is a graded-super derivation for the wedge product, i.e. $d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^k \omega \wedge d\omega'$, for any super k-form ω and any super form ω' over U.*

Let now $\mathcal{M}$ be a supermanifold of dimension $p|q$. Any k-form $\omega \in (\Omega^k\mathcal{M})(U)$ reads (nonuniquely)

$$\omega = \sum f\, df_1 \wedge \ldots \wedge df_k,$$

where the sum is finite and $f, f_i \in \mathcal{O}(U)$ and where $d = d_U$. Its exterior or de Rham derivative is necessarily given by

$$d_U\omega = \sum df \wedge df_1 \wedge \ldots \wedge df_k,$$

so that – although already proven – uniqueness is obvious. If U is a patch of local coordinates (x, ξ), we recover the fact, see Section 1.5, that a super 2-form locally reads

$$\sum_{i_1<i_2} dx^{i_1}\, dx^{i_2} f_{i_1 i_2}(x,\xi) + \sum_{i,a} dx^i\, d\xi^a g_{ia}(x,\xi) + \sum_{a_1 \le a_2} d\xi^{a_1}\, d\xi^{a_2} h_{a_1 a_2}(x,\xi).$$

Of course, more generally, a local super k-form is of the type

$$\sum_{\underline{I}+|J|=k} dx^I \wedge d\,\xi^J\, f_{IJ}(x,\xi), \tag{4.4}$$

where the sum is over all (strictly) increasing sequences $I_1, \ldots, I_r \in \{1, \ldots, p\}$, $0 \le r \le k$, and all multiindexes $J = (J_1, \ldots, J_q) \in \mathbb{N}^{\times q}$, such that the sum of $\underline{I} := r$ and $|J| := \sum_\ell J_\ell$ is k, where $dx^I = dx^{I_1} \wedge \ldots \wedge dx^{I_r}$, and where $d\xi^J = (d\xi^1)^{J_1} \wedge \ldots \wedge (d\xi^q)^{J_q}$.

Remark 18. *From what has been said it is clear that there exist no top forms: $(\Omega^k\mathcal{M})(U) \neq 0$, for all $k \in \mathbb{N}$.*

The local computation of the de Rham differential is obvious from Equation 4.4 and the properties of $d = d_U$.

The $(\Omega\mathcal{M})(U) = \wedge(\Omega^1\mathcal{M})(U)$, where U runs through the open subsets of M, form a presheaf and even a sheaf over M – the **sheaf $\Omega\mathcal{M}$ of super differential forms** of $\mathcal{M}$, the exterior sheaf $\wedge\,\Omega^1\mathcal{M}$ of the sheaf of super differential 1-forms. The d_U define a derivation $d : \Omega\mathcal{M} \to \Omega\mathcal{M}$ of this sheaf of $\mathcal{O}$-algebras.

4.7 Inner product and Lie derivative

In the following, we denote the parity of a symbol by the same character as the symbol itself.

Definition 47. *Let X be a vector field of a supermanifold $\mathcal{M} = (M, \mathcal{O})$. The* **inner product** *by X is the map $i_X : \Omega\mathcal{M} \to \Omega\mathcal{M}$ of cohomological degree -1 and parity X, which is defined by $i_X f = 0$, for any $f \in \Omega^0\mathcal{M} = \mathcal{O}$, by $i_X\alpha = (-1)^{X\alpha}\alpha(X)$, for any $\alpha \in \Omega^1\mathcal{M}$, and by the graded-super derivation property*

$$i_X(\omega \wedge \omega') = (i_X\omega) \wedge \omega' + (-1)^{-k+X\omega}\omega \wedge (i_X\omega'),$$

for any $\omega \in \Omega^k\mathcal{M}$ and any $\omega' \in \Omega\mathcal{M}$.

In particular,

Corollary 6. *For any $X \in T\mathcal{M}$ and $f \in \mathcal{O}$, we have $i_X(d\,f) = X(f)$.*

Further, it follows from the definition of i_X on 1-forms that $i_X(f\alpha) = (-1)^{Xf} i_X\alpha$. Existence and uniqueness of the extension of the inner product of 1-forms as derivation and super $\mathcal{O}$-module morphism to differential forms of higher degree is due to universality of the wedge product.

The next proposition details the behavior of the inner product with respect to the module structures of vector fields and differential forms.

Proposition 28. *For any $f \in \mathcal{O}$, $X \in T\mathcal{M}$, and $\omega \in \Omega\mathcal{M}$, we have*

$$i_{fX}\omega = f\,i_X\omega$$

and

$$i_X(f\omega) = (-1)^{Xf} f\,i_X\omega.$$

Proof. Since i_{fX} and $f\,i_X$ are two derivations of $\Omega\mathcal{M}$ of degree -1 and parity $f + X$, it suffices to show that they coincide on the generators of $\Omega\mathcal{M}$, i.e. on 0-forms s and on closed 1-forms ds. On s both derivations vanish, and on ds we get

$$i_{fX}(ds) = (fX)(s) = f\,X(s) = f\,i_X(ds).$$

The second part of the proposition is obvious. □

The inner product and the de Rham derivative allow defining the Lie derivative in the direction a vector field.

Definition 48. *Let X be a vector field of a supermanifold $\mathcal{M}$. The Lie derivative in the direction of X is the graded-super derivation L_X of $(\Omega\mathcal{M}, \wedge)$ of degree 0 and parity X, defined as the graded-super commutator of the graded-super derivation d of degree 1 and parity 0 and the graded-super derivation i_X of degree -1 and parity X, i.e.*

$$L_X := [d, i_X] = d \circ i_X - (-1)^{1(-1)+0\,X} i_X \circ d = d \circ i_X + i_X \circ d.$$

Remark 19. *We recall that the graded-super commutator* $[-,-]$ *of graded-super derivations is defined by*

$$[A, B] := A \circ B - (-1)^{ij+ab} B \circ A,$$

for A *a derivation of degree* i *and parity* a *and* B *a derivation of degree* j *and parity* b. *This bracket is a new derivation of degree* $i+j$ *and parity* $a+b$. *It is well-known that* $[-,-]$ *is a graded-super Lie bracket and hence satisfies the graded-super Jacobi identity*

$$[A, [B, C]] = [[A, B], C] + (-1)^{ij+ab}[B, [A, C]]. \tag{4.5}$$

The behavior of the Lie derivative with respect to the module structure of $\Omega\mathcal{M}$ is clear from the fact that the derivative in the direction of a vector field is a derivation for the wedge product. As for the module structure of $T\mathcal{M}$, we have the

Proposition 29. *For* $f \in \mathcal{O}$, $X \in T\mathcal{M}$, *and* $\omega \in \Omega\mathcal{M}$, $L_{fX}\omega = f\, L_X\omega + d\,f \wedge i_X\omega$.

Proof. Simple verification. □

Moreover, the following classical results hold true in the super context:

Proposition 30. *For any vector fields* $X, Y \in T\mathcal{M}$,

1. $[d, L_X] = d \circ L_X - L_X \circ d = 0\,,$

2. $[i_X, i_Y] = i_X \circ i_Y + (-1)^{XY} i_Y \circ i_X = 0\,,$

3. $[i_X, L_Y] = i_X \circ L_Y - (-1)^{XY} L_Y \circ i_X = i_{[X,Y]}\,,$

4. $[L_X, L_Y] = L_X \circ L_Y - (-1)^{XY} L_Y \circ L_X = L_{[X,Y]}\,.$

Proof. The LHS and the RHS of each equation are graded-super derivations of the same degree and the same parity of the algebra of differential forms. Hence, it suffices to prove that they coincide on the generators $f \in \mathcal{O}$ and $d\,f$ of $\Omega\mathcal{M}$.

1. This result is obvious from the definition of the Lie derivative.

2. Since $[i_X, i_Y]$ is a graded-super derivation of degree -2, it vanishes on 0-forms and 1-forms.

3. The brackets $[i_X, L_Y]$ and $i_{[X,Y]}$ are derivations of degree -1 and thus vanish on 0-forms. For $d\,f$, we get

$$\begin{aligned}
[i_X, L_Y]d\,f &= i_X L_Y d\,f - (-1)^{XY} L_Y i_X d\,f \\
&= i_X d\, i_Y d\,f + i_X i_Y d^2 f - (-1)^{XY} d\, i_Y i_X d\,f - (-1)^{XY} i_Y d\, i_X d\,f \\
&= i_X d\, i_Y d\,f - (-1)^{XY} i_Y d\, i_X d\,f \\
&= X(i_Y d\,f) - (-1)^{XY} Y(i_X d\,f) \\
&= XY(f) - (-1)^{XY} YX(f) \\
&= [X, Y](f) \\
&= i_{[X,Y]} d\,f,
\end{aligned}$$

since $d^2 = 0$ and $i_Y i_X d\,f = 0$.

4. In view of the Jacobi identity (4.5) and the results 1. and 3. of this proposition, we obtain

$$[L_X, L_Y] = [[d, i_X], L_Y] = [d, [i_X, L_Y]] - (-1)^{1(-1)+0\cdot X}[i_X, [d, L_Y]] =$$

$$[d, i_{[X,Y]}] = L_{[X,Y]}.$$

□

4.8 Cartan's formula

For any p-form ω and any vector fields $X_0, \ldots, X_p$ of a smooth manifold, Cartan's formula reads

$$\begin{aligned}
d\omega(X_0, \ldots, X_p) = &\sum_{\alpha=0}^{p} (-1)^\alpha L_{X_\alpha}\left(\omega(X_0, \ldots, \widehat{X_\alpha}, \ldots, X_p)\right) \\
&+ \sum_{\alpha<\beta} (-1)^{\alpha+\beta} \omega([X_\alpha, X_\beta], X_0, \ldots, \widehat{X_\alpha}, \ldots, \widehat{X_\beta}, \ldots, X_p),
\end{aligned} \tag{4.6}$$

where $\widehat{.}$ means that the considered term is omitted.

The next proposition is the super counterpart of this result.

Proposition 31. *For any p-superform ω and any super vector fields $(X_0, \ldots, X_p)$,*

$$d\omega(X_0, \ldots, X_p) = \sum_{\alpha=0}^{p} (-1)^{\alpha + X_\alpha(\sum_{\gamma=0}^{\alpha-1} X_\gamma) + X_\alpha \omega} L_{X_\alpha}\left(\omega(X_0, \ldots, \widehat{X_\alpha}, \ldots, X_p)\right)$$

$$+ \sum_{\alpha<\beta} (-1)^{\alpha+\beta+X_\alpha(\sum_{\gamma=0}^{\alpha-1} X_\gamma) + X_\beta(\sum_{\gamma=0}^{\beta-1} X_\gamma) - X_\alpha X_\beta} \omega([X_\alpha, X_\beta], X_0, \ldots, \widehat{X_\alpha}, \ldots, \widehat{X_\beta}, \ldots, X_p).$$

In this formula differential forms are viewed as super alternating $\mathcal{O}$-multilinear forms:

Definition 49. *For any $\omega \in \Omega^p\mathcal{M}$ and any $X_1, \ldots, X_p \in T\mathcal{M}$, we set*

$$\omega(X_1, \ldots, X_p) := (-1)^{\sum_{\ell=1}^{p} X_\ell(\omega+\sum_{\alpha=1}^{\ell-1} X_\alpha)} i_{X_p} \cdots i_{X_1}\omega \in \mathcal{O}.$$

Proposition 32. *The preceding definition allows viewing any $\omega \in \Omega^p\mathcal{M}$ as a super alternating and super $\mathcal{O}$-p-linear form $\omega : T\mathcal{M}^{\times p} \to \mathcal{O}$. More precisely, for any $X_k \in T\mathcal{M}$ and any $f \in \mathcal{O}$,*

1. $\omega(X_1, \ldots, fX_i, \ldots, X_p) = (-1)^{f(\omega+\sum_{\alpha=1}^{i-1} X_\alpha)} f\, \omega(X_1, \ldots, X_i, \ldots, X_p)$,
2. $\omega(X_1, \ldots, X_{i+1}, X_i, \ldots, X_p) = -(-1)^{X_i X_{i+1}} \omega(X_1, \ldots, X_i, X_{i+1}, \ldots, X_p)$.

Proof. Simple verification. □

Proof of Proposition 31. To avoid technical computations, we confine ourselves to the verification of Cartan's formula for 0-forms, which is obvious, and for 1-forms. If $\omega \in \Omega^1\mathcal{M}$ and $X_0, X_1 \in T\mathcal{M}$, we have

$$\begin{aligned} i_{X_0} i_{X_1} d\omega &= i_{X_0}[i_{X_1}, d]\omega - [i_{X_0}, d] i_{X_1}\omega \\ &= i_{X_0} L_{X_1}\omega - L_{X_0} i_{X_1}\omega \\ &= [i_{X_0}, L_{X_1}]\omega + (-1)^{X_0 X_1} L_{X_1} i_{X_0}\omega - L_{X_0} i_{X_1}\omega \\ &= i_{[X_0, X_1]}\omega + (-1)^{X_0 X_1} L_{X_1} i_{X_0}\omega - L_{X_0} i_{X_1}\omega, \end{aligned}$$

which is the Cartan formula for $p = 1$. The general case can be proven by induction. □

Chapter 5

Integral Calculus on Supermanifolds

5.1 Berezinian of a free supermodule

For convenience we reproduce here the last section of Chapter 1.

The Berezinian of a free supermodule is the superversion of the Determinant of a free module. If S is a free module of rank n over a commutative ring R, we set

$$\operatorname{Det} S := \wedge^n S.$$

To any basis (e_i) of S, corresponds a basis $e_1 \wedge \ldots \wedge e_n$ of $\operatorname{Det} S$, such that if (e'_i) is another basis of S with $e'_i = B^k_i e_k$, then

$$e'_1 \wedge \ldots \wedge e'_n = \operatorname{Det} B \; e_1 \wedge \ldots \wedge e_n.$$

When trying to extend this concept to the supercase, we note that the exterior product of odd vectors commutes, so that there is no top exterior power for an odd module. It is easily understood that the Berezinian of a free supermodule $S = S_0 \oplus S_1$ of rank $p|q$ over a supercommutative ring $R = R_0 \oplus R_1$ (in which 2 is invertible) is isomorphic to

$$\operatorname{Ber} S :\simeq \wedge^p S_0 \otimes \vee^q S_1^*.$$

If $(e_1, \ldots, e_{p+q})$ is a standard basis of S and $(\varepsilon^1, \ldots, \varepsilon^{p+q})$ denotes the dual basis, the vector

$$[e_1, \ldots, e_{p+q}] \simeq e_1 \wedge \ldots \wedge e_p \otimes \varepsilon^{p+1} \vee \ldots \vee \varepsilon^{p+q}$$

is a basis of $\operatorname{Ber} S$ (of cohomological degree p). It can be shown that if $(e'_1, \ldots, e'_{p+q})$ is a second basis of S related to the first by $e'_i = e_k B^k_i$, we have

$$[e'_1, \ldots, e'_{p+q}] = [e_1, \ldots, e_{p+q}] \operatorname{Ber} B. \tag{5.1}$$

It is clear that $\operatorname{Ber} S$ is a free super R-module of rank $1|0$, if q is even, and of rank $0|1$, if q is odd. Hence, since a change of basis $B : S \to S$ and its Berezinian $\operatorname{Ber} B : \operatorname{Ber} S \to$

Ber S are even automorphisms, the Berezinian Ber is an endofunctor of the category of free super R-modules of finite rank and corresponding even automorphisms.

For a precise treatment of these questions – even in the $\mathbb{Z}_2^n$-graded situation – we refer the reader to [Cov12].

5.2 Integration over a classical manifold

To integrate over classical manifolds, we use the theory of Radon measures.

Let M be a smooth n-dimensional [Hausdorff and second countable] manifold. Since M is a locally compact (metrizable [in fact this assumption of the theory of Radon measures is not needed for manifolds, as partitions of unity exist in this case]) topological space and also a countable union of compact subspaces, it suffices to define a *Radon measure* on M, i.e. a positive linear form μ of the space $C_c^0(M)$ of compactly supported continuous functions of M. The general theory then allows to extend μ to a bigger space $L^1_\mu(M) \supset C^0_c(M)$. The functions $f \in L^1_\mu(M)$ are said to be integrable over M with respect to the measure μ and their integral is defined by $\int_M f\,\mu := \mu(f)$.

The Euclidean space $\mathbb{R}^n$ admits a canonical measure, the Lebesgue measure, which we denote by $\delta_0 = |dx^1 \wedge \ldots \wedge dx^n|$. The theorem that allows to change coordinates in a Lebesgue integral reads as follows. Let $x = x(y) \rightleftharpoons y = y(x)$ be a diffeomorphism between two open subsets U and V of $\mathbb{R}^n$. We have $f = f(x) \in L^1_{\delta_0}(U)$ if and only if $f = f(x(y))|\det \partial_y x| \in L^1_{\delta_0}(V)$ and

$$\int_U f(x)\,|dx^1 \wedge \ldots \wedge dx^n| = \int_V f(x(y))|\det \partial_y x||dy^1 \wedge \ldots \wedge dy^n|. \tag{5.2}$$

The appropriate objects for integration over a classical manifold are 1-densities. Roughly, 1-densities are differential top forms up to sign. More precisely, a 1-density on the vector space T_mM is a map $\mathfrak{d} : \wedge^n T_mM \setminus \{0\} \to \mathbb{R}$, such that, for any $s \in \mathbb{R}\setminus\{0\}$ and any $\Pi \in \wedge^n T_mM \setminus \{0\}$, we have $\mathfrak{d}(s\Pi) = |s|^\lambda \mathfrak{d}(\Pi)$, with $\lambda = 1$. If λ is an arbitrary real number, $\mathfrak{d}$ is a λ-density of T_mM. It is clear that the set $D_\lambda(T_mM)$ of all λ-densities of T_mM is a real 1-dimensional vector space and that the disjoint union $D_\lambda(M) = \sqcup_m D_\lambda(T_mM)$ is a rank 1 vector bundle over M. Indeed, if $\omega \in \wedge^n T^*_mM$ is a nonzero top linear form of the tangent space, then $|\omega|^\lambda$ is a basis of $D_\lambda(T_mM)$. A λ-*density* field of M is then a smooth section $\delta \in \mathfrak{D}_\lambda(M) := \Gamma(D_\lambda(M))$ of the λ-density bundle. If no confusion is possible it is customary to speak about λ-densities instead of fields of such densities and about densities instead of 1-densities. From what has just been said it is obvious that over a coordinate chart $(U, \phi = (x^1, \ldots, x^n))$ of M a λ-density reads

$$\delta|_U = \delta(x)|dx^1 \wedge \ldots \wedge dx^n|^\lambda,$$

where $\delta = \delta(x)$ is smooth. Observe that if $(V, \psi = (y^1, \ldots, y^n))$ is another coordinate patch and if $\delta|_V = \delta'(y)|dy^1 \wedge \ldots \wedge dy^n|^\lambda$, then, obviously, the component transformation law is

$$\delta'(y) = \delta(x(y))|\det \partial_y x|^\lambda. \tag{5.3}$$

The point with densities is that for $\lambda = 1$ the basis vector $|\omega|$ is a volume element of the tangent space, viewed up to $\mathbb{Z}_2$-action. Whereas on nonorientable manifolds a global top differential form is either not smooth or has to vanish at some point, it is clear that a global smooth nowhere vanishing top differential form up to sign, i.e. a global smooth nevervanishing 1-density field, must exist even for nonorientable manifolds. It follows that the line bundle $\mathfrak{D}_1(M)$ is trivial. Although the following remark is not relevant for integration theory, let us mention that this nevervanishing 1-density, say ρ_0, provides a nevervanishing λ-density ρ_0^λ, which then gives a bijection between functions and λ-densities (however, when reading the Lie derivative of densities through this bijection we [obtain a 1-cocycle, called a divergence, and thus] see that densities can definitely not be identified with functions) [GP04].

Let us now come to the integral over a manifold M associated with a 1-density $\delta \in \mathfrak{D}_1(M)$. We know that it suffices to show that this density defines a positive linear form of $C_c^0(M)$.

Let us first consider a function $f \in C_c^0(M)$ that is compactly supported by a chart domain U with coordinates $\phi = (x^1, \ldots, x^n)$. If $f = f(x)$ is this function read in these coordinates and $\delta(x)|dx^1 \wedge \ldots \wedge dx^n|$ is the coordinate form of the density δ, we define the measure or integral associated to δ by

$$\int_M f\,\delta := \int_{\varphi(U)} f(x)\,\delta(x)|dx^1 \wedge \ldots \wedge dx^n|, \tag{5.4}$$

where the RHS Lebesgue integral makes sense as the integrated function is continuous and compactly supported in $\varphi(U)$.

We then pass to an arbitrary $f \in C_c^0(M)$ by means of a partition of unity $(U_i, \phi_i, \pi_i)_i$ subordinate to the charts of an atlas, i.e. we set

$$\int_M f\,\delta := \sum_i \int_M (\pi_i f)\,\delta. \tag{5.5}$$

It now suffices to prove that this defines a linear positive form (what is obvious) and that the integrals (5.4) and (5.5) only depend on f and neither on the chosen chart, nor on the considered partition. Independence of the partition follows from a well-known computation in integration theory, which we will not repeat here, whereas independence of the chart is readily checked. Indeed, if $(V, \psi = (y^1, \ldots, y^n))$ is another chart that contains the support of f, we may write as well

$$\int_M f\,\delta := \int_{\psi(V)} f'(y)\,\delta'(y)|dy^1 \wedge \ldots \wedge dy^n|. \tag{5.6}$$

However, when executing the coordinate change $x = x(y) \rightleftharpoons y = y(x)$ in the integral (5.4), see (5.2), we find

$$\int_{\psi(V)} f(x(y))\,\delta(x(y))|\det \partial_y x||dy^1 \wedge \ldots \wedge dy^n|,$$

which, in view of transformation law (5.3), coincides with the integral (5.6). This also allows to understand why compactly supported 1-densities can be integrated over any manifold, whereas integrable top differential forms can be defined and integrated only over oriented manifolds – for which there exists an atlas such that all Jacobian determinants are strictly positive. Of course, integration of compactly supported functions with respect to a 1-density can be viewed as a particular case of integration of compactly supported 1-densities.

We finish this section with two remarks that will be used in the following.

The sheaf of 1-densities of a manifold M can be viewed as

$$\mathfrak{D}_{1\ M} = \Omega^n_M \otimes \mathrm{or}_M,$$

where Ω^n_M denotes the sheaf of maximal forms and or_M the orientation sheaf of M. The set $\mathrm{or}_M(M) = \Gamma(M, \mathrm{or}_M)$ of global sections is the disjoint union of the sets made up by the two orientations of the tangent spaces. It is clear that this union is a smooth manifold and even a clothing of M with two leaves. The sheaf of densities is of course a sheaf of $\mathcal{C}^\infty_M$-modules.

Instead of defining a density intrinsically, we could have defined it, for a chosen atlas, as a family of functions $\delta = \delta(x)$, $\delta' = \delta'(y)$... associated with the atlas charts and that transform according to the rule $\delta'(y) = \delta(x(y))|\det \partial_y x|$.

5.3 Integration over a supermanifold

5.3.1 Berezinian sheaf – supervector bundle approach

Integration over supermanifolds is of course tightly connected with the superversions of top differential forms and 1-densities. As mentioned previously, the "sheaf of top forms $\Omega^{\mathrm{top}}_M = \wedge^{\mathrm{top}}\Omega^1_M = \mathrm{Det}\,\Omega^1_M$" has to be replaced by the Berezinian sheaf $\mathrm{Ber}\,\Omega^1_M$, which is defined as follows.

Let us fix an atlas of supercharts of the considered supermanifold $\mathcal{M}$.

Over a domain U with supercoordinates $(x^1, \ldots, \xi^q)$ the $O_M(U)$-module $(\mathrm{Ber}\,\Omega^1_M)(U)$ is given by

$$\Gamma(U, \mathrm{Ber}\,\Omega^1_M) := [dx^1, \ldots, d\xi^q]\, O_M(U) \simeq \left(dx^1 \wedge \ldots \wedge dx^p \otimes \partial_{\xi^1} \vee \ldots \vee \partial_{\xi^q}\right)\, O_M(U). \tag{5.7}$$

We know, see Equation (5.1), that if V is a second patch with supercoordinates $(y^1, \dots, \eta^q)$, then in the overlap $U \cap V$, we have

$$[dy^1, \dots, d\eta^q] = [dx^1, \dots, d\xi^q] \operatorname{Ber} J(x, \xi),$$

where $J(x, \xi)$ is the modified Jacobian matrix of the isomorphism $y = y(x, \xi), \eta = \eta(x, \xi)$. Let us mention here that the modified Jacobian $J(y, \eta)$ of the inverse isomorphism $x = x(y, \eta), \xi = \xi(y, \eta)$ is, as in the classical setting, of course the inverse of $J(x, \xi)$ (so that their Berezinians are inverses of each other as well). In parts of the literature the modified Jacobian matrix is replaced by its supertranspose, which is of course irrelevant, as the two Berezinians coincide.

A global section in $\Gamma(M, \operatorname{Ber} \Omega^1_M)$ is a family of local sections $[dx^1, \dots, d\xi^q]\omega(x, \xi)$,

$$[dy^1, \dots, d\eta^q]\omega'(y, \eta) = [dx^1, \dots, d\xi^q] \operatorname{Ber} J(x, \xi)\omega'(y(x, \xi), \eta(x, \xi)) \dots,$$

over all coordinate charts of the considered atlas, whose components verify the transformation rule $\omega(x, \xi) = \operatorname{Ber} J(x, \xi)\omega'(y(x, \xi), \eta(x, \xi))$ or, equivalently,

$$\omega'(y, \eta) = \operatorname{Ber} J(y, \eta)\omega(x(y, \eta), \xi(y, \eta)). \tag{5.8}$$

As the supervector bundles over a supermanifold $\mathcal{M} = (M, O_M)$ are defined as the locally free sheaves of O_M-modules, the preceding construction of the locally free Berezinian sheaf of O_M-modules (of rank $1|0$ [if q is even] or $0|1$ [if q is odd]) can be viewed, in the geometric language, as the gluing of a superbundle from trivial local line bundles by means of the Berezinian of the differential of the transition isomorphisms.

We finish this section with a remark on Symbol Calculus. Let $U \subset \mathbb{R}^n$ be an open subset and V, W two real finite-dimensional vector spaces. If $D \in \mathcal{D}^k(\mathcal{C}^\infty(U) \otimes V, \mathcal{C}^\infty(U) \otimes W)$ denotes a kth order differential operator, and if $f \in \mathcal{C}^\infty(U)$, $v \in V$, $x \in U$, and $p \in (\mathbb{R}^n)^*$, we write

$$\begin{aligned} D(fv) = \ & \textstyle\sum_{|\alpha| \le k} D_{\alpha, x}(v) \partial_{x^1}^{\alpha^1} \dots \partial_{x^n}^{\alpha^n} f \\ & \simeq \textstyle\sum_{|\alpha| \le k} D_{\alpha, x}(v) p_1^{\alpha^1} \dots p_n^{\alpha^n} \\ & =: D(p; v), \end{aligned}$$

where $D_{\alpha, x} \in \operatorname{Hom}_{\mathbb{R}}(V, W)$ is a linear map that depends smoothly on $x \in U$, p represents the derivatives of D acting on f and v represents the argument fv of D. Eventually, $D(p; v)$ is a degree k polynomial in $p \in (\mathbb{R}^n)^*$ with coefficients in $\mathcal{C}^\infty(U, \operatorname{Hom}_{\mathbb{R}}(V, W))$. A differential operator can thus be replaced by a polynomial – its total symbol. This method leads to a powerful nonstandard computing technique that has important applications especially in Homological Algebra [Pon04]. In particular, it entails that

$$\Gamma(\vee TU) \simeq \operatorname{Pol}(T^* U) \simeq \mathcal{D}(\mathcal{C}^\infty(U)),$$

so that, if $p \in (\mathbb{R}^n)^* \subset \Gamma(T^*U)$, we have for instance

$$\partial_{x^1} \vee \partial_{x^2} \simeq (p \mapsto (1/2)(\partial_{x^1} \vee \partial_{x^2})(p,p) = p_1 p_2) \simeq \partial_{x^1}\partial_{x^2}.$$

It is thus natural to view local Berezinian sections, see (5.7), as

$$\Gamma(U, \mathrm{Ber}\,\Omega^1_M) = [dx^1, \ldots, d\xi^q]\, O_M(U)$$

$$\simeq (dx^1 \wedge \ldots \wedge dx^p \otimes \partial_{\xi^1} \ldots \partial_{\xi^q})\; O_M(U) \simeq (dx^1 \wedge \ldots \wedge dx^p \otimes \partial_{\xi^q} \ldots \partial_{\xi^1})\; O_M(U). \tag{5.9}$$

5.3.2 Sheafs of super differential operators

In view of the description of an intrinsic approach to the Berezinian sheaf, we next detail the construction of differential operators on supermanifolds [GKP10].

Let us recall that a *smooth supermanifold* $\mathcal{M}$ of dimension $p|q$ is a (local) super ringed space $(M, \mathcal{A})$ over a topological space M that is locally isomorphic to $(\mathbb{R}^p, C^{\infty\, p|q})$, where, for any open subset $\mathfrak{U} \subset \mathbb{R}^p$, $C^{\infty\, p|q}(\mathfrak{U}) := C^\infty(\mathfrak{U})[\xi^1, \ldots, \xi^q]$ – the ξ^α being formal anticommuting generators. More precisely, we assume that $\mathcal{A}$ is a sheaf of associative supercommutative $\mathbb{R}$-algebras with unit. The superalgebra $\mathcal{A}(M) = \Gamma(M, \mathcal{A})$ of global sections of $\mathcal{A}$ is the algebra $C^\infty(\mathcal{M})$ of *functions* of the supermanifold $\mathcal{M}$. It is well-known that, due to the local model condition, the locality condition for the stalks is automatically satisfied. Further, the considered data induce a smooth manifold structure of dimension p on M and provide an embedding of the classical manifold M into the supermanifold $\mathcal{M}$.

For any open subset $U \subset M$, we denote by $(\mathrm{Der}\,\mathcal{A})(U)$ the $\mathcal{A}(U)$-module $\mathrm{Der}(\mathcal{A}(U))$ of derivations of the superalgebra $\mathcal{A}(U)$. If $X \in (\mathrm{Der}\,\mathcal{A})(U)$, there is, in view of the localization principle, for any open subset $V \subset U$, a unique derivation $X|_V \in (\mathrm{Der}\,\mathcal{A})(V)$ such that $(Xf)|_V = X|_V f|_V$, for all $f \in \mathcal{A}(U)$. The assignment $U \to (\mathrm{Der}\,\mathcal{A})(U)$ is actually a locally free sheaf of $\mathcal{A}$-modules, called the derivation sheaf $\mathrm{Der}\,\mathcal{A}$ of the structure sheaf $\mathcal{A}$, or, also, the tangent sheaf $T\mathcal{M}$ of the supermanifold $\mathcal{M}$. The module $(T\mathcal{M})(M)$ of global sections of the super vector bundle $T\mathcal{M}$ is the $C^\infty(\mathcal{M})$-module $\mathcal{X}(\mathcal{M})$ of *vector fields* of $\mathcal{M}$ – which carries an obvious Lie superalgebra structure.

In the following we denote by $\underline{\mathrm{End}}(\mathcal{A}(U))$ the $\mathcal{A}(U)$-module of even and odd $\mathbb{R}$-linear maps from $\mathcal{A}(U)$ to itself. The $\mathcal{A}(U)$-module of k-th order differential operators $\mathcal{D}^k(U)$, $k \in \mathbb{N}$, is then defined inductively by

$$\mathcal{D}^k(U) := \{D \in \underline{\mathrm{End}}(\mathcal{A}(U)) : [D, \mathcal{A}(U)] \subset \mathcal{D}^{k-1}(U)\},$$

where $[-,-]$ is the supercommutator and where $\mathcal{D}^{-1}(U) = \{0\}$.

Of course, $\mathcal{D}^0(U) = \mathcal{A}(U)$, and thus 0-order operators are local. This entails by induction that any super differential operator is local. Indeed, if $D \in \mathcal{D}^k(U)$, if the restriction $f|_V$ of $f \in \mathcal{A}(U)$ to an open $V \subset U$ vanishes, and if $v \in V$, let $\gamma \in \mathcal{A}_0(U)$ be a super bump function with support $\operatorname{supp}\gamma \subset V$ (in the supercontext the support can be defined as usual as the complement in U of the set of those points $u \in U$ for which the restriction of γ to some neighborhood of u vanishes) and restriction $\gamma|_W = 1$, for some neighborhood $W \subset V$ of v, see localization principle [Lei80, Corollary 3.1.8]. It then follows from the defining property of differential operators applied to $[D,\gamma]f$, the induction assumption, and the fact $\gamma f = 0$, that$(Df)|_W = 0$. We can now show that there exists, just as in the case of vector fields, for any $D \in \mathcal{D}^k(U)$ and any open $V \subset U$, a unique $D|_V \in \mathcal{D}^k(V)$ such that $(Df)|_V = D|_V f|_V$, for all $f \in \mathcal{A}(U)$. Indeed, if $f \in \mathcal{A}(V)$ and $v \in V$, it is possible to choose a function $F \in \mathcal{A}(U)$ (of the same parity as f) such that $\operatorname{supp} F \subset \operatorname{supp} f$ and $F|_W = f|_W$, for some neighborhood $W \subset V$ of v. Locality entails that $(DF)|_W \in \mathcal{A}(W)$ and $(DF')|_{W'} \in \mathcal{A}(W')$, defined for two points $v, v' \in V$, depend only on f and coincide in the intersection $W \cap W'$. Thus these local functions define a unique global function $D|_V f \in \mathcal{A}(V)$ such that

$$(D|_V f)|_W = (DF)|_W.$$

Since, obviously, $D|_V \in \underline{\mathrm{End}}(\mathcal{A}(V))$ (note that $D|_V$ has the same parity as D), it suffices – to prove the above claim – to observe that, for any $f_1, \ldots, f_{k+1} \in \mathcal{A}(V)$, we have

$$[\ldots[[D|_V, f_1], f_2], \ldots, f_{k+1}]|_W = [\ldots[[D, F_1], F_2], \ldots, F_{k+1}]|_W = 0,$$

with self-explaining notations.

In view of the just detailed restrictions of differential operators, the assignment $U \to \mathcal{D}^k(U)$ is a presheaf and obviously also a sheaf – as $\mathcal{A}$ is a sheaf.

Proposition 33. *For any $k \in \mathbb{N}$, the presheaf $\mathcal{D}^k$ of k-th order super differential operators over the base manifold M of a smooth supermanifold $\mathcal{M} = (M, \mathcal{A})$ of dimension $p|q$, is a locally free sheaf of $\mathcal{A}$-modules, with local basis*

$$\partial_x^\alpha \partial_\xi^\beta := \partial_{x^1}^{\alpha^1} \ldots \partial_{x^p}^{\alpha^p} \partial_{\xi^q}^{\beta^q} \ldots \partial_{\xi^1}^{\beta^1},$$

where (x, ξ) are local coordinates, $\beta_a \in \{0, 1\}$, and $|\alpha| + |\beta| \leq k$.

Proof. The method used to prove local freeness of the sheaf of vector fields goes through in the case of differential operators. Let us give some details because of the increased technicity.

If $M = U^{p|q}$ is a superdomain, if $D \in \mathcal{D}^k(U)$ is of the type

$$\sum_{i=0}^k D^i = \sum_{i=0}^k \sum_{|\alpha|+|\beta|=i} D^i_{\alpha\beta}(x,\xi)\, \partial_x^\alpha \partial_\xi^\beta \in \mathcal{D}^k(U), \tag{5.10}$$

and if $m_{\alpha\beta} = (1/\alpha!)x^\alpha\xi^\beta$, where the odd coordinates are written in increasing order, then necessarily

$$D^i_{\alpha\beta} = D^i m_{\alpha\beta} = D m_{\alpha\beta} - \sum_{j=0}^{i-1} D^j m_{\alpha\beta}, \tag{5.11}$$

and an induction on i immediately shows that the coefficients $D^i_{\alpha\beta}$, if they exist, are unique.

Take now an arbitrary $D \in \mathcal{D}^k(U)$ and set $\Delta = D - \sum \in \mathcal{D}^k(U)$, where $\sum$ denotes the RHS of (5.10) with the coefficients defined in (5.11). This operator Δ vanishes by construction on the polynomials of degree $\leq k$ in x, ξ.

For any $f_1, \ldots, f_{\ell-1}, h \in \mathcal{A}(U)$, $\ell \geq k+1$, we have

$$\Delta(f_1 \ldots f_{\ell-1} h) = \sum_{b=1}^{\ell-1} \sum \pm f_{i_1} \ldots f_{i_b} \Delta(f_{i_{b+1}} \ldots f_{i_{\ell-1}} h) + F(h), \tag{5.12}$$

as immediately seen when developing $F(h) := [\ldots[[\Delta, f_1], f_2], \ldots, f_{\ell-1}]h$. If $\ell > k+1$, the term $F(h)$ vanishes, whereas in the case $\ell = k+1$ it is given by $F(h) = F(1)h$. Equation (5.12) shows that $\Delta = 0$ on any polynomial of degree $k+1$, then, by induction, that $\Delta = 0$ on an arbitrary polynomial in x, ξ. Further, this equation entails that $\Delta \mathcal{I}_m^{k+c} \subset \mathcal{I}_m^c$, $m \in U$, $c \geq 1$, where $\mathcal{I}_m$ is the unique homogeneous maximal ideal of the stalk $\mathcal{A}_m$. However, in view of Hadamard's lemma, we can, for any $f \in \mathcal{A}(U)$ and any $m \in U$, find a polynomial $P_{f,m}$ in x, ξ such that $f - P_{f,m} \in \mathcal{I}_m^{k+q+1}$. It follows that $\Delta f = \Delta(f - P_{f,m}) \in \mathcal{I}_m^{q+1}$, for all $m \in U$, so that $\Delta f = 0$. $\square$ $\square$

The super $\mathbb{R}$-vector space $\underline{\mathrm{End}}(\mathcal{A}(U))$ carries natural associative and Lie superalgebra structures $\circ$ and $[-,-]$ (we often omit the symbol $\circ$). An induction on $k+\ell$ allows seeing that $\mathcal{D}^k(U) \circ \mathcal{D}^\ell(U) \subset \mathcal{D}^{k+\ell}(U)$ and $[\mathcal{D}^k(U), \mathcal{D}^\ell(U)] \subset \mathcal{D}^{k+\ell-1}(U)$, so that the super vector space $\mathcal{D}(U) := \cup_{k\in\mathbb{N}} \mathcal{D}^k(U)$ of all differential operators inherits associative and Lie superalgebra structures that have weight 0 and -1, respectively, with respect to the filtration degree. It is easily checked that $\mathcal{D} : U \to \mathcal{D}(U)$ (resp. $\mathcal{D}^1 : U \to \mathcal{D}^1(U)$) is a locally free sheaf of $\mathcal{A}$-modules and associative and Lie superalgebras (resp. of $\mathcal{A}$-modules and sub Lie superalgebras) over M. The algebra $\mathcal{D}(M)$ (resp. $\mathcal{D}^1(M)$) is the super Lie algebra of *differential operators* (resp. *first order differential operators*) of the supermanifold $\mathcal{M}$. In the sequel we denote this algebra also by $\mathcal{D}(\mathcal{M})$ or even by $\mathcal{D}$ (resp. by $\mathcal{D}^1(\mathcal{M})$ or $\mathcal{D}^1$).

5.3.3 Density bundle

We now construct the bundle $\mathfrak{D}_{1\ M} = \mathrm{Ber}\,\Omega^1_M \otimes \mathrm{or}_M$ of 1-densities of a supermanifold $\mathcal{M} = (M, O_M)$ as the Berezinian sheaf twisted by the orientation sheaf of the body.

Fix again an atlas of supercharts of $\mathcal{M}$.

Over a domain U with supercoordinates $(x^1, \ldots, \xi^q)$ the $O_M(U)$-module $\mathfrak{D}_{1\,M}(U)$ is given by

$$\Gamma(U, \mathfrak{D}_{1\,M}) := |[dx^1, \ldots, d\xi^q]|\, O_M(U) \simeq \left(|dx^1 \wedge \ldots \wedge dx^p| \otimes \partial_{\xi^q} \ldots \partial_{\xi^1}\right) O_M(U).$$

If V is a domain with coordinates $(y^1, \ldots, \eta^q)$, then

$$|[dy^1, \ldots, d\eta^q]| = |[dx^1, \ldots, d\xi^q]|\, \text{sign det}\left(\partial_x y|_{(x,0)}\right) \text{Ber}\, J(x, \xi)$$

in the overlap $U \cap V$.

A global section in $\mathfrak{D}_{1\,M}$ is then a family of local sections $|[dx^1, \ldots, d\xi^q]|\delta(x, \xi)$,

$$|[dy^1, \ldots, d\eta^q]|\delta'(y, \eta) =$$

$$|[dx^1, \ldots, d\xi^q]|\, \text{sign det}\left(\partial_x y|_{(x,0)}\right) \text{Ber}\, J(x, \xi)\delta'(y(x, \xi), \eta(x, \xi)) \ldots,$$

over all coordinate charts of the considered atlas, whose components verify the transformation rule

$$\delta'(y, \eta) = \text{sign det}\left(\partial_y x|_{(y,0)}\right) \text{Ber}\, J(y, \eta)\delta(x(y, \eta), \xi(y, \eta)). \tag{5.13}$$

Let us mention that a section of the density bundle $\mathfrak{D}_{1\,M}$ of $\mathcal{M}$ is said to be nondegenerate if its local forms have invertible components $\delta(x, \xi)$. Just as the 1-density line bundle of a classical manifold has always a nevervanishing section, there exists a nondegenerate global section of the 1-density line bundle of a supermanifold. In the literature the sections of $\mathfrak{D}_{1\,M}$ are sometimes referred to as nonoriented Berezinian sections.

5.3.4 Integration over a supermanifold

Integration over supermanifolds consists locally of integration with respect to even and odd variables. The theory is mainly due to Felix Berezin. The "defect" of the Berezinian integral is its confinement to compactly supported objects.

Let $\delta \in \Gamma(M, \mathfrak{D}_{1\,M})$ be a compactly supported global 1-density, i.e. a global density whose components $\delta(x, \xi) = \sum_I \delta_I(x)\xi^I$ are compactly supported, which means that they have compactly supported coefficients $\delta_I(x)$.

As in the classical setting, we first assume that δ is compactly supported in a superchart domain. Let $(U, \varphi = (x, \xi))$ and $(V, \psi = (y, \eta))$ be two charts such that $\text{supp}\, \delta_I(x) \subset U$, $\text{supp}\, \delta'_I(y) \subset V$, for all I. Of course, we write

$$\delta|_U = |[dx^1, \ldots, d\xi^q]|\delta(x, \xi) \simeq \left(|dx^1 \wedge \ldots \wedge dx^p| \otimes \partial_{\xi^q} \ldots \partial_{\xi^1}\right) \delta(x, \xi)$$

and

$$\delta|_V = |[dy^1, \ldots, d\eta^q]|\delta'(y,\eta) \simeq \left(|dy^1 \wedge \ldots \wedge dy^p| \otimes \partial_{\eta^q} \ldots \partial_{\eta^1}\right) \delta'(y,\eta).$$

It seems then quite natural to define the integral over $\mathcal{M}$ of δ by

$$\begin{gathered}\textstyle\int_{\mathcal{M}} \delta = \int_{\mathcal{U}^{p|q}} |[dx^1, \ldots, d\xi^q]|\delta(x,\xi) := \\ \textstyle\int_{\mathcal{U}^{p|q}} |dx^1 \wedge \ldots \wedge dx^p| \left(\partial_{\xi^q} \ldots \partial_{\xi^1} \delta(x,\xi)\right) = \int_{\varphi(U)} |dx^1 \wedge \ldots \wedge dx^p| \delta_{1\ldots q}(x) \in \mathbb{R},\end{gathered} \tag{5.14}$$

where $\delta_{1\ldots q}(x)$ denotes the coefficient of $\xi^1 \ldots \xi^q$ in $\delta(x,\xi)$ and where the RHS Lebesgue integral makes sense as $\delta_{1\ldots q}(x) \in \mathcal{C}^\infty(\varphi(U))$ has a compact support in $\varphi(U)$. This integral is independent of the considered coordinates, if

$$\begin{gathered}\textstyle\int_{\mathcal{M}} \delta = \int_{\mathcal{V}^{p|q}} |[dy^1, \ldots, d\eta^q]|\delta'(y,\eta) := \\ \textstyle\int_{\mathcal{V}^{p|q}} |dy^1 \wedge \ldots \wedge dy^p| \left(\partial_{\eta^q} \ldots \partial_{\eta^1} \delta'(y,\eta)\right) = \int_{\psi(V)} |dy^1 \wedge \ldots \wedge dy^p| \delta'_{1\ldots q}(y) \in \mathbb{R}\end{gathered} \tag{5.15}$$

leads to the same result.

To continue we need the change of variables formula for the just defined Berezinian integral. We only state the result here – its proof can be found in works of Leites, Berezin, Pakhomov, Voronov..., see e.g. [DSB03], Berezinian integral). Since this proof relies substantially on integration by parts, we understand that the requirement that *all* the coefficients be compactly supported is really needed. For the preceding Berezinian definition of the integral over a superdomain of a compactly supported superfunction, we get the

Theorem 6. *Let $x = x(y,\eta), \xi = \xi(y,\eta) \rightleftharpoons y = y(x,\xi), \eta = \eta(x,\xi)$ be an isomorphism of superdomains $\mathcal{U}^{p|q}$ and $\mathcal{V}^{p|q}$ and let $\delta(x,\xi)$ be a compactly supported superfunction of $\mathcal{U}^{p|q}$. Then,*

$$\begin{gathered}\int_{\mathcal{U}^{p|q}} |[dx^1, \ldots, d\xi^q]|\delta(x,\xi) = \\ \int_{\mathcal{V}^{p|q}} |[dy^1, \ldots, d\eta^q]| \operatorname{sign} \det \left(\partial_y x|_{(y,0)}\right) \operatorname{Ber} J(y,\eta)\delta(x(y,\eta),\xi(y,\eta)).\end{gathered} \tag{5.16}$$

When combining this result (valid for the Berezinian definition of the superintegral) with the transformation law (5.13) (valid for global 1-densities), we get

$$\int_{\mathcal{U}^{p|q}} |[dx^1, \ldots, d\xi^q]|\delta(x,\xi) = \int_{\mathcal{V}^{p|q}} |[dy^1, \ldots, d\eta^q]|\delta'(y,\eta),$$

so that the integral $\int_{\mathcal{M}} \delta$ is actually well-defined.

A partition of unity $(U_i, \varphi_i, \pi_i)_i$ subordinate to a coordinate cover allows to define the integral over a supermanifold $\mathcal{M}$ of an arbitrary compactly supported density δ:

$$\int_{\mathcal{M}} \delta := \sum_i \int_{\mathcal{M}} \pi_i \delta. \tag{5.17}$$

As for classical manifolds, this definition is independent of the choice of the partition.

Clearly, the *integration over supermanifolds* $\mathcal{M} = (M, O_M)$ *of compactly supported densities* (or nonoriented Berezinian sections) contains the *integration over* $\mathcal{M}$ *with respect to a fixed density* δ *of compactly supported superfunctions* $s \in O_M(M)$. Indeed, the product $s\delta$ is then a compactly supported density and it suffices to integrate that density:

$$\int_{\mathcal{M}} s\,\delta := \int_{\mathcal{M}} (s\delta). \tag{5.18}$$

Further, if $\mathcal{M}$ is orientable and *oriented*, i.e. if the body M is orientable and oriented by the choice of a classical volume form $\Omega \in \Omega^p(M)$, *we can integrate over* $(\mathcal{M}, \Omega)$ *compactly supported (oriented) Berezinian sections*. Indeed, if $\omega \in \Gamma(M, \mathrm{Ber}\,\Omega^1_M)$ is compactly supported, define $|\omega|$ over any coordinate chart $(U, (x, \xi))$ by

$$|\omega||_U := |[dx^1, \ldots, d\xi^q]|(\pm 1)_U \omega(x, \xi), \tag{5.19}$$

where $(\pm 1)_U$ is $+1$ if the form $(dx^1 \wedge \ldots \wedge dx^p)_m \in \wedge^p T^*_m M \setminus \{0\}$ belongs to the same orientation (connected component) as $\Omega_m \in \wedge^p T^*_m M \setminus \{0\}$, whereas it is -1 otherwise. Since, if $(V, (y, \eta))$ is another chart and $x = x(y, \eta), \xi = \xi(y, \eta) \rightleftharpoons y = y(x, \xi), \eta = \eta(x, \xi)$ denotes the transition isomorphism, we have

$$(\pm 1)_U \text{ sign det} \left(\partial_y x|_{(y,0)}\right) = (\pm 1)_V,$$

the global section $|\omega|$ verifies the transition law (5.13) and is thus a compactly supported density. We then define the integral of ω over $\mathcal{M}$ oriented by Ω as

$$\int_{(\mathcal{M}, \Omega)} \omega := \int_{\mathcal{M}} |\omega|. \tag{5.20}$$

5.3.5 Berezinian sheaf – intrinsic approach

Let as usual $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold of dimension $p|q$. The topological space M then carries a natural structure of smooth manifold, the smooth functions of M being the global sections of

$$\mathcal{C}^\infty_M = \mathcal{O}/\mathcal{J},$$

the quotient sheaf of the structure sheaf $\mathcal{O}$ by the subsheaf $\mathcal{J}$ of nilpotent sections of $\mathcal{O}$.

The projection

$$\varepsilon : \mathcal{C}^\infty_{\mathcal{M}} = \mathcal{O} \ \to \ \mathcal{O}/_{\mathcal{J}} = \mathcal{C}^\infty_M$$

is the evaluation map of sections of $\mathcal{O}$, which is defined locally, in a chart domain U with supercoordinates (x, ξ), as the map that sends a section $s \in \mathcal{O}(U)$ to the term of degree 0 of its local description, *i.e.*

$$\varepsilon \left(\sum_{0 \leq |I| \leq q} s_I(x) \xi^I \right) = s_0(x).$$

Since the underlying space has a smooth manifold structure, we can consider the sheaf of differential forms Ω_M over it, with the classical de Rham differential d_{dR}. The projection ε then admits an extension to the sheaf $\Omega_{\mathcal{M}}$ of super differential forms – with super de Rham differential d – that commutes with the differentials d and d_{dR}. More precisely, there is a sheaf morphism of parity 0 and degree 0

$$\tilde{} : \ \Omega_{\mathcal{M}} \ \to \ \Omega_M \ ,$$

such that, for any U open in M and any $\omega \in \Omega_{\mathcal{M}}(U)$,

$$\widetilde{d\,\omega} = d_{\mathrm{dR}}\, \widetilde{\omega} \ . \tag{5.21}$$

If U is a domain of supercoordinates (x, ξ), this property forces in particular the images of the super differential forms dx^i and $d\xi^a$ to be

$$\begin{aligned} \widetilde{d\,x^i} = & \ \ d_{\mathrm{dR}}\, x^i \ , \qquad 1 \leq i \leq p \ , \\ \widetilde{d\,\xi^a} = & \quad\ \ 0 \ , \qquad\quad 1 \leq a \leq q \ . \end{aligned}$$

Since the sheaf morphism $\tilde{}$ is made up by a family of algebra morphisms, the latter result implies that

$$\left(d\,x^I \wedge d\,\xi^J\right)^{\sim} = \begin{cases} 0 & , \text{ if } |J| \neq 0, \\ d_{\mathrm{dR}} x^I & , \text{ otherwise.} \end{cases}$$

Let now $\mathcal{D}^q_{\mathcal{M}}$ be the sheaf of super differential operators of order q of the $p|q$-dimensional supermanifold $\mathcal{M}$ – which is a sheaf of left and right $\mathcal{O}$-modules, where the right $\mathcal{O}$-module structure is, for $D \in \mathcal{D}^q_{\mathcal{M}}$ and $s, t \in \mathcal{O}$, given by $(D \cdot s)(t) = D(st)$. Note that this right and the natural left module structures are compatible, *i.e.* that $(s \cdot D) \cdot t = s \cdot (D \cdot t)$. Consider further the sheaf $\Omega^p_{\mathcal{M}}$ of differential superforms of degree p – a sheaf of right $\mathcal{O}$-modules, where the right super $\mathcal{O}$-module structure is induced by the canonical left one. We then associate to each open subset $U \subset M$ the right $\mathcal{O}$-module

$$\left(\Omega^p_{\mathcal{M}} \otimes \mathcal{D}^q_{\mathcal{M}}\right)(U) := \Omega^p_{\mathcal{M}}(U) \otimes \mathcal{D}^q_{\mathcal{M}}(U).$$

Set moreover

$$\mathcal{S}(U) := \left\{ \mathrm{D} \in \Omega^p_{\mathcal{M}}(U) \otimes \mathcal{D}^q_{\mathcal{M}}(U) : \forall s \in \mathcal{O}_c(U),\ \exists \omega \in \Omega^{p-1}_M(U) : (\mathrm{D}s)\tilde{} = d_{\mathrm{dR}}\omega \right\},$$

where $\mathcal{O}_c(U)$ denotes the sections of $\mathcal{O}$ over U with compact support. The right module structure of the tensor product, which is implemented by that of differential operators, induces a right module structure on $\mathcal{S}(U)$. Indeed, if $t \in \mathcal{O}(U)$ and $s \in \mathcal{O}_c(U)$, we have

$$((\mathrm{D} \cdot t)(s))\tilde{} = (\mathrm{D}(ts))\tilde{} = d_{\mathrm{dR}}\omega,$$

where $\omega \in \Omega^{p-1}_M(U)$. It follows that $\mathcal{S}(U)$ is a right submodule of $\Omega^p_{\mathcal{M}}(U) \otimes \mathcal{D}^q_{\mathcal{M}}(U)$, so that we can construct the quotient right module.

In fact, we have to take the sheaf $\mathcal{D}^q_{\mathcal{M}}(\mathcal{O}, \Omega^p)$ associated to the presheaf $\Omega^p_{\mathcal{M}} \otimes \mathcal{D}^q_{\mathcal{M}}$. Then $\mathcal{S}$ is to be defined as a subsheaf and the quotient is actually the quotient sheaf.

Definition 50. *The* **Berezinian sheaf** *is the quotient sheaf*

$$\mathrm{Ber}_{\mathcal{M}} := \mathcal{D}^q_{\mathcal{M}}(\mathcal{O}, \Omega^p)/_{\mathcal{S}}\,.$$

Proposition 34. *The Berezinian sheaf* $\mathrm{Ber}_{\mathcal{M}}$ *of a* $p|q$*-dimensional supermanifold* $\mathcal{M}$ *is a locally free sheaf of* $\mathcal{O}$*-modules of rank* $1|0$ *if* q *is even and of rank* $0|1$ *if* q *is odd. Over a coordinate superchart* $(U, (x, \xi))$*, its basis is*

$$\left[dx^1 \wedge \ldots \wedge dx^p \otimes \partial_{\xi^q} \ldots \partial_{\xi^1} \right],$$

where the bracket denotes the class modulo $\mathcal{S}(U)$*.*

Note first that it follows from this proposition that the intrinsically defined Berezinian sheaf coincides with the Berezinian sheaf glued from trival local line bundles.

Proof. Roughly, the sheaf associated with a presheaf consists of sections that locally coincide with sections of the presheaf. Therefore, we may in the following be a bit sketchy and avoid all complications due to sheafification.

Over a superchart $(U, (x, \xi))$, the right $\mathcal{O}(U)$-module $\mathcal{D}^q_{\mathcal{M}}(\mathcal{O}, \Omega^p)(U)$ admits the basis

$$dx^I \wedge d\xi^J \otimes \partial^K_x \partial^L_\xi, \tag{5.22}$$

where $I \in \{1, \ldots, p\}^{\times k}$ is a sequence $i_1 < \ldots < i_k$ and $J \in \{1, \ldots, q\}^{\times \ell}$ a sequence $j_1 \leq \ldots \leq j_\ell$, with $k + \ell = p$. Moreover, $K \in \mathbb{N}^{\times p}$ and $L \in \{0,1\}^{\times q}$, such that $|K| + |L| = q$. The derivatives in ∂^L_ξ are written in decreasing order. For any basis element and any

$$s = \sum_{0 \leq |I| \leq q} s_I(x) \xi^I \in \mathcal{O}_c(U),$$

$$\big(d\,x^I \wedge d\,\xi^J \otimes \partial_x^K \partial_\xi^L\, s\big)^{\sim} = 0 \in \mathcal{S}(U),$$

unless $|J| = 0$. Hence, we now consider

$$\big(dx^1 \wedge \ldots \wedge dx^p \otimes \partial_x^K \partial_\xi^L\, s\big)^{\sim} = d_{\mathrm{dR}}x^1 \wedge \ldots \wedge d_{\mathrm{dR}}x^p\, \big(\partial_x^K \partial_\xi^L\, s\big)^{\sim} = d_{\mathrm{dR}}x^1 \wedge \ldots \wedge d_{\mathrm{dR}}x^p\; \partial_x^K s_L$$

and remark that the RHS of the last equation is an exact classical differential p-form on U, if $|K| > 0$, so that in this case we deal again with an element of $\mathcal{S}(U)$. This shows that $dx^1 \wedge \ldots \wedge dx^p \otimes \partial_{\xi^q} \ldots \partial_{\xi^1}$ is the unique basis element of $\mathcal{D}^q_{\mathcal{M}}(\mathcal{O}, \Omega^p)(U)$ whose equivalence class does not vanish. Hence, the result. $\square$

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